# INSTABILITY OF THE SOLITARY WAVES FOR THE GENERALIZED BOUSSINESQ EQUATIONS

BING LI, MASAHITO OHTA, YIFEI WU, AND JUN XUE*

Abstract. In this work, we consider the generalized Boussinesq equation

$$\partial_t^2 u - \partial_x^2 u + \partial_x^2(\partial_x^2 u + |u|^p u) = 0, \qquad (t,x) \in \mathbb{R}\times\mathbb{R},$$

with $0 < p < \infty$. This equation has the traveling wave solutions $\phi_\omega(x-\omega t)$, with the frequency $\omega \in (-1,1)$ and $\phi_\omega$ satisfying

$$-\partial_{xx}\phi_\omega + (1-\omega^2)\phi_\omega - \phi_\omega^{p+1} = 0.$$

Bona and Sachs [2] proved that the traveling wave $\phi_\omega(x-\omega t)$ is orbitally stable when $0 < p < 4$, $\frac{p}{4} < \omega^2 < 1$. Liu [9] proved the orbital instability under the conditions $0 < p < 4$, $\omega^2 < \frac{p}{4}$ or $p \ge 4$, $\omega^2 < 1$. In this paper, we prove the orbital instability in the degenerate case $0 < p < 4, \omega^2 = \frac{p}{4}$ .

## 1. Introduction

In this paper, we consider the stability theory of the generalized Boussinesq equation

$$\partial_t^2 u - \partial_x^2 u + \partial_x^2(\partial_x^2 u + |u|^p u) = 0, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{1.1}$$

with the initial data

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \tag{1.2}$$

Here $0 < p < \infty$.

The Boussinesq equation was originally derived by Boussinesq [3]. It arises from studying an approximation to the evolution of the free surface of a water wave.

Equation (1.1) has the solitary wave solution $u(x,t) = \phi_\omega(x-\omega t)$, where $\phi_\omega$ is the ground state solution of the following elliptic equation

$$-\partial_{xx}\phi_\omega + (1-\omega^2)\phi_\omega - \phi_\omega^{p+1} = 0, \qquad |\omega| < 1. \tag{1.3}$$

The ground state solution $\phi_\omega$ is an even function and it has the property of exponential decay, that is, $|\phi_\omega| \le C_1 e^{-C_2|x|}$ for some $C_1, C_2 > 0$ and $|\partial_x\phi_\omega| \le C_3 e^{-C_4|x|}$ for some $C_3, C_4 > 0$.

2010 *Mathematics Subject Classification.* Primary 35B35; Secondary 35L70.

*Key words and phrases.* generalized Boussinesq equation, instability, critical frequency, traveling wave.

Funding: The work of the first and third authors was partially supported by National Natural Science Foundation of China grants 11771325 and 11571118. The work of the second author was partially supported by JSPS KAKENHI grant 15K04968. The work of the third author was also partially supported by National Youth Topnotch Talent Support Program in China. The work of the fourth author was partially supported by Research Council of Norway (No. 250070).

* Corresponding author.

SIAM Journal on Mathematical Analysis, Volume 52, Issue 4, Pages 3192–3221. This paper is the final version.

Equation (1.1) has the equivalent system form

$$\begin{cases} u_t = v_x, \\ v_t = (-u_{xx} + u - |u|^p u)_x. \end{cases} \tag{1.4}$$

Then the system (1.4) has the following solitary wave solution

$$\begin{pmatrix} u \\ v \end{pmatrix}(t,x) = \begin{pmatrix} \phi_\omega(x-\omega t) \\ -\omega\phi_\omega(x-\omega t) \end{pmatrix}.$$

For the $H^1 \times L^2$-solution $(u,v)^T$ of (1.1)–(1.2), the momentum Q and the energy E are conserved under the flow, where

$$Q\begin{pmatrix} u \\ v \end{pmatrix} = \int_{\mathbb{R}} uv \, \mathrm{d}x; \tag{1.5}$$

$$E\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\int_{\mathbb{R}} (|u_x|^2 + |u|^2 + |v|^2) \, \mathrm{d}x - \frac{1}{p+2}\int_{\mathbb{R}} |u|^{p+2} \, \mathrm{d}x. \tag{1.6}$$

There are several related results for the generalized Boussinesq equation. For a local existence result, Liu [9] proved that the system (1.4) is locally well-posed in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. For the stability theories, Bona and Sachs [2] proved that when $0 < p < 4$, $\frac{p}{4} < \omega^2 < 1$, the solitary wave solution is orbitally stable. Liu [9] proved the orbital instability if $0 < p < 4$ and $\omega^2 < \frac{p}{4}$ or $p \geq 4$ and $\omega^2 < 1$. Liu [10] proved that when the wave speed $\omega = 0$, the solitary wave solution is strongly unstable by blow-up. Liu, Ohta, and Todorova [11] showed that when $0 < p < \infty$ and $0 < 2(p+2)\omega^2 < p$, the solitary wave solution is strongly unstable by blow-up. For the abstract Hamiltonian systems, we refer the readers to Grillakis, Shatah, and Strauss [5, 6] for the stability/instability theories, in which the Vakhitov-Kolokolov stability criteria of the solitary waves were confirmed except the degenerate cases. In the degenerate cases, it was also proved by Comech and Pelinovsky [4] (see also [14]) that the solitary wave solution is orbitally unstable under some regularity restrictions in the nonlinearity (for example, $p$ should be suitably large in our cases). In this paper, we consider the stability theory on the solitary wave solutions of the generalized Boussinesq equation and aim to show its instability in the degenerate cases without any regularity restriction. It is worth noting that none of the above two frameworks of Grillakis, Shatah and Strauss [5, 6] and Comech and Pelinovsky [4] are available in our cases, either because of the degeneration or because of insufficient regularity of the nonlinearity.

Before starting our theorem, we give some definitions. Let $v_0 = \int_{-\infty}^{x} u_1(y) \, \mathrm{d}y$, $\vec{u} = (u,v)^T$, $\vec{u}_0 = (u_0, v_0)^T$, and $\overrightarrow{\Phi_\omega} = (\phi_\omega, -\omega\phi_\omega)^T$. For $\varepsilon > 0$, we denote the set $U_\varepsilon(\overrightarrow{\Phi_\omega})$ as

$$U_\varepsilon(\overrightarrow{\Phi_\omega}) = \{\vec{u} \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) : \inf_{y\in\mathbb{R}} \|\vec{u} - \overrightarrow{\Phi_\omega}(\cdot - y)\|_{H^1 \times L^2} < \varepsilon\}. \tag{1.7}$$

**Definition 1.1.** *We say that the solitary wave solution $\phi_\omega(x-\omega t)$ of (1.1) is orbitally stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\vec{u}_0 - \overrightarrow{\Phi_\omega}\|_{H^1 \times L^2} < \delta$, then the solution $\vec{u}(t)$ of (1.1) with $\vec{u}(0) = \vec{u}_0$ exists for all $t \in \mathbb{R}$, and $\vec{u}(t) \in U_\varepsilon(\overrightarrow{\Phi_\omega})$ for all $t \in \mathbb{R}$. Otherwise, $\phi_\omega(x-\omega t)$ is said to be orbitally unstable.*

Then the main result in the present paper is the following.

**Theorem 1.2.** *Let $0 < p < 4$, $\omega \in (-1,1)$ and $\phi_\omega$ be the solution of (1.3). If $|\omega| = \sqrt{\frac{p}{4}}$, then the solitary wave solution $\phi_\omega(x-\omega t)$ is orbitally unstable.*

The main method that we use in the present paper is from [19], in which the instability of the standing wave solutions of the Klein-Gordon equation in the degenerate cases was proved. Instead of construction of the Lyapunov functional, the argument in [19] is to use the monotonicity of the virial quantity to control the modulations. However, the details of this argument depend sensitively on the problem, and the key ingredients of our proof are the following.

(1) The nonstandard modulation and coercivity properties are given. More precisely, define the functional $S_\omega$ as

$$S_\omega(\vec{u}) = E(\vec{u}) + \omega Q(\vec{u}).$$

Inspired by [12, 13, 18], we establish the following nonstandard coercivity properties. We prove the existence of suitable directions $\overrightarrow{\Gamma_\omega}, \overrightarrow{\Psi_\omega} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ such that the following coercivity properties hold. Suppose that $\vec{\eta} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfies

$$\left\langle \vec{\eta}, \overrightarrow{\Gamma_\omega} \right\rangle = \left\langle \vec{\eta}, \overrightarrow{\Psi_\omega} \right\rangle = 0;$$

then

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta} \right\rangle \gtrsim \|\vec{\eta}\|^2_{H^1 \times L^2}.$$

The choices of $\overrightarrow{\Gamma_\omega}, \overrightarrow{\Psi_\omega}$ play important roles in our estimation. $\overrightarrow{\Psi_\omega}$ can be regarded as the negative direction, which satisfies $\langle S''_\omega(\overrightarrow{\Phi_\omega})\overrightarrow{\Psi_\omega}, \overrightarrow{\Psi_\omega}\rangle < 0$. However, we remark that $\overrightarrow{\Gamma_\omega} \notin Ker(S''_\omega(\overrightarrow{\Phi_\omega}))$, which is much different from the standard. Moreover, by suitably setting the translation and scaling parameters $y, \lambda$, we can establish the modulation by writing

$$\vec{u} = \left(\vec{\eta} + \overrightarrow{\Phi_{\lambda(t)}}\right)(\cdot - y(t))$$

such that $\vec{\eta}$ verifies similar orthogonal conditions above (by replacing $\overrightarrow{\Gamma_\omega}, \overrightarrow{\Psi_\omega}$ with $\overrightarrow{\Gamma_\lambda}, \overrightarrow{\Psi_\lambda}$, respectively).

(2) A subtle control on the modulated translation parameters is obtained. Instead of the rough control of the modulation parameter $y$ as $\dot{y} - \lambda = O(\|\vec{\eta}\|_{H^1 \times L^2})$, we obtain the following finer estimate:

$$\dot{y} - \lambda = \|\phi_\lambda\|^{-2}_{L^2}\left[Q(\overrightarrow{\Phi_\lambda}) - Q(\overrightarrow{\Phi_\omega})\right] - \|\phi_\lambda\|^{-2}_{L^2}\left[Q(\vec{u}_0) - Q(\overrightarrow{\Phi_\omega})\right] + O(\|\vec{\eta}\|^2_{H^1 \times L^2}).$$

The subtle estimate benefits from the choices of $\overrightarrow{\Gamma_\omega}, \overrightarrow{\Psi_\omega}$ in the first step and the dynamic of the solution. This estimate has great effects when we set up the structure of virial identity $I'(t)$ in the following.

(3) The monotonicity of the virial quantity is constructed. The key ingredient here is to suitably define a quantity $I(t)$ and obtain its monotonicity. To this end, the crucial issue is to prove the following structure of $I'(t)$ as

$$I'(t) = \rho(\vec{u}_0) + h(\lambda) + R(\vec{u}),$$

where

$$\rho(\vec{u}_0) \geq C_1 a, \quad C_1 > 0;$$
$$h(\lambda) \geq C_2(\lambda - \omega)^2 + C_3 a(\lambda - \omega)^2 + o(\lambda - \omega)^2, \quad C_2 > 0, C_3 > 0,$$

and $R(\vec{u})$ is an easy remainder term which can be dominated by $\rho$ and $h$. Here $a$ is the difference between the initial data and the soliton. The obstacles in the proof come from nonconservation terms among $I'(t)$ and how to eliminate the first-order terms about $\vec{\eta}$ and $\lambda$. These make much technical complexity. By a delicate analysis and the utilization of the estimates above, we overcome all difficulties and finally obtain the monotonicity of $I(t)$.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we show the coercivity property of the Hessian $S''_\omega(\overrightarrow{\Phi_\omega})$. In Section 4, we show the existence of modulation parameters. In Section 5, we control the modulation parameters obtained in Section 4. In Section 6, we show the localized virial identities. Finally, we prove the main theorem in Section 7.

## 2. Preliminary

2.1. **Notations.** For $f, g \in L^2(\mathbb{R}) = L^2(\mathbb{R}, \mathbb{R})$, we define

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)\,\mathrm{d}x$$

and regard $L^2(\mathbb{R})$ as a real Hilbert space. Similarly, for $\vec{f}, \vec{g} \in \big(L^2(\mathbb{R})\big)^2 = \big(L^2(\mathbb{R}, \mathbb{R})\big)^2$, we define

$$\langle \vec{f}, \vec{g}\rangle = \int_{\mathbb{R}} \vec{f}(x)^T \cdot \vec{g}(x)\,\mathrm{d}x.$$

For a function $f(x)$, its $L^q$-norm $\|f\|_{L^q} = \Big(\int_{\mathbb{R}} |f(x)|^q\,\mathrm{d}x\Big)^{\frac{1}{q}}$ and its $H^1$-norm $\|f\|_{H^1} = (\|f\|_{L^2}^2 + \|\partial_x f\|_{L^2}^2)^{\frac{1}{2}}$. For $\vec{f} = (f, g)^T$, its $H^1 \times L^2$-norm $\|\vec{f}\|_{H^1\times L^2} = (\|f\|_{H^1}^2 + \|g\|_{L^2}^2)^{\frac{1}{2}}$.

Further, we write $X \lesssim Y$ or $Y \gtrsim X$ to indicate $X \le CY$ for some constant $C > 0$. We use the notation $X \sim Y$ to denote $X \lesssim Y \lesssim X$. We also use $O(Y)$ to denote any quantity $X$ such that $|X| \lesssim Y$ and use $o(Y)$ to denote any quantity $X$ such that $X/Y \to 0$ if $Y \to 0$. Throughout the whole paper, the letter $C$ will denote various positive constants which are of no importance in our analysis.

2.2. **Some basic definitions and properties.** In the rest of this paper, we consider the case of $0 < p < 4$, and $\omega_c = \sqrt{\frac{p}{4}}$, $\omega = \pm\omega_c$. Let $\vec{u} = (u, v)^T$, $\overrightarrow{\Phi_\omega} = (\phi_\omega, -\omega\phi_\omega)^T$. Recall the conserved equalities,

$$Q(\vec{u}) = \int_{\mathbb{R}} uv\,\mathrm{d}x,$$

$$E(\vec{u}) = \frac{1}{2}(\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|v\|_{L^2}^2) - \frac{1}{p+2}\|u\|_{L^{p+2}}^{p+2}.$$

First, we give some basic properties on the momentum and energy.

**Lemma 2.1.** *Let $|\omega| = \sqrt{\frac{p}{4}}$; then the following equality holds:*

$$\partial_\lambda Q(\overrightarrow{\Phi_\lambda})\Big|_{\lambda=\omega} = 0.$$

*Proof.* Note that for $\lambda \in (-1,1)$, we have

$$Q\big(\overrightarrow{\Phi_\lambda}\big) = -\lambda \|\phi_\lambda\|_{L^2}^2. \tag{2.1}$$

By rescaling, we find

$$\phi_\lambda(x) = (1-\lambda^2)^{\frac{1}{p}} \phi_0\left(\sqrt{1-\lambda^2}x\right). \tag{2.2}$$

This implies that

$$Q\big(\overrightarrow{\Phi_\lambda}\big) = -\lambda(1-\lambda^2)^{\frac{2}{p}-\frac{1}{2}} \|\phi_0\|_{L^2}^2.$$

By a straightforward computation, we have

$$\partial_\lambda Q\big(\overrightarrow{\Phi_\lambda}\big) = -(1-\lambda^2)^{\frac{2}{p}-\frac{3}{2}}\Big(1-\frac{4}{p}\lambda^2\Big)\|\phi_0\|_{L^2}^2.$$

Finally, we substitute $\lambda^2 = \frac{p}{4}$ into the equality above and thus complete the proof. □

Now we define the functional $S_\omega$ as

$$S_\omega(\vec{u}) = E(\vec{u}) + \omega Q(\vec{u}). \tag{2.3}$$

Then we have

$$Q'(\vec{u}) = \begin{pmatrix} v \\ u \end{pmatrix}, \tag{2.4}$$

$$E'(\vec{u}) = \begin{pmatrix} -\partial_{xx}u + u - |u|^p u \\ v \end{pmatrix}, \tag{2.5}$$

$$S'_\omega(\vec{u}) = \begin{pmatrix} -u_{xx} + u - |u|^p u + \omega v \\ v + \omega u \end{pmatrix}.$$

Note that $S'_\omega\big(\overrightarrow{\Phi_\omega}\big) = \vec{0}$. Moreover, for the real-valued vector $\vec{f} = (f,g)^T$, a direct computation shows

$$S''_\omega\big(\overrightarrow{\Phi_\omega}\big)\vec{f} = \begin{pmatrix} -\partial_{xx}f + f - (p+1)\phi_\omega^p f + \omega g \\ g + \omega f \end{pmatrix}, \tag{2.6}$$

and for any vector $\vec{\xi}$, $\vec{\eta}$,

$$\Big\langle S''_\omega\big(\overrightarrow{\Phi_\omega}\big)\vec{\xi}, \vec{\eta}\Big\rangle = \Big\langle S''_\omega\big(\overrightarrow{\Phi_\omega}\big)\vec{\eta}, \vec{\xi}\Big\rangle.$$

Moreover, taking the derivative of $S'_\omega\big(\overrightarrow{\Phi_\omega}\big) = \vec{0}$ with respect to $\omega$ gives

$$S''_\omega\big(\overrightarrow{\Phi_\omega}\big)\partial_\omega \overrightarrow{\Phi_\omega} = -Q'\big(\overrightarrow{\Phi_\omega}\big). \tag{2.7}$$

Then a consequence of Lemma 2.1 is

**Corollary 2.2.** *Let $\lambda \in (-1,1)$, $|\omega| = \omega_c$; then*

$$S_\lambda\big(\overrightarrow{\Phi_\lambda}\big) - S_\lambda\big(\overrightarrow{\Phi_\omega}\big) = o\big((\lambda-\omega)^2\big).$$

*Proof.* From the definition of $S_\omega(\vec{u})$ in (2.3), we have

$$S_\lambda\big(\overrightarrow{\Phi_\lambda}\big) - S_\lambda\big(\overrightarrow{\Phi_\omega}\big) = S_\omega\big(\overrightarrow{\Phi_\lambda}\big) - S_\omega\big(\overrightarrow{\Phi_\omega}\big) + (\lambda-\omega)\Big(Q\big(\overrightarrow{\Phi_\lambda}\big) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big).$$

Recall that $S'_\omega\big(\overrightarrow{\Phi_\omega}\big) = \vec{0}$; then we use Taylor's expansion to calculate

$$S_\lambda\big(\overrightarrow{\Phi_\lambda}\big) - S_\lambda\big(\overrightarrow{\Phi_\omega}\big)$$

$$
\begin{aligned}
=&\frac{1}{2}\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\Big(\overrightarrow{\Phi_\lambda}-\overrightarrow{\Phi_\omega}\Big),\Big(\overrightarrow{\Phi_\lambda}-\overrightarrow{\Phi_\omega}\Big)\Big\rangle\\
&\qquad+(\lambda-\omega)\Big(Q(\overrightarrow{\Phi_\lambda})-Q(\overrightarrow{\Phi_\omega})\Big)+o\big((\lambda-\omega)^2\big). \qquad (2.8)
\end{aligned}
$$

Note that

$$\overrightarrow{\Phi_\lambda}-\overrightarrow{\Phi_\omega}=(\lambda-\omega)\partial_\omega\overrightarrow{\Phi_\omega}+o(\lambda-\omega);$$

then we find

$$
\begin{aligned}
&\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\Big(\overrightarrow{\Phi_\lambda}-\overrightarrow{\Phi_\omega}\Big),\Big(\overrightarrow{\Phi_\lambda}-\overrightarrow{\Phi_\omega}\Big)\Big\rangle\\
&\quad=(\lambda-\omega)^2\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\partial_\omega\overrightarrow{\Phi_\omega},\partial_\omega\overrightarrow{\Phi_\omega}\Big\rangle+o\big((\lambda-\omega)^2\big)\\
&\quad=-(\lambda-\omega)^2\Big\langle Q'(\overrightarrow{\Phi_\omega}),\partial_\omega\overrightarrow{\Phi_\omega}\Big\rangle+o\big((\lambda-\omega)^2\big)\\
&\quad=-(\lambda-\omega)^2\partial_\lambda Q(\overrightarrow{\Phi_\lambda})\Big|_{\lambda=\omega}+o\big((\lambda-\omega)^2\big),
\end{aligned}
$$

where we have used equality (2.7) in the second step. Using Lemma 2.1, we have

$$\partial_\lambda Q(\overrightarrow{\Phi_\lambda})\Big|_{\lambda=\omega}=0.$$

Hence,

$$Q(\overrightarrow{\Phi_\lambda})-Q(\overrightarrow{\Phi_\omega})=o(\lambda-\omega),$$

and

$$\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\Big(\overrightarrow{\Phi_\lambda}-\overrightarrow{\Phi_\omega}\Big),\Big(\overrightarrow{\Phi_\lambda}-\overrightarrow{\Phi_\omega}\Big)\Big\rangle=o\big((\lambda-\omega)^2\big).$$

Taking these two results into (2.8), we obtain the desired estimate. □

## 3. Coercivity

In this section, we prove a coercivity property on the Hessian of the action $S''_\omega(\overrightarrow{\Phi_\omega})$. First, we study the kernel of $S''_\omega(\overrightarrow{\Phi_\omega})$ in the following lemma. The proof is standard, and it is a consequence of the result from [17].

**Lemma 3.1.** *The kernel of $S''_\omega(\overrightarrow{\Phi_\omega})$ satisfies that*

$$Ker\Big(S''_\omega(\overrightarrow{\Phi_\omega})\Big)=\big\{C\partial_x\overrightarrow{\Phi_\omega}:C\in\mathbb{R}\big\}.$$

*Proof.* First, we need to show the relationship "⊃". For any $\vec{f}\in\big\{C\partial_x\overrightarrow{\Phi_\omega}:C\in\mathbb{R}\big\}$, using (1.3), we have

$$S''_\omega(\overrightarrow{\Phi_\omega})\vec{f}=S''_\omega(\overrightarrow{\Phi_\omega})\big(C\partial_x\overrightarrow{\Phi_\omega}\big)=C\begin{pmatrix}\partial_x\big(-\partial_{xx}\phi_\omega+(1-\omega^2)\phi_\omega-\phi_\omega^{p+1}\big)\\ -\omega\phi'_\omega+\omega\phi'_\omega\end{pmatrix}=\vec{0}. \qquad (3.1)$$

Then (3.1) implies that $\vec{f}$ is in the kernel of $S''_\omega(\overrightarrow{\Phi_\omega})$, and we have the conclusion

$$Ker\Big(S''_\omega(\overrightarrow{\Phi_\omega})\Big)\supset\big\{C\partial_x\overrightarrow{\Phi_\omega}:C\in\mathbb{R}\big\}.$$

Second, we prove the reverse relationship "$\subset$". For any $\vec{f} \in Ker\Big(S''_\omega(\overrightarrow{\Phi_\omega})\Big)$, by the expression of $S''_\omega(\overrightarrow{\Phi_\omega})$ in (2.6), we have

$$\begin{cases} -\partial_{xx} f + (1-\omega^2) f - (p+1)\phi_\omega^p f = 0, \\ g + \omega f = 0. \end{cases} \tag{3.2}$$

By the work of Weinstein [17], the only solutions to (3.2) are

$$\begin{cases} f = C\partial_x \phi_\omega, \\ g = -C\omega\partial_x\phi_\omega, \end{cases} \quad C \in \mathbb{R}.$$

This implies that $\vec{f} \in \big\{C\partial_x \overrightarrow{\Phi_\omega} : C \in \mathbb{R}\big\}$, and we have

$$Ker\Big(S''_\omega(\overrightarrow{\Phi_\omega})\Big) \subset \big\{C\partial_x \overrightarrow{\Phi_\omega} : C \in \mathbb{R}\big\}.$$

Finally, combining the two relationship gives us

$$Ker\Big(S''_\omega(\overrightarrow{\Phi_\omega})\Big) = \big\{C\partial_x \overrightarrow{\Phi_\omega} : C \in \mathbb{R}\big\}.$$

This gives the proof of the lemma. □

The second lemma is the uniqueness of the negative eigenvalue of $S''_\omega(\overrightarrow{\Phi_\omega})$.

**Lemma 3.2.** $S''_\omega(\overrightarrow{\Phi_\omega})$ *exists only one negative eigenvalue.*

*Proof.* It is known that the operator $-\partial_{xx} + (1-\omega^2) - (p+1)\phi_\omega^p$ has only one negative eigenvalue (see [17]), and we denote it by $\lambda_{-1}$. Then there exists a unique associated eigenvector $\zeta \in H^1(\mathbb{R})$ such that

$$-\partial_{xx}\zeta + (1-\omega^2)\zeta - (p+1)\phi_\omega^p\zeta = \lambda_{-1}\zeta. \tag{3.3}$$

Using the expression of $S''_\omega(\overrightarrow{\Phi_\omega})$ in (2.6), we have

$$\begin{aligned} &\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\overrightarrow{\Phi_\omega}, \overrightarrow{\Phi_\omega}\Big\rangle \\ &\quad = \int_{\mathbb{R}} (-\partial_{xx}\phi_\omega + \phi_\omega - (p+1){\phi_\omega}^{p+1} - \omega^2\phi_\omega, -\omega\phi_\omega + \omega\phi_\omega) \cdot \begin{pmatrix} \phi_\omega \\ -\omega\phi_\omega \end{pmatrix} \mathrm{d}x \\ &\quad = -\,p\|\phi_\omega\|_{L^{p+2}}^{p+2} < 0. \end{aligned}$$

This implies that $S''_\omega(\overrightarrow{\Phi_\omega})$ has at least one negative eigenvalue, say, $\mu_0$. Assume its associated eigenvector $\vec{\eta}_0 = (\xi_0, \eta_0)^T$, that is,

$$S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}_0 = \mu_0\vec{\eta}_0.$$

Using the expression of $S''_\omega(\overrightarrow{\Phi_\omega})$ in (2.6) again, the last equality yields

$$\begin{cases} -\partial_{xx}\xi_0 + \xi_0 - (p+1)\phi_\omega^p\xi_0 + \omega\eta_0 = \mu_0\xi_0, \\ \eta_0 + \omega\xi_0 = \mu_0\eta_0. \end{cases}$$

From the second equality, we have $\eta_0 = -\frac{\omega}{1-\mu_0}\xi_0$. Then we substitute it into the first equality to get

$$-\partial_{xx}\xi_0 + (1-\omega^2)\xi_0 - (p+1)\phi_\omega^p\xi_0 = \mu_0\Big(\frac{\omega^2}{1-\mu_0} + 1\Big)\xi_0.$$

Hence, by (3.3), $(\mu_0, \vec{\eta}_0)$ is exactly the pair satisfying

$$\mu_0 = \frac{1}{2}\left(\lambda_{-1} + \omega^2 + 1 - \sqrt{\lambda_{-1}^2 + 2(\omega^2 - 1)\lambda_{-1} + (\omega^2+1)^2}\right), \qquad \vec{\eta}_0 = \begin{pmatrix} \zeta \\ \dfrac{\omega\zeta}{\mu_0 - 1} \end{pmatrix}. \tag{3.4}$$

This implies that $S''_\omega(\overrightarrow{\Phi_\omega})$ has exactly one simple negative eigenvalue. This completes the proof of Lemma 3.2. □

The next lemma gives one of the negative direction of $S''_\omega(\overrightarrow{\Phi_\omega})$.

**Lemma 3.3.** *Let*

$$\vec{\psi}_\omega = \frac{1}{2\omega}\begin{pmatrix} \partial_\omega\phi_\omega \\ -\omega\partial_\omega\phi_\omega \end{pmatrix}, \qquad \overrightarrow{\Psi_\omega} = \begin{pmatrix} \phi_\omega \\ 0 \end{pmatrix}.$$

*Then*

$$S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega = \overrightarrow{\Psi_\omega}. \tag{3.5}$$

*Moreover, we have*

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega \right\rangle < 0.$$

*Proof.* Taking the derivative of (1.3) with respect to $\omega$, we have

$$-\partial_{xx}(\partial_\omega\phi_\omega) + (1-\omega^2)\partial_\omega\phi_\omega - (p+1)\phi_\omega^p\partial_\omega\phi_\omega = 2\omega\phi_\omega. \tag{3.6}$$

Using the expression of $S''_\omega(\overrightarrow{\Phi_\omega})$ in (2.6), we have

$$S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega = \frac{1}{2\omega}\begin{pmatrix} -\partial_{xx}(\partial_\omega\phi_\omega) + (1-\omega^2)\partial_\omega\phi_\omega - (p+1)\phi_\omega^p\partial_\omega\phi_\omega \\ 0 \end{pmatrix}.$$

This combined with (3.6) gives

$$S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega = \begin{pmatrix} \phi_\omega \\ 0 \end{pmatrix} = \overrightarrow{\Psi_\omega}. \tag{3.7}$$

Now we show $\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega \right\rangle < 0$. From (3.7), we have

$$\begin{aligned}\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega \right\rangle &= \left\langle \overrightarrow{\Psi_\omega}, \vec{\psi}_\omega \right\rangle = \int_{\mathbb{R}} (\phi_\omega, 0)\cdot \frac{1}{2\omega}\begin{pmatrix} \partial_\omega\phi_\omega \\ -\omega\partial_\omega\phi_\omega \end{pmatrix} \mathrm{d}x \\ &= \frac{1}{2\omega}\int_{\mathbb{R}} \phi_\omega\, \partial_\omega\phi_\omega \,\mathrm{d}x = \frac{1}{4\omega}\partial_\omega\|\phi_\omega\|_{L^2}^2. \end{aligned} \tag{3.8}$$

Note that, by (2.2),

$$\|\phi_\omega\|_{L^2}^2 = (1-\omega^2)^{\frac{2}{p}-\frac{1}{2}}\|\phi_0\|_{L^2}^2.$$

Hence,

$$\partial_\omega\|\phi_\omega\|_{L^2}^2 = -\Big(\frac{4}{p} - 1\Big)\frac{\omega}{1-\omega^2}\|\phi_\omega\|_{L^2}^2 < 0.$$

This completes the proof. □

Now we prove the following coercivity property.

**Proposition 3.4.** *Let $|\omega|<1$. Suppose that $\vec{\eta}=(\xi,\eta)^T\in H^1(\mathbb{R})\times L^2(\mathbb{R})$ satisfies*

$$\left\langle \vec{\eta}, \partial_x \overrightarrow{\Phi_\omega}\right\rangle = \left\langle \vec{\eta}, \overrightarrow{\Psi_\omega}\right\rangle = 0, \tag{3.9}$$

*where $\overrightarrow{\Psi_\omega}=(\phi_\omega,0)^T$. Then*

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\right\rangle \gtrsim \|\vec{\eta}\|^2_{H^1\times L^2}.$$

*Proof.* From the expression of $S''_\omega(\overrightarrow{\Phi_\omega})$ in (2.6), we can write $S''_\omega(\overrightarrow{\Phi_\omega})$ as

$$S''_\omega(\overrightarrow{\Phi_\omega}) = L+V,$$

where $L=\begin{pmatrix} -\partial_{xx}+1 & \omega \\ \omega & 1\end{pmatrix}$, and $V=\begin{pmatrix} -(p+1)\phi_\omega^p & 0 \\ 0 & 0\end{pmatrix}$. Hence $V$ is a compact perturbation of the self-adjoint operator $L$.

*Step 1. Analyse the spectrum of $S''_\omega(\overrightarrow{\Phi_\omega})$.*

We first compute the essential spectrum of $L$. Note that for any $\vec{f}=(f,g)^T\in H^1(\mathbb{R})\times L^2(\mathbb{R})$,

$$\begin{aligned}
\langle L\vec{f},\vec{f}\rangle &= \left\langle \begin{pmatrix} -\partial_{xx}+1 & \omega \\ \omega & 1\end{pmatrix}\begin{pmatrix} f \\ g\end{pmatrix}, \begin{pmatrix} f \\ g\end{pmatrix}\right\rangle \\
&= \int_{\mathbb{R}} (-\partial_{xx}f+f+\omega g, \omega f+g)\cdot \begin{pmatrix} f \\ g\end{pmatrix} \mathrm{d}x \\
&= \|\partial_x f\|^2_{L^2}+\|f\|^2_{L^2}+2\omega\langle f,g\rangle+\|g\|^2_{L^2} \\
&= \|\vec{f}\|^2_{H^1\times L^2}+2\omega\langle f,g\rangle. 
\end{aligned}\tag{3.10}$$

For the term $2\omega\langle f,g\rangle$, applying Hölder's and Young's inequalities, we have

$$|2\omega\langle f,g\rangle|\le|\omega|\|\vec{f}\|^2_{H^1\times L^2}.$$

Taking this estimate into (3.10), we have

$$\langle L\vec{f},\vec{f}\rangle \ge (1-|\omega|)\|\vec{f}\|^2_{H^1\times L^2}.$$

Since $|\omega|<1$, we get

$$\langle L\vec{f},\vec{f}\rangle \gtrsim \|\vec{f}\|^2_{H^1\times L^2}.$$

This means that there exists $\delta>0$ such that the essential spectrum of $L$ is $[\delta,+\infty)$. By Weyl's Theorem, $S''_\omega(\overrightarrow{\Phi_\omega})$ and $L$ share the same essential spectrum. So we obtain the essential spectrum of $S''_\omega(\overrightarrow{\Phi_\omega})$. Recall that we have obtained the only one negative eigenvalue $\mu_0$ of $S''_\omega(\overrightarrow{\Phi_\omega})$ in Lemma 3.2 and the kernel of $S''_\omega(\overrightarrow{\Phi_\omega})$ in Lemma 3.1. So the discrete spectrum of $S''_\omega(\overrightarrow{\Phi_\omega})$ is $\mu_0$, 0, and the essential spectrum is $[\delta,+\infty)$.

*Step 2. Positivity.*

The argument here is inspired by [1, 8]. By Lemma 3.2, we have the unique negative eigenvalue $\mu_0$ and eigenvector $\vec{\eta}_0$ of $S''_\omega(\overrightarrow{\Phi_\omega})$. For convenience, we normalize the eigenvector $\vec{\eta}_0$ such that $\|\vec{\eta}_0\|_{L^2\times L^2}=1$. Hence, for vector $\vec{\eta}\in H^1(\mathbb{R})\times L^2(\mathbb{R})$, by the spectral decomposition theorem we can write the decomposition of $\vec{\eta}$ along the spectrum of $S''_\omega(\overrightarrow{\Phi_\omega})$,

$$\vec{\eta}=a_\eta\vec{\eta}_0+b_\eta\partial_x\overrightarrow{\Phi_\omega}+\vec{g}_\eta,$$

where $a_\eta$, $b_\eta \in \mathbb{R}$, $\partial_x \overrightarrow{\Phi_\omega} \in Ker\Big(S''_\omega(\overrightarrow{\Phi_\omega})\Big)$ and $\vec{g}_\eta$ lies in the positive eigenspace of $S''_\omega(\overrightarrow{\Phi_\omega})$, that is, $\vec{g}_\eta$ satisfies

$$\langle \vec{g}_\eta, \vec{\eta}_0\rangle = \langle \vec{g}_\eta, \partial_x\overrightarrow{\Phi_\omega}\rangle = 0,$$

and there exists an absolute constant $\sigma > 0$ such that

$$\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta\Big\rangle \geq \sigma\|\vec{g}_\eta\|^2_{L^2\times L^2}. \tag{3.11}$$

Since $\vec{\eta}$ satisfies the orthogonality condition $\Big\langle \vec{\eta}, \partial_x\overrightarrow{\Phi_\omega}\Big\rangle = 0$ in (3.9) and $\langle \vec{\eta}_0, \partial_x\overrightarrow{\Phi_\omega}\rangle = 0$, we have $b_\eta = 0$, and thus

$$\vec{\eta} = a_\eta\vec{\eta}_0 + \vec{g}_\eta. \tag{3.12}$$

Substituting (3.12) into $\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta}\Big\rangle$, we get

$$\begin{aligned}\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta}\Big\rangle &= \Big\langle S''_\omega(\overrightarrow{\Phi_\omega})(a_\eta\vec{\eta}_0 + \vec{g}_\eta), a_\eta\vec{\eta}_0 + \vec{g}_\eta\Big\rangle\\ &= a_\eta^2\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}_0, \vec{\eta}_0\Big\rangle + 2\mu_0 a_\eta\Big\langle \vec{g}_\eta, \vec{\eta}_0\Big\rangle + \Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta\Big\rangle.\end{aligned}$$

Due to the orthogonality property of eigenvector $\langle \vec{g}_\eta, \vec{\eta}_0\rangle = 0$, we have

$$\begin{aligned}\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta}\Big\rangle &= a_\eta^2\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}_0, \vec{\eta}_0\Big\rangle + \Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta\Big\rangle\\ &= \mu_0 a_\eta^2\langle \vec{\eta}_0, \vec{\eta}_0\rangle + \Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta\Big\rangle\\ &= \mu_0 a_\eta^2 + \Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta\Big\rangle.\end{aligned} \tag{3.13}$$

To $\vec{\psi}_\omega$, by spectral decomposition theorem again, we may write

$$\vec{\psi}_\omega = a\vec{\eta}_0 + b\partial_x\overrightarrow{\Phi_\omega} + \vec{g},$$

where $a, b \in \mathbb{R}$, and $\vec{g}$ lies in the positive eigenspace of $S''_\omega(\overrightarrow{\Phi_\omega})$. We note that $\Big\langle \vec{\psi}_\omega, \partial_x\overrightarrow{\Phi_\omega}\Big\rangle = 0$. Indeed, since $\phi_\omega$ is an even function, we have that $\partial_\omega\phi_\omega$ is even and $\partial_x\phi_\omega$ is odd. Hence, we get

$$\Big\langle \vec{\psi}_\omega, \partial_x\overrightarrow{\Phi_\omega}\Big\rangle = \frac{1+\omega^2}{2\omega}\int_{\mathbb{R}} \partial_\omega\phi_\omega\, \partial_x\phi_\omega\, dx = 0.$$

Then $b = 0$, and thus

$$\vec{\psi}_\omega = a\vec{\eta}_0 + \vec{g}.$$

Therefore, a similar computation as above shows that

$$\begin{aligned}\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega\Big\rangle &= \Big\langle S''_\omega(\overrightarrow{\Phi_\omega})(a\vec{\eta}_0 + \vec{g}), a\vec{\eta}_0 + \vec{g}\Big\rangle\\ &= \Big\langle S''_\omega(\overrightarrow{\Phi_\omega})(a\vec{\eta}_0), a\vec{\eta}_0\Big\rangle + \Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g}\Big\rangle\\ &= \mu_0 a^2 + \Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g}\Big\rangle.\end{aligned}$$

For convenience, let $-\delta_0 = \Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega\Big\rangle$. Then by Lemma 3.3, we know that $\delta_0 > 0$. Moreover, we have

$$-\delta_0 = \mu_0 a^2 + \Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g}\Big\rangle. \tag{3.14}$$

Using the orthogonality assumption $\left\langle \vec{\eta}, \overrightarrow{\Psi_\omega} \right\rangle = 0$ in (3.9) and (3.5), we have

$$\begin{aligned}
0 = \left\langle \vec{\eta}, \overrightarrow{\Psi_\omega} \right\rangle &= \left\langle a_\eta \vec{\eta}_0 + \vec{g}_\eta, S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega \right\rangle \\
&= \left\langle a_\eta \vec{\eta}_0 + \vec{g}_\eta, S''_\omega(\overrightarrow{\Phi_\omega})(a\vec{\eta}_0 + \vec{g}) \right\rangle \\
&= \left\langle a_\eta \vec{\eta}_0, S''_\omega(\overrightarrow{\Phi_\omega})(a\vec{\eta}_0) \right\rangle + \left\langle \vec{g}_\eta, S''_\omega(\overrightarrow{\Phi_\omega})\vec{g} \right\rangle \\
&= \mu_0 a a_\eta \langle \vec{\eta}_0, \vec{\eta}_0 \rangle + \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g}_\eta \right\rangle \\
&= \mu_0 a a_\eta + \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g}_\eta \right\rangle.
\end{aligned}$$

So we get the equality

$$0 = \mu_0 a a_\eta + \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g}_\eta \right\rangle.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
(\mu_0 a a_\eta)^2 &= \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g}_\eta \right\rangle^2 \\
&\le \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle.
\end{aligned}$$

This gives

$$(-\mu_0 a^2)(-\mu_0 a_\eta^2) \le \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle. \tag{3.15}$$

The last equality combining with (3.14) implies that

$$-\mu_0 a_\eta^2 \le \frac{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle}{-\mu_0 a^2} = \frac{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle}{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle + \delta_0},$$

that is,

$$\mu_0 a_\eta^2 \ge -\frac{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle}{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle + \delta_0}. \tag{3.16}$$

Inserting (3.16) into (3.13), we obtain

$$\begin{aligned}
\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta} \right\rangle &\ge \Big(1 - \frac{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle}{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle + \delta_0}\Big) \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle \\
&= \frac{\delta_0}{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle + \delta_0} \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle.
\end{aligned}$$

Recalling that $\vec{g}_\eta$ satisfies (3.11), we have

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta} \right\rangle \ge \frac{\delta_0 \sigma}{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle + \delta_0} \|\vec{g}_\eta\|^2_{L^2 \times L^2}, \quad \sigma > 0. \tag{3.17}$$

From the expression of $\vec{\eta}$ in (3.12) and the inequality (3.16), we have

$$\|\vec{\eta}\|^2_{L^2 \times L^2} = \|a_\eta \vec{\eta}_0 + \vec{g}_\eta\|^2_{L^2 \times L^2} = a_\eta^2 + \|\vec{g}_\eta\|^2_{L^2 \times L^2}$$

$$\le -\frac{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g},\vec{g}\right\rangle}{\mu_0\delta_0}\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\right\rangle + \|\vec{g}_\eta\|^2_{L^2\times L^2}$$
$$\lesssim \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\right\rangle.$$

Therefore, this gives

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\right\rangle \gtrsim \|\vec{\eta}\|^2_{L^2\times L^2}. \tag{3.18}$$

To obtain the final conclusion, we still need to estimate

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\right\rangle \gtrsim \|\vec{\eta}\|^2_{H^1\times L^2}.$$

Using the expression of $S''_\omega(\overrightarrow{\Phi_\omega})$ in (2.6), we have

$$\begin{aligned}\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\rangle &= \int_{\mathbb{R}}(-\partial_{xx}\xi+\xi-(p+1)\phi_\omega^p\xi+\omega\eta,\eta+\omega\xi)\cdot\begin{pmatrix}\xi\\ \eta\end{pmatrix}\mathrm{d}x\\ &= \|\partial_x\xi\|^2_{L^2}+\|\vec{\eta}\|^2_{L^2\times L^2}+2\omega\int_{\mathbb{R}}\xi\eta\,\mathrm{d}x-(p+1)\int_{\mathbb{R}}|\phi_\omega|^p\xi^2\,\mathrm{d}x.\end{aligned}$$

Thus by Hölder's and Young's inequalities and (3.18), we get

$$\begin{aligned}\|\partial_x\xi\|^2_{L^2} &= \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\right\rangle - 2\omega\int_{\mathbb{R}}\xi\eta\,\mathrm{d}x+(p+1)\int_{\mathbb{R}}|\phi_\omega|^p\xi^2\,\mathrm{d}x-\|\vec{\eta}\|^2_{L^2\times L^2}\\ &\le \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\right\rangle + 2|\omega|\|\xi\|_{L^2}\|\eta\|_{L^2}+(p+1)\|\phi_\omega\|^p_{L^\infty}\|\xi\|^2_{L^2}\\ &\le \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\right\rangle + \Big(|\omega|+(p+1)\|\phi_\omega\|^p_{L^\infty}\Big)\|\vec{\eta}\|^2_{L^2\times L^2}\\ &\lesssim \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\right\rangle + \|\vec{\eta}\|^2_{L^2\times L^2} \lesssim \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\right\rangle.\end{aligned} \tag{3.19}$$

Therefore, together (3.18) and (3.19), we obtain

$$\|\vec{\eta}\|^2_{H^1\times L^2} = \|\partial_x\xi\|^2_{L^2}+\|\vec{\eta}\|^2_{L^2\times L^2} \lesssim \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\right\rangle.$$

Thus we obtain the desired result. □

Applying Proposition 3.4, we obtain the following corollary, which is the nonstandard coercivity property and one of the key ingredients in our proof. Corollary 3.5 shows that we can replace the element $\partial_x\overrightarrow{\Phi_\omega}$ in the orthogonal condition (3.20) by a suitably defined vector $\overrightarrow{\Gamma_\omega}$. The new orthogonal condition $\left\langle\vec{\eta},\overrightarrow{\Gamma_\omega}\right\rangle = 0$ has an essential effect on the estimates of the translation parameter $y$ and $\lambda$ in Section 5.

**Corollary 3.5.** *Let $|\omega|<1$. Suppose that $\vec{\eta}\in H^1(\mathbb{R})\times L^2(\mathbb{R})$ satisfies*

$$\left\langle\vec{\eta},\overrightarrow{\Gamma_\omega}\right\rangle = \left\langle\vec{\eta},\overrightarrow{\Psi_\omega}\right\rangle = 0, \tag{3.20}$$

*where $\overrightarrow{\Gamma_\omega}\in H^1(\mathbb{R})\times L^2(\mathbb{R})$ and $\partial_x\overrightarrow{\Gamma_\omega}=\overrightarrow{\Psi_\omega}=(\phi_\omega,0)^T$. Then*

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\right\rangle \gtrsim \|\vec{\eta}\|^2_{H^1\times L^2}. \tag{3.21}$$

*Proof.* We define

$$\vec{\xi} = \vec{\eta} + b\partial_x \overrightarrow{\Phi_\omega}, \qquad \vec{\xi} \in H^1(\mathbb{R}) \times L^2(\mathbb{R}).$$

If we choose

$$b = -\frac{\left\langle \vec{\eta}, \partial_x \overrightarrow{\Phi_\omega} \right\rangle}{\|\partial_x \overrightarrow{\Phi_\omega}\|_{L^2 \times L^2}^2},$$

then

$$\left\langle \vec{\xi}, \partial_x \overrightarrow{\Phi_\omega} \right\rangle = 0.$$

Moreover, by (3.20), we have

$$\left\langle \vec{\xi}, \overrightarrow{\Psi_\omega} \right\rangle = \left\langle \vec{\eta} + b\partial_x \overrightarrow{\Phi_\omega}, \overrightarrow{\Psi_\omega} \right\rangle = \left\langle \vec{\eta}, \overrightarrow{\Psi_\omega} \right\rangle + b\left\langle \partial_x \overrightarrow{\Phi_\omega}, \overrightarrow{\Psi_\omega} \right\rangle. \tag{3.22}$$

Note that

$$b\left\langle \partial_x \overrightarrow{\Phi_\omega}, \overrightarrow{\Psi_\omega} \right\rangle = b\int_{\mathbb{R}} \Big( \partial_x \phi_\omega, (-\omega)\partial_x \phi_\omega \Big) \cdot \begin{pmatrix} \phi_\omega \\ 0 \end{pmatrix} \mathrm{d}x = b\int_{\mathbb{R}} \partial_x \phi_\omega \phi_\omega \, \mathrm{d}x = 0.$$

Hence, $\left\langle \vec{\xi}, \overrightarrow{\Psi_\omega} \right\rangle = 0$. Therefore, $\vec{\xi}$ satisfies the orthogonality condition (3.9) in Proposition 3.4. Then using the conclusion of Proposition 3.4 and $S''_\omega(\overrightarrow{\Phi_\omega})\partial_x \overrightarrow{\Phi_\omega} = \vec{0}$, we get

$$\begin{aligned}
\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta} \right\rangle &= \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\Big(\vec{\xi} - b\partial_x \overrightarrow{\Phi_\omega}\Big), \Big(\vec{\xi} - b\partial_x \overrightarrow{\Phi_\omega}\Big) \right\rangle \\
&= \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\xi}, \vec{\xi} \right\rangle - 2b\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\partial_x \overrightarrow{\Phi_\omega}, \vec{\xi} \right\rangle + b^2\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\partial_x \overrightarrow{\Phi_\omega}, \partial_x \overrightarrow{\Phi_\omega} \right\rangle \\
&= \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\xi}, \vec{\xi} \right\rangle \gtrsim \|\vec{\xi}\|_{H^1 \times L^2}^2,
\end{aligned}$$

where we have used the self-adjoint property of the operator $S''_\omega(\overrightarrow{\Phi_\omega})$ in the second step.

Now we claim that $\|\vec{\xi}\|_{H^1 \times L^2}^2 \gtrsim \|\vec{\eta}\|_{H^1 \times L^2}^2$. Indeed, using the orthogonality assumption (3.20), we have

$$\left\langle \vec{\xi}, \overrightarrow{\Gamma_\omega} \right\rangle = \left\langle \vec{\eta} + b\partial_x \overrightarrow{\Phi_\omega}, \overrightarrow{\Gamma_\omega} \right\rangle = -b\int_{\mathbb{R}} (\phi_\omega, -\omega\phi_\omega) \cdot \begin{pmatrix} \phi_\omega \\ 0 \end{pmatrix} = -b\|\phi_\omega\|_{L^2}^2.$$

Thus, by Hölder's inequality, we have

$$|b| = \frac{\left| \left\langle \vec{\xi}, \overrightarrow{\Gamma_\omega} \right\rangle \right|}{\|\phi_\omega\|_{L^2}^2} \lesssim \|\vec{\xi}\|_{H^1 \times L^2}. \tag{3.23}$$

Now from (3.23),

$$\|\vec{\eta}\|_{H^1 \times L^2} = \left\| \vec{\xi} - b\partial_x \overrightarrow{\Phi_\omega} \right\|_{H^1 \times L^2} \leq \|\vec{\xi}\|_{H^1 \times L^2} + |b| \left\| \partial_x \overrightarrow{\Phi_\omega} \right\|_{H^1 \times L^2} \lesssim \|\vec{\xi}\|_{H^1 \times L^2}.$$

This completes the proof. □

## 4. Modulation

We now suppose for contradiction that the solitary wave solution is stable; that is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that when

$$\|\vec{u}_0 - \overrightarrow{\Phi_\omega}\|_{H^1\times L^2} < \delta,$$

we have

$$\vec{u} \in U_\varepsilon\big(\overrightarrow{\Phi_\omega}\big). \tag{4.1}$$

Then the modulation theory shows that by choosing suitable parameters, the orthogonality conditions in Corollary 3.5 can be verified. The modulation is obtained via the standard implicit function theorem.

**Proposition 4.1.** *(Modulation). Let $|\omega| = \omega_c$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\vec{u} \in U_\varepsilon\big(\overrightarrow{\Phi_\omega}\big)$, the following properties are verified. There exist $C^1$-functions*

$$y : \mathbb{R} \to \mathbb{R}, \quad \lambda : \mathbb{R} \to \mathbb{R}^+$$

*such that if we define $\vec{\eta}$ by*

$$\vec{\eta}(t) = \vec{u}\big(t, \cdot + y(t)\big) - \overrightarrow{\Phi_{\lambda(t)}}, \tag{4.2}$$

*then $\vec{\eta}$ satisfies the following orthogonality conditions for any $t \in \mathbb{R}$:*

$$\Big\langle \vec{\eta}, \overrightarrow{\Gamma_{\lambda(t)}} \Big\rangle = \Big\langle \vec{\eta}, \overrightarrow{\Psi_{\lambda(t)}} \Big\rangle = 0, \tag{4.3}$$

*where $\overrightarrow{\Gamma_\lambda} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and $\partial_x \overrightarrow{\Gamma_\lambda} = \overrightarrow{\Psi_\lambda} = \begin{pmatrix} \phi_\lambda \\ 0 \end{pmatrix}$. Moreover, the following estimate verifies that*

$$\|\vec{\eta}\|_{H^1\times L^2} + |\lambda - \omega| \lesssim \varepsilon. \tag{4.4}$$

*Proof.* We use the implicit function theorem to prove this proposition. Here we only give the important steps of the proof and refer the reader to [17, 18, 12, 13] for the similar argument. Define

$$p = (\vec{u}; \lambda, y), \qquad p_0 = (\overrightarrow{\Phi_\omega}; \omega, 0).$$

Let $\varepsilon$ be the parameter decided later, and define the functional pair $(F_1, F_2) : U_\varepsilon\big(\overrightarrow{\Phi_\omega}\big) \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^2$ as

$$F_1(p) = \Big\langle \vec{\eta}, \overrightarrow{\Gamma_\lambda} \Big\rangle, \quad F_2(p) = \Big\langle \vec{\eta}, \overrightarrow{\Psi_\lambda} \Big\rangle.$$

We claim that there exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a unique $C^1$ map: $U_\varepsilon\big(\overrightarrow{\Phi_\omega}\big) \to \mathbb{R}^+ \times \mathbb{R}$ such that $(F_1(p), F_2(p)) = 0$.

Indeed, firstly we have

$$F_1(p_0) = F_2(p_0) = 0.$$

Second, we prove that

$$|J| = \left| \begin{matrix} \partial_\lambda F_1 & \partial_y F_1 \\ \partial_\lambda F_2 & \partial_y F_2 \end{matrix} \right|_{p=p_0} \neq 0.$$

Indeed, a direct calculation gives that

$$\partial_\lambda F_1(p) = \partial_\lambda \Big\langle \vec{\eta}, \overrightarrow{\Gamma_\lambda} \Big\rangle = \partial_\lambda \Big\langle \vec{u}\big(t, x + y(t)\big) - \overrightarrow{\Phi_{\lambda(t)}}, \overrightarrow{\Gamma_\lambda} \Big\rangle$$

$$= \left\langle \vec{u}\big(t, x + y(t)\big) - \overrightarrow{\Phi_{\lambda(t)}}, \partial_\lambda \overrightarrow{\Gamma_\lambda} \right\rangle - \left\langle \partial_\lambda \overrightarrow{\Phi_{\lambda(t)}}, \overrightarrow{\Gamma_\lambda} \right\rangle.$$

When $p = p_0$, we observe that $\vec{u}\big(t, x + y(t)\big) - \overrightarrow{\Phi_{\lambda(t)}} = 0$, and the first term vanishes. For the second term, we note that $\overrightarrow{\Gamma_\lambda}$ is an odd vector and $\partial_\lambda \overrightarrow{\Phi_{\lambda(t)}}$ is an even vector, so we get

$$\partial_\lambda F_1(p)\Big|_{p=p_0} = 0.$$

A similar computation shows that

$$\partial_y F_1(p)\Big|_{p=p_0} = \left\langle \partial_x \vec{u}(x+y), \overrightarrow{\Gamma_\lambda} \right\rangle\Big|_{p=p_0} = \left\langle \partial_x \overrightarrow{\Phi_\lambda}, \overrightarrow{\Gamma_\lambda} \right\rangle\Big|_{p=p_0} = -\|\phi_\omega\|_{L^2}^2;$$

$$\partial_\lambda F_2(p)\Big|_{p=p_0} = -\left\langle \partial_\lambda \overrightarrow{\Phi_\lambda}, \overrightarrow{\Psi_\lambda} \right\rangle\Big|_{p=p_0} = -\left\langle \partial_\lambda \phi_\lambda, \phi_\lambda \right\rangle\Big|_{p=p_0} = -\frac{1}{2}\partial_\lambda \|\phi_\lambda\|_{L^2}^2\Big|_{p=p_0} = \frac{1}{2\omega}\|\phi_\omega\|_{L^2}^2;$$

$$\partial_y F_2(p)\Big|_{p=p_0} = \left\langle \partial_x \overrightarrow{\Phi_\lambda}, \overrightarrow{\Psi_\lambda} \right\rangle\Big|_{p=p_0} = \int_{\mathbb{R}} \partial_x \phi_\lambda \phi_\lambda \,\mathrm{d}x\Big|_{p=p_0} = 0.$$

Then we find that

$$\begin{vmatrix} \partial_\lambda F_1 & \partial_y F_1 \\ \partial_\lambda F_2 & \partial_y F_2 \end{vmatrix}_{p=p_0} = \frac{1}{2\omega}\|\phi_\omega\|_{L^2}^4 \neq 0.$$

Therefore, the implicit function theorem implies that there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\vec{u} \in U_\varepsilon\big(\overrightarrow{\Phi_\omega}\big)$, there exist unique $C^1$-functions

$$y : U_\varepsilon\big(\overrightarrow{\Phi_\omega}\big) \to \mathbb{R}, \quad \lambda : U_\varepsilon\big(\overrightarrow{\Phi_\omega}\big) \to \mathbb{R}^+,$$

such that

$$\left\langle \vec{\eta}, \overrightarrow{\Gamma_\lambda} \right\rangle = \left\langle \vec{\eta}, \overrightarrow{\Psi_\lambda} \right\rangle = 0. \tag{4.5}$$

Furthermore,

$$\begin{pmatrix} \partial_u \lambda & \partial_v \lambda \\ \partial_u y & \partial_v y \end{pmatrix} = J^{-1} \begin{pmatrix} \partial_u F_1 & \partial_v F_1 \\ \partial_u F_2 & \partial_v F_2 \end{pmatrix}.$$

This implies that

$$|\lambda - \omega| \lesssim \|\vec{u} - \overrightarrow{\Phi_\omega}\|_{H^1 \times L^2} < \varepsilon.$$

This finishes the proof of the proposition. □

## 5. Dynamic of the parameters

In this section, we control the modulation parameters $y$ and $\lambda$. The effect of giving a precise control on modulation parameters is to obtain the structure of $I'(t)$ in Section 7. The main result is the following.

**Proposition 5.1.** *Let $\vec{u} = (u, v)^T$ be the solution of (1.4) with $\vec{u} \in U_\varepsilon(\overrightarrow{\Phi_\omega})$, where $\varepsilon$ is obtained in Proposition 4.1. Let $y, \lambda$, $\vec{\eta} = (\xi, \eta)^T$ be the parameters and vector obtained in Proposition 4.1; then*

$$\dot{y} - \lambda = \|\phi_\lambda\|_{L^2}^{-2}\Big[Q\big(\overrightarrow{\Phi_\lambda}\big) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] - \|\phi_\lambda\|_{L^2}^{-2}\Big[Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] + O(\|\vec{\eta}\|_{H^1\times L^2}^2)$$

*and*

$$\dot{\lambda} = O\big(\|\vec{\eta}\|_{H^1\times L^2}\big).$$

The proof of the proposition is split into the following two lemmas. The first lemma is

**Lemma 5.2.** *Under the same assumption in Proposition 5.1, we have*

$$\dot{y} - \lambda = -\|\phi_\lambda\|_{L^2}^{-2}\langle \eta, \phi_\lambda\rangle + O\big(\|\vec{\eta}\|_{H^1\times L^2}^2\big),$$

*and*

$$\dot{\lambda} = O\big(\|\vec{\eta}\|_{H^1\times L^2}\big).$$

*Proof.* Recall the definition $\vec{\eta}(t) = \vec{u}\big(t, \cdot + y(t)\big) - \overrightarrow{\Phi_{\lambda(t)}}$ in (4.2), that is,

$$\begin{cases} u(t,x) = \phi_\lambda\big(x - y(t)\big) + \xi\big(t, x - y(t)\big), \\ v(t,x) = -\lambda\phi_\lambda\big(x - y(t)\big) + \eta\big(t, x - y(t)\big). \end{cases} \tag{5.1}$$

Using the first equation of the equivalent system (1.4), we have

$$\dot{\lambda}\partial_\lambda\phi_\lambda - (\dot{y} - \lambda)\partial_x\phi_\lambda = -\dot{\xi} + (\dot{y} - \lambda)\partial_x\xi + \lambda\partial_x\xi + \partial_x\eta. \tag{5.2}$$

We recall the definition of $\overrightarrow{\Gamma_\lambda}$ in Proposition 4.1 and denote $\gamma_\lambda$ as the first component of $\overrightarrow{\Gamma_\lambda}$. Now we multiply both sides of equality (5.2) by $\gamma_\lambda$ and integrate to obtain

$$\begin{aligned} &\langle \dot{\lambda}\partial_\lambda\phi_\lambda, \gamma_\lambda\rangle - \langle (\dot{y} - \lambda)\partial_x\phi_\lambda, \gamma_\lambda\rangle \\ &\qquad = \langle -\dot{\xi}, \gamma_\lambda\rangle + (\dot{y} - \lambda)\langle \partial_x\xi, \gamma_\lambda\rangle + \lambda\langle \partial_x\xi, \gamma_\lambda\rangle + \langle \partial_x\eta, \gamma_\lambda\rangle. \end{aligned} \tag{5.3}$$

We know that $\phi_\lambda$ is an even function and $\gamma_\lambda$ is an odd function, so $\langle \dot{\lambda}\partial_\lambda\phi_\lambda, \gamma_\lambda\rangle = 0$. By the orthogonality condition (4.3), we have

$$\langle \partial_x\xi, \gamma_\lambda\rangle = -\langle \vec{\eta}, \overrightarrow{\Psi_\lambda}\rangle = 0,$$

so we get

$$\langle \dot{\xi}, \gamma_\lambda\rangle = \partial_t\langle \xi, \gamma_\lambda\rangle - \langle \xi, \partial_t\gamma_\lambda\rangle = \partial_t\langle \vec{\eta}, \overrightarrow{\Gamma_\lambda}\rangle - \langle \xi, \partial_t\gamma_\lambda\rangle = -\langle \xi, \partial_t\gamma_\lambda\rangle = -\dot{\lambda}\langle \xi, \partial_\lambda\gamma_\lambda\rangle.$$

Thus, we simplify equality (5.3) to obtain

$$(\dot{y} - \lambda)\|\phi_\lambda\|_{L^2}^2 - \dot{\lambda}\langle \xi, \partial_\lambda\gamma_\lambda\rangle = -\langle \eta, \phi_\lambda\rangle. \tag{5.4}$$

Next we multiply both sides of equality (5.2) by the first component of $\overrightarrow{\Psi_\lambda}$ and integrate to obtain

$$\begin{aligned} &\langle \dot{\lambda}\partial_\lambda\phi_\lambda, \phi_\lambda\rangle - (\dot{y} - \lambda)\langle \partial_x\phi_\lambda, \phi_\lambda\rangle \\ &\qquad = \langle -\dot{\xi}, \phi_\lambda\rangle + (\dot{y} - \lambda)\langle \partial_x\xi, \phi_\lambda\rangle + \langle \lambda\partial_x\xi, \phi_\lambda\rangle + \langle \partial_x\eta, \phi_\lambda\rangle. \end{aligned} \tag{5.5}$$

Now we consider the term in (5.5) one by one. From Lemma 2.1, we have $\partial_\lambda \|\phi_\lambda\|_{L^2}^2 = -\frac{\|\phi_\lambda\|_{L^2}^2}{\lambda}$, so

$$\langle \dot\lambda \partial_\lambda \phi_\lambda, \phi_\lambda \rangle = \dot\lambda \int_{\mathbb{R}} \phi_\lambda \partial_\lambda \phi_\lambda = \frac{1}{2}\dot\lambda \partial_\lambda \|\phi_\lambda\|_{L^2}^2 = -\frac{\dot\lambda}{2\lambda}\|\phi_\lambda\|_{L^2}^2.$$

The term $-(\dot y - \lambda)\langle \partial_x \phi_\lambda, \phi_\lambda\rangle$ vanishes as $\phi_\lambda$ is an even function. By the orthogonality condition (4.3), we have

$$\langle \dot\xi, \phi_\lambda\rangle = \partial_t \langle \xi, \phi_\lambda\rangle - \langle \xi, \partial_t \phi_\lambda\rangle = \partial_t \langle \vec\eta, \overrightarrow{\Psi_\lambda}\rangle - \langle \xi, \partial_t\phi_\lambda\rangle = -\langle \xi, \partial_t \phi_\lambda\rangle.$$

Thus we simplify equality (5.5) to obtain

$$\dot\lambda\Big[-\frac{1}{2\lambda}\|\phi_\lambda\|_{L^2}^2 - \langle \xi, \partial_\lambda \phi_\lambda\rangle\Big] + (\dot y - \lambda)\langle \xi, \partial_x\phi_\lambda\rangle = -\langle \lambda\xi + \eta, \partial_x \phi_\lambda\rangle. \tag{5.6}$$

Since $\overrightarrow{\Psi_\lambda}, \overrightarrow{\Gamma_\lambda}, \overrightarrow{\Phi_\lambda}$ are smooth functions with exponential decay, combining (5.4) and (5.6), we get

$$\begin{cases} (\dot y - \lambda)\|\phi_\lambda\|_{L^2}^2 - \dot\lambda\langle \xi, \partial_\lambda \gamma_\lambda\rangle = -\langle \eta, \phi_\lambda\rangle, \\ \dot\lambda\Big[-\frac{1}{2\lambda}\|\phi_\lambda\|_{L^2}^2 - \langle \xi, \partial_\lambda\phi_\lambda\rangle\Big] + (\dot y - \lambda)\langle \xi, \partial_x \phi_\lambda\rangle = O(\|\vec\eta\|_{H^1\times L^2}). \end{cases} \tag{5.7}$$

We denote

$$A = \begin{pmatrix} -\langle \xi, \partial_\lambda \gamma_\lambda\rangle & \|\phi_\lambda\|_{L^2}^2 \\ -\frac{1}{2\lambda}\|\phi_\lambda\|_{L^2}^2 - \langle \xi, \partial_\lambda \phi_\lambda\rangle & \langle \xi, \partial_x \phi_\lambda\rangle \end{pmatrix}.$$

Then by a direct calculation, we get

$$\begin{pmatrix} \dot\lambda \\ \dot y - \lambda \end{pmatrix} = A^{-1}\begin{pmatrix} -\langle \eta, \phi_\lambda\rangle \\ O\big(\|\vec\eta\|_{H^1\times L^2}\big) \end{pmatrix} = \begin{pmatrix} O\big(\|\vec\eta\|_{H^1\times L^2}\big) \\ -\|\phi_\lambda\|_{L^2}^{-2}\langle \eta, \phi_\lambda\rangle + O\big(\|\vec\eta\|_{H^1\times L^2}^2\big) \end{pmatrix}.$$

This proves the lemma. □

The second lemma we need is the following.

**Lemma 5.3.** *Under the same assumption in Proposition 5.1, we have*

$$\int_{\mathbb{R}} \eta\phi_\lambda \,\mathrm{d}x = \Big[Q(\vec u_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] + \Big[Q\big(\overrightarrow{\Phi_\omega}\big) - Q\big(\overrightarrow{\Phi_\lambda}\big)\Big] + O\big(\|\vec\eta\|_{H^1\times L^2}^2\big).$$

*Proof.* Using equality (5.1) and the expression $Q(\vec u) = \int_{\mathbb{R}} uv\,\mathrm{d}x$, we have

$$\begin{aligned} Q(\vec u) &= Q\begin{pmatrix} \phi_\lambda + \xi \\ -\lambda\phi_\lambda + \eta \end{pmatrix} \\ &= \int_{\mathbb{R}} -\lambda\phi_\lambda^2\,\mathrm{d}x - \lambda\int_{\mathbb{R}} \xi\phi_\lambda\,\mathrm{d}x + \int_{\mathbb{R}} \eta\phi_\lambda\,\mathrm{d}x + \int_{\mathbb{R}} \xi\eta\,\mathrm{d}x. \end{aligned}$$

Now we analyse the last equality one by one. By (2.1), we have $Q\big(\overrightarrow{\Phi_\lambda}\big) = \int_{\mathbb{R}} -\lambda\phi_\lambda^2\,\mathrm{d}x$. Recall that we have the orthogonality condition $\Big\langle \vec\eta, \overrightarrow{\Psi_{\lambda(t)}}\Big\rangle = 0$ in (4.3), then

$$-\lambda\int_{\mathbb{R}} \xi\phi_\lambda\,\mathrm{d}x = -\lambda\int_{\mathbb{R}} \vec\eta\cdot\overrightarrow{\Psi_{\lambda(t)}}\,\mathrm{d}x = 0.$$

The final term gives $\int_{\mathbb{R}} \xi\eta\,\mathrm{d}x = O\big(\|\vec\eta\|_{H^1\times L^2}^2\big)$. Therefore,

$$Q(\vec u) = Q\big(\overrightarrow{\Phi_\lambda}\big) + \int_{\mathbb{R}} \eta\phi_\lambda\,\mathrm{d}x + O(\|\vec\eta\|_{H^1\times L^2}^2).$$

From the conservation law of momentum, we know

$$\begin{aligned}\int_{\mathbb{R}} \eta\phi_\lambda \,\mathrm{d}x &= Q(\vec{u}) - Q\big(\overrightarrow{\Phi_\lambda}\big) + O\big(\|\vec{\eta}\|^2_{H^1\times L^2}\big)\\ &= \Big[Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] + \Big[Q\big(\overrightarrow{\Phi_\omega}\big) - Q\big(\overrightarrow{\Phi_\lambda}\big)\Big] + O\big(\|\vec{\eta}\|^2_{H^1\times L^2}\big).\end{aligned}$$

This proves the lemma. □

Now we are ready to prove Proposition 5.1.

*Proof of Proposition 5.1.* Combining the estimates obtained in Lemmas 5.2 and 5.3, we have

$$\begin{aligned}\dot{y} - \lambda &= -\|\phi_\lambda\|^{-2}_{L^2} \int_{\mathbb{R}} \eta\phi_\lambda \,\mathrm{d}x + O(\|\vec{\eta}\|^2_{H^1\times L^2})\\ &= \|\phi_\lambda\|^{-2}_{L^2}\Big[Q\big(\overrightarrow{\Phi_\lambda}\big) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] - \|\phi_\lambda\|^{-2}_{L^2}\Big[Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] + O\big(\|\vec{\eta}\|^2_{H^1\times L^2}\big).\end{aligned}$$

This gives the proof of the proposition. □

## 6. Localized virial identities

The following lemmas are the localized virial identities. One can see [11] for the details of the proof.

Let $\nu$ is a $H^2$-solution of $\partial_x \nu = u$, and

$$I_1(t) = \int_{\mathbb{R}} \nu\partial_t\nu \,\mathrm{d}x.$$

**Lemma 6.1.** *Let $\vec{u} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ be the solution of the system* (1.4)*, then*

$$I_1'(t) = \|v\|^2_{L^2} - \|u\|^2_{L^2} - \|u_x\|^2_{L^2} + \|u\|^{p+2}_{L^{p+2}}.$$

Let

$$I_2(t) = \int_{\mathbb{R}} \varphi\big(x - y(t)\big)uv \,\mathrm{d}x,$$

then we have the following lemma.

**Lemma 6.2.** *Let $\varphi \in C^3(\mathbb{R})$, $\vec{u} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ be the solution of* (1.4)*, then*

$$\begin{aligned}I_2'(t) = &-\dot{y}\int_{\mathbb{R}} \varphi'\big(x - y(t)\big)uv \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi'\big(x - y(t)\big)\Big(3|u_x|^2 + v^2 + u^2 - \frac{2(p+1)}{p+2}|u|^{p+2}\Big)\,\mathrm{d}x\\ &+ \frac{1}{2}\int_{\mathbb{R}} \varphi'''\big(x - y(t)\big)u^2 \,\mathrm{d}x.\end{aligned}$$

## 7. Proof of the main theorem

This section is devoted to prove our main theorem.

**7.1. Virial identities.** Let $\varphi(x)$ be a smooth cutoff function, where

$$\varphi(x)=\begin{cases}x, & |x|\le R,\\ 0, & |x|\ge 2R,\end{cases} \tag{7.1}$$

$0\le\varphi'\le 1$, $|\varphi'''|\lesssim\frac{1}{R^2}$ for any $x\in\mathbb{R}$. Moreover, we denote

$$I(t)=\Big(\frac{4}{p}-2\Big)I_1(t)+2I_2(t).$$

Then we have the following lemma.

**Lemma 7.1.** *Let $R>0$, $y$, $\lambda$, $\vec{\eta}=(\xi,\eta)^T$ be the parameters and vector obtained in Proposition 4.1. Then*

$$\begin{aligned}I'(t)=&-2\Big(\frac{4}{p}+1\Big)E(\vec{u}_0)-\Big(4\lambda\frac{4-p}{p}+2\lambda\Big)Q(\vec{u}_0)+\Big(2-2\lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2\\&-2\Big(\dot y-\lambda\Big)Q(\vec{u}_0)+\Big(2-2\lambda^2\frac{4-p}{p}\Big)\|\xi\|_{L^2}^2+2\frac{4-p}{p}\|\lambda\xi+\eta\|_{L^2}^2+R(\vec{u}),\end{aligned}\tag{7.2}$$

*where*

$$\begin{aligned}R(\vec{u})=2\int_{\mathbb{R}}\Big[1-\varphi'\big(x-y(t)\big)\Big]\Big(\dot y uv+\frac{3}{2}u_x^2+\frac{1}{2}u^2+\frac{1}{2}v^2-\frac{p+1}{p+2}|u|^{p+2}\Big)\,\mathrm{d}x\\+\int_{\mathbb{R}}\varphi'''\big(x-y(t)\big)u^2\,\mathrm{d}x.\end{aligned}\tag{7.3}$$

*Proof.* From Lemma 6.2 and the conservation law of momentum, we change the form of $I_2'(t)$ as

$$\begin{aligned}I_2'(t)=&-\dot y\int_{\mathbb{R}}\Big[\varphi'\big(x-y(t)\big)-1+1\Big]uv\,\mathrm{d}x+\frac12\int_{\mathbb{R}}\varphi'''\big(x-y(t)\big)u^2\,\mathrm{d}x\\&-\frac12\int_{\mathbb{R}}\Big[\varphi'\big(x-y(t)\big)-1+1\Big]\Big[3|u_x|^2+v^2+u^2-\frac{2(p+1)}{p+2}|u|^{p+2}\Big]\,\mathrm{d}x\\=&-\dot yQ(\vec{u}_0)-\frac12\Big[3\|u_x\|_{L^2}^2+\|u\|_{L^2}^2+\|v\|_{L^2}^2-\frac{2(p+1)}{p+2}\|u\|_{L^{p+2}}^{p+2}\Big]+\frac12\int_{\mathbb{R}}\varphi'''\big(x-y(t)\big)u^2\,\mathrm{d}x\\&+\int_{\mathbb{R}}\Big[1-\varphi'\big(x-y(t)\big)\Big]\Big(\dot yuv+\frac32|u_x|^2+\frac12v^2+\frac12u^2-\frac{p+1}{p+2}|u|^{p+2}\Big)\,\mathrm{d}x.\end{aligned}$$

Then a direct computation gives

$$\begin{aligned}I'(t)=&\Big(\frac{4}{p}-2\Big)I_1'(t)+2I_2'(t)\\=&-\Big(\frac4p+1\Big)\|u_x\|_{L^2}^2+\Big(\frac4p-3\Big)\|v\|_{L^2}^2+\Big(-\frac4p+1\Big)\|u\|_{L^2}^2+\frac{2(p+4)}{p(p+2)}\|u\|_{L^{p+2}}^{p+2}\\&+2\int_{\mathbb{R}}\Big[1-\varphi'\big(x-y(t)\big)\Big]\Big(\dot yuv+\frac32|u_x|^2+\frac12v^2+\frac12u^2-\frac{p+1}{p+2}|u|^{p+2}\Big)\,\mathrm{d}x\\&+\int_{\mathbb{R}}\varphi'''\big(x-y(t)\big)u^2\,\mathrm{d}x-2\dot yQ(\vec{u}_0).\end{aligned}$$

From the conservation law of energy, we have

$$2E(\vec{u}_0) = \|u_x\|_{L^2}^2 + \|v\|_{L^2}^2 + \|u\|_{L^2}^2 - \frac{2}{p+2}\|u_x\|_{L^{p+2}}^{p+2}.$$

Then

$$\begin{aligned}
&-\Big(\frac{4}{p}+1\Big)\|u_x\|_{L^2}^2 + \Big(\frac{4}{p}-3\Big)\|v\|_{L^2}^2 + \Big(-\frac{4}{p}+1\Big)\|u\|_{L^2}^2 + \frac{2(p+4)}{p(p+2)}\|u\|_{L^{p+2}}^{p+2} \\
=&-2\Big(\frac{4}{p}+1\Big)E(\vec{u}_0) + \frac{2(4-p)}{p}\Big[\frac{p}{4-p}\|u\|_{L^2}^2 + \|v\|_{L^2}^2\Big] \\
=&-2\Big(\frac{4}{p}+1\Big)E(\vec{u}_0) + \frac{2(4-p)}{p}\Big[\lambda^2\|u\|_{L^2}^2 + \|v\|_{L^2}^2\Big] + 2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|u\|_{L^2}^2 \\
=&-2\Big(\frac{4}{p}+1\Big)E(\vec{u}_0) + \frac{2(4-p)}{p}\|v+\lambda u\|_{L^2}^2 - 4\lambda\frac{4-p}{p}Q(\vec{u}_0) + 2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|u\|_{L^2}^2.
\end{aligned}$$

By orthogonality condition (4.3) and using formula (5.1), we have the following two equalities:

$$\begin{aligned}
\|u\|_{L^2}^2 &= \|\phi_\lambda\|_{L^2}^2 + 2\langle\phi_\lambda,\xi\rangle + \|\xi\|_{L^2}^2 \\
&= \|\phi_\lambda\|_{L^2}^2 + 2\langle\overrightarrow{\Psi}_\lambda,\vec{\eta}\rangle + \|\xi\|_{L^2}^2 = \|\phi_\lambda\|_{L^2}^2 + \|\xi\|_{L^2}^2, \\
\|v+\lambda u\|_{L^2}^2 &= \|-\lambda\phi_\lambda + \eta + \lambda\phi_\lambda + \lambda\xi\|_{L^2}^2 = \|\lambda\xi+\eta\|_{L^2}^2.
\end{aligned}$$

Hence, using the equalities above, we obtain

$$\begin{aligned}
I'(t) =&-2\Big(\frac{4}{p}+1\Big)E(\vec{u}_0) + \frac{2(4-p)}{p}\|v+\lambda u\|_{L^2}^2 - 4\lambda\frac{4-p}{p}Q(\vec{u}_0) + 2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|u\|_{L^2}^2 \\
&+ 2\int_{\mathbb{R}}\Big[1-\varphi'\big(x-y(t)\big)\Big]\Big(\dot{y}uv + \frac{3}{2}u_x^2 + \frac{1}{2}v^2 + \frac{1}{2}u^2 - \frac{p+1}{p+2}|u|^{p+2}\Big)\,\mathrm{d}x \\
&+ \int_{\mathbb{R}}\varphi'''\big(x-y(t)\big)u^2\,\mathrm{d}x - 2\dot{y}Q(\vec{u}_0) \\
=&-2\Big(\frac{4}{p}+1\Big)E(\vec{u}_0) + \frac{2(4-p)}{p}\|\eta+\lambda\xi\|_{L^2}^2 - 4\lambda\frac{4-p}{p}Q(\vec{u}_0) \\
&+ 2\Big(1-\lambda^2\frac{4-p}{p}\Big)\big(\|\phi_\lambda\|_{L^2}^2 + \|\xi\|_{L^2}^2\big) \\
&+ 2\int_{\mathbb{R}}\Big[1-\varphi'\big(x-y(t)\big)\Big]\Big(\dot{y}uv + \frac{3}{2}u_x^2 + \frac{1}{2}v^2 + \frac{1}{2}u^2 - \frac{p+1}{p+2}|u|^{p+2}\Big)\,\mathrm{d}x \\
&+ \int_{\mathbb{R}}\varphi'''\big(x-y(t)\big)u^2\,\mathrm{d}x - 2(\dot{y}-\lambda+\lambda)Q(\vec{u}_0) \\
=&-2\Big(\frac{4}{p}+1\Big)E(\vec{u}_0) - 2\lambda\Big(2\frac{4-p}{p}+1\Big)Q(\vec{u}_0) + 2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2 \\
&- 2(\dot{y}-\lambda)Q(\vec{u}_0) + 2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\xi\|_{L^2}^2 + 2\frac{4-p}{p}\|\lambda\xi+\eta\|_{L^2}^2 + R(\vec{u}).
\end{aligned}$$

This proves the lemma. □

Now we consider $R(\vec{u})$ in (7.3).

**Lemma 7.2.** *Let $R(\vec{u})$ be defined in (7.3); then*

$$R(\vec{u}) = O(\|\vec{\eta}\|_{H^1\times L^2}^2 + \frac{1}{R}).$$

*Proof.* Using the definition of the cutoff function $\varphi$ in (7.1), we have

$$\begin{aligned}|R(\vec{u})| =& \bigg| \int_{\{|x-y(t)|>R\}} 2\Big[1-\varphi'\big(x-y(t)\big)\Big] \\ & \Big(\dot{y}uv + \frac{3}{2}|u_x|^2 + \frac{1}{2}u^2 + \frac{1}{2}v^2 - \frac{p+1}{p+2}|u|^{p+2}\Big)\,\mathrm{d}x + \int_{\mathbb{R}} \varphi'''\big(x-y(t)\big)u^2\,\mathrm{d}x \bigg| \\ \lesssim & \int_{\{|x-y(t)|>R\}} \Big(1+|\varphi'\big(x-y(t)\big)|\Big)\Big(|\dot{y}||u||v| + |u_x|^2 + u^2 + v^2 + |u|^{p+2}\Big)\,\mathrm{d}x + \frac{1}{R^2}.\end{aligned}$$

By Hölder's inequality, $|\varphi'| \le 1$, and $|\dot{y}| \lesssim 1$ (from Lemma 5.2), we have

$$\begin{aligned}|R(\vec{u})| &\lesssim \int_{\{|x-y(t)|>R\}} \big(|u_x|^2 + u^2 + v^2 + |u|^{p+2}\big)\,\mathrm{d}x + \frac{1}{R^2} \\ &\lesssim \int_{\{|x|>R\}} \Big[(\partial_x\phi_\lambda + \partial_x\xi)^2 + (\phi_\lambda + \xi)^2 + (\lambda\phi_\lambda - \eta)^2 + |\phi_\lambda + \xi|^{p+2}\Big]\,\mathrm{d}x + \frac{1}{R^2},\end{aligned}$$

where we have used equality (4.2) in the last step. Further, using the property of exponential decay of $\partial_x\phi_\lambda$, we have

$$\int_{\{|x|>R\}} (\partial_x\phi_\lambda)^2\,\mathrm{d}x \le C\int_{\{|x|>R\}} e^{-C|x|}\,\mathrm{d}x \le \frac{C}{R}.$$

Then Young's inequality gives

$$\begin{aligned}&\int_{\{|x|>R\}} (\partial_x\phi_\lambda + \partial_x\xi)^2\,\mathrm{d}x \\ &\qquad\lesssim \int_{\{|x|>R\}} \left[(\partial_x\phi_\lambda)^2 + (\partial_x\xi)^2\right]\,\mathrm{d}x \\ &\qquad\lesssim \frac{1}{R} + \|\partial_x\xi\|_{L^2}^2. \qquad (7.4)\end{aligned}$$

Using a similar method, we can prove

$$\int_{\{|x|>R\}} (\phi_\lambda + \xi)^2\,\mathrm{d}x \le C(\frac{1}{R} + \|\xi\|_{L^2}^2), \qquad (7.5)$$

$$\int_{\{|x|>R\}} (\lambda\phi_\lambda - \eta)^2\,\mathrm{d}x \le C(\frac{1}{R} + \|\eta\|_{L^2}^2), \qquad (7.6)$$

$$\int_{\{|x|>R\}} |\phi_\lambda + \xi|^{p+2}\,\mathrm{d}x \le C(\frac{1}{R} + \|\xi\|_{H^1}^2). \qquad (7.7)$$

Thus, we combine (7.4)-(7.7) to obtain

$$|R(\vec{u})| \le C(\frac{1}{R} + \|\vec{\eta}\|_{H^1\times L^2}^2).$$

This implies that

$$R(\vec{u}) = O(\|\vec{\eta}\|_{H^1\times L^2}^2 + \frac{1}{R}).$$

This proves the lemma. ∎

7.2. **Structure of $I'(t)$.** Our purpose is to control the difference between $u$ and the modulated solitons and the modulated scaling parameter. Note that the quantities involved in $I'(t)$ are nonconserved; the main issue is to analyse the quantities in detail. In particular, we structure $I'(t)$ as follows.

Denote

$$\begin{aligned}\rho(\vec{u_0}) =& -2\Big(\frac{4}{p}+1\Big)\Big[E(\vec{u_0})-E\big(\overrightarrow{\Phi_\omega}\big)\Big]-2\lambda\Big(2\frac{4-p}{p}+1\Big)\Big[Q(\vec{u_0})-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]\\ &+2\|\phi_\lambda\|_{L^2}^{-2}Q(\vec{u_0})\Big[Q(\vec{u_0})-Q\big(\overrightarrow{\Phi_\omega}\big)\Big], \qquad (7.8)\\ h(\lambda) =& -2\Big(\frac{4}{p}+1\Big)E\big(\overrightarrow{\Phi_\omega}\big)-2\lambda\Big(2\frac{4-p}{p}+1\Big)Q\big(\overrightarrow{\Phi_\omega}\big)+2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2\\ &-2\|\phi_\lambda\|_{L^2}^{-2}Q(\vec{u_0})\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big], \qquad (7.9)\\ \widetilde{R}(\vec{u}) =& R(\vec{u})+2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\xi\|_{L^2}^2+2\frac{4-p}{p}\|\lambda\xi+\eta\|_{L^2}^2\\ &-2Q(\vec{u_0})\Bigg\{(\dot{y}-\lambda)-\frac{1}{\|\phi_\lambda\|_{L^2}^2}\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]\\ &+\|\phi_\lambda\|_{L^2}^{-2}\Big[Q(\vec{u_0})-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]\Bigg\}. \qquad (7.10)\end{aligned}$$

Now we rewrite $I'(t)$ as follows. In particular, we remark that there are no one-order terms with respect to $\vec{\eta}$ and $\lambda$.

**Lemma 7.3.**

$$I'(t)=\rho(\vec{u_0})+h(\lambda)+\widetilde{R}(\vec{u}).$$

*Proof.* We will make a direct calculation. From (7.2), we know that

$$\begin{aligned}I'(t) =& -2\Big(\frac{4}{p}+1\Big)E(\vec{u_0})-2\lambda\Big(2\frac{4-p}{p}+1\Big)Q(\vec{u_0})+2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2\\ &-2(\dot{y}-\lambda)Q(\vec{u_0})+2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\xi\|_{L^2}^2+2\frac{4-p}{p}\|\lambda\xi+\eta\|_{L^2}^2+R(\vec{u})\\ =& -2\Big(\frac{4}{p}+1\Big)\Big[E(\vec{u_0})-E\big(\overrightarrow{\Phi_\omega}\big)\Big]-2\lambda\Big(2\frac{4-p}{p}+1\Big)\Big[Q(\vec{u_0})-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]\\ &+2\|\phi_\lambda\|_{L^2}^{-2}Q(\vec{u_0})\Big[Q(\vec{u_0})-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]\\ &-2\Big(\frac{4}{p}+1\Big)E\big(\overrightarrow{\Phi_\omega}\big)-2\lambda\Big(2\frac{4-p}{p}+1\Big)Q\big(\overrightarrow{\Phi_\omega}\big)+2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2\\ &-2\|\phi_\lambda\|_{L^2}^{-2}Q(\vec{u_0})\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]\\ &+R(\vec{u})+2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\xi\|_{L^2}^2+2\frac{4-p}{p}\|\lambda\xi+\eta\|_{L^2}^2\\ &-2Q(\vec{u_0})\Bigg\{(\dot{y}-\lambda)-\|\phi_\lambda\|_{L^2}^{-2}\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]+\|\phi_\lambda\|_{L^2}^{-2}\Big[Q(\vec{u_0})-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]\Bigg\}\\ =& \rho(\vec{u_0})+h(\lambda)+\widetilde{R}(\vec{u}).\end{aligned}$$

This completes the proof. □

By Lemma 7.2 and Proposition 5.1, we obtain

$$\widetilde{R}(\vec{u}) = O\big(\|\vec{\eta}\|^2_{H^1\times L^2} + \frac{1}{R}\big). \tag{7.11}$$

7.3. **Positivity of the main parts.** The main parts of $I'(t)$, $\rho(\vec{u}_0)$, and $h(\lambda)$ are considered in this subsection. We shall prove their positivity in the following.

**Lemma 7.4.** *Let $\vec{u}_0 = (1+a)\overrightarrow{\Phi_\omega}$ for some small positive constant $a$. Then*

1) $\rho(\vec{u}_0) \geq C_1 a, \quad$ *for some* $\quad C_1 > 0;$

2) $h(\lambda) \geq C_2(\lambda-\omega)^2 + O(a(\lambda-\omega)^2) + o((\lambda-\omega)^2) \quad$ *for some* $\quad C_2 > 0.$

*Proof.* 1) Recall the definition of $\rho(\vec{u}_0)$ in (7.8):

$$\begin{aligned}\rho(\vec{u}_0) =& -2\Big(\frac{4}{p}+1\Big)\Big[E(\vec{u}_0) - E\big(\overrightarrow{\Phi_\omega}\big)\Big] - 2\lambda\Big(2\frac{4-p}{p}+1\Big)\Big[Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big]\\ &+ 2\|\phi_\lambda\|^{-2}_{L^2} Q(\vec{u}_0)\Big[Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big].\end{aligned} \tag{7.12}$$

First, by Taylor's type expansion, we have

$$\begin{aligned}E(\vec{u}_0) - E\big(\overrightarrow{\Phi_\omega}\big) &= \Big\langle E'\big(\overrightarrow{\Phi_\omega}\big), \vec{u}_0 - \overrightarrow{\Phi_\omega}\Big\rangle + O\big(\|\vec{u}_0 - \overrightarrow{\Phi_\omega}\|^2_{H^1\times L^2}\big)\\ &= a\Big\langle E'\big(\overrightarrow{\Phi_\omega}\big), \overrightarrow{\Phi_\omega}\Big\rangle + O(a^2).\end{aligned}$$

Using the expression of $E'\big(\overrightarrow{\Phi_\omega}\big)$ in (2.5), we have

$$\begin{aligned}E(\vec{u}_0) - E\big(\overrightarrow{\Phi_\omega}\big) &= a\int_{\mathbb{R}} (-\partial_{xx}\phi_\omega + \phi_\omega - \phi_\omega^{p+1}, -\omega\phi_\omega)\cdot\begin{pmatrix}\phi_\omega\\ -\omega\phi_\omega\end{pmatrix}\,\mathrm{d}x + O(a^2)\\ &= a\int_{\mathbb{R}} (-\partial_{xx}\phi_\omega + (1-\omega^2)\phi_\omega - \phi_\omega^{p+1} + \omega^2\phi_\omega, -\omega\phi_\omega)\cdot\begin{pmatrix}\phi_\omega\\ -\omega\phi_\omega\end{pmatrix}\,\mathrm{d}x + O(a^2)\\ &= 2a\omega^2\|\phi_\omega\|^2_{L^2} + O(a^2),\end{aligned} \tag{7.13}$$

where we have used equation (1.3) in the last step. Next, we compute the term $Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)$ in (7.12):

$$\begin{aligned}Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big) &= \Big\langle Q'\big(\overrightarrow{\Phi_\omega}\big), \vec{u}_0 - \overrightarrow{\Phi_\omega}\Big\rangle + O\big(\|\vec{u}_0 - \overrightarrow{\Phi_\omega}\|^2_{H^1\times L^2}\big)\\ &= a\Big\langle Q'\big(\overrightarrow{\Phi_\omega}\big), \overrightarrow{\Phi_\omega}\Big\rangle + O(a^2).\end{aligned}$$

Using the expression of $Q'\big(\overrightarrow{\Phi_\omega}\big)$ in (2.4), we have

$$\begin{aligned}Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big) &= a\int_{\mathbb{R}} (-\omega\phi_\omega, \phi_\omega)\cdot\begin{pmatrix}\phi_\omega\\ -\omega\phi_\omega\end{pmatrix}\,\mathrm{d}x + O(a^2)\\ &= -2a\omega\|\phi_\omega\|^2_{L^2} + O(a^2).\end{aligned} \tag{7.14}$$

Then we put (7.13) and (7.14) into the expression of $\rho(\vec{u}_0)$:

$$\rho(\vec{u}_0) = -2\Big(\frac{4}{p}+1\Big)\Big[E(\vec{u}_0) - E\big(\overrightarrow{\Phi_\omega}\big)\Big] - 2\lambda\Big(2\frac{4-p}{p}+1\Big)\Big[Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big]$$

$$
\begin{aligned}
&+ 2\|\phi_\lambda\|_{L^2}^{-2} Q(\vec{u}_0)\Big[Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] \\
=&- 2\Big(\frac{4}{p}+1\Big)\Big[2a\omega^2\|\phi_\omega\|_{L^2}^2 + O(a^2)\Big] - 2\lambda\Big(2\frac{4-p}{p}+1\Big)\Big[-2a\omega\|\phi_\omega\|_{L^2}^2 + O(a^2)\Big] \\
&+ 2\|\phi_\lambda\|_{L^2}^{-2} Q(\vec{u}_0)\Big[-2a\omega\|\phi_\omega\|_{L^2}^2 + O(a^2)\Big] \\
=&- 4a\omega^2\Big(\frac{4}{p}+1\Big)\|\phi_\omega\|_{L^2}^2 + 4a\omega\lambda\Big(2\frac{4-p}{p}+1\Big)\|\phi_\omega\|_{L^2}^2 - 4a\omega Q(\vec{u}_0)\frac{\|\phi_\omega\|_{L^2}^2}{\|\phi_\lambda\|_{L^2}^2} \\
&+ O(a^2). \qquad (7.15)
\end{aligned}
$$

For the term $4a\omega\lambda\Big(2\frac{4-p}{p}+1\Big)\|\phi_\omega\|_{L^2}^2$, we have

$$
4a\omega\lambda\Big(2\frac{4-p}{p}+1\Big)\|\phi_\omega\|_{L^2}^2 = 4a\omega^2\Big(2\frac{4-p}{p}+1\Big)\|\phi_\omega\|_{L^2}^2 + O(a|\lambda-\omega|). \qquad (7.16)
$$

For the term $-4a\omega Q(\vec{u}_0)\frac{\|\phi_\omega\|_{L^2}^2}{\|\phi_\lambda\|_{L^2}^2}$, we use the expression $\phi_\omega(x) = (1-\omega^2)^{\frac{1}{p}}\phi_0\left(\sqrt{1-\omega^2}x\right)$ in (2.2) and Taylor's type expansion again to calculate

$$
\begin{aligned}
-4a\omega Q(\vec{u}_0)\frac{\|\phi_\omega\|_{L^2}^2}{\|\phi_\lambda\|_{L^2}^2} &= -4a\omega Q(\vec{u}_0)\frac{(1-\omega^2)^{\frac{2}{p}-\frac{1}{2}}\|\phi_0\|_{L^2}^2}{(1-\lambda^2)^{\frac{2}{p}-\frac{1}{2}}\|\phi_0\|_{L^2}^2} = -4a\omega Q(\vec{u}_0)\frac{(1-\omega^2)^{\frac{2}{p}-\frac{1}{2}}}{(1-\lambda^2)^{\frac{2}{p}-\frac{1}{2}}} \\
&= -4a\omega Q(\vec{u}_0)(1-\omega^2)^{\frac{2}{p}-\frac{1}{2}}\Big[(1-\omega^2)^{\frac{1}{2}-\frac{2}{p}} + O(|\lambda-\omega|)\Big] \\
&= -4a\omega Q(\vec{u}_0) + Q(\vec{u}_0)O(a|\lambda-\omega|).
\end{aligned}
$$

From the definition of $Q(\vec{u})$ in (1.5), we have

$$
Q(\vec{u}_0) = Q\Big((1+a)\overrightarrow{\Phi_\omega}\Big) = -\omega(1+a)^2\|\phi_\omega\|_{L^2}^2.
$$

Combining the last two estimates, we obtain

$$
-4a\omega Q(\vec{u}_0)\frac{\|\phi_\omega\|_{L^2}^2}{\|\phi_\lambda\|_{L^2}^2} = 4a\omega^2\|\phi_\omega\|_{L^2}^2 + O(a^2) + O(a|\lambda-\omega|). \qquad (7.17)
$$

Finally we put (7.16) and (7.17) into (7.15) to obtain

$$
\begin{aligned}
\rho(\vec{u}_0) =&- 4a\omega^2\Big(\frac{4}{p}+1\Big)\|\phi_\omega\|_{L^2}^2 + 4a\omega^2\frac{8-p}{p}\|\phi_\omega\|_{L^2}^2 \\
&+ 4a\omega^2\|\phi_\omega\|_{L^2}^2 + O(a|\lambda-\omega|) + O(a^2) \\
=&4a\omega^2\frac{4-p}{p}\|\phi_\omega\|_{L^2}^2 + O(a|\lambda-\omega|) + O(a^2).
\end{aligned}
$$

Choosing $a$ and $\varepsilon_0$ small enough, where $\varepsilon_0$ is the constant in Proposition 4.1, and by (4.4), we obtain conclusion 1) of this lemma.

2) Recall the definition of $h(\lambda)$ from (7.9):

$$
\begin{aligned}
h(\lambda) = -2\Big(\frac{4}{p}+1\Big)E\big(\overrightarrow{\Phi_\omega}\big) - 2\lambda\Big(2\frac{4-p}{p}+1\Big)Q\big(\overrightarrow{\Phi_\omega}\big) + 2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2 \\
- 2\|\phi_\lambda\|_{L^2}^{-2}Q(\vec{u}_0)\Big[Q\big(\overrightarrow{\Phi_\lambda}\big) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big]. \qquad (7.18)
\end{aligned}
$$

First, we consider the last term and claim that

$$-2\|\phi_\lambda\|_{L^2}^{-2}Q(\vec{u}_0)\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big] \\ =2\omega\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]+o\big((\lambda-\omega)^2\big)+O\big(a(\lambda-\omega)^2\big). \tag{7.19}$$

To prove (7.19), we need the following equalities, which can be obtained by Taylor's type expansion and Lemma 2.1:

$$Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)=\partial_\lambda Q\big(\overrightarrow{\Phi_\lambda}\big)\Big|_{\lambda=\omega}(\lambda-\omega)+O((\lambda-\omega)^2) \\ =O((\lambda-\omega)^2), \tag{7.20}$$

$$Q(\vec{u}_0)-Q\big(\overrightarrow{\Phi_\omega}\big)=O(a), \tag{7.21}$$

$$\|\phi_\lambda\|_{L^2}^{-2}-\|\phi_\omega\|_{L^2}^{-2}=O(|\lambda-\omega|). \tag{7.22}$$

Using (7.20)–(7.22), we obtain

$$-2\|\phi_\lambda\|_{L^2}^{-2}Q(\vec{u}_0)\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big] \\ =-2\|\phi_\omega\|_{L^2}^{-2}Q\big(\overrightarrow{\Phi_\omega}\big)\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]+o((\lambda-\omega)^2)+O\big(a(\lambda-\omega)^2\big).$$

Further, from (2.1), we get

$$-2\|\phi_\omega\|_{L^2}^{-2}Q\big(\overrightarrow{\Phi_\omega}\big)\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big] \\ =-2\|\phi_\omega\|_{L^2}^{-2}\cdot(-\omega\|\phi_\omega\|_{L^2}^2)\cdot\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big] \\ =2\omega\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big].$$

Thus, we obtain

$$-2\|\phi_\lambda\|_{L^2}^{-2}Q(\vec{u}_0)\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big] \\ =2\omega\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]+o\big((\lambda-\omega)^2\big)+O\big(a(\lambda-\omega)^2\big).$$

This proves (7.19).

Inserting (7.19) into (7.18), we get

$$h(\lambda)=-2\Big(\frac{4}{p}+1\Big)E\big(\overrightarrow{\Phi_\omega}\big)-2\lambda\Big(2\frac{4-p}{p}+1\Big)Q\big(\overrightarrow{\Phi_\omega}\big)+2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2 \\ +2\omega\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]+o\big((\lambda-\omega)^2\big)+O\big(a(\lambda-\omega)^2\big).$$

Let

$$h_1(\lambda)=-2\Big(\frac{4}{p}+1\Big)E\big(\overrightarrow{\Phi_\omega}\big)-2\lambda\Big(2\frac{4-p}{p}+1\Big)Q\big(\overrightarrow{\Phi_\omega}\big) \\ +2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2+2\omega\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]. \tag{7.23}$$

Then

$$h(\lambda)=h_1(\lambda)+o\big((\lambda-\omega)^2\big)+O\big(a(\lambda-\omega)^2\big). \tag{7.24}$$

Now we claim that

$$h_1(\omega) = 0, \quad h_1'(\omega) = 0, \quad h_1''(\omega) > 0. \tag{7.25}$$

We prove the claim by the following three steps.

*Step 1.* $h_1(\omega) = 0$.

By the definition of $h_1(\lambda)$, we have

$$h_1(\omega) = -2\Big(\frac{4}{p} + 1\Big)E\big(\overrightarrow{\Phi_\omega}\big) - 2\omega\Big(2\frac{4-p}{p} + 1\Big)Q\big(\overrightarrow{\Phi_\omega}\big) + 2\Big(1 - \omega^2\frac{4-p}{p}\Big)\|\phi_\omega\|_{L^2}^2.$$

By (2.1) and $E\big(\overrightarrow{\Phi_\omega}\big)$ in (1.6), we have

$$\begin{aligned}
h_1(\omega) = &-2\Big(\frac{4}{p} + 1\Big)\Big(\frac{1}{2}\int_{\mathbb{R}} \big(|\partial_x\phi_\omega|^2 + |\phi_\omega|^2 + |-\omega\phi_\omega|^2\big)\,\mathrm{d}x - \frac{1}{p+2}\int_{\mathbb{R}} |\phi_\omega|^{p+2}\,\mathrm{d}x\Big)\\
&- 2\omega\Big(2\frac{4-p}{p} + 1\Big)\Big(\int_{\mathbb{R}} -\omega\phi_\omega^2\,\mathrm{d}x\Big) + 2\Big(1 - \omega^2\frac{4-p}{p}\Big)\|\phi_\omega\|_{L^2}^2\\
= &-\frac{8}{p}\omega^2\|\phi_\omega\|_{L^2}^2 + 2\|\phi_\omega\|_{L^2}^2 = 0,
\end{aligned}$$

where we have used $\omega^2 = \frac{p}{4}$ in the above computation. Therefore, we have $h_1(\omega) = 0$.

*Step 2.* $h_1'(\omega) = 0$.

Using the expression of $h_1(\lambda)$ in (7.23), we have

$$\begin{aligned}
h_1'(\lambda) = &-2\Big(2\frac{4-p}{p} + 1\Big)Q\big(\overrightarrow{\Phi_\omega}\big) - 4\lambda\frac{4-p}{p}\|\phi_\lambda\|_{L^2}^2\\
&+ 2\Big(1 - \lambda^2\frac{4-p}{p}\Big)\partial_\lambda(\|\phi_\lambda\|_{L^2}^2) + 2\omega\partial_\lambda Q\big(\overrightarrow{\Phi_\lambda}\big).
\end{aligned} \tag{7.26}$$

By (2.1) and Lemma 2.1, we have

$$\begin{aligned}
h_1'(\omega) &= -2\Big(2\frac{4-p}{p} + 1\Big)Q\big(\overrightarrow{\Phi_\omega}\big) + 4\frac{4-p}{p}Q\big(\overrightarrow{\Phi_\omega}\big) + 2\Big(1 - \omega^2\frac{4-p}{p}\Big)\partial_\lambda(\|\phi_\lambda\|_{L^2}^2)\Big|_{\lambda=\omega}\\
&= -2Q\big(\overrightarrow{\Phi_\omega}\big) + 2\Big(1 - \omega^2\frac{4-p}{p}\Big)\partial_\lambda(\|\phi_\lambda\|_{L^2}^2)\Big|_{\lambda=\omega}.
\end{aligned} \tag{7.27}$$

Now we compute the term $\partial_\lambda(\|\phi_\lambda\|_{L^2}^2)\Big|_{\lambda=\omega}$. Note that

$$\partial_\lambda Q\big(\overrightarrow{\Phi_\lambda}\big) = \partial_\lambda(-\lambda\|\phi_\lambda\|_{L^2}^2) = -\|\phi_\lambda\|_{L^2}^2 - \lambda\partial_\lambda(\|\phi_\lambda\|_{L^2}^2);$$

then Lemma 2.1 gives

$$\partial_\lambda(\|\phi_\lambda\|_{L^2}^2)\Big|_{\lambda=\omega} = -\frac{1}{\omega}\|\phi_\omega\|_{L^2}^2. \tag{7.28}$$

Taking (7.28) into (7.27), we get

$$\begin{aligned}
h_1'(\omega) &= 2\omega\|\phi_\omega\|_{L^2}^2 + 2\Big(1 - \omega^2\frac{4-p}{p}\Big)\Big(-\frac{1}{\omega}\|\phi_\omega\|_{L^2}^2\Big)\\
&= \frac{2}{\omega}\Big(\omega^2 - 1 + \omega^2\frac{4-p}{p}\Big)\|\phi_\omega\|_{L^2}^2\\
&= \frac{2}{\omega}\Big(\frac{4}{p}\omega^2 - 1\Big)\|\phi_\omega\|_{L^2}^2 = 0.
\end{aligned}$$

Thus, we prove the result $h_1'(\omega)=0$.

*Step 3.* $h_1''(\omega)>0$.

Taking the derivative of (7.26) with respect to $\lambda$, we have

$$\begin{aligned} h_1''(\lambda)=&-4\frac{4-p}{p}\|\phi_\omega\|_{L^2}^2-8\lambda\frac{4-p}{p}\partial_\lambda\big(\|\phi_\lambda\|_{L^2}^2\big)\\ &+2\Big(1-\frac{4-p}{p}\lambda^2\Big)\partial_\lambda^2\big(\|\phi_\lambda\|_{L^2}^2\big)+2\omega\partial_\lambda^2Q\big(\overrightarrow{\Phi_\lambda}\big). \end{aligned}$$

Since

$$\begin{aligned} \partial_\lambda^2Q\big(\overrightarrow{\Phi_\lambda}\big)&=-\partial_\lambda^2\big(\lambda\|\phi_\lambda\|_{L^2}^2\big)\\ &=-2\partial_\lambda\big(\|\phi_\lambda\|_{L^2}^2\big)-\lambda\partial_\lambda^2\big(\|\phi_\lambda\|_{L^2}^2\big), \end{aligned}$$

we have

$$\begin{aligned} h_1''(\lambda)=&-4\frac{4-p}{p}\|\phi_\omega\|_{L^2}^2-4\Big(2\frac{4-p}{p}\lambda+\omega\Big)\partial_\lambda\big(\|\phi_\lambda\|_{L^2}^2\big)\\ &+2\Big(1-\frac{4-p}{p}\lambda^2-\lambda\omega\Big)\partial_\lambda^2\big(\|\phi_\lambda\|_{L^2}^2\big). \end{aligned}$$

Hence,

$$h_1''(\omega)=-4\frac{4-p}{p}\|\phi_\omega\|_{L^2}^2-4\omega\frac{8-p}{p}\partial_\lambda\big(\|\phi_\lambda\|_{L^2}^2\big)\Big|_{\lambda=\omega}+2\Big(1-\frac{4}{p}\omega^2\Big)\partial_\lambda^2\|\phi_\lambda\|_{L^2}^2\Big|_{\lambda=\omega}.$$

Using (7.28) and $\omega^2=\frac{p}{4}$, we have

$$h_1''(\omega)=-4\frac{4-p}{p}\|\phi_\omega\|_{L^2}^2+4\frac{8-p}{p}\|\phi_\omega\|_{L^2}^2=\frac{16}{p}\|\phi_\omega\|_{L^2}^2>0.$$

Thus, we prove the result $h_1''(\omega)>0$. This proves the claim (7.25).

Using (7.25) and Taylor's type extension, we get

$$\begin{aligned} h_1(\lambda)&=h_1(\omega)+h_1'(\omega)(\lambda-\omega)+\frac12h_1''(\omega)(\lambda-\omega)^2+o\big((\lambda-\omega)^2\big)\\ &\geq C_2(\lambda-\omega)^2+o(\lambda-\omega)^2, \end{aligned}$$

where $C_2=\frac12h_1''(\omega)>0$. Putting this into (7.24), we obtain the conclusion 2) of this lemma. □

Hence, combining Lemmas 7.3 and 7.4, and (7.11), we have

$$I'(t)\geq C_1a+C_2(\lambda-\omega)^2+O\Big(\|\vec\eta\|_{H^1\times L^2}^2+a(\lambda-\omega)^2+\frac1R\Big). \tag{7.29}$$

7.4. **Upper control of $\|\vec\eta\|_{H^1\times L^2}$.** From (7.29), to prove the monotonicity of $I'(t)$, we only need to estimate $\|\vec\eta\|_{H^1\times L^2}$. In this subsection, we give the following estimate on $\|\vec\eta\|_{H^1\times L^2}$.

**Lemma 7.5.** *Let $\vec\eta$ be defined in (4.2); then*

$$\|\vec\eta\|_{H^1\times L^2}^2\lesssim O(a|\lambda-\omega|+a^2)+o\big((\lambda-\omega)^2\big).$$

*Proof.* First, since $\vec{u}=\left(\overrightarrow{\Phi_\lambda}+\vec{\eta}\right)(x-y)$ in (5.1), we have

$$S_\lambda(\vec{u})-S_\lambda\big(\overrightarrow{\Phi_\lambda}\big)=\Big\langle S_\lambda'\big(\overrightarrow{\Phi_\lambda}\big),\vec{\eta}\Big\rangle+\frac{1}{2}\Big\langle S_\lambda''\big(\overrightarrow{\Phi_\lambda}\big)\vec{\eta},\vec{\eta}\Big\rangle+o(\|\vec{\eta}\|^2_{H^1\times L^2}).$$

Using $S_\omega'\big(\overrightarrow{\Phi_\omega}\big)=\vec{0}$ and Taylor's type extension, we have

$$S_\lambda(\vec{u})-S_\lambda\big(\overrightarrow{\Phi_\lambda}\big)=\frac{1}{2}\Big\langle S_\lambda''\big(\overrightarrow{\Phi_\lambda}\big)\vec{\eta},\vec{\eta}\Big\rangle+o(\|\vec{\eta}\|^2_{H^1\times L^2}).$$

Then by the estimate (3.21) in Corollary 3.5, we get

$$S_\lambda(\vec{u})-S_\lambda\big(\overrightarrow{\Phi_\lambda}\big)\gtrsim\|\vec{\eta}\|^2_{H^1\times L^2}.$$

Second, note that

$$S_\lambda(\vec{u})-S_\lambda\big(\overrightarrow{\Phi_\lambda}\big)=S_\lambda(\vec{u}_0)-S_\lambda\big(\overrightarrow{\Phi_\omega}\big)+S_\lambda\big(\overrightarrow{\Phi_\omega}\big)-S_\lambda\big(\overrightarrow{\Phi_\lambda}\big),$$

and Taylor's type extension gives

$$\begin{aligned}S_\lambda(\vec{u}_0)-S_\lambda\big(\overrightarrow{\Phi_\omega}\big)&=E(\vec{u}_0)-E\big(\overrightarrow{\Phi_\omega}\big)+\lambda\Big(Q(\vec{u}_0)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big)\\&=S_\omega(\vec{u}_0)-S_\omega\big(\overrightarrow{\Phi_\omega}\big)+(\lambda-\omega)\Big(Q(\vec{u}_0)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big)\\&=\Big\langle S_\omega'\big(\overrightarrow{\Phi_\omega}\big),\vec{u}_0-\overrightarrow{\Phi_\omega}\Big\rangle+O\Big(\|\vec{u}_0-\overrightarrow{\Phi_\omega}\|^2_{H^1\times L^2}\Big)+(\lambda-\omega)O\Big(\|\vec{u}_0-\overrightarrow{\Phi_\omega}\|_{H^1\times L^2}\Big)\\&=O(a^2+a|\lambda-\omega|).\end{aligned}$$

By Corollary 2.2, we have

$$S_\lambda\big(\overrightarrow{\Phi_\omega}\big)-S_\lambda\big(\overrightarrow{\Phi_\lambda}\big)=o((\lambda-\omega)^2).$$

Finally, we get the desired result:

$$\begin{aligned}\|\vec{\eta}\|^2_{H^1\times L^2}\lesssim S_\lambda(\vec{u})-S_\lambda\big(\overrightarrow{\Phi_\lambda}\big)&=S_\lambda(\vec{u}_0)-S_\lambda\big(\overrightarrow{\Phi_\omega}\big)+S_\lambda\big(\overrightarrow{\Phi_\omega}\big)-S_\lambda\big(\overrightarrow{\Phi_\lambda}\big)\\&=O(a|\lambda-\omega|+a^2)+o\big((\lambda-\omega)^2\big).\end{aligned}$$

This completes the proof. □

7.5. **Proof of Theorem 1.2.** As in the discussion above, we assume that $\vec{u}\in U_\varepsilon\big(\overrightarrow{\Phi_\omega}\big)$, and thus $|\lambda-\omega|\lesssim\varepsilon$. First, we note that from the definition of $I(t)$ and Young's inequality, we have the time uniform boundedness of $I(t)$:

$$\sup_{t\in\mathbb{R}}I(t)\lesssim R\Big(\|\overrightarrow{\Phi_\omega}\|^2_{H^1\times L^2}+1\Big). \tag{7.30}$$

Now we estimate on $I'(t)$. From (7.29) and Lemma 7.5,

$$\begin{aligned}I'(t)&\geq C_1a+C_2(\lambda-\omega)^2+O(\|\vec{\eta}\|^2_{H^1\times L^2})+O\Big(a(\lambda-\omega)^2+\frac{1}{R}\Big)\\&\geq\frac{1}{2}C_1a+C_2(\lambda-\omega)^2+O(a|\lambda-\omega|+a^2)+o\big((\lambda-\omega)^2\big)+O\Big(\frac{1}{R}\Big).\end{aligned}$$

By (4.4), choosing R satisfying $\frac{1}{R}\leq a^2$, and choosing $\varepsilon$, $a_0$ small enough, we obtain that for any $a\in(0,a_0)$,

$$I'(t)\geq\frac{1}{2}C_1a+C_2(\lambda-\omega)^2+O(a^2+a|\lambda-\omega|)+o(\lambda-\omega)^2$$

$$\geq \frac{1}{4}C_1 a + \frac{1}{2}C_2(\lambda - \omega)^2.$$

This implies $I(t) \to +\infty$ when $t \to +\infty$, which is contradicted with (7.30). Hence we prove the instability of the solitary wave $\phi_\omega(x - \omega t)$ and thus give the proof of Theorem 1.2.

## Acknowledgements

The authors are grateful to the associate editor who gave useful notes on grammatical/typographical errors and to the anonymous referees who carefully read the paper and gave many helpful comments and suggestions.

## References

[1] J. Bellazzini, M. Ghimenti, and S. Le Coz, *Multi–solitary waves for the nonlinear Klein–Gordon equation*, Comm. Partial Differential Equations, 39 (2014), pp. 1479–1522.

[2] J. Bona and R. Sachs, *Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation*, Comm. Math. Phys., 118 (1988), pp. 15–29.

[3] J. Boussinesq, *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond*, J. Math. Pures Appl., 17 (1872), pp. 55–108.

[4] A. Comech and D. Pelinovsky, *Purely nonlinear instability of standing waves with minimal energy*, Comm. Pure Appl. Math., 56 (2003), pp. 1565–1607.

[5] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry, I*, J. Funct. Anal., 74 (1987), pp. 160–197.

[6] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry, II*, J. Funct. Anal., 94 (1990), pp. 308–348.

[7] M. K. Kwong, *Uniqueness of positive solutions of $\triangle u - u + u^p = 0$ in $\mathbb{R}^n$*, Arch. Ration. Mech. Anal., 105 (1989), pp. 234–266.

[8] S. Le Coz and Y. F. Wu, *Stability of multisolitons for the derivative nonlinear Schrödinger equation*, Int. Math. Res. Not. IMRN, 13 (2018), pp. 4120–4170.

[9] Y. Liu, *Instability of solitary waves for generalized Boussinesq equations*, J. Dynam. Differential Equations., 5 (1993), pp. 537–558.

[10] Y. Liu, *Instability and blow–up of solutions to a generalized Boussinesq equation*, SIAM. J. Math. Anal., 26 (1995), pp. 1527–1546.

[11] Y. Liu, M. Ohta., and G. Todorova, *Strong instability of solitary waves for nonlinear Klein–Gordon equations and generalized Boussinesq equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 24 (2007), pp. 539–548.

[12] F. Merle and P. Raphael, *On universality of blow–up profile for $L^2$ critical nonlinear Schrödinger equation*, Invent. Math., 156 (2004), pp. 565–672.

[13] F. Merle and P. Raphael, *The blow–up dynamic and upper bound on the blow–up rate for critical nonlinear Schrödinger equation*, Ann. of Math., 161 (2005), pp. 157–222.

[14] M. Ohta, *Instability of bound states for abstract nonlinear Schrödinger equations*, J. Funct. Anal., 261 (2011), pp. 90–110.

[15] M. Ohta and G. Todorova, *Strong instability of standing waves for the nonlinear Klein–Gordon equation and the Klein–Gordon–Zakharov system*, SIAM J. Math. Anal., 38 (2007), pp. 1912–1931.

[16] W. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys., 55 (1977), pp. 149–162.

[17] M. Weinstein, *Modulational stability of ground states of nonlinear Schrödinger equations*, SIAM J. Math. Anal., 16 (1985), pp. 472–491.

[18] M. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math., 39 (1986), pp. 51–67.

[19] Y. F. Wu, *Instability of the standing waves for the nonlinear Klein–Gordon equations in one dimension*, https://arXiv.org/abs/1705.04216v3.

Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

*Email address*: binglimath@gmail.com

Department of Mathematics, Tokyo University of Science, Tokyo 162-8601, Japan

*Email address*: mohta@rs.tus.ac.jp

Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

*Email address*: yerfmath@gmail.com

Department of Mathematical Sciences, Norwegian University of Science and Technology, Trondheim 7491, Norway

*Email address*: jxuemath@hotmail.com