Controlled Singular Volterra Integral Equations and Pontryagin Maximum Principle *

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Abstract. This paper is concerned with a class of controlled singular Volterra integral equations, which could be used to describe problems involving memories. The well-known fractional order ordinary differential equations of the Riemann–Liouville or Caputo types are strictly special cases of the equations studied in this paper. Well-posedness and some regularity results in proper spaces are established for such kind of questions. For the associated optimal control problem, by using a Liapounoff's type theorem and the spike variation technique, we establish a Pontryagin's type maximum principle for optimal controls. Different from the existing literature, our method enables us to deal with the problem without assuming regularity conditions on the controls, the convexity condition on the control domain, and some additional unnecessary conditions on the nonlinear terms of the integral equation and the cost functional.

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1 Introduction.

In this paper, we consider the following controlled Volterra integral equation:

$$y(t)=\eta(t)+\int_0^t f(t,s,y(s),u(s))ds,\qquad t\in[0,T]. \tag{1.1}$$

We call the above the *state equation*, where $\eta(\cdot)$ and $f(\cdot,\cdot,\cdot,\cdot)$ are given maps, called the *free term* and the *generator* of the state equation, respectively, $y(\cdot)$ is called the *state trajectory* taking values in the Euclidean space $\mathbb{R}^n$, and $u(\cdot)$ is called the *control* taking values in some separable metric space U. To measure the performance of the control, we introduce the cost functional

$$J(u(\cdot))=\int_0^T g(t,y(t),u(t))dt+\sum_{j=1}^m h^j(y(t_j)), \tag{1.2}$$

with the two terms on the right hand representing the *running cost* and the *specific instant costs* (at $0\leqslant t_1<t_2<\cdots<t_m\leqslant T$), respectively.

Equations like (1.1) can be used to describe some dynamics involving memories. In the classical situations of optimal control for Volterra integral equations, people usually assume that the map $f(\cdot,\cdot,\cdot,\cdot)$ is continuous, together with some further smoothness/differentiability conditions. Relevant works can be traced back to those by Vinokurov in the later 1960s [45], followed by the works of Angell [4], Kamien-Muller [27], Medhin [33], Carlson [15], Burnap-Kazemi [12], and some recent works by de la Vega [19], Belbas [6, 7], and Bonnans–de la Vega–Dupuis [9]. On the other hand, in the past several decades, fractional differential equations have attracted quite a few researchers' attention due to some very interesting applications in physics,

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chemistry, engineering, population dynamics, finance and other sciences; See Oldham–Spanier [37] for some early examples of diffusion processes, Torvik–Bagley [44], Caputo [13], and Caputo–Mainardi [14] for modeling of the mechanical properties of materials, Benson [8] for the advection and the dispersion of solutes in natural porous or fractured media, Chern [16], Diethelm–Freed [22] for the modeling behavior of viscoelastic and viscoplastic materials under external influences, Scalas–Gorenflo–Mainardi [41] for the mathematical models in finance, Das–Gupta [18], Demirci–Unal–Özalp [20], Arafa–Rida–Khalil [5], Diethelm [21] for some population and epidemic models, and Metzler et al. [34] for the relaxation in filled polymer networks. An extensive survey on fractional differential equations can be found in the book by Kilbas–Srivastava–Trujillo [31]. In the recent years, optimal control problems have been studied for fractional differential equations by a number of authors. We mention the works of Agrawal [1, 2], Agrawal–Defterli–Baleanu [3], Bourdin [11], Frederico–Torres [23], Hasan–Tangpong–Agrawal [25] and Kamocki [28, 29].

The most popular fractional differential equations are those in the sense of Riemann–Liouville or in the sense of Caputo (See Section 3 for some details). It turns out that these equations (of the order no more than 1, for scalar functions) are equivalent to Volterra integral equations with the generator being singular along $s = t$, and the free term $\eta(\cdot)$ being possibly discontinuous (blowing up) at $t = 0$. More precisely, the corresponding controlled state equation of form (1.1) could have the feature that

$$\eta(t) = \frac{c}{t^{1-\alpha}} \text{ (or } c), \qquad f(t,s,y,u) = \frac{\widetilde{f}(s,y,u)}{(t-s)^{1-\alpha}}, \qquad 0 \leqslant s < t \leqslant T, \quad \forall (y,u), \tag{1.3}$$

for some map $\widetilde{f}(\cdot\,,\cdot\,,\cdot)$ and constants $\alpha \in (0,1)$, $c \in \mathbb{R}$. Such kind of singularity makes the optimal control problems for fractional differential equations different from the classical optimal control problems for Volterra integral equations as in the above-mentioned literature.

The purpose of this paper is to study an optimal control problem with the state equation (1.1) allowing $(t,s) \mapsto f(t,s,y,u)$ to have some singularity along $t = s$ and allowing the free term $\eta(\cdot)$ to be (unboundedly) discontinuous. We point out that our state equation (1.1) could cover a much wider class of dynamic systems with various type memories than the ones described by fractional differential equations (with the conditions like (1.3)). Let us make a little more comments on our state equation. Since the free term $\eta(\cdot)$ is allowed to have some singularities, a natural class for $\eta(\cdot)$ should be L^p functions. Then we expect, under suitable conditions, the state trajectory $y(\cdot)$ will also be a function in the same class. On the other hand, in the cost functional, we need $y(t_j)$ to be defined. Therefore, we need to have certain continuity of the state trajectory. Then it is necessary to narrow the L^p space by adding certain continuity. This will lead to some difficulties in establishing the well-posedness of the state equation in the correct class of functions that the state trajectories will belong to. To overcome the difficulty, we introduce certain weighted function spaces, and extend some classical results, such as Gronwall's inequality, etc. to the form that will make our procedure works.

The rest of the paper is organized as follows. In Section 2, necessary preliminaries will be presented. Some results are interesting by themselves. Well-posedness of the state equation, together with the continuity of the solutions, will be established in Section 3. Section 4 is devoted to a proof of Pontryagin's type maximum principle for our optimal control problem of singular integral equations. As a special case, the maximum principles for fractional differential equations in the sense of Riemann–Liouville, and Caputo, will be briefly described. Some concluding remarks will be collected in Section 5.

2 Preliminary

In this section, we will present some preliminary results which will be useful later. First of all, let $T > 0$ be a fixed time horizon. We introduce the following spaces:

$$\begin{aligned}
&L^p(0,T;\mathbb{R}^n) = \Big\{\varphi : [0,T] \to \mathbb{R}^n \bigm| \|\varphi(\cdot)\|_{L^p(0,T;\mathbb{R}^n)} \equiv \Big(\int_0^T |\varphi(t)|^p dt\Big)^{\frac{1}{p}} < \infty\Big\}, \quad 1 \leqslant p < \infty,\\
&L^\infty(0,T;\mathbb{R}^n) = \Big\{\varphi : [0,T] \to \mathbb{R}^n \bigm| \|\varphi(\cdot)\|_{L^\infty(0,T;\mathbb{R}^n)} \equiv \operatorname*{esssup}_{t\in[0,T]} |\varphi(t)| < \infty\Big\},\\
&C([0,T];\mathbb{R}^n) = \Big\{\varphi : [0,T] \to \mathbb{R}^n \bigm| \varphi(\cdot) \text{ is continuous}\Big\}.
\end{aligned}$$

We denote

$$\|\varphi(\cdot)\|_p = \|\varphi(\cdot)\|_{L^p(0,T;\mathbb{R}^n)}, \qquad \forall \varphi(\cdot) \in L^p(0,T;\mathbb{R}^n), \qquad p \in [1,\infty],$$
$$\|\varphi(\cdot)\|_C = \max_{t\in[0,T]} |\varphi(t)| = \|\varphi(\cdot)\|_\infty, \qquad \forall \varphi(\cdot) \in C([0,T];\mathbb{R}^n).$$

Also, we define

$$L^{p+}(0,T;\mathbb{R}^n) = \bigcup_{q>p} L^q(0,T;\mathbb{R}^n), \qquad p \in [1,\infty),$$
$$L^{p-}(0,T;\mathbb{R}^n) = \bigcap_{q<p} L^q(0,T;\mathbb{R}^n), \qquad p \in (1,\infty).$$

Next, for any continuous function $w : [0,T] \to [0,\infty)$, called a *weight function*, we define

$$L^p_{w(\cdot)}(0,T;\mathbb{R}^n) = \Big\{\varphi : [0,T] \to \mathbb{R}^n \bigm| w(\cdot)\varphi(\cdot) \in L^p(0,T;\mathbb{R}^n)\Big\}, \qquad p \in [1,\infty],$$
$$C_{w(\cdot)}([0,T];\mathbb{R}^n) = \Big\{\varphi : [0,T] \to \mathbb{R}^n \bigm| w(\cdot)\varphi(\cdot) \in C([0,T];\mathbb{R}^n)\Big\}.$$

Clearly, if $\operatorname{meas}\{t \in [0,T] \bigm| w(t) = 0\} = 0$, then $L^p_{w(\cdot)}(0,T;\mathbb{R}^n)$ and $C_{w(\cdot)}([0,T];\mathbb{R}^n)$ are normed linear spaces, under the following norms, respectively:

$$\|\varphi(\cdot)\|_{L^p_{w(\cdot)}} = \|w(\cdot)\varphi(\cdot)\|_p, \qquad \|\varphi(\cdot)\|_{C_{w(\cdot)}} = \|w(\cdot)\varphi(\cdot)\|_C.$$

Note that for any $\varphi(\cdot) \in C_{w(\cdot)}([0,T];\mathbb{R}^n)$, $\varphi(\cdot)$ is continuous on the set $\{t \in [0,T] \bigm| w(t) > 0\}$. If $w(s) = |s - s_0|^\gamma$ for some $s_0 \in [0,T]$ and $\gamma > 0$, then

$$\frac{a(\cdot)}{|\cdot - s_0|^\alpha} \in C_{w(\cdot)}([0,T];\mathbb{R}^n),$$

for any $\alpha \leqslant \gamma$, $a(\cdot) \in C([0,T];\mathbb{R}^n)$. From the above, we should have some feeling about the space $C_{w(\cdot)}([0,T];\mathbb{R}^n)$.

We denote

$$\Delta = \Big\{(t,s) \in [0,T]^2 \bigm| 0 \leqslant s < t \leqslant T\Big\}. \tag{2.1}$$

Note that the "diagonal line" $\{(t,t)\,|\,t \in [0,T]\}$ is not contained in Δ. Thus if $\varphi : \Delta \to \mathbb{R}^n$ with $(t,s) \mapsto \varphi(t,s)$ being continuous, then $\varphi(\cdot)$ could be unbounded as $|t - s| \to 0$.

Before going further, let us first recall the Young's inequality for convolution (Theorem 3.9.4 in [10]).

Lemma 2.1. *Let $p, q, r \geqslant 1$ satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Then for any $f(\cdot) \in L^p(\mathbb{R}^n)$, $g(\cdot) \in L^q(\mathbb{R}^n)$,

$$\|f * g\|_r \leqslant \|f\|_p \|g\|_q. \tag{2.2}$$

We now present several results. Some of them should be standard. However, we will provide the proofs for readers' convenience.

Lemma 2.2. *Let $\beta \in (0,1)$ and $\varphi : \Delta \to \mathbb{R}^n$. Define*

$$\psi(t) = \int_0^t \frac{\varphi(t,s)}{(t-s)^{1-\beta}} ds, \qquad t \in [0,T]. \tag{2.3}$$

(i) *Suppose for some $p \in [1,\infty)$,*

$$\int_0^T \sup_{t\in[s,T]} |\varphi(t,s)|^p ds < \infty. \tag{2.4}$$

Then

$$\|\psi(\cdot)\|_p \leqslant \frac{T^\beta}{\beta}\Big(\int_0^T \sup_{t\in[s,T]} |\varphi(t,s)|^p ds\Big)^{\frac{1}{p}}. \tag{2.5}$$

(ii) *Suppose, in addition,* $t\mapsto\varphi(t,s)$ *satisfies the following*

$$|\varphi(t,s)-\varphi(t',s)| \leqslant \omega(|t-t'|), \qquad \forall t,t'\in[t_0-\sigma,t_0+\sigma],\ \ s\in[0,t\wedge t'), \tag{2.6}$$

for some modulus of continuity $\omega:[0,\infty)\to[0,\infty)$*, and for some* $q>\frac{1}{\beta}$*,* $\sigma>0$*, with* $(t_0-\sigma,t_0+\sigma)\subseteq[0,T]$*, the following holds:*

$$\int_{t_0-\sigma}^{t_0+\sigma} \sup_{t\in[s,t_0+\sigma]} |\varphi(t,s)|^q ds < \infty. \tag{2.7}$$

Then $\psi(\cdot)$ *is continuous at* t_0*. Consequently, if* $t\mapsto\varphi(t,s)$ *is continuous uniformly in* $s\in[0,T]$ *and* (2.4) *holds for some* $p>\frac{1}{\beta}$*, then* $t\mapsto\psi(t)$ *is continuous on* $[0,T]$*.*

Proof. (i) Let

$$\bar\varphi(s)=\sup_{t\in(s,T]}|\varphi(t,s)|, \qquad s\in[0,T].$$

Then $\bar\varphi(\cdot)\in L^p(0,T;\mathbb{R})$. Hence, (2.5) follows from Young's inequality for convolution.

(ii) Let $q>\frac{1}{\beta}$ which is equivalent to $\kappa\equiv(1-\beta)\frac{q}{q-1}<1$ and let

$$\widehat\varphi(s)=\sup_{t\in(s,t_0+\sigma]}|\varphi(t,s)|, \qquad s\in[0,t_0+\sigma].$$

For any $t_0-\frac{\sigma}{m}<t<t'<t_0+\frac{\sigma}{m}$ with $m\geqslant2$ large enough, we look at the following:

$$\begin{aligned}
&|\psi(t)-\psi(t')| = \Big|\int_0^t \frac{\varphi(t,s)}{(t-s)^{1-\beta}}ds - \int_0^{t'}\frac{\varphi(t',s)}{(t'-s)^{1-\beta}}ds\Big| \\
&\leqslant \int_0^{t-\frac{\sigma}{m}}\Big|\frac{\varphi(t,s)}{(t-s)^{1-\beta}}-\frac{\varphi(t',s)}{(t'-s)^{1-\beta}}\Big|ds + \int_{t-\frac{\sigma}{m}}^t \frac{|\varphi(t,s)|}{(t-s)^{1-\beta}}ds + \int_{t-\frac{\sigma}{m}}^{t'}\frac{|\varphi(t',s)|}{(t'-s)^{1-\beta}}ds \\
&\leqslant \int_0^{t-\frac{\sigma}{m}}|\varphi(t,s)|\Big(\frac{1}{(t-s)^{1-\beta}}-\frac{1}{(t'-s)^{1-\beta}}\Big)ds + \int_0^{t-\frac{\sigma}{m}}\frac{|\varphi(t,s)-\varphi(t',s)|}{(t'-s)^{1-\beta}}ds \\
&\qquad + \int_{t-\frac{\sigma}{m}}^t \frac{\widehat\varphi(s)}{(t-s)^{1-\beta}}ds + \int_{t-\frac{\sigma}{m}}^{t'}\frac{\widehat\varphi(s)}{(t'-s)^{1-\beta}}ds \\
&\leqslant (t'-t)^{1-\beta}\int_0^{t-\frac{\sigma}{m}}\frac{\widehat\varphi(s)}{(t-s)^{1-\beta}(t'-s)^{1-\beta}}ds + \omega(|t-t'|)\int_0^{t-\frac{\sigma}{m}}\frac{ds}{(t'-s)^{1-\beta}} \\
&\qquad + \|\widehat\varphi(\cdot)\|_{L^q(t-\frac{\sigma}{m},t;\mathbb{R}^n)}\Big(\int_{t-\frac{\sigma}{m}}^t\frac{ds}{(t-s)^\kappa}\Big)^{\frac{q-1}{q}} + \|\widehat\varphi(\cdot)\|_{L^q(t-\frac{\sigma}{m},t';\mathbb{R}^n)}\Big(\int_{t-\frac{\sigma}{m}}^{t'}\frac{ds}{(t'-s)^\kappa}\Big)^{\frac{q-1}{q}} \\
&\leqslant \frac{(t'-t)^{1-\beta}}{(\frac{\sigma}{m})^{2(1-\beta)}}\|\widehat\varphi(\cdot)\|_1 + \omega(|t-t'|)\frac{T^\beta}{\beta} + \|\widehat\varphi(\cdot)\|_{L^q(t_0-\sigma,t_0+\sigma;\mathbb{R}^n)}\Big[\Big(\frac{(\frac{\sigma}{m})^{1-\kappa}}{1-\kappa}\Big)^{\frac{q-1}{q}} + \Big(\frac{(t'-t+\frac{\sigma}{m})^{1-\kappa}}{1-\kappa}\Big)^{\frac{q-1}{q}}\Big].
\end{aligned}$$

Hence, for any $\varepsilon>0$, we first take $m\geqslant1$ sufficiently large so that

$$\|\widehat\varphi(\cdot)\|_{L^q(t_0-\sigma,t_0+\sigma;\mathbb{R}^n)}\Big[\Big(\frac{(\frac{\sigma}{m})^{1-\kappa}}{1-\kappa}\Big)^{\frac{q-1}{q}} + \Big(\frac{(\frac{3\sigma}{m})^{1-\kappa}}{1-\kappa}\Big)^{\frac{q-1}{q}}\Big] < \frac{\varepsilon}{2}.$$

Then let $\delta\in(0,\sigma)$ be small enough so that

$$\frac{\delta^{1-\beta}}{(\frac{\sigma}{m})^{2(1-\beta)}}\|\widehat\varphi(\cdot)\|_1 + \omega(\delta)\frac{T^\beta}{\beta} < \frac{\varepsilon}{2}.$$

Combining the above, we see that $\psi(\cdot)$ is continuous at t_0. The last conclusion follows easily from what we just proved. $\square$

The above lemma show that for any $p \in [1,\infty)$, under condition (2.4), one has $\psi(\cdot) \in L^p(0,T;\mathbb{R}^n)$. To guarantee the continuity of $\psi(\cdot)$ at $t_0 \in (0,T]$, we need to assume the continuity of $t \mapsto \varphi(t,s)$ for $t \in [t_0-\sigma, t_0+\sigma]$, uniformly in $s \in [0,t)$, together with L^q integrability of $s \mapsto \sup\limits_{t\in(s,t_0+\sigma]} |\varphi(t,s)|$ for $q > \frac{1}{\beta}$. The following example shows that continuity of $\psi(\cdot)$ might fail $\bar\varphi(\cdot)$ does not have a good enough integrability.

Example 2.3. Let

$$\varphi(t,s) = \bar\varphi(s) = \frac{1}{|s-s_1|^{\frac{5}{6}}}, \quad \beta = \frac{1}{2},$$

with $s_1 \in (0,T)$. Then $\bar\varphi(\cdot) \in L^q(0,T;\mathbb{R})$ with $q < \frac{6}{5} < 2 = \frac{1}{\beta}$. Note that

$$\psi(t) = \int_0^t \frac{\varphi(t,s)}{(t-s)^{1-\beta}} ds = \int_0^t \frac{ds}{|s-s_1|^{\frac{5}{6}}(t-s)^{\frac{1}{2}}}, \qquad t \geqslant 0.$$

By Young's inequality, we know that the above $\psi(\cdot) \in L^p(0,T;\mathbb{R})$ for some $p > 1$. Also, one sees that

$$\lim_{t\uparrow s_1} \psi(t) = \int_0^{s_1} \frac{ds}{|s-s_1|^{\frac{4}{3}}} = \infty.$$

Thus, $\psi(\cdot)$ is discontinuous at s_1.

It is natural to ask if we relax the L^q-integrability of $\varphi(\cdot)$, what can we say about the continuity of $\psi(\cdot)$ defined by (2.3)? Let us make it more precise now. Let $\alpha_i, \beta \in (0,1)$, $0 \leqslant i \leqslant \ell$ and $0 \leqslant s_0 < s_1 < \cdots < s_\ell \leqslant T$. Define

$$w(s) = \prod_{i=0}^{\ell} |s-s_i|^{1-\alpha_i}, \qquad s \in [0,T]. \tag{2.8}$$

Let $\widetilde\varphi : \Delta \to \mathbb{R}^n$ and define

$$\widetilde\psi(t) = \int_0^t \frac{\widetilde\varphi(t,s)}{w(s)(t-s)^{1-\beta}} ds, \qquad t \in [0,T]. \tag{2.9}$$

Comparing the above with (2.3), we see that $\widetilde\psi(\cdot)$ would be the same as $\psi(\cdot)$ provided

$$\varphi(t,s) = \frac{\widetilde\varphi(t,s)}{w(s)}, \qquad (t,s) \in \Delta. \tag{2.10}$$

We have the following result.

Lemma 2.4. *Let $\alpha_i, \beta \in (0,1)$, $0 \leqslant i \leqslant \ell$, and $w(\cdot)$ be defined by (2.8) with $0 \leqslant s_0 < s_1 < \cdots < s_\ell \leqslant T$. Let $\widetilde\varphi : \Delta \to \mathbb{R}^n$ satisfy*

$$|\widetilde\varphi(t,s) - \widetilde\varphi(t',s)| \leqslant \omega(|t-t'|), \qquad \forall (t,s),\ (t',s) \in \Delta, \tag{2.11}$$

for some modulus of continuity $\omega : [0,\infty) \to [0,\infty)$, and

$$\int_0^T \max_{t\in(s,T]} |\widetilde\varphi(t,s)|^q ds < \infty, \tag{2.12}$$

with some

$$q > \frac{1}{\beta} \vee \frac{1}{\alpha_i}, \qquad 0 \leqslant i \leqslant \ell. \tag{2.13}$$

Define $\widetilde\psi(\cdot)$ *by* (2.9). *Then*

$$\widetilde\psi(\cdot)\in L^\infty_{\bar w(\cdot)}(0,T;\mathbb{R}^n)\bigcap\Big(\bigcap_{i=1}^{\ell}C\big((s_{i-1},s_i);\mathbb{R}^n\big)\Big),\tag{2.14}$$

where

$$\bar w(s)=\prod_{i=0}^{\ell}|s-s_i|^{(1+\frac1q-\alpha_i-\beta)^+}.\tag{2.15}$$

Consequently, for any $\varepsilon>0$,

$$\widetilde\psi(\cdot)\in C_{\bar w^\varepsilon(\cdot)}([0,T];\mathbb{R}^n),\tag{2.16}$$

where

$$\bar w^\varepsilon(s)=\prod_{i=0}^{\ell}|s-s_i|^{(1+\frac1q-\alpha_i-\beta)^++\varepsilon}.\tag{2.17}$$

Further, if

$$\alpha_i+\beta>1+\frac1q,\tag{2.18}$$

then $\widetilde\psi(\cdot)$ *is continuous at* s_i. *Consequently, if*

$$\min_{0\leqslant i\leqslant\ell}\alpha_i+\beta>1+\frac1q,\tag{2.19}$$

then $\widetilde\psi(\cdot)\in C([0,T];\mathbb{R}^n)$.

Proof. Denote

$$\bar\varphi(s)=\max_{t\in[s,T]}|\widetilde\varphi(t,s)|,\qquad s\in[0,T].$$

Define

$$w_i(s)=\prod_{j\neq i}|s-s_j|^{1-\alpha_j},\qquad s\in[0,T],\ 0\leqslant i\leqslant\ell,$$

and

$$\delta_0=\min_{0\leqslant i\leqslant\ell-1}\frac{s_{i+1}-s_i}{2}>0,\qquad \bar s_i=\frac{s_i+s_{i+1}}{2},\quad 0\leqslant i\leqslant\ell-1.\tag{2.20}$$

Then

$$s_i+\delta_0\leqslant\bar s_i\leqslant s_{i+1}-\delta_0,\qquad i=0,1,\cdots,\ell-1.\tag{2.21}$$

We establish some estimates for $\widetilde\psi(\cdot)$ on some time intervals, more precisely, on $[0,s_0)$, $(s_0,\bar s_0]$, $(0,\bar s_0]$, $[\bar s_0,s_1)$, $(s_1,\bar s_1]$, $[\bar s_1,s_2)$. Then induction will apply.

(i) For $t\in[0,s_0)$, one has

$$\begin{aligned}|\widetilde\psi(t)|&=\Big|\int_0^t\frac{\widetilde\varphi(t,s)}{w_0(s)(s_0-s)^{1-\alpha_0}(t-s)^{1-\beta}}ds\Big|\leqslant K\int_0^t\frac{\bar\varphi(s)}{(s_0-s)^{1-\alpha_0}(t-s)^{1-\beta}}ds\\&\leqslant K\Big(\int_0^t\bar\varphi(s)^qds\Big)^{\frac1q}\Big(\int_0^t\frac{ds}{(s_0-s)^{(1-\alpha_0)\frac{q}{q-1}}(t-s)^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}\\&\leqslant K\Big(\int_0^t\bar\varphi(s)^qds\Big)^{\frac1q}(s_0-t)^{\alpha_0+\beta-2}\Big(\int_0^t\frac{ds}{(1+\frac{t-s}{s_0-t})^{(1-\alpha_0)\frac{q}{q-1}}(\frac{t-s}{s_0-t})^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}.\end{aligned}$$

Here and throughout the paper, K is a positive constant, which may be different when appears at different places. Now, we let $\tau = \frac{t-s}{s_0-t}$. Then $s = t - (s_0 - t)\tau$, $ds = (t - s_0)d\tau$, and

$$\begin{aligned}
|\widetilde{\psi}(t)| &\leqslant K\Big(\int_0^t \bar{\varphi}(s)^q ds\Big)^{\frac{1}{q}}(s_0-t)^{\alpha_0+\beta-1-\frac{1}{q}}\Big(\int_0^{\frac{t}{s_0-t}} \frac{d\tau}{(1+\tau)^{(1-\alpha_0)\frac{q}{q-1}}\tau^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}} \\
&\le K(s_0-t)^{\alpha_0+\beta-1-\frac{1}{q}}\Big[\Big(\int_0^1 \frac{d\tau}{\tau^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}} + \Big(\int_1^{\frac{s_0}{s_0-t}} \frac{d\tau}{\tau^{(2-\alpha_0-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}\Big] \\
&\leqslant K(s_0-t)^{\alpha_0+\beta-1-\frac{1}{q}}\Big[1 + \Big(\frac{s_0}{s_0-t}\Big)^{[1-(2-\alpha_0-\beta)\frac{q}{q-1}]\frac{q-1}{q}}\Big] \\
&\leqslant K(s_0-t)^{\alpha_0+\beta-1-\frac{1}{q}}\Big[1 + \Big(\frac{1}{s_0-t}\Big)^{\alpha_0+\beta-1-\frac{1}{q}}\Big] \leqslant K + K(s_0-t)^{\alpha_0+\beta-1-\frac{1}{q}}, \quad t \in [0, s_0).
\end{aligned}$$

Hence,

$$(s_0-t)^{(1-\alpha_0-\beta+\frac{1}{q})^+}|\widetilde{\psi}(t)| \leqslant K, \qquad t \in [0, s_0). \tag{2.22}$$

(ii) For $t \in (s_0, \bar{s}_0]$, one has

$$|\widetilde{\psi}(t)| \leqslant \int_0^{s_0} \frac{\bar{\varphi}(s)}{w(s)(t-s)^{1-\beta}}ds + \int_{s_0}^t \frac{\bar{\varphi}(s)}{w(s)(t-s)^{1-\beta}}ds \equiv I_1 + I_2.$$

For I_1, we have

$$\begin{aligned}
I_1 &\leqslant K\Big(\int_0^{s_0} \frac{ds}{(s_0-s)^{(1-\alpha_0)\frac{q}{q-1}}(t-s)^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}} \\
&\leqslant K(t-s_0)^{\alpha_0+\beta-2}\Big(\int_0^{s_0} \frac{ds}{(\frac{s_0-s}{t-s_0})^{(1-\alpha_0)\frac{q}{q-1}}(1+\frac{s_0-s}{t-s_0})^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}.
\end{aligned}$$

Now, we let $\tau = \frac{s_0-s}{t-s_0}$. Then $s = s_0 - (t - s_0)\tau$, $ds = (s_0 - t)d\tau$, and

$$\begin{aligned}
I_1 &\leqslant K(t-s_0)^{\alpha_0+\beta-1-\frac{1}{q}}\Big(\int_0^{\frac{s_0}{t-s_0}} \frac{d\tau}{\tau^{(1-\alpha_0)\frac{q}{q-1}}(1+\tau)^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}} \\
&\leqslant K(t-s_0)^{\alpha_0+\beta-1-\frac{1}{q}}\Big[\Big(\int_0^1 \frac{d\tau}{\tau^{(1-\alpha_0)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}} + \Big(\int_1^{\frac{s_0}{t-s_0}} \frac{d\tau}{\tau^{(2-\alpha_0-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}\Big] \\
&\leqslant K(t-s_0)^{\alpha_0+\beta-1-\frac{1}{q}}\Big[1 + \Big(\frac{s_0}{t-s_0}\Big)^{[1-(2-\alpha_0-\beta)\frac{q}{q-1}]\frac{q-1}{q}}\Big] \\
&\leqslant K(t-s_0)^{\alpha_0+\beta-1-\frac{1}{q}}\Big[1 + \Big(\frac{1}{t-s_0}\Big)^{\alpha_0+\beta-1-\frac{1}{q}}\Big] \leqslant K + K(t-s_0)^{\alpha_0+\beta-1-\frac{1}{q}}, \quad t \in (s_0, \bar{s}_0].
\end{aligned}$$

Now, we look at I_2, noting $s_0 + \delta_0 \leqslant \bar{s}_0 \leqslant s_1 - \delta_0$,

$$\begin{aligned}
I_2 &= \int_{s_0}^t \frac{\bar{\varphi}(s)}{w(s)(t-s)^{1-\beta}}ds = \int_{s_0}^t \frac{\bar{\varphi}(s)}{w_0(s)(s-s_0)^{1-\alpha_0}(t-s)^{1-\beta}}ds \\
&\leqslant K\Big(\int_{s_0}^t \bar{\varphi}(s)^q ds\Big)^{\frac{1}{q}}\Big(\int_{s_0}^t \frac{ds}{(s-s_0)^{(1-\alpha_0)\frac{q}{q-1}}(t-s)^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}} \\
&\leqslant K(t-s_0)^{\alpha_0+\beta-2}\Big(\int_{s_0}^t \frac{ds}{(\frac{s-s_0}{t-s_0})^{(1-\alpha_0)\frac{q}{q-1}}(1-\frac{s-s_0}{t-s_0})^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}.
\end{aligned}$$

Note that by (2.13), we have

$$(1-\alpha_0)\frac{q}{q-1} < 1, \quad (1-\beta)\frac{q}{q-1} < 1,$$

which are equivalent to the following:

$$1-(1-\alpha_0)\frac{q}{q-1}=\frac{\alpha_0 q-1}{q-1}>0, \qquad 1-(1-\beta)\frac{q}{q-1}=\frac{\beta q-1}{q-1}>0.$$

Let $\tau=\frac{s-s_0}{t-s_0}$. Then $s=s_0+(t-s_0)\tau$, $ds=(t-s_0)d\tau$, and

$$I_2\leqslant K(t-s_0)^{\alpha_0+\beta-1-\frac{1}{q}}\Big(\int_0^1\frac{d\tau}{\tau^{1-\frac{\alpha_0 q-1}{q-1}}(1-\tau)^{1-\frac{\beta q-1}{q-1}}}\Big)^{\frac{q-1}{q}}=K(t-s_0)^{\alpha_0+\beta-1-\frac{1}{q}}B\Big(\frac{\alpha_0 q-1}{q-1},\frac{\beta q-1}{q-1}\Big)^{\frac{q-1}{q}}.$$

Here, $(a,b)\mapsto B(a,b)$ is the Beta function. Hence,

$$(t-s_0)^{(1-\alpha_0-\beta+\frac{1}{q})^+}|\widetilde{\psi}(t)|\leqslant K, \qquad t\in(s_0,\bar{s}_0]. \tag{2.23}$$

(iii) For $t\in[\bar{s}_0,s_1)$, we have

$$|\widetilde{\psi}(t)|\leqslant\Big|\int_0^{\bar{s}_0}\frac{\widetilde{\varphi}(t,s)}{w(s)(t-s)^{1-\beta}}ds\Big|+\Big|\int_{\bar{s}_0}^{t}\frac{\widetilde{\varphi}(t,s)}{w(s)(t-s)^{1-\beta}}ds\Big|\equiv I_3+I_4.$$

For I_3, we have

$$\begin{aligned}
I_3&\leqslant\int_0^{s_0}\frac{\bar{\varphi}(s)}{w(s)(t-s)^{1-\beta}}ds+\int_{s_0}^{\bar{s}_0}\frac{\bar{\varphi}(s)}{w(s)(t-s)^{1-\beta}}ds\\
&\leqslant\int_0^{s_0}\frac{\bar{\varphi}(s)}{(s-s_0)^{1-\alpha_0}}ds+\int_{s_0}^{\bar{s}_0}\frac{\bar{\varphi}(s)}{(s-s_0)^{1-\alpha_0}(\bar{s}_0-s)^{1-\beta}}ds\\
&\leqslant K\Big(\int_0^{s_0}\bar{\varphi}(s)^q ds\Big)^{\frac{1}{q}}\Big(\int_0^{s_0}\frac{ds}{(s-s_0)^{(1-\alpha_0)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}\\
&\quad+K\Big(\int_{s_0}^{\bar{s}_0}\bar{\varphi}(s)^q ds\Big)^{\frac{1}{q}}\Big(\int_{s_0}^{\bar{s}_0}\frac{ds}{(s-s_0)^{(1-\alpha_0)\frac{q}{q-1}}(\bar{s}_0-s)^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}\\
&\leqslant K+K(\bar{s}_0-s_0)^{\alpha_0+\beta-2}\Big(\int_{s_0}^{\bar{s}_0}\frac{ds}{(\frac{s-s_0}{\bar{s}_0-s_0})^{(1-\alpha_0)\frac{q}{q-1}}(1-\frac{s-s_0}{\bar{s}_0-s_0})^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}.
\end{aligned}$$

Let $\tau=\frac{s-s_0}{\bar{s}_0-s_0}$. Then $s=s_0+(\bar{s}_0-s_0)\tau$, $ds=(\bar{s}_0-s_0)d\tau$, and

$$\begin{aligned}
I_3&\leqslant K+K(\bar{s}_0-s_0)^{\alpha_0+\beta-1-\frac{1}{q}}\Big(\int_0^1\frac{d\tau}{\tau^{1-\frac{\alpha_0 q-1}{q-1}}(1-\tau)^{1-\frac{\beta q-1}{q-1}}}\Big)^{\frac{q-1}{q}}\\
&=K+K(\bar{s}_0-s_0)^{\alpha_0+\beta-1-\frac{1}{q}}B\Big(\frac{\alpha_0 q-1}{q-1},\frac{\beta q-1}{q-1}\Big)^{\frac{q-1}{q}}\leqslant K.
\end{aligned}$$

For I_4, we have

$$\begin{aligned}
I_4&\leqslant\int_{\bar{s}_0}^{t}\frac{\bar{\varphi}(s)}{w_1(s)(s_1-s)^{1-\alpha_1}(t-s)^{1-\beta}}ds\\
&\leqslant K\Big(\int_{\bar{s}_0}^{t}\bar{\varphi}(s)^q ds\Big)^{\frac{1}{q}}\Big(\int_{\bar{s}_0}^{t}\frac{ds}{(s_1-s)^{(1-\alpha_1)\frac{q}{q-1}}(t-s)^{(1-\beta)\frac{q}{q-1}}}ds\Big)^{\frac{q-1}{q}}\\
&\leqslant K(s_1-t)^{\alpha_1+\beta-2}\Big(\int_{\bar{s}_0}^{t}\frac{ds}{(1+\frac{t-s}{s_1-t})^{(1-\alpha_1)\frac{q}{q-1}}(\frac{t-s}{s_1-t})^{(1-\beta)\frac{q}{q-1}}}ds\Big)^{\frac{q-1}{q}}.
\end{aligned}$$

Let $\tau=\frac{t-s}{s_1-t}$. Then $s=t-(s_1-t)\tau$, $ds=-(s_1-t)d\tau$, and

$$
\begin{aligned}
I_4&\leqslant K(s_1-t)^{\alpha_1+\beta-1-\frac{1}{q}}\Big(\int_0^{\frac{t-\bar s_0}{s_1-t}}\frac{d\tau}{(1+\tau)^{(1-\alpha_1)\frac{q}{q-1}}\tau^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}\\
&\leqslant K(s_1-t)^{\alpha_1+\beta-1-\frac{1}{q}}\Big[\Big(\int_0^1\frac{d\tau}{\tau^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}+\Big(\int_1^{\frac{s_1-\bar s_0}{s_1-t}}\frac{d\tau}{\tau^{(2-\alpha_1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}\Big]\\
&\leqslant K(s_1-t)^{\alpha_1+\beta-1-\frac{1}{q}}\Big[1+\Big(\frac{s_1-\bar s_0}{s_1-t}\Big)^{[1-(2-\alpha_1-\beta)\frac{q}{q-1}]\frac{q-1}{q}}\Big]\\
&\leqslant K(s_1-t)^{\alpha_1+\beta-1-\frac{1}{q}}\Big[1+\Big(\frac{1}{s_1-t}\Big)^{\alpha_1+\beta-1-\frac{1}{q}}\Big]\leqslant K(s_1-t)^{\alpha_1+\beta-1-\frac{1}{q}},\qquad t\in[\bar s_0,s_1).
\end{aligned}
$$

Then we have

$$
(s_1-t)^{(1-\alpha_1-\beta+\frac{1}{q})^+}|\widetilde\psi(t)|\leqslant K,\qquad t\in[\bar s_0,s_1). \tag{2.24}
$$

(iv) For $t\in(s_1,\bar s_1]$, we have

$$
|\widetilde\psi(t)|\leqslant\int_0^{s_1}\frac{\bar\varphi(s)}{w(s)(t-s)^{1-\beta}}ds+\int_{s_1}^t\frac{\bar\varphi(s)}{w(s)(t-s)^{1-\beta}}ds\equiv I_5+I_6.
$$

Note that $\bar s_0\leqslant s_1-\delta_0$,

$$
\begin{aligned}
I_5&=\int_0^{s_1}\frac{\bar\varphi(s)}{w(s)(t-s)^{1-\beta}}ds=\int_0^{\bar s_0}\frac{\bar\varphi(s)}{w_0(s)|s-s_0|^{1-\alpha_0}(t-s)^{1-\beta}}ds+\int_{\bar s_0}^{s_1}\frac{\bar\varphi(s)}{w_1(s)(s_1-s)^{1-\alpha_1}(t-s)^{1-\beta}}ds\\
&\leqslant K\int_0^{\bar s_0}\frac{\bar\varphi(s)}{|s-s_0|^{1-\alpha_0}}ds+K\int_{\bar s_0}^{s_1}\frac{\bar\varphi(s)}{(s_1-s)^{1-\alpha_1}[(t-s_1)+(s_1-s)]^{1-\beta}}ds\\
&\leqslant K\Big(\int_0^{\bar s_0}\bar\varphi(s)^qds\Big)^{\frac{1}{q}}\Big[\Big(\int_0^{\bar s_0}\frac{ds}{|s-s_0|^{(1-\alpha_0)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}\\
&\qquad+(t-s_1)^{\alpha_1+\beta-2}\Big(\int_{\bar s_0}^{s_1}\frac{ds}{(\frac{s_1-s}{t-s_1})^{(1-\alpha_1)\frac{q}{q-1}}(1+\frac{s_1-s}{t-s_1})^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}\Big].
\end{aligned}
$$

Let $\tau=\frac{s_1-s}{t-s_1}$. Then $s=s_1-(t-s_1)\tau$, $ds=-(t-s_1)d\tau$, and

$$
\begin{aligned}
I_5&\leqslant K\Big[1+(t-s_1)^{\alpha_1+\beta-2}\Big(\int_{\bar s_0}^{s_1}\frac{ds}{(\frac{s_1-s}{t-s_1})^{(1-\alpha_1)\frac{q}{q-1}}(1+\frac{s_1-s}{t-s_1})^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}\Big]\\
&\leqslant K\Big[1+(t-s_1)^{\alpha_1+\beta-1-\frac{1}{q}}\Big(\int_0^{\frac{s_1-\bar s_0}{t-s_1}}\frac{d\tau}{\tau^{(1-\alpha_1)\frac{q}{q-1}}(1+\tau)^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}\Big]\\
&\leqslant K\Big[1+(t-s_1)^{\alpha_1+\beta-1-\frac{1}{q}}\Big(\int_0^1\frac{d\tau}{\tau^{(1-\alpha_1)\frac{q}{q-1}}}+\int_1^{\frac{s_1-\bar s_0}{t-s_1}}\frac{d\tau}{\tau^{(2-\alpha_1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}\Big]\\
&\leqslant K\Big[1+(t-s_1)^{\alpha_1+\beta-1-\frac{1}{q}}+(t-s_1)^{\alpha_1+\beta-1-\frac{1}{q}}\Big(\frac{s_1-\bar s_0}{t-s_1}\Big)^{[1-(2-\alpha_1-\beta)\frac{q}{q-1}]\frac{q-1}{q}}\Big]\\
&\leqslant K+K(t-s_1)^{\alpha_1+\beta-1-\frac{1}{q}}.
\end{aligned}
$$

Now, we look at I_6, noting $s_1+\delta_0\leqslant\bar s_1\leqslant s_2-\delta_0$,

$$
\begin{aligned}
I_6&=\int_{s_1}^t\frac{\bar\varphi(s)}{w(s)(t-s)^{1-\beta}}ds=\int_{s_1}^t\frac{\bar\varphi(s)}{w_1(s)(s-s_1)^{1-\alpha_1}(t-s)^{1-\beta}}ds\\
&\leqslant K\Big(\int_{s_1}^t\bar\varphi(s)^qds\Big)^{\frac{1}{q}}\Big(\int_{s_1}^t\frac{ds}{(s-s_1)^{(1-\alpha_1)\frac{q}{q-1}}(t-s)^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}\\
&\leqslant K(t-s_1)^{\alpha_1+\beta-2}\Big(\int_{s_1}^t\frac{ds}{(\frac{s-s_1}{t-s_1})^{(1-\alpha_1)\frac{q}{q-1}}(1-\frac{s-s_1}{t-s_1})^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}.
\end{aligned}
$$

Let $\tau = \frac{s-s_1}{t-s_1}$. Then $s = s_1 + (t-s_1)\tau$, $ds = (t-s_1)d\tau$, and

$$I_6 \leqslant K(t-s_1)^{\alpha_1+\beta-1-\frac{1}{q}}\Big(\int_0^1 \frac{d\tau}{\tau^{1-\frac{\alpha_1 q-1}{q-1}}(1-\tau)^{1-\frac{\beta q-1}{q-1}}}\Big)^{\frac{q-1}{q}} = K(t-s_1)^{\alpha_1+\beta-1-\frac{1}{q}} B\Big(\frac{\alpha_1 q-1}{q-1}, \frac{\beta q-1}{q-1}\Big)^{\frac{q-1}{q}}.$$

Hence,

$$(t-s_1)^{(1+\frac{1}{q}-\alpha_1-\beta)^+}|\widetilde{\psi}(t)| \leqslant K, \qquad t \in (s_1, \bar{s}_1]. \tag{2.25}$$

(v) For $t \in [\bar{s}_1, s_2)$,

$$|\widetilde{\psi}(t)| \leqslant \int_0^{s_1} \frac{\bar{\varphi}(s)}{w(s)(t-s)^{1-\beta}}ds + \int_{s_1}^{\bar{s}_1} \frac{\bar{\varphi}(s)}{w(s)(t-s)^{1-\beta}}ds + \int_{\bar{s}_1}^{t} \frac{\bar{\varphi}(s)}{w(s)(t-s)^{1-\beta}}ds \equiv I_7 + I_8 + I_9.$$

We look at the three terms one-by-one. Since $\bar{s}_1 \geqslant s_1 + \delta_0$, one has

$$\begin{aligned} I_7 = \int_0^{s_1} \frac{\bar{\varphi}(s)}{w(s)(t-s)^{1-\beta}}ds &\leqslant K\Big(\int_0^{\bar{s}_0} \frac{\bar{\varphi}(s)}{|s-s_0|^{1-\alpha_0}(t-s)^{1-\beta}}ds + \int_{\bar{s}_0}^{s_1} \frac{\bar{\varphi}(s)}{(s_1-s)^{1-\alpha_1}(t-s)^{1-\beta}}ds\Big) \\ &\leqslant K\Big(\int_0^{\bar{s}_0} \bar{\varphi}(s)^q ds\Big)^{\frac{1}{q}}\Big(\int_0^{\bar{s}_0} \frac{ds}{|s-s_0|^{(1-\alpha_0)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}} + K\Big(\int_{\bar{s}_0}^{s_1} \bar{\varphi}(s)^q ds\Big)^{\frac{1}{q}}\Big(\int_{\bar{s}_0}^{s_1} \frac{ds}{(s_1-s)^{(1-\alpha_1)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}} \leqslant K. \end{aligned}$$

For I_8, since $(t-s) \geqslant (s_1-s)$, one has

$$\begin{aligned} I_8 = \int_{s_1}^{\bar{s}_1} \frac{\bar{\varphi}(s)}{w_1(s)(s-s_1)^{1-\alpha_1}(t-s)^{1-\beta}}ds &\leqslant K\Big(\int_{s_1}^{\bar{s}_1} \bar{\varphi}(s)^q ds\Big)^{\frac{1}{q}}\Big(\int_{s_1}^{\bar{s}_1} \frac{ds}{(s-s_1)^{(1-\alpha_1)\frac{q}{q-1}}(\bar{s}_1-s)^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}} \\ &\leqslant K(\bar{s}_1-s_1)^{\alpha_1+\beta-2}\Big(\int_{s_1}^{\bar{s}_1} \frac{ds}{(\frac{s-s_1}{\bar{s}_1-s_1})^{(1-\alpha_1)\frac{q}{q-1}}(\frac{\bar{s}_1-s}{\bar{s}_1-s_1})^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}. \end{aligned}$$

Let $\tau = \frac{s-s_1}{\bar{s}_1-s_1}$. Then $s = s_1 + (\bar{s}_1-s_1)\tau$, $ds = (\bar{s}_1-s_1)d\tau$, and

$$I_8 \leqslant K(\bar{s}_1-s_1)^{\alpha_1+\beta-1-\frac{1}{q}}\Big(\int_0^1 \frac{d\tau}{\tau^{1-\frac{\alpha_1 q-1}{q-1}}(1-\tau)^{1-\frac{\beta q-1}{q-1}}}\Big)^{\frac{q-1}{q}} \leqslant K.$$

Finally, for I_9, one has

$$\begin{aligned} I_9 = \int_{\bar{s}_1}^{t} \frac{\bar{\varphi}(s)}{w_2(s)(s_2-s)^{1-\alpha_2}(t-s)^{1-\beta}}ds &\leqslant K\Big(\int_{\bar{s}_1}^{t} \bar{\varphi}(s)^q ds\Big)^{\frac{1}{q}}\Big(\int_{\bar{s}_1}^{t} \frac{ds}{(s_2-s)^{(1-\alpha_2)\frac{q}{q-1}}(t-s)^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}} \\ &\leqslant K(s_2-t)^{\alpha_2+\beta-2}\Big(\int_{\bar{s}_1}^{t} \frac{ds}{(1+\frac{t-s}{s_2-t})^{(1-\alpha_2)\frac{q}{q-1}}(\frac{t-s}{s_2-t})^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}}. \end{aligned}$$

Let $\tau = \frac{t-s}{s_2-t}$. Then $s = t - (s_2-t)\tau$, $ds = -(s_2-t)d\tau$, and

$$\begin{aligned} I_9 = \int_{\bar{s}_1}^{t} \frac{\bar{\varphi}(s)}{w(s)(t-s)^{1-\beta}}ds &\leqslant K(s_2-t)^{\alpha_2+\beta-1-\frac{1}{q}}\Big(\int_0^{\frac{t-\bar{s}_1}{s_2-t}} \frac{d\tau}{(1+\tau)^{(1-\alpha_2)\frac{q}{q-1}}\tau^{(1-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}} \\ &\leqslant K(s_2-t)^{\alpha_2+\beta-1-\frac{1}{q}}\Big(\int_0^1 \frac{ds}{\tau^{(1-\beta)\frac{q}{q-1}}}ds + \int_1^{\frac{s_2-\bar{s}_1}{s_2-t}} \frac{ds}{\tau^{(2-\alpha_2-\beta)\frac{q}{q-1}}}\Big)^{\frac{q-1}{q}} \\ &\leqslant K(s_2-t)^{\alpha_2+\beta-1-\frac{1}{q}}\Big[1 + \Big(\frac{s_2-\bar{s}_1}{s_2-t}\Big)^{[1-(2-\alpha_2-\beta)\frac{q}{q-1}]\frac{q-1}{q}}\Big] \leqslant K + K(s_2-t)^{\alpha_2+\beta-1-\frac{1}{q}}. \end{aligned}$$

Hence,

$$(s_2-t)^{(1-\alpha_2-\beta+\frac{1}{q})^+}|\widetilde{\psi}(t)| \leqslant K, \qquad t \in [\bar{s}_1, s_2). \tag{2.26}$$

By induction, we can obtain the following:

$$
\begin{cases}
(s_0-t)^{(1+\frac{1}{q}-\alpha_0-\beta)^+}|\widetilde\psi(t)|\leqslant K, & t\in(0,s_0),\\
(t-s_i)^{(1+\frac{1}{q}-\alpha_i-\beta)^+}|\widetilde\psi(t)|\leqslant K, & t\in(s_i,\bar s_i], \qquad 0\leqslant i\leqslant \ell-1,\\
(s_{i+1}-t)^{(1+\frac{1}{q}-\alpha_{i+1}-\beta)^+}|\widetilde\psi(t)|\leqslant K, & t\in[\bar s_i,s_{i+1}), \qquad 0\leqslant i\leqslant \ell-1,\\
(t-s_l)^{(1+\frac{1}{q}-\alpha_l-\beta)^+}|\widetilde\psi(t)|\leqslant K, & t\in(s_l,T].
\end{cases}
\tag{2.27}
$$

Hence, for $\bar w(\cdot)$, we have

$$
\bar w(t)|\widetilde\psi(t)|\leqslant K, \qquad \forall t\in[0,T]\setminus\{s_0,s_1,\cdots,s_\ell\}.
$$

Then (2.16) follows easily. On the other hand, if $s_0=0$ and $s_l=T$, then for any $t_0\in[0,T]\setminus\{s_0,s_1,s_2,\cdots,s_\ell\}$, one can find a $\sigma>0$ such that for some $i=0,1,2,\cdots,\ell-1$, $[t_0-\sigma,t_0+\sigma]\subseteq(s_i,s_{i+1})$. Then

$$
\int_{t_0-\sigma}^{t_0+\sigma}\sup_{t\in[s,t_0+\sigma]}\Big|\frac{\widetilde\varphi(t,s)}{w(s)}\Big|^q ds\leqslant K\int_{t_0-\sigma}^{t_0+\sigma}\bar\varphi(s)^q ds<\infty.
$$

Hence, by the similar argument of Lemma 2.2, $\widetilde\psi(\cdot)$ is continuous at any such a t_0. If $0<s_0$ or $s_\ell<T$, we can use the similar argument to show the continuity of $\widetilde\psi(\cdot)$ at $t_0\in[0,T]\setminus\{s_0,s_1,s_2,\cdots,s_\ell\}$.

Finally, if (2.18) holds, then for $\sigma>0$ small enough, and for $r<q$, with $1+\frac{1}{q}-\alpha_i<\frac{1}{r}<\beta$,

$$
\int_{s_i-\sigma}^{s_i+\sigma}\frac{\bar\varphi(s)^r}{w(s)^r}ds\leqslant K\Big(\int_{s_i-\sigma}^{s_i+\sigma}\bar\varphi(s)^q ds\Big)^{\frac{r}{q}}\Big(\int_{s_i-\sigma}^{s_i+\sigma}\frac{ds}{|s-s_i|^{(1-\alpha_i)\frac{rq}{q-r}}}\Big)^{\frac{q-r}{q}}<\infty,
$$

since

$$
1+\frac{1}{q}-\alpha_i<\frac{1}{r} \qquad\Longleftrightarrow\qquad (1-\alpha_i)\frac{rq}{q-r}<1.
$$

Thus, by the similar argument of Lemma 2.2, we have the continuity of $\widetilde\psi(\cdot)$ at s_i. The last conclusion is clear. $\square$

From the above, we see that if $\alpha_i+\beta>1$ for all $0\leqslant i\leqslant\ell$ and

$$
\bar\varphi(\cdot)\in L^{\frac{1}{\alpha_i+\beta-1}+}(0,T;\mathbb{R}), \qquad 0\leqslant i\leqslant\ell,
\tag{2.28}
$$

then $\widetilde\psi(\cdot)\in C([0,T];\mathbb{R}^n)$. On the other hand, if $\alpha_i+\beta<1$, and $\bar\varphi(\cdot)$ is essentially non-zero near s_i, then $\widetilde\psi(\cdot)$ will be blow-up near s_i, and roughly it will grow no more than $|t-s_i|^{\alpha_i+\beta-1-\frac{1}{q}}$.

The following result is a kind of Gronwall's inequality with a singular kernel.

Lemma 2.5. *Let $\beta\in(0,1)$ and $q>\frac{1}{\beta}$. Let $L(\cdot),a(\cdot),y(\cdot)$ be nonnegative functions with*

$$
L(\cdot)\in L^q(0,T;\mathbb{R}), \quad a(\cdot),\ y(\cdot)\in L^{\frac{q}{q-1}}(0,T;\mathbb{R}).
$$

Suppose

$$
y(t)\leqslant a(t)+\int_0^t\frac{L(s)y(s)}{(t-s)^{1-\beta}}ds, \qquad \text{a.e. } t\in[0,T].
\tag{2.29}
$$

Then

$$
y(t)\leqslant a(t)+\sum_{i=0}^{k-1}c_i\int_0^t\frac{L(s)a(s)}{(t-s)^{1-\beta_i}}ds+c_k\int_0^t L(s)a(s)ds, \qquad \text{a.e. } t\in[0,T],
\tag{2.30}
$$

for some constants $c_i>0$ and $\beta_i\in(0,1)$ defined by

$$
\beta_i=\beta+i\Big(\beta-\frac{1}{q}\Big), \qquad 0\leqslant i\leqslant k-1,
$$

with k being the smallest integer satisfying

$$
\beta+k\Big(\beta-\frac{1}{q}\Big)\geqslant 1.
$$

Proof. First of all, since $L(\cdot)\in L^q(0,T;\mathbb{R})$ and $y(\cdot)\in L^{q'}(0,T;\mathbb{R})$, $q'=\frac{q}{q-1}$, we know that $L(\cdot)y(\cdot)\in L^1(0,T;\mathbb{R})$. Hence, the integral on the right hand side of (2.29) is well-defined, as a function in $L^1(0,T;\mathbb{R})$. Now, we observe the following:

$$\begin{aligned}y(t)&\leqslant a(t)+\int_0^t\frac{L(s)y(s)}{(t-s)^{1-\beta}}ds\leqslant a(t)+\int_0^t\frac{L(s)a(s)}{(t-s)^{1-\beta}}ds+\int_0^t\frac{L(s)}{(t-s)^{1-\beta}}\int_0^s\frac{L(\tau)y(\tau)}{(s-\tau)^{1-\beta}}d\tau ds\\&\leqslant a(t)+\int_0^t\frac{L(s)a(s)}{(t-s)^{1-\beta}}ds+\int_0^t L(\tau)\Big[\int_\tau^t\frac{L(s)}{(t-s)^{1-\beta}(s-\tau)^{1-\beta}}ds\Big]y(\tau)d\tau.\end{aligned}$$

Let $r=\dfrac{s-\tau}{t-\tau}$. Then $s=\tau+(t-\tau)r$ and $ds=(t-\tau)dr$. Thus,

$$\begin{aligned}&\int_\tau^t\frac{L(s)}{(t-s)^{1-\beta}(s-\tau)^{1-\beta}}ds\leqslant\Big(\int_\tau^t L(s)^q ds\Big)^{\frac1q}\Big(\int_\tau^t\frac{ds}{(t-s)^{(1-\beta)q'}(s-\tau)^{(1-\beta)q'}}ds\Big)^{\frac{1}{q'}}\\&\leqslant\|L(\cdot)\|_q\Big(\int_0^1\frac{(t-\tau)dr}{[(t-\tau)(1-r)]^{(1-\beta)q'}[(t-\tau)r]^{(1-\beta)q'}}\Big)^{\frac{1}{q'}}\\&=\|L(\cdot)\|_q\frac{1}{(t-\tau)^{2(1-\beta)-\frac{1}{q'}}}\Big(\int_0^1\frac{dr}{(1-r)^{(1-\beta)q'}r^{(1-\beta)q'}}\Big)^{\frac{1}{q'}}.\end{aligned}$$

Since $q>\frac{1}{\beta}$ which is equivalent to

$$0<1-(1-\beta)q'=1-\frac{(1-\beta)q}{q-1}=\frac{q-1-q+\beta q}{q-1}=\frac{\beta q-1}{q-1},\tag{2.31}$$

we obtain

$$\int_\tau^t\frac{L(s)}{(t-s)^{1-\beta}(s-\tau)^{1-\beta}}ds\leqslant\frac{\|L(\cdot)\|_q}{(t-\tau)^{2(1-\beta)-\frac{1}{q'}}}B\Big(\frac{\beta q-1}{q-1},\frac{\beta q-1}{q-1}\Big)^{\frac{1}{q'}}\equiv\frac{c_1}{(t-\tau)^{1-\beta_1}},$$

with $B(\cdot\,,\cdot)$ being the Beta function and

$$\begin{aligned}&c_1=\|L(\cdot)\|_qB\Big(\frac{\beta q-1}{q-1},\frac{\beta q-1}{q-1}\Big)^{\frac{1}{q'}},\\&\beta_1=1-\Big(2(1-\beta)-\frac{1}{q'}\Big)=2\beta+\frac{1}{q'}-1=\beta+\Big(\beta-\frac1q\Big)>\beta.\end{aligned}$$

Consequently,

$$\begin{aligned}y(t)&\leqslant a(t)+\int_0^t\frac{L(s)a(s)}{(t-s)^{1-\beta}}ds+c_1\int_0^t\frac{L(s)y(s)}{(t-s)^{1-\beta_1}}ds\\&\leqslant a(t)+\int_0^t\frac{L(s)a(s)}{(t-s)^{1-\beta}}ds+c_1\int_0^t\frac{L(s)a(s)}{(t-s)^{1-\beta_1}}ds+c_1\int_0^t\frac{L(s)}{(t-s)^{1-\beta_1}}\int_0^s\frac{L(\tau)y(\tau)}{(s-\tau)^{1-\beta}}d\tau ds\\&=a(t)+\sum_{i=0}^1c_i\int_0^t\frac{L(s)a(s)}{(t-s)^{1-\beta_i}}ds+c_1\int_0^tL(\tau)\Big[\int_\tau^t\frac{L(s)}{(t-s)^{1-\beta_1}(s-\tau)^{1-\beta}}ds\Big]y(\tau)d\tau,\end{aligned}$$

with $c_0=1$ and $\beta_0=\beta$. Let $r=\dfrac{s-\tau}{t-\tau}$. Then $s=\tau+(t-\tau)r$ and $ds=(t-\tau)dr$. Thus,

$$\begin{aligned}&\int_\tau^t\frac{L(s)}{(t-s)^{1-\beta_1}(s-\tau)^{1-\beta}}ds\leqslant\Big(\int_\tau^tL(s)^qds\Big)^{\frac1q}\Big(\int_\tau^t\frac{ds}{(t-s)^{(1-\beta_1)q'}(s-\tau)^{(1-\beta)q'}}ds\Big)^{\frac{1}{q'}}\\&\leqslant\|L(\cdot)\|_q\Big(\int_0^1\frac{(t-\tau)dr}{[(t-\tau)(1-r)]^{(1-\beta_1)q'}[(t-\tau)r]^{(1-\beta)q'}}\Big)^{\frac{1}{q'}}\\&=\|L(\cdot)\|_q\frac{1}{(t-\tau)^{2-\beta-\beta_1-\frac{1}{q'}}}\Big(\int_0^1\frac{dr}{(1-r)^{(1-\beta_1)q'}r^{(1-\beta)q'}}\Big)^{\frac{1}{q'}}.\end{aligned}$$

Since $q > \frac{1}{\beta} > \frac{1}{\beta_1}$, we have

$$0 < 1-(1-\beta_1)q' = 1-\frac{(1-\beta_1)q}{q-1} = \frac{q-1-q+\beta_1 q}{q-1} = \frac{\beta_1 q-1}{q-1}. \tag{2.32}$$

Hence, we obtain

$$c_1\int_\tau^t \frac{L(s)}{(t-s)^{1-\beta_1}(s-\tau)^{1-\beta}}ds \leqslant \frac{c_1\|L(\cdot)\|_q}{(t-\tau)^{2-\beta-\beta_1-\frac{1}{q'}}}B\Big(\frac{\beta q-1}{q-1},\frac{\beta_1 q-1}{q-1}\Big)^{\frac{1}{q'}} \equiv \frac{c_2}{(t-\tau)^{1-\beta_2}},$$

with

$$\begin{aligned}
&c_2 = c_1\|L(\cdot)\|_q B\Big(\frac{\beta q-1}{q-1},\frac{\beta_1 q-1}{q-1}\Big)^{\frac{1}{q'}},\\
&\beta_2 = 1-\Big(2-\beta-\beta_1-\frac{1}{q'}\Big) = \beta+\beta_1+\frac{1}{q'}-1 = \beta+2\Big(\beta-\frac{1}{q}\Big) > \beta.
\end{aligned}$$

Consequently,

$$y(t) \leqslant a(t)+\sum_{i=0}^{1}c_i\int_0^t\frac{L(s)a(s)}{(t-s)^{1-\beta_i}}ds + c_2\int_0^t\frac{L(s)y(s)}{(t-s)^{1-\beta_2}}ds.$$

By induction, we are able to show that

$$y(t) \leqslant a(t)+\sum_{i=0}^{k-1}c_i\int_0^t\frac{L(s)a(s)}{(t-s)^{1-\beta_i}}ds + c_k\int_0^t\frac{L(s)y(s)}{(t-s)^{1-\beta_k}}ds,$$

with

$$\beta_i = \beta + i\Big(\beta-\frac{1}{q}\Big), \qquad 0\leqslant i\leqslant k,$$

and recursively defined $c_i > 0$:

$$c_i = c_{i-1}\|L(\cdot)\|_q B\Big(\frac{\beta q-1}{q-1},\frac{\beta_{i-1}q-1}{q-1}\Big)^{\frac{1}{q'}}, \qquad 1\leqslant i\leqslant k.$$

We let $k\geqslant 1$ be the smallest integer that $\beta_k\geqslant 1$. Then the above implies

$$y(t) \leqslant a(t)+\sum_{i=0}^{k-1}c_i\int_0^t\frac{L(s)a(s)}{(t-s)^{1-\beta_i}}ds + c_kT^{\beta_k-1}\int_0^t L(s)y(s)ds, \qquad \text{a.e. } t\in[0,T].$$

Now, let

$$z(t) = \int_0^t L(s)y(s)ds, \qquad t\in[0,T].$$

Then

$$\dot z(t) = L(t)y(t) \leqslant L(t)a(t)+\sum_{i=0}^{k-1}c_iL(t)\int_0^t\frac{L(s)a(s)}{(t-s)^{1-\beta_i}}ds + c_kT^{\beta_k-1}L(t)z(t).$$

Hence,

$$\begin{aligned}
z(t) &\leqslant \int_0^t e^{c_kT^{\beta_k-1}\int_s^t L(\tau)d\tau}L(s)a(s)ds + \sum_{i=0}^{k-1}c_i\int_0^t e^{c_kT^{\beta_k-1}\int_s^t L(\tau)d\tau}L(s)\int_0^s\frac{L(\tau)a(\tau)}{(s-\tau)^{1-\beta_i}}d\tau ds\\
&\leqslant \int_0^t e^{c_kT^{\beta_k-1}\int_s^t L(\tau)d\tau}L(s)a(s)ds + \sum_{i=0}^{k-1}c_iK\int_0^t L(\tau)a(\tau)d\tau\int_\tau^t\frac{L(s)}{(s-\tau)^{1-\beta_i}}ds\\
&\leqslant \int_0^t e^{c_kT^{\beta_k-1}\int_s^t L(\tau)d\tau}L(s)a(s)ds + \sum_{i=0}^{k-1}c_iK\Big(\int_\tau^t L(s)^qds\Big)^{\frac{1}{q}}\Big(\int_\tau^t\frac{1}{(s-\tau)^{(1-\beta_i)\frac{q}{q-1}}}ds\Big)^{\frac{q-1}{q}}\int_0^t L(s)a(s)ds\\
&\leqslant c_k\int_0^t L(s)a(s)ds,
\end{aligned}$$

for a properly redefined constant $c_k > 0$. Hence,

$$y(t) \leqslant a(t) + \sum_{i=0}^{k-1} c_i \int_0^t \frac{L(s)a(s)}{(t-s)^{1-\beta_i}} ds + c_k \int_0^t L(s)a(s)ds, \qquad \text{a.e. } t \in [0,T],$$

proving our conclusion. $\square$

Comparing with the Gronwall type inequality appearing in literature on fractional differential equations (see [26], for example), our inequality only involves a finite sum, instead of an infinite series.

3 State Equation

In this section, we discuss our state equation (1.1), together with the cost functional (1.2). In what follows, U will be a separable metric space with the metric ρ, which could be a non-empty bounded or unbounded set in $\mathbb{R}^m$ with the metric induced by the usual Euclidean norm. Let $u_0 \in U$ be fixed. For any $p \geqslant 1$, we define

$$\mathscr{U}^p[0,T] = \big\{ u : [0,T] \to U \bigm| u(\cdot) \text{ is measurable, } \rho(u(\cdot), u_0) \in L^p(0,T;\mathbb{R}) \big\}.$$

3.1 Well-posedness in L^p space

We introduce the following assumptions for the generator $f(\cdot\,,\cdot\,,\cdot\,,\cdot)$ of our state equation.

(H1) Let the map $f : \Delta \times \mathbb{R}^n \times U \to \mathbb{R}^n$ be measurable. There are nonnegative functions $L(\cdot), L_0(\cdot)$ with

$$L(\cdot) \in L^{(\frac{1}{\beta} \vee \frac{p}{p-1})+}(0,T;\mathbb{R}), \qquad L_0(\cdot) \in L^{(\frac{p}{1+\beta p} \vee 1)+}(0,T;\mathbb{R}), \tag{3.1}$$

for some $p \geqslant 1$ (with the convention that $\frac{1}{0} = \infty$) and $\beta \in (0,1)$ such that

$$\begin{cases} |f(t,s,y_1,u) - f(t,s,y_2,u)| \leqslant \dfrac{L(s)|y_1 - y_2|}{(t-s)^{1-\beta}}, & \forall (t,s,u) \in \Delta \times U, \ y_1, \ y_2 \in \mathbb{R}^n, \\ |f(t,s,0,u)| \leqslant \dfrac{L(s)\rho(u,u_0) + L_0(s)}{(t-s)^{1-\beta}}, & \forall (t,s,u) \in \Delta \times U. \end{cases} \tag{3.2}$$

Note that the larger the $\beta \in (0,1)$, the weaker the singularity of the generator $f(\cdot\,,\cdot\,,\cdot\,,\cdot)$. Also, (3.2) imply

$$|f(t,s,y,u)| \leqslant \frac{L(s)\big[|y| + \rho(u,u_0)\big] + L_0(s)}{(t-s)^{1-\beta}}, \qquad \forall (t,s,y,u) \in \Delta \times \mathbb{R}^n \times U. \tag{3.3}$$

We now present the well-posedness of the state equation (1.1) in L^p spaces.

Theorem 3.1. *Let* (H1) *hold with some $p \geqslant 1$ and $\beta \in (0,1)$. Then for any $\eta(\cdot) \in L^p(0,T;\mathbb{R}^n)$ and $u(\cdot) \in \mathscr{U}^p[0,T]$,* (1.1) *admits a unique solution $y(\cdot) \equiv y(\cdot\,;\eta(\cdot),u(\cdot)) \in L^p(0,T;\mathbb{R}^n)$, and the following estimates hold*

$$\|y(\cdot)\|_p \leqslant \|\eta(\cdot)\|_p + K\Big(1 + \|\rho(u(\cdot),u_0)\|_p\Big). \tag{3.4}$$

If $(\eta_1(\cdot),u_1(\cdot))$, $(\eta_2(\cdot),u_2(\cdot)) \in L^p(0,T;\mathbb{R}^n) \times \mathscr{U}^p[0,T]$ and $y_1(\cdot)$, $y_2(\cdot)$ are the solutions of (1.1) *corresponding to $(\eta_1(\cdot),u_1(\cdot))$ and $(\eta_2(\cdot),u_2(\cdot))$, respectively, then*

$$\begin{aligned} \|y_1(\cdot) - y_2(\cdot)\|_p \leqslant K\Big\{ &\|\eta_1(\cdot) - \eta_2(\cdot)\|_p \\ &+ \Big[\int_0^T \Big(\int_0^t |f(t,s,y_1(s),u_1(s)) - f(t,s,y_1(s),u_2(s))| ds \Big)^p dt \Big]^{\frac{1}{p}} \Big\}. \end{aligned} \tag{3.5}$$

Proof. Fix any $\eta(\cdot)\in L^p(0,T;\mathbb{R}^n)$ and $u(\cdot)\in\mathcal{U}^p[0,T]$. For any $z(\cdot)\in L^p(0,T;\mathbb{R}^n)$, define

$$\mathscr{T}[z(\cdot)](t)=\eta(t)+\int_0^t f(t,s,z(s),u(s))ds,\qquad t\in[0,T].$$

Denote $\theta(t)=t^{\beta-1}I_{(0,\infty)}(t)$, where $I_{(0,\infty)}$ is the characteristic function of $(0,\infty)$. Then

$$\begin{aligned}\|\mathscr{T}[z(\cdot)]\|_p&\leqslant\|\eta(\cdot)\|_p+\Big(\int_0^T\Big|\int_0^t f(t,s,z(s),u(s))ds\Big|^p dt\Big)^{\frac1p}\\&\leqslant\|\eta(\cdot)\|_p+\Big[\int_0^T\Big(\int_0^t\frac{L(s)\big[|z(s)|+\rho(u(s),u_0)\big]+L_0(s)}{(t-s)^{1-\beta}}ds\Big)^p dt\Big]^{\frac1p}\\&\leqslant\|\eta(\cdot)\|_p+\|\theta(\cdot)*\big\{L(\cdot)\big[\rho(u(\cdot),u_0)+|z(\cdot)|\big]\big\}\|_p+\|\theta(\cdot)*L_0(\cdot)\|_p\equiv\|\eta(\cdot)\|_p+I_1+I_0.\end{aligned}\tag{3.6}$$

Now, we split the proof into three cases.

Case 1. $p>\frac{1}{1-\beta}$. In this case,

$$\frac1\beta>\frac{p}{p-1},\qquad\frac{p}{1+\beta p}>1.$$

For any $\varepsilon\in(0,\frac{\beta}{1-\beta})$, which is equivalent to $(1-\beta)(1+\varepsilon)<1$, define q through the following:

$$\frac1q=\frac1p+1-\frac{1}{1+\varepsilon}<\frac1p+1-\frac{1}{1+\frac{\beta}{1-\beta}}=\frac1p+\beta<1.$$

The last inequality in the above follows from $p>\frac{1}{1-\beta}$. Thus, $1<q<p$ and

$$q\searrow\frac{p}{1+\beta p}>1,\qquad\text{as }\varepsilon\nearrow\frac{\beta}{1-\beta}.$$

Since $L_0(\cdot)\in L^{\frac{p}{1+\beta p}+}(0,T;\mathbb{R})$, we may assume that $L_0(\cdot)\in L^q(0,T;\mathbb{R})$ (for an ε being close enough to $\frac{\beta}{1-\beta}$). Hence, by Young's inequality,

$$I_0\equiv\|\theta(\cdot)*L_0(\cdot)\|_p\leqslant\|\theta(\cdot)\|_{1+\varepsilon}\|L_0(\cdot)\|_q.$$

Also,

$$\frac1q-\frac1p=1-\frac{1}{1+\varepsilon}<\beta.$$

Thus,

$$\frac{p-q}{pq}\nearrow\beta,\qquad\text{as }\varepsilon\nearrow\frac{\beta}{1-\beta},$$

which is equivalent to

$$\frac{pq}{p-q}\searrow\frac1\beta,\qquad\text{as }\varepsilon\nearrow\frac{\beta}{1-\beta}.$$

Hence, by $L(\cdot)\in L^{\frac1\beta+}(0,T;\mathbb{R})$, we could find ε which is close enough to $\frac{\beta}{1-\beta}$ so that $L(\cdot)\in L^{\frac{pq}{p-q}}(0,T;\mathbb{R})$. Then

$$\begin{aligned}I_1&\equiv\|\theta(\cdot)*\big\{L(\cdot)\big[|z(\cdot)|+\rho(u(\cdot),u_0)\big]\big\}\|_p\\&\leqslant\|\theta\|_{1+\varepsilon}\|L(\cdot)\big[|z(\cdot)|+\rho(u(\cdot),u_0)\big]\|_q\leqslant\|\theta(\cdot)\|_{1+\varepsilon}\|L(\cdot)\|_{\frac{pq}{p-q}}\|\,|z(\cdot)|+\rho(u(\cdot),u_0)\|_p.\end{aligned}$$

Then we see that $\mathscr{T}:L^p(0,T;\mathbb{R}^n)\to L^p(0,T;\mathbb{R}^n)$. Next, let $z_1(\cdot),z_2(\cdot)\in L^p(0,T;\mathbb{R}^n)$, we look at the following:

$$\begin{aligned}&\|\mathscr{T}[z_1(\cdot)]-\mathscr{T}[z_2(\cdot)]\|_{L^p(0,\delta;\mathbb{R}^n)}\equiv\Big(\int_0^\delta|\mathscr{T}[z_1(\cdot)](t)-\mathscr{T}[z_2(\cdot)](t)|^p dt\Big)^{\frac1p}\\&\leqslant\Big[\int_0^\delta\Big|\int_0^t\Big(f(t,s,z_1(s),u(s))-f(t,s,z_2(s),u(s))\Big)ds\Big|^p dt\Big]^{\frac1p}\\&\leqslant\Big[\int_0^\delta\Big(\int_0^t\frac{L(s)|z_1(s)-z_2(s)|}{(t-s)^{1-\beta}}ds\Big)^p dt\Big]^{\frac1p}\le\|\theta(\cdot)*\big[L(\cdot)|z_1(\cdot)-z_2(\cdot)|\big]\|_{L^p(0,\delta;\mathbb{R})}\\&\leqslant\|\theta(\cdot)\|_{L^{1+\varepsilon}(0,\delta;\mathbb{R})}\|L(\cdot)|z_1(\cdot)-z_2(\cdot)|\,\|_{L^q(0,\delta;\mathbb{R})}\leqslant\Big(\frac{\delta^{\beta-(1-\beta)\varepsilon}}{\beta-(1-\beta)\varepsilon}\Big)^{\frac{1}{1+\varepsilon}}\|L(\cdot)\|_{L^{\frac{pq}{p-q}}(0,\delta;\mathbb{R})}\|z_1(\cdot)-z_2(\cdot)\|_{L^p(0,\delta;\mathbb{R}^n)}.\end{aligned}$$

Clearly, for $\delta > 0$ small, the map $\mathscr{T} : L^p(0, \delta; \mathbb{R}^n) \to L^p(0, \delta; \mathbb{R}^n)$ is a contraction. Hence, it admits a unique fixed point on $L^p(0, \delta; \mathbb{R}^n)$, which is the unique solution of the state equation (1.1) on $[0, \delta]$.

Next, we look (1.1) on $[\delta, T]$, which can be written as

$$y(t) = \eta(t) + \int_0^\delta f(t, s, y(s), u(s))ds + \int_\delta^t f(t, s, y(s), u(s))ds, \qquad t \in [\delta, T].$$

Since (similar to (3.6))

$$\begin{aligned}
&\Big\|\eta(\cdot) + \int_0^\delta f(\cdot\,, s, y(s), u(s))ds\Big\|_{L^p(\delta,T;\mathbb{R}^n)} = \Big[\int_\delta^T \Big|\eta(t) + \int_0^\delta f(t, s, y(s), u(s))ds\Big|^p dt\Big]^{\frac{1}{p}} \\
&\leqslant \Big(\int_\delta^T |\eta(t)|^p dt\Big)^{\frac{1}{p}} + \Big[\int_\delta^T \Big(\int_0^\delta |f(t, s, y(s), u(s))|ds\Big)^p dt\Big]^{\frac{1}{p}} \\
&\leqslant \|\eta(\cdot)\|_{L^p(\delta,T;\mathbb{R}^n)} + \Big[\int_\delta^T \Big(\int_0^\delta \frac{L(s)\big[|y(s)| + \rho(u(s), u_0)\big] + L_0(s)}{(t-s)^{1-\beta}} ds\Big)^p dt\Big]^{\frac{1}{p}} \\
&\leqslant \|\eta(\cdot)\|_{L^p(\delta,T;\mathbb{R}^n)} + \Big[\int_\delta^T \Big(\int_0^\delta \frac{L(s)\big[|y(s)| + \rho(u(s), u_0)\big]}{(t-s)^{1-\beta}} ds\Big)^p dt\Big]^{\frac{1}{p}} + \Big[\int_\delta^T \Big(\int_0^\delta \frac{L_0(s)}{(t-s)^{1-\beta}} ds\Big)^p dt\Big]^{\frac{1}{p}} \\
&\leqslant \|\eta(\cdot)\|_{L^p(\delta,T;\mathbb{R}^n)} + K\Big(\|y(\cdot)\|_{L^p(0,\delta;\mathbb{R}^n)} + \|\rho(u(\cdot), u_0)\|_{L^p(0,\delta;\mathbb{R})} + 1\Big).
\end{aligned}$$

Then using the same argument as above, we obtain the existence and uniqueness of the solution to the state equation on $[0, 2\delta]$. By induction, we could get the solvability of the state equation on $[0, T]$.

Now, let $(\eta_1(\cdot), u_1(\cdot))$, $(\eta_2(\cdot), u_2(\cdot)) \in L^p(0, T; \mathbb{R}^n) \times \mathscr{U}^p[0, T]$ and $y_1(\cdot)$, $y_2(\cdot)$ be the corresponding solutions. Then

$$\begin{aligned}
|y_1(t) - y_2(t)| &\leqslant |\eta_1(t) - \eta_2(t)| + \int_0^t |f(t, s, y_1(s), u_1(s)) - f(t, s, y_1(s), u_2(s))|ds + \int_0^t \frac{L(s)|y_1(s) - y_2(s)|}{(t-s)^{1-\beta}} ds \\
&\equiv a(t) + \int_0^t \frac{L(s)|y_1(s) - y_2(s)|}{(t-s)^{1-\beta}} ds.
\end{aligned}$$

Hence, by Lemma 2.5,

$$|y_1(t) - y_2(t)| \leqslant a(t) + \sum_{i=1}^{k-1} c_i \int_0^t \frac{L(s)a(s)}{(t-s)^{1-\beta_i}} ds + c_k \int_0^t L(s)a(s)ds, \qquad \text{a.e. } t \in [0, T], \tag{3.7}$$

for some constants $c_i > 0$ and $\beta_i \in [\beta, 1)$. Consequently, similar to (3.6),

$$\begin{aligned}
&\|y_1(\cdot) - y_2(\cdot)\|_p \leqslant K\Big(\int_0^T a(t)^p dt\Big)^{\frac{1}{p}} \\
&\leqslant K\Big\{\|\eta_1(\cdot) - \eta_2(\cdot)\|_p + \Big[\int_0^T \Big(\int_0^t |f(t, s, y_1(s), u_1(s)) - f(t, s, y_1(s), u_2(s))|ds\Big)^p dt\Big]^{\frac{1}{p}}\Big\},
\end{aligned}$$

proving the stability estimate. We can use the similar argument to prove this estimate to get (3.4).

Case 2. $1 < p \leqslant \frac{1}{1-\beta}$. In this case,

$$\frac{1}{\beta} \leqslant \frac{p}{p-1}, \qquad \frac{p}{1+\beta p} \leqslant 1.$$

Also, since $1 - \beta \leqslant \frac{1}{p} < 1$, for any $\varepsilon \in (0, p-1)$, the following holds:

$$1 - \beta \leqslant \frac{1}{p} < \frac{1}{1+\varepsilon}.$$

This implies $(1-\beta)(1+\varepsilon)<1$. Define q through the following:

$$\frac{1}{p}<\frac{1}{q}=\frac{1}{p}+1-\frac{1}{1+\varepsilon}\nearrow 1,\qquad \text{as } \varepsilon\nearrow p-1.$$

Then

$$\frac{1}{q}-\frac{1}{p}=1-\frac{1}{1+\varepsilon}\nearrow 1-\frac{1}{p}\qquad \text{as } \varepsilon\nearrow p-1.$$

Thus,

$$\frac{pq}{p-q}\searrow\frac{p}{p-1},\qquad \text{as } \varepsilon\nearrow p-1.$$

Consequently, by choosing $\varepsilon>0$ close enough to $p-1$, we have $q>1$ close enough to 1 and $\frac{pq}{p-1}$ close enough to $\frac{p}{p-1}$. Hence,

$$\|\theta(\cdot)*L_0(\cdot)\|_p\leqslant\|\theta(\cdot)\|_{1+\varepsilon}\|L_0(\cdot)\|_q,$$

and

$$\|\theta(\cdot)*\big\{L(\cdot)\big[|z(\cdot)|+\rho(u(\cdot),u_0)\big]\big\}\|_p\leqslant\|\theta(\cdot)\|_{1+\varepsilon}\|L(\cdot)\|_{\frac{pq}{p-q}}\|\,|z(\cdot)|+\rho(u(\cdot),u_0)\|_p.$$

The rest of the proof is similar to that of Case 1.

Case 3. $p=1$. In this case, the condition reads $L(\cdot)\in L^\infty(0,T;\mathbb{R})$ and $L_0(\cdot)\in L^{1+}(0,T;\mathbb{R})$. Then

$$\|\theta(\cdot)*L_0(\cdot)\|_1\leqslant\|\theta(\cdot)\|_1\|L_0(\cdot)\|_1,$$

and

$$\|\theta(\cdot)*\big\{L(\cdot)\big[|z(\cdot)|+\rho(u(\cdot),u_0)\big]\big\}\|_1\leqslant\|\theta(\cdot)\|_1\|L(\cdot)\|_\infty\|\,|z(\cdot)|+\rho(u(\cdot),u_0)\|_1.$$

The rest of the proof is similar to that of Case 1. $\square$

Let us make some comments and observations on the above theorem. First of all, the above theorem gives some sufficient conditions under which for $(\eta(\cdot),u(\cdot))\in L^p(0,T;\mathbb{R}^n)\times\mathscr{U}^p[0,T]$, equation (1.1) admits a unique solution $y(\cdot)\in L^p(0,T;\mathbb{R}^n)$. The conditions we imposed in (H1) are compatibility conditions of the integrability for the free term $\eta(\cdot)$, the control $u(\cdot)$, and the coefficients $L(\cdot)$ and $L_0(\cdot)$. From the above, we see that if $(\eta(\cdot),u(\cdot))\in L^p(0,T;\mathbb{R})\times\mathcal{U}^p[0,T]$ with $p>\frac{1}{1-\beta}$, then by assuming $L(\cdot)\in L^{\frac{1}{\beta}+}(0,T;\mathbb{R})$ and $L_0(\cdot)\in L^{\frac{1}{\beta}-}(0,T;\mathbb{R})$ (note that $\frac{p}{1+\beta p}<\frac{1}{\beta}$), the equation has a unique solution $y(\cdot)\in L^p(0,T;\mathbb{R}^n)$. This is the case, in particular, if $\eta(\cdot)\in L^\infty(0,T;\mathbb{R}^n)$ and U is bounded (under the metric ρ). We will come back to this later. On the other hand, if $1\leqslant p<\frac{1}{1-\beta}$, that is, say, the free term and/or the control have weaker integrability, then we need to strengthen the integrability condition for $L(\cdot)$ from $L^{\frac{1}{\beta}+}$ to $L^{\frac{p}{p-1}+}$ (in the current case, $\frac{p}{p-1}>\frac{1}{\beta}$) to get L^p solution $y(\cdot)$. But, the integrability of $L_0(\cdot)$ is only required to be $L^{1+}(0,T;\mathbb{R})$. Finally, since we have used the contraction mapping theorem to establish the well-posedness of the state equation, one can see that the solution to the state equation can be obtained by a Picard iteration.

Let us present an example from which we could get some feeling about the above result.

Example 3.2. Consider the following Volterra integral equation

$$y(t)=\frac{1}{|t-1|^{1-\gamma}}+\int_0^t\frac{\sqrt{(s-1)^{2\delta-2}+y(s)^2}}{|s-1|^{1-\alpha}(t-s)^{1-\beta}}ds,\qquad \text{a.e. } t\in[0,T],\tag{3.8}$$

for some $\alpha,\beta\in(0,1)$, $\gamma,\delta\in(0,1]$, and with $T>1$. In this case, we have/can take

$$\eta(t)=\frac{1}{|t-1|^{1-\gamma}},\quad L(s)=\frac{1}{|s-1|^{1-\alpha}},\quad L_0(s)=\frac{1}{|s-1|^{2-\alpha-\delta}}.$$

In order $\eta(\cdot)\in L^p(0,T;\mathbb{R})$, we need

$$p(1-\gamma)<1\qquad\Longleftrightarrow\qquad p<\frac{1}{1-\gamma}\quad(\frac{1}{0}\triangleq\infty).$$

In order $L(\cdot)\in L^{(\frac{1}{\beta}\vee\frac{p}{p-1})+}(0,T;\mathbb{R})$, one needs

$$\frac{1-\alpha}{\beta}<1\qquad\iff\qquad\alpha+\beta>1,$$

and

$$(1-\alpha)\frac{p}{p-1}<1\qquad\iff\qquad 1-\alpha<1-\frac{1}{p}\qquad\iff\qquad p>\frac{1}{\alpha}.$$

Finally, in order $L_0(\cdot)\in L^{(\frac{p}{1+\beta p}\vee 1)+}(0,T;\mathbb{R})$, one needs

$$(2-\alpha-\delta)\frac{p}{1+\beta p}<1\qquad\iff\qquad 2-\alpha-\beta-\delta<\frac{1}{p}\qquad\iff\qquad p<\frac{1}{(2-\alpha-\beta-\delta)^+},$$

and

$$2-\alpha-\delta<1\qquad\iff\qquad\alpha+\delta>1.$$

Hence, equation (3.8) has a unique solution $y(\cdot)\in L^p(0,T;\mathbb{R}^n)$ for any $p\in[\frac{1}{\alpha},\frac{1}{(1-\gamma)\vee(2-\alpha-\beta-\delta)^+})$, provided

$$\alpha+\beta>1,\qquad\alpha+\delta>1. \tag{3.9}$$

We point out that in general, the solution $y(\cdot)$ of the equation (3.8) is not necessarily continuous, even if the free term $\eta(\cdot)$ is continuous. In fact, let $\gamma=1$. Then $\eta(t)\equiv 1$ which is continuous. It is seen that the solution $y(\cdot)$ is positive (which can be seen from a Picard iteration). Consequently,

$$\lim_{t\to 1}y(t)\geqslant 1+\lim_{t\to 1}\int_0^t\frac{ds}{|s-1|^{2-\alpha-\delta}(t-s)^{1-\beta}}=\int_0^1\frac{ds}{(1-s)^{3-\alpha-\beta-\delta}}=\infty,$$

provided

$$3-\alpha-\beta-\delta>1\qquad\iff\qquad\alpha+\beta+\delta<2. \tag{3.10}$$

This will be the case if we take

$$\alpha=\frac{2}{3},\quad\beta=\delta=\frac{1}{2}.$$

In this case, the solution $y(\cdot)\in L^p(0,T;\mathbb{R})$ exists with $p\in(\frac{3}{2},3)$ and it is discontinuous at $t=1$.

Note that in the above example, the solution $y(\cdot)$ is discontinuous at $t=1$ only, which is the singularity of $L(\cdot)$ and $L_0(\cdot)$. It is natural to ask what will be the result for the general situation? Such a question has its own interest. And also since the values $y(t_i)$ of $y(\cdot)$ are needed in the cost functional (1.2), we would like to locate the discontinuity points of the solution $y(\cdot)$ a priori based on the information of $L(\cdot)$ and $L_0(\cdot)$. This leads to the following subsection.

3.2 Continuity of the solution

In this subsection, we would like to explore the continuity of the solution $y(\cdot)$ to the state equation (1.1). Let us begin with some observations. Suppose $y(\cdot)\in L^p(0,T;\mathbb{R}^n)$ is the unique solution to the state equation (1.1) which is rewritten here:

$$y(t)=\eta(t)+\int_0^t f(t,s,y(s),u(s))ds,\qquad t\in[0,T]. \tag{3.11}$$

Then the continuity of $y(\cdot)$ is determined by that of $\eta(\cdot)$ and

$$\psi(\cdot)\equiv\int_0^{\cdot}f(\cdot\,,s,y(s),u(s))ds.$$

The continuity of $\eta(\cdot)$ should be given a priori. Thus, we need to look at the continuity of the above-defined function $\psi(\cdot)$. Hence, the preliminary results presented in Section 2 will play an interesting role here. To make it precise, we introduce the following hypothesis.

(H2) Let $w(\cdot)$ be given by (2.8) with $\alpha_i \in (0,1)$, $0 \leqslant i \leqslant \ell$ and $0 \leq s_0 < s_1 < \cdots < s_\ell \leqslant T$. Let $f : \Delta \times \mathbb{R}^n \times U \to \mathbb{R}^n$ be given by the following:

$$f(t,s,y,u) = \frac{f_0(t,s,y,u)}{w(s)(t-s)^{1-\beta}}, \qquad (t,s,y,u) \in \Delta \times \mathbb{R}^n \times U, \tag{3.12}$$

with $\beta \in (0,1)$ and $f_0 : \Delta \times \mathbb{R}^n \times U \to \mathbb{R}^n$ being measurable such that

$$|f_0(t,s,y,u) - f_0(t',s,y,u)| \leqslant \omega(|t-t'|), \qquad \forall (t,s),\ (t',s) \in \Delta,\ (y,u) \in \mathbb{R}^n \times U, \tag{3.13}$$

for some modulus of continuity $\omega : [0,\infty) \to [0,\infty)$, and

$$|f_0(t,s,y,u)| \leqslant \bar{\varphi}(s), \qquad \forall (t,s,y,u) \in \Delta \times \mathbb{R}^n \times U, \tag{3.14}$$

and

$$|f_0(t,s,y_1,u) - f_0(t,s,y_2,u)| \leqslant \bar{L}(s)(|y_1 - y_2|), \qquad (t,s,u) \in \Delta \times U,\ y_1,\ y_2 \in \mathbb{R}^n, \tag{3.15}$$

for some measurable functions $\bar{\varphi},\ \bar{L} : [0,T] \to [0,\infty)$.

Note that under (H2), we will have (H1) if one takes the following:

$$L_0(s) = \frac{\bar{\varphi}(s)}{w(s)}, \qquad L(s) = \frac{\bar{L}(s)}{w(s)}, \qquad s \in [0,T].$$

Thus, according to Theorem 3.1, state equation (1.1) admits a unique solution in $L^p(0,T;\mathbb{R}^n)$, under (H2), for any $\eta(\cdot) \in L^p(0,T;\mathbb{R}^n)$ with some $p \in [1,\infty)$, if

$$\frac{\bar{\varphi}(\cdot)}{w(\cdot)} \in L^{(\frac{p}{1+\beta p} \vee 1)+}(0,T;\mathbb{R}), \qquad \frac{\bar{L}(\cdot)}{w(\cdot)} \in L^{(\frac{1}{\beta} \vee \frac{p}{p-1})+}(0,T;\mathbb{R}). \tag{3.16}$$

Let us make a simple observation on the above condition. Recall the definition of δ_0 and $\bar{s}_i$ from (2.20). It is not hard to see that (3.16) holds if $(1-\alpha_i)(\frac{1}{\beta} \vee \frac{p}{p-1}) < 1$, $0 \leqslant i \leqslant \ell$ and for some $\varepsilon \in (0,\delta_0)$, the following holds:

$$\begin{cases} \bar{\varphi}(\cdot) \in L^{(\frac{p}{1+\beta p} \vee 1)+}([0,T] \setminus \bigcup\limits_{i=0}^{\ell}((s_i - \varepsilon) \vee 0, (s_i + \varepsilon) \wedge T);\mathbb{R}), \\ \bar{L}(\cdot) \in L^{(\frac{1}{\beta} \vee \frac{p}{p-1})+}([0,T] \setminus \bigcup\limits_{i=0}^{\ell}((s_i - \varepsilon) \vee 0, (s_i + \varepsilon) \wedge T);\mathbb{R}), \\ \bar{\varphi}(\cdot), \bar{L}(\cdot) \in L^\infty((s_i - \varepsilon) \vee 0, (s_i + \varepsilon) \wedge T;\mathbb{R}), \qquad 0 \leqslant i \leqslant \ell. \end{cases}$$

Namely, due to the special structure of $w(\cdot)$, it suffices to have boundedness of $\bar{\varphi}(\cdot)$ and $\bar{L}(\cdot)$ near s_i $(0 \leqslant i \leqslant \ell)$ and proper integrability of these functions away from the points s_i. Therefore, the condition (3.16) is very mild.

We have the following result which is a direct consequence of Lemma 2.4.

Proposition 3.3. *Let* (H2) *hold with* (3.16) *for some* $p \in [1,\infty)$, *and* $\bar{\varphi}(\cdot) \in L^q(0,T;\mathbb{R})$, $q > \dfrac{1}{\beta} \vee \dfrac{1}{\alpha_i}$, *for all* $i = 0,1,\cdots,\ell$. *Then for any* $\eta(\cdot) \in L^p(0,T;\mathbb{R}^n)$ *and any* $u(\cdot) \in \mathscr{U}^p[0,T]$, *state equation* (1.1) *admits a unique solution* $y(\cdot) \in L^p(0,T;\mathbb{R}^n)$ *such that*

$$y(\cdot) - \eta(\cdot) \in L^\infty_{\bar{w}(\cdot)}((0,T);\mathbb{R}^n) \bigcap C_{\bar{w}^\varepsilon(\cdot)}([0,T];\mathbb{R}^n), \tag{3.17}$$

where $\bar{w}(\cdot)$ *and* $\bar{w}^\varepsilon(\cdot)$ *are given in* (2.15) *and* (2.17).

3.3 Special cases

In this subsection, we look at some special cases.

1. Linear Volterra integral equations. Consider the following equation:

$$y(t)=\eta(t)+\int_0^t \frac{A(t,s)y(s)}{w(s)(t-s)^{1-\beta}}ds, \qquad t\in[0,T], \tag{3.18}$$

where $\beta\in(0,1)$, $w(\cdot)$ is a weight function defined by (2.8) and $A:\Delta\to\mathbb{R}^n$ satisfies

$$|A(t,s)|\leqslant \bar L(s), \qquad \forall (t,s)\in\Delta, \tag{3.19}$$

for some measurable function $\bar L(\cdot)$ satisfying

$$\frac{\bar L(\cdot)}{w(\cdot)}\in L^{(\frac{1}{\beta}\vee\frac{p}{p-1})+}(0,T;\mathbb{R}), \tag{3.20}$$

with some $p\in[1,\infty)$. Then, by Theorem 3.1, for any $\eta(\cdot)\in L^p(0,T;\mathbb{R}^n)$, equation (3.18) admits a unique solution $y(\cdot)\in L^p(0,T;\mathbb{R}^n)$. Moreover, if we define operator $\mathcal{A}$ by

$$\mathcal{A}[y(\cdot)](t)=\int_0^t \frac{A(t,s)y(s)}{w(s)(t-s)^{1-\beta}}ds, \qquad t\in[0,T],$$

then, thanks to (3.20), by the proof of Theorem 3.1, we see that $\mathcal{A}:L^p(0,T;\mathbb{R}^n)\to L^p(0,T;\mathbb{R}^n)$ is a linear bounded operator. Our linear integral equation (3.18) reads

$$y(\cdot)=\eta(\cdot)+\mathcal{A}[y(\cdot)].$$

Therefore, the unique solution $y(\cdot)$ admits the following (abstract) representation:

$$y(\cdot)=(I-\mathcal{A})^{-1}\eta(\cdot)=\sum_{k=0}^\infty \mathcal{A}^k\eta(\cdot).$$

Now, let $(t,\tau)\mapsto\Phi(t,\tau)$ be the unique solution to the following equation:

$$\Phi(t,\tau)=\frac{A(t,\tau)}{w(\tau)(t-\tau)^{1-\beta}}+\int_\tau^t \frac{A(t,s)\Phi(s,\tau)}{w(s)(t-s)^{1-\beta}}ds, \qquad 0\leqslant\tau<t\leqslant T. \tag{3.21}$$

Then one has

$$y(t)=\eta(t)+\int_0^t \Phi(t,s)\eta(s)ds, \quad \text{a.e. } t\in[0,T]. \tag{3.22}$$

This is called the *variation of constant formula.*

2. Fractional differential equations. Let us first recall some basic notions of fractional integrals and derivatives. For $\alpha\in(0,1)$, let

$$[I^\alpha f(\cdot)](t)=\frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds, \qquad t\geqslant 0, \tag{3.23}$$

and as long as the right hand side is well-defined, where $\Gamma(\cdot)$ is the Gamma function. We call I^α the α-th order integral operator. Let

$$[D^\alpha y(\cdot)](t)=\frac{d}{dt}[I^{1-\alpha}y(\cdot)](t)\equiv\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{y(s)}{(t-s)^\alpha}ds, \tag{3.24}$$

and

$$[D^\alpha_* y(\cdot)](t)=[D^\alpha(y(\cdot)-y(0))](t)=[D^\alpha(y(\cdot)](t)-\frac{y(0)}{\Gamma(1-\alpha)}t^{-\alpha}. \tag{3.25}$$

In particular, when $y(\cdot) \in AC([0,T];\mathbb{R})$, the set of all absolutely continuous functions defined on $[0,T]$, one has

$$[D_*^\alpha y(\cdot)](t) = [I^{1-\alpha} y'(\cdot)](t) \equiv \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(s)}{(t-s)^\alpha} ds, \tag{3.26}$$

We call D^α and D_*^α the α-th order *Riemann-Liouville* and *Caputo differential operators*, respectively. We have the following standard result (see [31], Lemmas 2.5 and 2.22).

Proposition 3.4. *Let $\alpha \in (0,1)$. Then for any $y(\cdot) \in L^1(0,T;\mathbb{R})$ with $[I^{1-\alpha} y(\cdot)](\cdot) \in AC([0,T];\mathbb{R})$.*

$$I^\alpha \{D^\alpha [y(\cdot)]\}(t) = y(t) - \frac{I^{1-\alpha}[y(\cdot)](0)}{\Gamma(\alpha) t^{1-\alpha}}, \qquad \text{a.e. } t \in (0,T]; \tag{3.27}$$

and for $y(\cdot) \in AC([0,T];\mathbb{R})$,

$$I^\alpha \{D_*^\alpha [y(\cdot)]\}(t) = y(t) - y(0). \tag{3.28}$$

Now, let us consider the following fractional differential equation of Riemann-Liouville type:

$$D^\alpha [y(\cdot)](t) = f(t, y(t), u(t)), \qquad t \in [0,T]. \tag{3.29}$$

Applying the operator I^α to the above, we obtain

$$y(t) = \frac{I^{1-\alpha}[y(\cdot)](0)}{\Gamma(\alpha) t^{1-\alpha}} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s, y(s), u(s))}{(t-s)^{1-\alpha}} ds, \qquad t \in [0,T]. \tag{3.30}$$

We refer the readers to Theorem 3.1 in [31] for the equivalence of (3.29) and (3.30).

Likewise, if we consider the following fractional differential equation of Caputo type:

$$D_*^\alpha [y(\cdot)](t) = f(t, y(t), u(t)), \qquad t \in [0,T], \tag{3.31}$$

applying the operator I^α to the above, we obtain

$$y(t) = y(0) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s, y(s), u(s))}{(t-s)^{1-\alpha}} ds, \qquad t \in [0,T]. \tag{3.32}$$

We refer the readers to Theorem 3.24 in [31] for the equivalence of (3.31) and (3.32).

From the above, we see that fractional differential equations of Riemann-Liouville and Caputo types are special cases of (1.1).

3.4 A backward linear Volterra integral equation

In this subsection, we consider the following linear backward Volterra integral equation:

$$\psi(t) = \xi(t) + \int_t^T \frac{A(s,t)^\top \psi(s)}{w(t)(s-t)^{1-\beta}} ds, \qquad t \in [0,T], \tag{3.33}$$

where $A : \Delta \to \mathbb{R}^{n\times n}$ satisfies (3.19)–(3.20). Such an equation will play an important role in the next section. Let $1 < p < \frac{1}{1-\beta}$. We claim that for any $\xi(\cdot) \in L^{\frac{p}{p-1}}(0,T;\mathbb{R}^n)$, the above equation admits a unique solution $\psi(\cdot) \in L^{\frac{p}{p-1}}(0,T;\mathbb{R}^n)$. In fact, by condition (3.20), we can find an $r > \frac{1}{\beta} \vee \frac{p}{p-1}$ such that $\frac{\bar{L}(\cdot)}{w(\cdot)} \in L^r(0,T;\mathbb{R})$. By $r > \frac{1}{\beta}$, we can find an $\varepsilon > 0$ such that

$$\frac{1}{1+\varepsilon} = 1 - \frac{1}{r} > 1 - \beta \quad \Rightarrow \quad (1+\varepsilon)(1-\beta) < 1.$$

Then, for any $\psi(\cdot) \in L^{\frac{p}{p-1}}(0,T;\mathbb{R}^n)$, we have (denoting $p' = \frac{p}{p-1}$)

$$
\begin{aligned}
&\Big\|\int_\cdot^T \frac{A(s,\cdot)^\top \psi(s)}{w(\cdot)(s-\cdot)^{1-\beta}}ds\Big\|_{p'} = \Big\{\int_0^T\Big|\int_t^T \frac{A(s,t)^\top\psi(s)}{w(t)(s-t)^{1-\beta}}ds\Big|^{p'}dt\Big\}^{\frac{1}{p'}} \leqslant \Big\{\int_0^T\Big(\frac{\bar L(t)}{w(t)}\int_t^T\frac{|\psi(s)|}{(s-t)^{1-\beta}}ds\Big)^{p'}dt\Big\}^{\frac{1}{p'}}\\
&\leqslant \Big[\int_0^T\Big(\frac{\bar L(t)}{w(t)}\Big)^r dt\Big]^{\frac{1}{r}}\Big[\int_0^T\Big(\int_t^T\frac{|\psi(s)|}{(s-t)^{1-\beta}}ds\Big)^{\frac{p'r}{r-p'}}dt\Big]^{\frac{r-p'}{p'r}}\\
&\leqslant \Big\|\frac{\bar L(\cdot)}{w(\cdot)}\Big\|_r \|\theta(\cdot) * |\psi(\cdot)|\,\|_{\frac{p'r}{r-p'}} \leqslant \Big\|\frac{\bar L(\cdot)}{w(\cdot)}\Big\|_r \|\theta(\cdot)\|_{1+\varepsilon}\|\psi(\cdot)\|_{p'}.
\end{aligned}
$$

Here, the Young's inequality for convolution is used with

$$
\frac{1}{1+\varepsilon}+\frac{1}{p'} = \frac{r-p'}{p'r}+1 = \frac{1}{p'}-\frac{1}{r}+1 \qquad \Longleftrightarrow \qquad \frac{1}{1+\varepsilon} = 1-\frac{1}{r}.
$$

By a similar argument used in the proof of Theorem 3.1, we get the well-posedness of equation (3.33).

4 Pontryagin's Maximum Principle

In this section, we discuss the optimal control problem for equation (1.1) with cost functional (1.2). To begin with, let us introduce the following assumptions. The conditions assumed are more than sufficient. But for the simplicity of presentation, we prefer to use these stronger conditions.

(H3) Let $h^j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, \cdots, m$ be continuously differentiable, and $g : [0,T]\times\mathbb{R}^n\times U \to \mathbb{R}$ be measurable with $y \mapsto g(t,y,u)$ being continuously differentiable. There exist a constant $L > 0$ and a modulus of continuity $\omega : [0,+\infty) \to [0,+\infty)$ such that

$$
\begin{aligned}
&|g(t,y_1,u_1)-g(t,y_2,u_2)| \leqslant L|y_1-y_2| + \omega\big(\rho(u_1,u_2)\big), \qquad \forall\,(t,y_1,u_1),(t,y_2,u_2)\in[0,T]\times\mathbb{R}^n\times U,\\
&|g(t,0,u)| \leqslant L, \qquad \forall (t,u)\in[0,T]\times U,\\
&|g_y(t,y_1,u_1)-g_y(t,y_2,u_2)| \leqslant \omega(|y_1-y_2|+\rho(u_1,u_2)), \qquad \forall (t,y_1,u_1),(t,y_2,u_2)\in[0,T]\times\mathbb{R}^n\times U.
\end{aligned}
$$

Suppose $0 \leqslant s_0 < s_1 < \cdots < s_\ell \leqslant T$ are given as in (H2), and $0 < t_1 < t_2 < \cdots < t_m \leqslant T$ such that

$$
t_j \notin \{s_0, s_1, \cdots, s_\ell\}, \qquad \forall j = 1,2,\cdots,m. \tag{4.1}
$$

Clearly, under (H2)–(H3), our cost functional (1.2) is well-defined. Hence, we can formulate the following optimal control problem.

Problem (P) Find a $u^*(\cdot) \in \mathscr{U}^p[0,T]$ such that

$$
J(u^*(\cdot)) = \inf_{u(\cdot)\in\mathscr{U}^p[0,T]} J(u(\cdot)). \tag{4.2}
$$

Any $u^*(\cdot)$ satisfying (4.2) is called an *optimal control* of Problem (P), the corresponding state $y^*(\cdot)$ is called an *optimal state* and $(y^*(\cdot), u^*(\cdot))$ is called an *optimal pair*.

In this section, we shall first give a set of necessary conditions for optimal pairs of Problem (P). Usually, such a result is referred to as a *Pontryagin's maximum principle*. Then, we shall show some examples.

4.1 Pontryagin's maximum principle for Problem (P)

In establishing the Pontryagin's maximum principle for the case that U is not assumed to be convex, we need the following Liapunoff type theorem (see, Corollary 3.8 of Chapter 4 in [32]).

Lemma 4.1. *Let X be a Banach space. For any $\delta > 0$, let*

$$
\mathscr{E}_\delta = \big\{E \in [0,T] \bigm| |E| = \delta T\big\},
$$

where $|E|$ stands for the Lebesgue measure of E. Then for any $h(\cdot) \in C([0,T];L^1(0,T;X))$,

$$
\inf_{E\in\mathscr{E}_\delta}\Big\|\int_0^T\Big(\frac{1}{\delta}\mathbf{1}_E(s)-1\Big)h(\cdot,s)ds\Big\|_{C([0,T];X)} = 0. \tag{4.3}
$$

The following is our main result of this section, which is called Pontryagin's maximum principle for Problem (P).

Theorem 4.2. *Let* (H2)–(H3) *hold for some* $p \geqslant 1$, $\bar{\varphi}(\cdot) \in L^q(0,T;\mathbb{R})$, $q > \frac{1}{\beta} \vee \frac{1}{\alpha_i}$, *for all* $i = 0, 1, \cdots, \ell$ *and* (3.16) *holds. Let* $f^0 : \Delta \times \mathbb{R}^n \times U \to \mathbb{R}$ *be measurable with* $y \mapsto f^0(t,s,y,u)$ *being differentiable. Let* $\eta(\cdot) \in L^p(0,T;\mathbb{R}^n)$ *and* $\eta(\cdot)$ *be continuous at* t_j, $j = 1, 2, \cdots, m$. *Suppose* $(y^*(\cdot), u^*(\cdot))$ *is an optimal pair of Problem* (P). *Then there exists a solution* $\psi(\cdot) \in L^{\frac{p}{p-1}}(0,T;\mathbb{R}^n)$ *of the following adjoint equation*

$$
\begin{aligned}
\psi(t) = & -g_y(t, y^*(t), u^*(t))^\top - \sum_{j=1}^m \mathbf{1}_{[0,t_j)}(t) f_y(t_j, t, y^*(t), u^*(t))^\top h_y^j\big((y^*(t_j)\big)^\top \\
& + \int_t^T f_y(s, t, y^*(t), u^*(t))^\top \psi(s) ds, \qquad t \in [0,T],
\end{aligned} \tag{4.4}
$$

such that the following maximum condition holds:

$$
\begin{aligned}
& \int_s^T \psi(t)^\top f(t, s, y^*(s), u^*(s)) dt - g(s, y^*(s), u^*(s)) - \sum_{j=1}^m h_y^j\big(y^*(t_j)\big) \mathbf{1}_{[0,t_j]}(s) f(t_j, s, y^*(s), u^*(s)) \\
& = \min_{u \in U} \Big[\int_s^T \psi(t)^\top f(t, s, y^*(s), u) dt - g(s, y^*(s), u) - \sum_{j=1}^m h_y^j\big(y^*(t_j)\big) \mathbf{1}_{[0,t_j]}(s) f(t_j, s, y^*(s), u) \Big], \\
& \hspace{22em} \text{a.e. } s \in [0,T].
\end{aligned} \tag{4.5}
$$

Proof. We split the proof into several steps.

Step 1. A variational inequality. Let $(y^*(\cdot), u^*(\cdot))$ be an optimal pair of Problem (P). Fix any $u(\cdot) \in \mathscr{U}^p[0,T]$. Denote

$$
u^\delta(t) = \begin{cases} u^*(t), & t \in [0,T] \setminus E_\delta, \\ u(t), & t \in E_\delta, \end{cases} \tag{4.6}
$$

with $E_\delta \subseteq [0,T]$ being measurable and undetermined (see *Step 2*). It is obvious that the control $u^\delta(\cdot)$ is in $\mathscr{U}^p[0,T]$. Let $y^\delta(\cdot) = y(\cdot\,; \eta(\cdot), u^\delta(\cdot))$ be the corresponding solution, and let

$$
Y^\delta(t) = \frac{y^\delta(t) - y^*(t)}{\delta}, \qquad t \in [0,T].
$$

Then, $Y^\delta(\cdot)$ satisfies

$$
\begin{aligned}
Y^\delta(t) = & \frac{1}{\delta} \Big\{ \int_0^t \Big[f(t, s, y^\delta(s), u^\delta(s)) - f(t, s, y^*(s), u^\delta(s)) \Big] ds \\
& \qquad + \int_0^t \Big[f(t, s, y^*(s), u^\delta(s)) - f(t, s, y^*(s), u^*(s)) \Big] ds \Big\} \\
= & \int_0^t \Big[\int_0^1 f_y(t, s, y^*(s) + \tau \delta Y^\delta(s), u^\delta(s)) d\tau \Big] Y^\delta(s) ds \\
& \qquad + \frac{1}{\delta} \int_0^t \mathbf{1}_{E_\delta}(s) \Big[f(t, s, y^*(s), u(s)) - f(t, s, y^*(s), u^*(s)) \Big] ds \\
\equiv & \int_0^t f_y^\delta(t,s) Y^\delta(s) ds + \frac{1}{\delta} \int_0^t \mathbf{1}_{E_\delta}(s) \widehat{f}(t,s) ds,
\end{aligned}
$$

with

$$
\begin{cases} f_y^\delta(t,s) = \displaystyle\int_0^1 f_y(t, s, y^*(s) + \tau \delta Y^\delta(s), u^\delta(s)) d\tau, \\ \widehat{f}(t,s) = f(t, s, y^*(s), u(s)) - f(t, s, y^*(s), u^*(s)). \end{cases} \qquad (t,s) \in \Delta. \tag{4.7}
$$

By the optimality of $(y^*(\cdot),u^*(\cdot))$, one has the following variational inequality:

$$
\begin{aligned}
0\leqslant \frac{J(u^\delta(\cdot))-J(u^*(\cdot))}{\delta}&=\frac{1}{\delta}\Big\{\int_0^T\Big[g(t,y^\delta(t),u^\delta(t))-g(t,y^*(t),u^*(t))\Big]ds\\
&\qquad+\sum_{j=1}^m\Big[h^j(y^\delta(t_j))-h^j(y^*(t_j))\Big]\Big\}\\
=&\int_0^T\Big[\int_0^1 g_y(t,y^*(t)+\tau\delta Y^\delta(t),u^\delta(t))d\tau\Big]Y^\delta(t)dt+\frac{1}{\delta}\int_{E_\delta}\big[g(t,y^*(t),u(t))-g(t,y^*(t),u^*(t))\big]dt\\
&+\sum_{j=1}^m\Big[\int_0^1 h_y^j(y^*(t_j)+\tau\delta Y^\delta(t_j))d\tau\Big]Y^\delta(t_j).
\end{aligned}\tag{4.8}
$$

Step 2. Convergence of $Y^\delta(\cdot)$ *and so on.* We introduce the following integral equation:

$$
Y(t)=\int_0^t\Big[f_y(t,s,y^*(s),u^*(s))Y(s)+\widehat f(t,s)\Big]ds,\qquad t\in[0,T],\tag{4.9}
$$

where $\widehat f(\cdot\,,\cdot)$ is given by (4.7). Under our conditions, the above admits a unique solution $Y(\cdot)$ such that

$$
Y(\cdot)\in L^p(0,T;\mathbb{R}^n)\bigcap\Big(\bigcap_{i=1}^{\ell}C\big((s_{i-1},s_i);\mathbb{R}^n\big)\Big),
$$

and

$$
Y(\cdot)\in L^\infty_{\bar w(\cdot)}(0,T;\mathbb{R}^n)\bigcap C_{\bar w^\varepsilon(\cdot)}([0,T];\mathbb{R}^n).
$$

We now show that for a suitable choice of E_δ, the following holds:

$$
\lim_{\delta\to0}\|Y^\delta(\cdot)-Y(\cdot)\|_p=0,\qquad \lim_{\delta\to0}Y^\delta(t_j)=Y(t_j),\quad 1\leqslant j\leqslant m.
$$

Note that

$$
\begin{aligned}
&\int_0^t\mathbf{1}_{E_\delta}(s)\Big[f(t,s,y^*(s),u(s))-f(t,s,y^*(s),u^*(s))\Big]ds\\
&=\int_0^t\mathbf{1}_{E_\delta}(s)\frac{f^0(t,s,y^*(s),u(s))-f^0(t,s,y^*(s),u^*(s))}{w(s)(t-s)^{1-\beta}}ds\equiv\int_0^t\mathbf{1}_{E_\delta}(s)\frac{\widehat f^0(t,s)}{w(s)(t-s)^{1-\beta}}ds,
\end{aligned}
$$

with

$$
\widehat f^0(t,s)=f^0(t,s,y^*(s),u(s))-f^0(t,s,y^*(s),u^*(s)),\qquad (t,s)\in\Delta.
$$

By (3.14), we have

$$
|\widehat f^0(t,s)|\leqslant 2\bar\varphi(s),\qquad (t,s)\in\Delta,
$$

with $\bar\varphi(\cdot)\in L^q(0,T;\mathbb{R})$, where q satisfies $q>\dfrac{1}{\beta}\vee\dfrac{1}{\alpha_i}$, for all $i=0,1,\cdots,\ell$. Let

$$
h(t,s)=\mathbf{1}_{[0,t)}(s)\frac{\widehat f^0(t,s)}{w(s)(t-s)^{1-\beta}},\qquad (t,s)\in[0,T]^2.
$$

Then for any $\bar p>1$ sufficiently large,

$$
\begin{aligned}
&\Big[\int_0^T\Big(\int_0^T|h(t,s)|ds\Big)^{\bar p}dt\Big]^{\frac{1}{\bar p}}\leqslant\Big[\int_0^T\Big(\int_0^t\frac{2\bar\varphi(s)}{w(s)(t-s)^{1-\beta}}ds\Big)^{\bar p}dt\Big]^{\frac{1}{\bar p}}\\
&\equiv\Big\|\theta(\cdot)*\frac{2\bar\varphi(\cdot)}{w(\cdot)}\Big\|_{\bar p}\leqslant 2\|\theta(\cdot)\|_{1+\varepsilon}\Big\|\frac{\bar\varphi(\cdot)}{w(\cdot)}\Big\|_q<\infty,
\end{aligned}
$$

where $\varepsilon > 0$ is chosen so that

$$\frac{1}{1+\varepsilon} + \frac{1}{q} = \frac{1}{\bar{p}} + 1, \qquad (1+\varepsilon)(1-\beta) < 1.$$

Then

$$\frac{1}{\bar{p}} = \frac{1}{q} + \frac{1}{1+\varepsilon} - 1 = -\Big(\beta - \frac{1}{q}\Big) + \frac{1-(1+\varepsilon)(1-\beta)}{1+\varepsilon}.$$

Note that on the right hand side of the above, the first term is valued in $(-\beta, 0)$ since $q > \frac{1}{\beta}$; and the second term is positive with the range

$$\Big\{\frac{1-(1+\varepsilon)(1-\beta)}{1+\varepsilon} \mid \varepsilon \in [0, \frac{\beta}{1-\beta}]\Big\} = [0, \beta].$$

Thus, by suitably choosing $\varepsilon > 0$, we may make $\bar{p} > 0$ as large as we wish. Consequently, for our given $p \geq 1$, by choosing $\varepsilon > 0$ properly, we may have $\bar{p} \geqslant p$. Hence,

$$\Big[\int_0^T \Big(\int_0^T |h(t,s)| ds\Big)^p dt\Big]^{\frac{1}{p}} < \infty.$$

Clearly, there exists a sequence of continuous functions $h_k(\cdot\,,\cdot)$ such that

$$\Big[\int_0^T \Big(\int_0^T |h(t,s) - h_k(t,s)| ds\Big)^p dt\Big]^{\frac{1}{p}} < \frac{1}{k}, \qquad \forall k \geqslant 1.$$

Now, for each $h_k(\cdot\,,\cdot)$, applying Lemma 4.1, we have that for any fixed $\delta > 0$, there exists some $E^k \subseteq \mathscr{E}_\delta$ such that

$$\sup_{t\in[0,T]} \Big| \int_0^T \Big(\frac{1}{\delta}\mathbf{1}_{E^k}(s) - 1\Big) h_k(t,s) ds\Big| < \frac{1}{k}.$$

Then

$$\begin{aligned}
&\Big\{\int_0^T \Big|\int_0^t \Big(\frac{1}{\delta}\mathbf{1}_{E^k}(s) - 1\Big)\frac{\widehat{f}^0(t,s)}{w(s)(t-s)^{1-\beta}} ds\Big|^p dt\Big\}^{\frac{1}{p}} = \Big\{\int_0^T \Big|\int_0^T \Big(\frac{1}{\delta}\mathbf{1}_{E^k}(s) - 1\Big) h(t,s) ds\Big|^p dt\Big\}^{\frac{1}{p}} \\
&\leqslant \Big\{\int_0^T \Big|\int_0^T \Big(\frac{1}{\delta}\mathbf{1}_{E^k}(s) - 1\Big) h_k(t,s) ds\Big|^p dt\Big\}^{\frac{1}{p}} \\
&\quad + \Big\{\int_0^T \Big|\int_0^T \Big(\frac{1}{\delta}\mathbf{1}_{E^k}(s) - 1\Big)\big[h(t,s) - h_k(t,s)\big] ds\Big|^p dt\Big\}^{\frac{1}{p}} \leqslant \frac{1}{k}\Big(\frac{1}{\delta} + 2\Big).
\end{aligned}$$

Hence, for any fixed $\delta > 0$,

$$\inf_{E\in\mathscr{E}_\delta} \Big\{\int_0^T \Big|\int_0^t \Big(\frac{1}{\delta}\mathbf{1}_E(s) - 1\Big)\frac{\widehat{f}^0(t,s)}{w(s)(t-s)^{1-\beta}} ds\Big|^p dt\Big\}^{\frac{1}{p}} = 0.$$

Next, for any t_j, let us observe

$$\int_0^T \mathbf{1}_{[0,t_j]}(s)\frac{|\widehat{f}^0(t_j,s)|}{w(s)(t_j-s)^{1-\beta}} ds = \int_0^{t_j} \frac{|\widehat{f}^0(t_j,s)|}{w(s)(t_j-s)^{1-\beta}} ds \leqslant \int_0^{t_j} \frac{2\bar{\varphi}(s)}{w(s)(t_j-s)^{1-\beta}} ds < \infty.$$

Hence,

$$\inf_{E\in\mathscr{E}_\delta} \Big|\int_0^{t_j} \Big(\frac{1}{\delta}\mathbf{1}_E(s) - 1\Big)\frac{\widehat{f}^0(t_j,s)}{w(s)(t_j-s)^{1-\beta}} ds\Big| = 0, \qquad 1 \leqslant j \leqslant m.$$

Likewise, we also have

$$\inf_{E\in\mathscr{E}_\delta} \Big|\int_0^T \Big(\frac{1}{\delta}\mathbf{1}_E(s) - 1\Big)\big[g(s, y^*(s), u(s)) - g(s, y^*(s), u^*(s))\big] ds\Big| = 0.$$

Now, we consider the following map

$$H(t,s)=\begin{pmatrix}\dfrac{\widehat{f}^0(t,s)}{w(s)(t-s)^{1-\beta}}\mathbf{1}_{[0,t)}(s)\\ \dfrac{\widehat{f}^0(t_1,s)}{w(s)(t_1-s)^{1-\beta}}\mathbf{1}_{[0,t_1)}(s)\\ \vdots\\ \dfrac{\widehat{f}^0(t_m,s)}{w(s)(t_m-s)^{1-\beta}}\mathbf{1}_{[0,t_m)}(s)\\ g(s,y^*(s),u(s))-g(s,y^*(s),u^*(s))\end{pmatrix}$$

Then applying Lemma 4.1 to the above function in a proper product space, we obtain that for any $\delta>0$, there exists an $E_\delta\in\mathscr{E}_\delta$ such that

$$\begin{cases}\Big\{\displaystyle\int_0^T\Big|\int_0^t\Big(\frac{1}{\delta}\mathbf{1}_{E_\delta}(s)-1\Big)\frac{\widehat{f}^0(t,s)}{w(s)(t-s)^{1-\beta}}ds\Big|^p dt\Big\}^{\frac{1}{p}}=o(1),\\ \Big|\displaystyle\int_0^{t_j}\Big(\frac{1}{\delta}\mathbf{1}_{E_\delta}(s)-1\Big)\frac{\widehat{f}^0(t_j,s)}{w(s)(t_j-s)^{1-\beta}}ds\Big|=o(1),\qquad 1\leqslant j\leqslant m,\\ \Big|\displaystyle\int_0^T\Big(\frac{1}{\delta}\mathbf{1}_{E_\delta}(s)-1\Big)\big[g(s,y^*(s),u(s))-g(s,y^*(s),u^*(s))\big]ds\Big|=o(1).\end{cases}\tag{4.10}$$

By choosing such a family of E_δ, $\delta>0$, we see that the following convergence hold:

$$\begin{cases}\displaystyle\lim_{\delta\to0}\|Y^\delta(\cdot)-Y(\cdot)\|_p=0,\\ \displaystyle\lim_{\delta\to0}Y^\delta(t_j)=Y(t_j),\qquad 1\leqslant j\leqslant m,\\ \displaystyle\lim_{\delta\to0}\int_0^T\Big(\frac{1}{\delta}\mathbf{1}_{E_\delta}(s)-1\Big)\big[g(s,y^*(s),u(s))-g(s,y^*(s),u^*(s))\big]ds=0.\end{cases}$$

Hence, we end up with the following variational inequality

$$\begin{aligned}0&\leqslant\int_0^T\Big(g_y(t,y^*(t),u^*(t))Y(t)+g(t,y^*(t),u(t))-g(t,y^*(t),u^*(t))\Big)dt+\sum_{j=1}^m h_y^j\big(y^*(t_j)\big)Y(t_j)\\ &=\int_0^T\Big(g_y(s,y^*(s),u^*(s))Y(s)+g(s,y^*(s),u(s))-g(s,y^*(s),u^*(s))\Big)ds\\ &\quad+\int_0^T\sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j]}(s)\Big(f_y(t_j,s,y^*(s),u^*(s))Y(s)+f(t_j,s,y^*(s),u(s))-f(t_j,s,y^*(s),u^*(s))\Big)ds\\ &=\int_0^T\Big(g_y(s,y^*(s),u^*(s))+\sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j]}(s)f_y(t_j,s,y^*(s),u^*(s))\Big)Y(s)ds\\ &\quad+\int_0^T\Big[g(s,y^*(s),u(s))-g(s,y^*(s),u^*(s))\\ &\qquad+\sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j]}(s)\Big(f(t_j,s,y^*(s),u(s))-f(t_j,s,y^*(s),u^*(s))\Big)\Big]ds,\end{aligned}$$

with $Y(\cdot)$ being the solution to the variational equation (4.9).

Step 3. Duality. Let $\psi(\cdot)$ be the solution to the adjoint equation (4.4). Then we have

$$
\begin{aligned}
0 \leqslant & \int_0^T \Big(g_y(s,y^*(s),u^*(s)) + \sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j]}(s) f_y(t_j,s,y^*(s),u^*(s))\Big) Y(s)ds \\
& + \int_0^T \Big[g(s,y^*(s),u(s)) - g(s,y^*(s),u^*(s)) \\
& \qquad + \sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j]}(s)\Big(f(t_j,s,y^*(s),u(s)) - f(t_j,s,y^*(s),u^*(s))\Big)\Big] ds \\
= & \int_0^T \Big(-\psi(s) + \int_s^T f_y(t,s,y^*(s),u^*(s))^\top \psi(t)dt\Big)^\top Y(s)ds \\
& + \int_0^T \Big[g(s,y^*(s),u(s)) - g(s,y^*(s),u^*(s)) \\
& \qquad + \sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j]}(s)\Big(f(t_j,s,y^*(s),u(s)) - f(t_j,s,y^*(s),u^*(s))\Big)\Big] ds \\
= & \int_0^T \psi(t)^\top \Big(-Y(t) + \int_0^t f_y(t,s,y^*(s),u^*(s))Y(s)ds\Big)dt \\
& + \int_0^T \Big[g(s,y^*(s),u(s)) - g(s,y^*(s),u^*(s)) \\
& \qquad + \sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j]}(s)\Big(f(t_j,s,y^*(s),u(s)) - f(t_j,s,y^*(s),u^*(s))\Big)\Big] ds \\
= & \int_0^T \Big[-\psi(t)^\top \int_0^t \Big(f(t,s,y^*(s),u(s)) - f(t,s,y^*(s),u^*(s))\Big)ds\Big]dt \\
& \qquad + \int_0^T \Big[g(s,y^*(s),u(s)) - g(s,y^*(s),u^*(s)) \\
& \qquad + \sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j]}(s)\Big(f(t_j,s,y^*(s),u(s)) - f(t_j,s,y^*(s),u^*(s))\Big)\Big] ds \\
= & \int_0^T \Big[\int_s^T -\psi(t)^\top \Big(f(t,s,y^*(s),u(s)) - f(t,s,y^*(s),u^*(s))\Big)dt \\
& \qquad + g(s,y^*(s),u(s)) - g(s,y^*(s),u^*(s)) \\
& \qquad + \sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j]}(s)\Big(f(t_j,s,y^*(s),u(s)) - f(t_j,s,y^*(s),u^*(s))\Big)\Big] ds.
\end{aligned}
$$

Hence, using the Lebesgue point theorem for integrable functions, we reach the following:

$$
\begin{aligned}
& \int_s^T \psi(t)^\top f(t,s,y^*(s),u^*(s))dt - g(s,y^*(s),u^*(s)) - \sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j]}(s) f(t_j,s,y^*(s),u^*(s)) \\
& \geqslant \Big[\int_s^T \psi(t)^\top f(t,s,y^*(s),u)dt - g(s,y^*(s),u) - \sum_{j=1}^m h_y^j\big(y^*(t_j)\big)\mathbf{1}_{[0,t_j]}(s) f(t_j,s,y^*(s),u)\Big], \text{ a.e. } s \in [0,T].
\end{aligned}
$$

This gives the maximum condition (4.5). □

4.2 Special cases in the sense of Riemann-Liouville and Caputo senses

In recent years, optimal control problems for fractional differential equations have attracted the attention of some researchers. However, most of the works on maximum principles for fractional differential equations were established by convex perturbation technique. See, for instance, Agrawal [1], Agrawal–Defterli–Baleanu

[3], Frederico–Torres [23] and Kamocki [29] in the sense of Riemann-Liouville case, and Agrawal [2], Bourdin [11] and Hasan–Tangpong–Agrawal [25] in the sense of Caputo case.

Let us take a look a recent work [29], in which Kamocki considered the fractional differential equation of Riemann-Liouville type (3.29) with $\alpha \in (0, 1)$ and some convex assumptions. The control $u(\cdot)$ takes value in a compact set U in $\mathbb{R}^m$ and f satisfies

$$\begin{aligned}&|f(t, y_1, u) - f(t, y_2, u)| \leqslant N|y_1 - y_2|, &&\forall\, y_1, y_2 \in \mathbb{R}^n,\ t \in [0, T],\ u \in U,\\ &|f(t, 0, u)| \leqslant r(t) + \gamma|u|, &&\forall (t, u) \in [0, T] \times U,\end{aligned}$$

where $N > 0$ and $\gamma \geqslant 0$ are two constants and $r(\cdot) \in L^p(0, T; \mathbb{R})$. The corresponding solution belongs to $L^p(0, T; \mathbb{R}^n)$ for some $p \geqslant 1$. When $p > 1$ and $I^{1-\alpha}[y(\cdot)](0) = 0$, a Pontryagin's maximum principle for Problem (P) was proved. For the case $I^{1-\alpha}[y(\cdot)](0) \neq 0$, maximum principle was obtained only for $1 < p < \frac{1}{1-\alpha}$.

It is easy to check that all the above-mentioned results for fractional differential equations are the special cases of what we presented in the previous subsection.

5 Concluding Remarks

This paper presented some analysis of singular Volterra integral equations, and established a Pontryagin type maximum principle for an optimal control of such kind of equations. Here are some remarks in order.

• As we have indicated, the fractional differential equations of Riemann-Liouville or Caputo types of order no more than one are fully covered by our results. For fractional differential equations of higher order, similar results can be obtained by properly modifying our approach.

• It is easy to see that all the results that we presented will remain true for non-singular Volterra integral equations.

• We have allowed to have very general singularity in the free term and the generator. Therefore, our results can apply to a much wider class of problems than those covered by fractional differential equations and non-singular Volterra integral equations.

References

[1] O. P. Agrawal, *A general formulation and solution scheme for fractional optimal control problems,* Nonlinear Dynamics, 38 (2004), 323-337.

[2] O. P. Agrawal, *A formulation and numerical scheme for fractional optimal control problems,* Journal of Vibration and Control, 14(9-10) (2008), 1291–1299.

[3] O. P. Agrawal, O. Defterli, and D. Baleanu, *Fractional optimal control problems with several state and control variables, J. Vibration & Control,* 16 (2010), 1967–1976.

[4] T. S. Angell, *On the optimal control of systems governed by nonlinear integral equations, J. Optim. Theory Appl.*, 19 (1976), 63–79.

[5] A. A. M. Arafa, S. Z. Rida, and M. Khalil, *Solutions of fractional order model of childhook diseases with constant vaccination strategy, Math. Sci. Lett.*, 1 (2012), 17–23.

[6] S. A. Belbas, *A new method for optimal control of Volterra integral equations, Applied Math. Comp.*, 189 (2007), 1902–1915.

[7] S. A. Belbas, *A reduction method for optimal control of Volterra integral equations, Appl. Math. Comp.*, 197 (2008), 880–890.

[8] D. A. Benson, *The fractional advection-dispersion equation: development and application, Ph.D. thesis,* University of Nevada at Reno, 1998.

[9] J. F. Bonnans, C. de la Vega, and X. Dupuis, *First- and second-order optimality conditions for optimal control problems of state constrained integral equations, J. Optim. Theory Appl.*, 159 (2013), 1–40.

[10] V. I. Bogachev, *Measure Theory, I,* Springer-Verlag, New York, 2007.

[11] L. Bourdin, *A class of fractional optimal control problems and fractional Pontryagin's systems. Existence of a fractional Noether's theorem,* arXive:1203.1422v1, 2012.

[12] C. Burnap and M. A. Kazemi, *Optimal control of a system governed by nonlinear Volterra integral equations with delay, IMA J. Math. Control Inform.*, 16 (1999), 73–89.

[13] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent-II, Geophys. J. Roy. Astron. Soc.*, 13 (1967), 529–539; reprinted in *Fract. Calc. Appl. Anal.*, 11 (2008), 4–14.

[14] M. Caputo and F. Mainardi, *A new dissipation model based on memory mechanism, Pure Appl. Geophys.*, 91 (1971), 134–147; reprinted in *Fract. Calc. Appl. Anal.*, 10 (2007), 310–323.

[15] D. A. Carlson, *An elementary proof of the maximum principle for optimal control problems governed by a Volterra integral equation, J. Optim. Theory Appl.*, 54 (1987), 43–61.

[16] J. T. Chern, *Finite element modeling of viscoelastic materials on the theory of fractional calculus, Ph.D. thesis,* Pennsylvania State University, 1993.

[17] M. Dalir and M. Bashour, *Applications of fractional calculus, Appl. Math. Sci.*, 4 (2010), 1021–1032.

[18] S. Das and P. K. Gupta, *A mathematical model on fractional Lotka-Volterra equations, J. Theoretical Biology,* 277 (2011), 1–6.

[19] C. de la Vega, *Necessary conditions for optimal terminal time control problems governed by a Volterra integral equation, J. Optim. Theory Appl.*, 130 (2006), 79–93.

[20] E. Demirci, A. Unal, and N. Özalp, *A fractional order SEIR model with density dependent death rate, Hacettepe J. Math. Stat.*, 40 (2011), 287–295.

[21] K. Diethelm, *A fractional calculus based model for the simulation of an outbreak of dengue fever, Nonlinear Dynamics,* 71 (2013), 613–619.

[22] K. Diethelm and A. D. Freed, *On the solution of nonlinear fractional differential equations used in the modeling of viscoplasticity,* in: *F. Keil, W. Mackens, H. Voβ, J. Werther (eds.), Scientific Computing in Chemical Engineering II: Computational Fluid Dynamics, Reaction Engineering, and Molecular Properties,* 217–224, Springer, Heidelberg 1999.

[23] G. S. F. Frederico and D. F. M. Torres, *Fractional conservation laws in optimal control theory, Nonlinear Dynamics,* 53 (2008), 215–222.

[24] J. F. Gómez-Aguilar, R. Razo-Hernandez, and D. Granados-Lieberman, *A physical interpretation of fractional calculus in observables terms: analysis of the fractional time constant and the transitory response, Revista Mexicana de Fisica,* 60 (2014), 32–38.

[25] M. M. Hasan, X. W. Tangpong, and O. P. Agrawal, *Fractional optimal control of distributed systems in spherical and cylindrical coordinates, Journal of Vibration and Control,* 18(10) (2011), 1506–1525.

[26] D. Henry, *Geometric Theory of Semilinear Parabolic Equations,* Springer-Verlag, New York, 1981.

[27] M. I. Kamien and E. Muller, *Optimal control with integral state equations, Rev. Econ. Stud.*, 43 (1976), 469–473.

[28] R. Kamocki, *On the existence of optimal solutions to fractional optimal control problems, Appl. Math. & Computation,* 235 (2014), 94–104.

[29] R. Kamocki, *Pontryagin maximum principle for fractional ordinary optimal control problems,* Math. Meth. Appl. Sci., 37 (2014), 1668–1686.

[30] T. Kisela, *Fractional differential equations and their applications,* Diploma Thesis, BRNO Univ. of Technology, 2008.

[31] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Eqautions,* North-Holland, Amsterdam, 2006.

[32] X. Li and J. Yong, *Optimal Control Theory for Infinite Dimensional Systems,* Birkhäuser, Boston, 1995.

[33] N. G. Medhin, *Optimal process governed by integral equations, J. Math. Anal. Appl.,* 120 (1986), 1–12.

[34] R. Metzler, W. Schick, H. G. Kilian, and T. F. Nonnenmacher, *Relaxation in filled polymers: a fractional calculus approach, J. Chem. Phys.,* 103 (1995), 7180–7186.

[35] M. Moshrefi-Torbati and J. K. Hommond, *Physical and geometrical interpretation of fractional operators, J. Franklin Inst.,* 335B (1998), 1077–1086.

[36] E. Okyere, F. T. Oduro, S. K. Amponsah, I. K. Dontwi, and N. K. Frempng, *Fractional order SIR model with constant population, British J. Math. & Computer Sci.,* 14(2) (2016), 1–12.

[37] K. B. Oldham, J. Spanier, *The Fractional Calculus,* Academic Press, New York, 1974.

[38] M. Rahimy, *Applications of fractional differential equations, Appl. Math. Sci.,* 4 (2010), no.50, 2453–2461.

[39] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Intgrals and Applications, Theory and Applications,* Gordon and Breach Science Publishers, 1987.

[40] E. Scalas, R. Gorenflo, and F. Mainardi, *Fractional calculus and continuous-time finance, Physica A,* 284 (2000), 376–384.

[41] E. Scalas, R. Gorenflo, and F. Mainardi, *Uncoupled continuous-time random walks: analytic solution and limiting behaviour of the master equation, Phys. Rev. E,* 69 (2004), 011107.

[42] V. E. Tarasov, *Review of some promising fractional physical models, Int. J. Modern Physics B,* 27 (2013), 1330005.

[43] J. A. Tenreiro Machado, M. F. Silva, R. S. Barbosa, I. S. Jesus, C. M. Reis, M. G. Marcos, and A. F. Galhano, *Some applications of fractional calculus in engineering, Math. Prob. Engineering,* 2010, Article ID 639801.

[44] P. J. Torvik and R. L. Bagley, *On the appearance of the fractional derivative in the behavior of real materials, J. Appl. Mech.,* 51 (1984), 294–298.

[45] V. R. Vinokurov, *Optimal control of processes describted by integra equations, parts I, II, and III, SIAM J. Control,* 7 (1969), 324–355; comments by L. W. Neustadt and J. Warga, *SIAM J. Control,* 8 (1970), 572.

[46] X. Zhang, X. Li, and Z. Chen, *Differential Equation Theory of Optimal Control Systems (in Chinese),* Higher Education Press, Beijing, 1989.