ON THE FULL SPACE–TIME DISCRETIZATION OF THE GENERALIZED STOKES EQUATIONS: THE DIRICHLET CASE

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Abstract. In this work we treat the space-time discretization of the generalized Stokes equations in the case of Dirichlet boundary conditions. We prove error estimates in the case $p \in [\frac{2d}{d+2}, \infty)$ that are independent of the degeneracy parameter $\delta \in [0, \delta_0]$. For $p \leq 2$, our convergence rate is optimal.

Key words. generalized Stokes equations, space-time discretization, error estimates

AMS subject classifications. 65M15, 65M60, 76A05, 35Q35.

1. Introduction.

The purpose of this paper is to establish an error analysis for the space-time discretization of the generalized Stokes system

$$
\begin{aligned}
\partial_t \mathbf{u} - \operatorname{div} \mathbf{S}(\mathbf{Du}) + \nabla q &= \mathbf{f} && \text{in } I \times \Omega, \\
\operatorname{div} \mathbf{u} &= 0 && \text{in } I \times \Omega, \\
\mathbf{u}(0) &= \mathbf{u}_0 && \text{in } \Omega \\
\mathbf{u} &= \mathbf{0} && \text{at } I \times \partial\Omega,
\end{aligned}
\tag{1.1}
$$

for given external body force $\mathbf{f} = (f_1, ..., f_d)$ and initial velocity $\mathbf{u}_0$, where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded, polygonal domain and $I = (0, T)$, $T > 0$, is a bounded time interval. The unknown functions are the velocity field $\mathbf{u} = (u_1, ..., u_d)$ and the pressure q. The function $\mathbf{S}$ is the extra stress tensor, whose structure is given by characteristic properties of the examined fluid. Here, $\mathbf{S}$ depends on the symmetric part of the gradient of $\mathbf{u}$, $\mathbf{Du} := \frac{1}{2}(\partial_i u_j + \partial_j u_i)_{i,j=1,...,d}$. The special case $\mathbf{S} = \mathbf{Id}$, i.e. $-\operatorname{div} \mathbf{S}(\mathbf{Du}) = -\Delta \mathbf{u}$, which leads to the Stokes equations. In this work, we will consider a more general situation. A typical example is given by

$$
\mathbf{S}(\mathbf{Du}) := \varphi'(|\mathbf{Du}|) \frac{\mathbf{Du}}{|\mathbf{Du}|}, \tag{1.2}
$$

where $\varphi'(t) := (\delta + t)^{p-2} t$ for some $p \in (1, \infty)$, $\delta \in [0, \delta_0]$. Note that our results carry over to the case $\mathbf{S}(\mathbf{Du}) := \psi'(|\mathbf{Du}|) \frac{\mathbf{Du}}{|\mathbf{Du}|}$, where ψ is an N-function that fulfills the equivalence $\psi'(t) \sim \varphi'(t)$.

The system (1.1) is a simplification of the generalized Navier-Stokes equations For a broader discussion of these models we refer to [29] and [28].

Our goal is to present a complete analysis for the space-time discretization of the generalized Stokes system (1.1). Our main result will be the error estimate

$$
\|\mathbf{u} - \mathbf{U}\|_{L^\infty(I, L^2(\Omega))} + \|\mathbf{F}(\mathbf{Du}) - \mathbf{F}(\mathbf{DU})\|_{L^2(I, L^2(\Omega))} \leq c \left(\Delta t + h^{\min\{1, \frac{2}{p}\}} \right), \tag{1.3}
$$

for $p \in [\frac{2d}{d+2}, \infty)$ (see Theorem 4.10).

Let us summarize some previous results. For the special case $p = 2$, i.e. the Navier-Stokes equations, Heywood and Rannacher established in a series of papers

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[21, 22, 23, 24] complete existence and regularity results as well as an error analysis for the time-space discretization. Regarding (1.1), one of the main difficulties lies in the treatment of the stress tensor $\mathbf{S}$. Barrett and Liu [1] introduced a quasi-norm technique to prove error estimates for the p-Laplacian. Later on, they also treated the parabolic p-Laplacian [2] and p-fluids [3]. In [10], Diening, Ebmeyer and Růžička adapted this technique for N-functions and proved optimal error estimates for parabolic systems with p-structure.

Considering the treatment of the generalized Navier-Stokes equations, there are various results for the case $p \leq 2$ given periodic boundary conditions. In [30] Prohl and Růžička proved a first result for the space-time discretization of the generalized Navier-Stokes equations for some $p \in (p_0, 2]$. In a series of papers together with Diening [12, 13], the authors improved the sub-optimal results for the time discretization and increased the range of admissible p's. In [5], Berselli, Diening and Růžička proved optimal error estimates for the time discretization of the generalized Navier-Stokes equations in the case $p \in (\frac{3}{2}, 2]$ and, together with Belenki, the authors also proved error estimates for the finite element approximation of the stationary generalized Stokes system, cf. [4]. In [7], Berselli, Diening and Růžička were finally able to prove the optimal estimate $E_{h,\Delta t} \leq c(h + \Delta t)$ for $p \in (\frac{3}{2}, 2]$.

Previous results for the generalized Navier-Stokes equations discretize first in time and then in space. Therefore, spatial regularity of the semi-discrete solution is needed. This regularity can so far only be obtained in the setting of periodic boundary conditions. In this work, we discretize first in space and then in time, as in [21, 22]. Therefore, we need time regularity of the semi-discrete solution, which we are able to prove even in the setting of Dirichlet boundary conditions. Moreover, our treatment includes for the first time also the case $p > 2$. For $p \in [\frac{2d}{d+2}, 2]$, our error estimates are optimal. The results of this paper are based on the PhD thesis of S. Eckstein, cf. [16].

This paper is organized as follows: In Section 2, we provide the necessary technical tools. We introduce N-functions and operators with N-potential. This provides the technical tools needed for handling the stress tensor. Moreover, we look into the finite element approximation of divergence-free fields. We introduce suitable function spaces, discuss several interpolation results as well as the discrete inf-sup condition, which is necessary for the spatial approximation of the generalized Stokes system. In Section 3, we briefly discuss existence and regularity results for (1.1). Then we introduce the corresponding spatial approximation. In Subsection 3.2, we show existence and regularity for the spatial approximation $\mathbf{u}_h$ of $\mathbf{u}$. Choosing a suitable approximation $\mathbf{u}_0^h$ for the initial value $\mathbf{u}_0$, we are able to prove time regularity of $\mathbf{u}_h$. Afterwards, we derive error estimates for the spatial error. Section 4 treats the fully discretized solution. We consider an implicit scheme and show existence and regularity as well as error estimates. We show that for the above-mentioned choice for the initial value, we can finally prove the error estimate (1.3) for $p \in [\frac{2d}{d+2}, \infty)$ (see Theorem 4.10).

2. Technical Tools.

2.1. Function Spaces. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open, bounded domain. By $L^p(\Omega)$ and $W^{k,p}(\Omega)$, $1 \leq p \leq \infty$, $k \in \mathbb{N}$, we denote the classical Lebesgue and Sobolev spaces, respectively. An element of a d-dimensional function space is distinguished from a scalar function by bold print, i.e. $\mathbf{u} = (u_1, ..., u_d) \in W^{k,p}(\Omega)$ means $u_i \in W^{k,p}(\Omega)$, $i = 1, ..., d$. We also use bold print to indicate tensor-valued functions. We define $W_0^{k,p}(\Omega)$ as the closure of compactly supported functions $w \in C_0^\infty(\Omega)$ with respect to $\| \cdot \|_{W^{k,p}(\Omega)}$. By $L_0^p(\Omega)$ we define the subspace of $L^p(\Omega)$ consisting of functions with vanishing mean value $\langle w\rangle_\Omega := \frac{1}{|\Omega|}\int_\Omega w\,dx = 0$ and $W^{1,p}_{0,\operatorname{div}}(\Omega)$ is defined as $W^{1,p}_{0,\operatorname{div}}(\Omega) := \{\mathbf{w}\in W^{1,p}_0(\Omega)\big|\operatorname{div}\mathbf{w}=0 \text{ a.e. in } \Omega\}$. The space $L^p_{\operatorname{div}}(\Omega)$ is defined as the closure of $C^\infty_{0,\operatorname{div}}(\Omega)$ with respect to the L^p-norm. For a Banach space X, we denote by $L^p(I,X)$, $p\in[1,\infty]$, the classical Bochner spaces, cf. [17].

By C, c we denote generic constants, which may change from line to line. We say that two functions f and g are equivalent and use the notation $f\sim g$, if there exist constants $c, C\geq 0$ such that $cf\leq g\leq Cf$. For normed vector spaces X we denote the dual space by X^* and the duality product between $f\in X^*$ and $u\in X$ by $\langle f,u\rangle_{X^*,X} := f(u)$ or simply by $\langle f,u\rangle$, if there is no risk of confusion. We will use the notation

$$(f,u) := \int\limits_\Omega f u\,dx,$$

whenever the right-hand side is well-defined.

The scalar product of two vectors $\mathbf{u}$, $\mathbf{v}\in\mathbb{R}^d$ is denoted by $\mathbf{u}\cdot\mathbf{v}$. For a tensor $\mathbf{A}\in\mathbb{R}^{d\times d}$ we denote its symmetric part by $\mathbf{A}^{\text{sym}} := \frac{1}{2}(\mathbf{A}+\mathbf{A}^\top)\in\mathbb{R}^{d\times d}_{\text{sym}} := \{\mathbf{A}\in\mathbb{R}^{d\times d}\big|\,\mathbf{A}=\mathbf{A}^\top\}$. For $\mathbf{A},\ \mathbf{B}\in\mathbb{R}^{d\times d}$ we denote by $\mathbf{A}:\mathbf{B}$ the component-wise inner product and $|\mathbf{A}|$ denotes the Hilbert-Schmidt norm.

We will also use Orlicz and Sobolev-Orlicz spaces, cf. [25]. To this end, we use N-functions $\psi:\mathbb{R}^{\geq 0}\to\mathbb{R}^{\geq 0}$, as defined in [31]. We denote by ψ^* its complementary function. We say that ψ fulfills the Δ_2-condition, if there exists a constant $c>0$, such that for all $t\geq 0$, there holds $\psi(2t)\leq c\psi(t)$. By $\Delta_2(\psi)$ we denote the smallest such constant. In the following we work solely with N-functions ψ, such that ψ and ψ^* satisfy the Δ_2-condition. Under this condition we have

$$\psi^*(\psi'(t))\sim\psi(t).$$

We denote by $L^\psi(\Omega)$ and $W^{1,\psi}(\Omega)$ the classical Orlicz and Sobolev-Orlicz spaces, i.e., $f\in L^\psi(\Omega)$ if the modular $\rho(f) := \int_\Omega \psi(|f|)\,dx$ is finite and $f\in W^{1,\psi}(\Omega)$ if $f,\nabla f\in L^\psi(\Omega)$. Note that the dual space $(L^\psi(\Omega))^*$ can be identified with the space $L^{\psi^*}(\Omega)$.

2.2. Basic properties of the extra stress tensor. In the whole paper we assume that the extra stress tensor $\mathbf{S}$ has N-potential, which will be defined now. A detailed discussion and full proofs can be found in [31].

Definition 2.1 (Operators with N-Potential). *Let ψ be an N-function. We say that the operator $\mathbf{S}:\mathbb{R}^{d\times d}\to\mathbb{R}^{d\times d}_{\text{sym}}$ possesses N-potential ψ, if $\mathbf{S}(\mathbf{0})=\mathbf{0}$ and if for all $\mathbf{P}\in\mathbb{R}^{d\times d}\setminus\{\mathbf{0}\}$ there holds*

$$\mathbf{S}(\mathbf{P}) = \mathbf{S}_\psi(\mathbf{P}) := \frac{\psi'(|\mathbf{P}^{\text{sym}}|)}{|\mathbf{P}^{\text{sym}}|}\mathbf{P}^{\text{sym}}. \tag{2.2}$$

We want to concentrate on a special N-function with (p,δ)-structure, which is for $t\geq 0$ given by

$$\varphi(t) = \int_0^t \varphi'(s)\,ds \qquad \text{with } \varphi'(s) := (\delta+s)^{p-2}s. \tag{2.3}$$

The function φ satisfies uniformly in t the important equivalence

$$\varphi'(t)t\sim\varphi''(t),$$

since $\min\{1,p-1\}(\delta+t)^{p-2} \le \varphi''(t) \le \max\{1,p-1\}(\delta+t)^{p-2}$. Moreover, φ satisfies the Δ_2-condition with $\Delta_2(\varphi) \le c2^{\max\{2,p\}}$, hence independent of δ. This implies that, uniformly in t, δ, we have

$$\varphi'(t)t \sim \varphi(t).$$

The conjugate function φ^* satisfies $\varphi^*(t) \sim (\delta^{p-1}+t)^{p'-2}t^2$. Also φ^* satisfies the Δ_2-condition with $\Delta_2(\varphi^*) \le c2^{\max\{2,p'\}}$. If φ is given by (2.3) the spaces $L^\varphi(\Omega)$ and $L^p(\Omega)$ coincide with uniform equivalence of the corresponding norms. The constants only depend on p and Ω.

Throughout this paper, we are going to assume

ASSUMPTION 2.4. *The stress tensor* $\mathbf{S} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ *possesses* N*-potential* φ*, where* φ *is given by* (2.3)*, with* $p \in (1,\infty)$ *and* $\delta \in [0,1]$.

REMARK 2.5. *Throughout this paper, the assumption* $\delta \in [0,1]$ *can be replaced by the assumption* $\delta \in [0,\delta_0]$ *for given* $\delta_0 > 0$*. The estimates will then depend on* δ_0.

REMARK 2.6. *All results of this paper remain true, if we replace Assumption 2.4 by the assumption that* $\mathbf{S}$ *has* N*-potential* ψ *for an* N*-function* ψ *such that* $\psi \sim \varphi$.

For an N-function ψ, we define the family of shifted N-functions $\{\psi_a\}_{a\ge 0}$ for $t \ge 0$ by $\psi_a(t) := \int\limits_0^t \psi_a'(s)\,ds$, where

$$\psi_a'(t) := \psi'(a+t)\frac{t}{a+t}. \tag{2.7}$$

For the N-function defined in (2.3) we have that $\varphi_a(t) \sim (\delta+a+t)^{p-2}t^2$ and also $(\varphi_a)^*(t) \sim ((\delta+a)^{p-1}+t)^{p'-2}t^2$. The families $\{\varphi_a\}_{a\ge0}$ and $\{(\varphi_a)^*\}_{a\ge0}$ satisfy the Δ_2-condition uniformly in $a \ge 0$, with $\Delta_2(\varphi_a) \le c2^{\max\{2,p\}}$ and $\Delta_2((\varphi_a)^*) \le c2^{\max\{2,p'\}}$, respectively.

We need the following refined version of Young's inequality, cf. [31]:

LEMMA 2.8 (Young's inequality). *Let* ψ *be an* N*-function with* $\Delta_2(\psi) < \infty$ *and* $\Delta_2(\psi^*) < \infty$*. Then, for every* $\varepsilon > 0$*, there exists* $c_\varepsilon > 0$ *only depending on* ε*,* $\Delta_2(\psi)$*, and* $\Delta_2(\psi^*)$ *such that for all* s*,* t*,* $a \ge 0$

$$st \le \varepsilon\psi_a(s) + c_\varepsilon\psi_a^*(t) \tag{2.9}$$

and

$$s\psi_a'(t) + t\psi_a'(s) \le \varepsilon\psi_a(s) + c_\varepsilon\psi_a(t). \tag{2.10}$$

Closely related to the extra stress tensor $\mathbf{S}$ is the function $\mathbf{F} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ defined through

$$\mathbf{F}(\mathbf{P}) := (\delta + |\mathbf{P}^{\mathrm{sym}}|)^{\frac{p-2}{2}}\mathbf{P}^{\mathrm{sym}}. \tag{2.11}$$

The connection between $\mathbf{S}$, $\mathbf{F}$, and $\{\varphi_a\}_{a\ge0}$ is best explained by the following lemma (cf. [31, Lemma 6.16]).

LEMMA 2.12. *Let Assumption 2.4 be fulfilled and let* $\mathbf{F}$ *be defined as in* (2.11)*. Then there holds for all* $\mathbf{P}$*,* $\mathbf{Q} \in \mathbb{R}^{d\times d}$

$$\begin{aligned}(\mathbf{S}(\mathbf{P})-\mathbf{S}(\mathbf{Q})) : (\mathbf{P}-\mathbf{Q}) &\sim |\mathbf{F}(\mathbf{P})-\mathbf{F}(\mathbf{Q})|^2\\ &\sim \varphi_{|\mathbf{P}^{\mathrm{sym}}|}(|\mathbf{P}^{\mathrm{sym}}-\mathbf{Q}^{\mathrm{sym}}|)\\ &\sim \varphi_{|\mathbf{Q}^{\mathrm{sym}}|}(|\mathbf{P}^{\mathrm{sym}}-\mathbf{Q}^{\mathrm{sym}}|)\\ &\sim \varphi''(|\mathbf{P}^{\mathrm{sym}}|+|\mathbf{Q}^{\mathrm{sym}}|)|\mathbf{P}^{\mathrm{sym}}-\mathbf{Q}^{\mathrm{sym}}|^2,\end{aligned} \tag{2.13}$$

where the constants only depend on p*. Furthermore, we have*

$$\mathbf{S}(\mathbf{P}) : \mathbf{P} \sim |\mathbf{F}(\mathbf{P})|^{\mathbf{2}} \sim \varphi(|\mathbf{P}^{\mathrm{sym}}|). \tag{2.14}$$

In this case we have

$$(\mathbf{S}(\mathbf{Du}), \mathbf{Du}) \sim \int_\Omega \varphi(|\mathbf{Du}|)\, dx \sim \int_\Omega (\delta + |\mathbf{Du}|)^{p-2}|\mathbf{Du}|^2\, dx.$$

Moreover, the following estimate follows directly from Lemma 2.12 and Young's inequality (2.10).

LEMMA 2.15. *Let Assumption 2.4 be fulfilled and let* $\mathbf{F}$ *be defined as in (2.11). For all* $\varepsilon > 0$ *exists* $c_\varepsilon > 0$*, depending on* ε *and the* Δ_2*-constants such that for all sufficiently smooth vector fields* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, *we have*

$$\begin{aligned}(\mathbf{S}(\mathbf{Du}) - \mathbf{S}(\mathbf{Dv}), \mathbf{Dw} - \mathbf{Dv}) &\le \varepsilon\|\mathbf{F}(\mathbf{Du}) - \mathbf{F}(\mathbf{Dv})\|_2^2 + c_\varepsilon\|\mathbf{F}(\mathbf{Dw}) - \mathbf{F}(\mathbf{Dv})\|_2^2\,,\\ (\mathbf{S}(\mathbf{Du}) - \mathbf{S}(\mathbf{Dv}), \mathbf{Dw} - \mathbf{Dv}) &\le \varepsilon\|\mathbf{F}(\mathbf{Dw}) - \mathbf{F}(\mathbf{Dv})\|_2^2 + c_\varepsilon\|\mathbf{F}(\mathbf{Du}) - \mathbf{F}(\mathbf{Dv})\|_2^2.\end{aligned}$$

LEMMA 2.16. *Let* $\Omega \subset \mathbb{R}^d$ *be an open, bounded domain and let* $\mathbf{S}$ *fulfill Assumption 2.4. For* $p \in (1, 2)$ *we have*

$$\begin{aligned}\|\mathbf{F}(\mathbf{Du}) - \mathbf{F}(\mathbf{Dv})\|_2^{\frac{4}{p}} &\le \|\mathbf{Du} - \mathbf{Dv}\|_p^2\\ &\le c\big(K + \|\mathbf{Du}\|_p + \|\mathbf{Du} - \mathbf{Dv}\|_p\big)^{2-p}\|\mathbf{F}(\mathbf{Du}) - \mathbf{F}(\mathbf{Dv})\|_2^2\end{aligned} \tag{2.17}$$

and for $p \in [2, \infty)$ *we have*

$$\begin{aligned}\|\mathbf{Du} - \mathbf{Dv}\|_p^p &\le \|\mathbf{F}(\mathbf{Du}) - \mathbf{F}(\mathbf{Dv})\|_2^2\\ &\le c\big(K + \|\mathbf{Du}\|_p + \|\mathbf{Du} - \mathbf{Dv}\|_p\big)^{p-2}\|\mathbf{Du} - \mathbf{Dv}\|_p^2,\end{aligned} \tag{2.18}$$

with constants c *independent of* $\delta \in [0, 1]$*. The constant* K *is given by*

$$K := \delta|\Omega|^{\frac{1}{p}} \le |\Omega|^{\frac{1}{p}}.$$

PROOF : See [1, Lemma 2.2] and [16, Lemma 2.80]. ∎

COROLLARY 2.19. *Assume that the assumptions of Lemma 2.16 are satisfied. Then* $\mathbf{u} \in W^{1,p}(\Omega)$ *implies* $\mathbf{F}(\mathbf{Du}) \in L^2(\Omega)$.

We also use

THEOREM 2.20. *Let* $I \subset \mathbb{R}$ *be an interval,* $f \in L^p(I, L^q(\Omega))$, $1 < p, q < \infty$*. Suppose there exists a constant* $K > 0$ *such that there holds* $d_\tau f \in L^p(I', L^q(\Omega))$ *and* $\|d_\tau f\|_{L^p(I',L^q(\Omega))} \le K$ *for all* $0 < \tau < \operatorname{dist}(I', \partial I)$*. Then the weak derivative* $\partial_t f$ *exists and we have the estimate* $\|\partial_t f\|_{L^p(I,L^q(\Omega))} \le K$.

PROOF : The proof adapts the classical results for Sobolev spaces and can be found in [16, Theorem 2.1]. ∎

2.3. Finite Element Approximation. For the spatial approximation of the generalized Stokes equations, we will need two different finite element spaces: one for the approximation of the divergence-free velocity field and one for the approximation of the pressure. The choice of these two spaces is not arbitrary. In fact, they need to fulfill the *discrete inf-sup condition*.

We start by explaining the triangulation of the domain. From now on we assume that $\Omega \subset \mathbb{R}^d$, $d \ge 2$, is a bounded domain with polyhedral Lipschitz boundary. Furthermore, we assume that for fixed $h > 0$, $\mathcal{T}_h = \{T_i\}_{i=1,\dots,N}$ is a finite decomposition of Ω into simplices T_i. Let $h_T := \operatorname{diam}(T), h := \max_{T\in\mathcal{T}_h} h_T$ and let ρ_T denote the diameter of the largest closed ball contained in $\overline{T}$. We assume that the mesh is such that any two elements of $\mathcal{T}_h$ meet only in entire common faces or sides or vortices (i.e. there are no hanging nodes), that the mesh is non-degenerate, i.e. there exists a constant $\sigma_0 > 0$ independent of h, such that $\max_{T\in\mathcal{T}_h} \frac{h_T}{\rho_T} \leq \sigma_0$. Let N_T be the neighborhood of T, $N_T := \bigcup\{T' \,|\, \overline{T} \cap \overline{T'} \neq \emptyset\}$, and $S_T := \operatorname{int}\big(\cup_{T'\in N_T} \overline{T'}\big)$. Under the above assumptions, it is clear that the number of simplices in every S_T is bounded by a constant independent of h_T and therefore

$$|T| \sim h_T^d \sim |S_T|. \tag{2.21}$$

For $l \in \mathbb{N}_0$, let $\mathcal{P}_l(T)$ be the space of polynomials of degree less or equal to l on T and let

$$\mathcal{P}_l(\mathcal{T}_h) := \{v \in C^0(\overline{\Omega}) \big| v|_T \in \mathcal{P}_l(T) \text{ for all } T \in \mathcal{T}_h\}$$

be the space of piecewise polynomials.

Now we are ready to introduce suitable finite element spaces for the spatial approximation of the generalized Stokes equations. The natural setting for the continuous equation is to seek the velocity in $W^{1,p}_{0,\mathrm{div}}(\Omega)$ and the pressure in $L_0^{p'}(\Omega)$, hence our finite element spaces should be approximations of these spaces. We approximate $W_0^{1,p}(\Omega)$ by

$$X_h := \{\mathbf{v}_h \in W_0^{1,p}(\Omega) \,\big|\, \mathbf{v}_h|_T \in \mathcal{P}_k(T) \ \forall\, T \in \mathcal{T}_h,\ \mathbf{v}_h|_{\partial\Omega} = 0\}$$

and $L^{p'}(\Omega)$ by

$$Y_h := \{q_h \in L^{p'}(\Omega) \,\big|\, q_h|_T \in \mathcal{P}_r(T) \ \forall\, T \in \mathcal{T}_h\}$$

for some $k,\ r \in \mathbb{N}_0$. For the approximation of the pressure, we then define the space

$$Q_h := Y_h \cap L_0^{p'}(\Omega).$$

The discrete divergence-free space V_h is then defined by

$$V_h := \{\mathbf{v}_h \in X_h \big| (q_h, \operatorname{div} \mathbf{v}_h) = 0 \ \forall\, q_h \in Y_h\}.$$

The choice of the polynomial degrees k and r is not arbitrary and plays an important role for the solvability of the discretized problem. The existence of stable pairings X_h, Y_h has been widely discussed, see [19].

The existence of interpolation operators for X_h and Y_h is quite standard. Typical examples would be the Scott-Zhang operator [32] for X_h and the Clément operator [9] or also a version of the Scott-Zhang operator for Y_h. However, we need to introduce additional assumptions in order to guarantee well-posedness of the discretized problem as well as interpolation results in V_h and Q_h.

Assumption 2.22. *Let X_h, V_h, Y_h and Q_h be defined as above with $\mathcal{P}_1(\mathcal{T}_h) \subset X_h$ and $\mathcal{P}_0(\mathcal{T}_h) \subset Y_h$. We assume that there exist linear projection operators*

$$\Pi_h^{\mathrm{div}} : W_0^{1,p}(\Omega) \to X_h,$$
$$\Pi_h^{Y} : L^{p'}(\Omega) \to Y_h,$$

which fulfill the following assumptions

(i) Π_h^{div} is divergence-preserving in the sense that for all $\mathbf{w} \in W_0^{1,p}(\Omega)$, $\eta_h \in Y_h$

$$(\operatorname{div} \mathbf{w}, \eta_h) = (\operatorname{div} \Pi_h^{\mathrm{div}} \mathbf{w}, \eta_h). \tag{2.23}$$

(ii) Π_h^{div} *is locally* $W^{1,1}$*-stable in the sense that for all* $\mathbf{w} \in W_0^{1,p}(\Omega),\ T \in \mathcal{T}_h$,

$$\fint_T |\Pi_h^{\mathrm{div}} \mathbf{w}|\, dx \le c \fint_{S_T} |\mathbf{w}|\, dx + c \fint_{S_T} h_T |\nabla \mathbf{w}|\, dx. \tag{2.24}$$

(iii) There holds

$$\Pi_h^{\mathrm{div}} \mathbf{w} = \mathbf{w} \qquad \forall\ \mathbf{w} \in \mathcal{P}_1(\mathcal{T}_h). \tag{2.25}$$

(iv) Π_h^Y *is locally* L^1*-stable in the sense that for all* $q \in L^{p'}(\Omega),\ T \in \mathcal{T}_h$,

$$\fint_T |\Pi_h^Y q|\, dx \le c \fint_{S_T} |q|\, dx. \tag{2.26}$$

REMARK 2.27. *In order for Assumption 2.22 to be fulfilled, there are only certain admissible pairings of polynomial degrees* k *and* r *for the spaces* X_h *and* Y_h*, respectively, cf. [19].*

REMARK 2.28. *Note that since our choice of* X_h *already includes zero boundary values,* Π_h^{div} *also needs to preserve boundary values. The Scott–Zhang operator [32] is one example for such an interpolation operator, but it needs to be modified in order to be divergence-preserving. Following [18], we show how this is done for the MINI element in three space dimensions.*
Let $\Omega \subset \mathbb{R}^3$*. Assume that for every element* $\eta_h \in Y_h$ *holds* $\eta_h|_T \in \mathcal{P}_1(T)$, $T \in \mathcal{T}_h$ *and that for every* $\mathbf{w}_h \in X_h$*, the restriction* $\mathbf{w}_h|_T$ *is the sum of a polynomial of* $\mathcal{P}_1(T)$ *and a bubble function* $b_T \in \mathcal{P}_4(T) \cap W_0^{1,p}(T)$*. For each simplex* $T \in \mathcal{T}_h$ *we define the constant*

$$c_T := \frac{\fint_T \Pi_h^{SZ} \mathbf{w} - \mathbf{w}\, dx}{\fint_T b_T\, dx},$$

where Π_h^{SZ} *is the Scott–Zhang operator. Now, we can show that the operator*

$$\Pi_h^{\mathrm{div}} \mathbf{w} := \Pi_h^{SZ} \mathbf{w} - \sum_{T \in \mathcal{T}_h} c_T b_T$$

satisfies Assumption 2.22. Since Π_h^{SZ} *preserves zero boundary values and* b_T *vanishes at every edge of our triangulation and therefore particularly on* $\partial\Omega$, Π_h^{div} *maps* $W_0^{1,p}(\Omega)$ *to* X_h*. The definition of* c_T *gives the divergence-preserving property (2.23): Since* $\Pi_h^{\mathrm{div}} \mathbf{w} - \mathbf{w}$ *has zero boundary values on* Ω *and* $\nabla \eta_h|_T$ *is constant for every* $T \in \mathcal{T}_h$*, we have*

$$\begin{aligned}
\int_\Omega \operatorname{div}(\Pi_h^{\mathrm{div}} \mathbf{w} - \mathbf{w}) \eta_h\, dx &= -\int_\Omega (\Pi_h^{\mathrm{div}} \mathbf{w} - \mathbf{w}) \cdot \nabla \eta_h\, dx \\
&= -\sum_{T \in \mathcal{T}_h} \int_T (\Pi_h^{\mathrm{div}} \mathbf{w} - \mathbf{w})\, dx \cdot \nabla \eta_h|_T \\
&= -\sum_{T \in \mathcal{T}_h} \nabla \eta_h|_T \cdot \left(\int_T (\Pi_h^{SZ} \mathbf{w} - \mathbf{w})\, dx - \int_T b_T\, dx \frac{\fint_T \Pi_h^{SZ} \mathbf{w} - \mathbf{w}\, dx}{\fint_T b_T\, dx} \right) \\
&= 0.
\end{aligned}$$

The local $W^{1,1}$-stability follows since the Scott–Zhang operator fulfills (2.24). We have, using also (2.21),

$$\begin{aligned}\fint_T |\Pi_h^{\operatorname{div}} \mathbf{w}|\,dx &\le \fint_T |\Pi_h^{SZ} \mathbf{w}|\,dx + |c_T|\Big|\fint_T b_T\,dx\Big| \\ &\le c\fint_{S_T} |\mathbf{w}|\,dx + c\fint_{S_T} h_T|\nabla \mathbf{w}|\,dx + \Big|\fint_T \Pi_h^{SZ}\mathbf{w} - \mathbf{w}\,dx\Big| \\ &\le c\fint_{S_T} |\mathbf{w}|\,dx + c\fint_{S_T} h_T|\nabla \mathbf{w}|\,dx,\end{aligned}$$

which is (2.24). Other examples for $\Pi_h^{\operatorname{div}}$ are given in [18], [9], [4] and [20].

The results of [15] stated in the theorem below clarify that (2.24) and (2.26) already provide sufficient approximability results.

Theorem 2.29. *Let $Z_h := \{\mathbf{w} \in L^1_{loc}(\Omega) \,\big|\, \mathbf{w}|_T \in \mathcal{P}(T) \text{ for all } T \in \mathcal{T}_h\}$ be a finite element space, where $\mathcal{P}_{r_0}(T) \subset \mathcal{P}(T) \subset \mathcal{P}_{r_1}(T)$ for $r_0 \le r_1 \in \mathbb{N}_0$ and assume that for $l_0 \in \mathbb{N}_0$ there exists an interpolation operator $\Pi_h : W^{l_0,1}(\Omega) \to Z_h$ such that*

a) Π_h is $W^{l,1}$-stable in the sense that for some $l_0 \le l \le r_0+1$ and $m \in \mathbb{N}_0$ holds uniformly in $T \in \mathcal{T}_h$ and $\mathbf{w} \in W^{l,1}(\Omega)$

$$\sum_{j=0}^{m} \fint_T |h_T^j \nabla^j \Pi_h \mathbf{w}|\,dx \le c(m,l) \sum_{k=0}^{l} h_T^k \fint_{S_T} |\nabla^k \mathbf{w}|\,dx. \tag{2.30}$$

b) For all $\mathbf{w} \in \mathcal{P}_{r_0}(\mathcal{T}_h)$ holds $\Pi_h \mathbf{w} = \mathbf{w}$.

Let further $\Psi : [0,\infty) \to [0,\infty)$ be an N-function which satisfies the Δ_2-condition. Then there holds uniformly in $T \in \mathcal{T}_h$ and $\mathbf{w} \in W^{l,1}(\Omega)$

(i) Π_h is Orlicz-stable in the sense that

$$\sum_{j=0}^{m} \fint_T \Psi(h_T^j |\nabla^j \Pi_h \mathbf{w}|)\,dx \le c(m,l,\Delta_2(\Psi)) \sum_{k=0}^{l} \fint_{S_T} \Psi(h_T^k |\nabla^k \mathbf{w}|)\,dx, \tag{2.31}$$

(ii) Π_h possesses an Orlicz-approximability property:

$$\sum_{j=0}^{l} \fint_T \Psi(h_T^j |\nabla^j (\mathbf{w} - \Pi_h \mathbf{w})|)\,dx \le c(l,\Delta_2(\Psi),\sigma_0) \fint_{S_T} \Psi(h_T^l |\nabla^l \mathbf{w}|)\,dx, \tag{2.32}$$

(iii) Π_h is Orlicz-continuous:

$$\fint_T \Psi(h_T^l |\nabla^l \Pi_h \mathbf{w}|)\,dx \le c(l,\Delta_2(\Psi),\sigma_0) \fint_{S_T} \Psi(h_T^l |\nabla^l \mathbf{w}|)\,dx. \tag{2.33}$$

Proof : The proof can be found in [15]. ■

Remark 2.34. *Assumption 2.22 guarantees that $\Pi_h^{\operatorname{div}}$ fulfills the requirements of Theorem 2.29 with $Z_h = X_h$, $r_0 = 1$, $m = 0$, $l = 1$ and that Π_h^Y fulfills the requirements of Theorem 2.29 with $r_0 = l_0 = l = m = 0$. Moreover, due to the choice of Y_h, Π_h^Y fulfills the requirements with $r_0 = l_0 = m = 0$ and $l = 1$. Using inverse estimates, we can show that $\Pi_h^{\operatorname{div}}$ also fulfills the requirements in the case $r_0 = l_0 = 1$, $m = l = 2$.* From Theorem 2.29 we deduce

LEMMA 2.35. *Let Assumption 2.22 be fulfilled, let* $\mathbf{S}$ *satisfy Assumption 2.4 and let the associated operator* $\mathbf{F}$ *be defined as in (2.11). Then we have for* $1 < q < \infty$ *and for all sufficiently smooth enough vector fields* $\mathbf{v}$

$$\|\mathbf{v} - \Pi_h^{\mathrm{div}}\mathbf{v}\|_q + h\|\nabla\Pi_h^{\mathrm{div}}\mathbf{v}\|_q \le ch\|\nabla\mathbf{v}\|_q, \tag{2.36}$$

$$\|\mathbf{v} - \Pi_h^{\mathrm{div}}\mathbf{v}\|_q + h\|\nabla(\mathbf{v} - \Pi_h^{\mathrm{div}}\mathbf{v})\|_q \le ch^2\|\nabla^2\mathbf{v}\|_q, \tag{2.37}$$

$$\|\mathbf{F}(\mathbf{D}\Pi_h^{\mathrm{div}}\mathbf{v})\|_2 \le c\|\mathbf{F}(\mathbf{D}\mathbf{v})\|_2, \tag{2.38}$$

$$\|\mathbf{F}(\mathbf{D}\mathbf{v}) - \mathbf{F}(\mathbf{D}\Pi_h^{\mathrm{div}}\mathbf{v})\|_2 \le ch\|\nabla\mathbf{F}(\mathbf{D}\mathbf{v})\|_2. \tag{2.39}$$

For the interpolation operator Π_h^Y *and for every* N*-function* ψ *with* $\Delta_2(\psi) < \infty$ *we have*

$$\int_T \psi(|\Pi_h^Y q|)\,dx \le c\int_{S_T} \psi(|q|)\,dx \tag{2.40}$$

$$\int_T \psi(|q - \Pi_h^Y q|)\,dx \le c\int_{S_T} \psi(h_T|\nabla q|)\,dx. \tag{2.41}$$

PROOF : See [15] and [4]. ■

From Assumption 2.22 it can be shown that the existence of a divergence-preserving interpolation operator as in (2.23) ensures that a discrete inf-sup condition is fulfilled by X_h and Q_h. This is needed for the existence of a discrete pressure, cf. Remark 3.11 below.

LEMMA 2.42. *Let Assumption 2.22 be fulfilled and let* φ *be defined by (2.3) with* $\delta \in [0,1]$, $p \in (1,\infty)$. *Then there exists a constant* $c > 0$, *depending only on* p *and* Ω, *such that*

$$\|q_h\|_{L^{p'}(\Omega)} \le c \sup_{\boldsymbol{\xi}_h \in X_h\setminus\{0\}} \frac{(q_h, \operatorname{div}\boldsymbol{\xi}_h)}{\|\boldsymbol{\xi}_h\|_{W_0^{1,p}(\Omega)}} \tag{2.43}$$

holds for any $q_h \in Q_h$ *and*

$$\int_\Omega \varphi^*(|q_h|)\,dx \le \sup_{\boldsymbol{\xi}_h \in X_h} \Big(\int_\Omega q_h \operatorname{div}\boldsymbol{\xi}_h\,dx - \frac{1}{c}\int_\Omega \varphi(|\nabla\boldsymbol{\xi}_h|)\,dx\Big) \tag{2.44}$$

holds for any $q_h \in Q_h$.

PROOF : See [4, Lemma 4.1]. ■

3. Spatial Approximation.

3.1. The continuous solution. Before we discuss the spatial approximation of system (1.1), we will discuss some existence and regularity results for the continuous problem. We assume that $\mathbf{S}$ fulfills Assumption 2.4.

The first approach to show existence of a unique solution of (1.1) is by using a Galerkin ansatz, solving the emerging ordinary differential equations, establishing a priori estimates and then passing to the limit in the approximate system using monotone operator techniques and Minty's trick [1]. For $p \in (1,\infty)$, $\delta \in [0,1]$, this leads to the existence of a unique weak solution $\mathbf{u}$ satisfying the energy estimate

$$\|\mathbf{u}\|_{L^\infty(I,L^2(\Omega))} + \|\mathbf{F}(\mathbf{D}\mathbf{u})\|_{L^2(I,L^2(\Omega))} \le c$$

[1]In view of Korn's inequality, this follows from standard monotone operator theory for evolution equations (cf. [33]) for $p \ge \frac{2d}{d+2}$. The case $p > 1$ can be handled as in [11]. The case of generalized Navier-Stokes equations is treated in [26] and [27].

uniformly in $\delta \in [0,1]$, where the constant c only depends on the data.

Using a completely different technique, namely linearization, maximal regularity, and a fixed point argument, Bothe und Prüß [8, Theorem 4.1] were able to show that for $\delta > 0$ and $p \in (1,\infty)$, there exists a unique, strong solution on a maximal time interval, provided that the data is smooth enough. More precisely, it is proved:

THEOREM 3.1. *Let* $\mathbf{f} \in L^r(J\times\Omega)$, $\mathbf{u}_0 \in W^{2-\frac{2}{r}}(\Omega)\cap W^{1,r}_{0,\mathrm{div}}(\Omega)$ *with* $d+2 < r < \infty$. *Moreover, assume that* $\mathbf{S}$ *fulfills Assumption 2.4 with* $p \in (1,\infty)$ *and* $\delta \in (0,1]$. *Then, there exists a maximal time interval* $I \subset J$ *and a unique velocity field*

$$\mathbf{u} \in L^r(I, W^{2,r}(\Omega)) \cap W^{1,r}(I, L^r(\Omega))$$

and a unique scalar function

$$q \in L^r(I, W^{1,r}(\Omega)) \cap L^r(I, L^r_0(\Omega)),$$

that solve (1.1).

REMARK 3.2. *Under the same assumptions, Bothe and Prüß [8, Theorem 2.1] proved this result for the generalized Navier-Stokes equations. Their work also covers more general boundary conditions as well as other structures for* φ'. *However, the case* $\delta = 0$ *is not included.*

3.2. Existence and Regularity of the Finite Element Solution. We now focus on the spatial approximation of the generalized Stokes system. In Theorem 3.4, we show existence of weak solutions $\mathbf{u}_h$ of the spatial discretization. In order to estimate the error between space- and space-time approximation in Section 4, we need a certain time regularity of $\mathbf{u}_h$. This is accomplished for a special approximation $\mathbf{u}_0^h$ for the initial value $\mathbf{u}_0$ in Theorem 3.17.

The weak formulation of (1.1) suggests the discrete analogue: For a sufficiently smooth field $\mathbf{f} : I \times \Omega \to \mathbb{R}^d$ and $\mathbf{u}_0^h \in V_h$ find $\mathbf{u}_h \in C^1(\overline{I}, X_h)$ such that for every $t \in \overline{I}$ there holds

$$\begin{aligned} (\partial_t \mathbf{u}_h(t), \boldsymbol{\xi}_h) + (\mathbf{S}(\mathbf{D}\mathbf{u}_h(t)), \mathbf{D}\boldsymbol{\xi}_h) &= (\mathbf{f}(t), \boldsymbol{\xi}_h) \qquad \forall\, \boldsymbol{\xi}_h \in V_h, \\ \mathbf{u}_h(0) &= \mathbf{u}_0^h \qquad \text{in}\, \Omega. \end{aligned} \tag{3.3}$$

We will formulate the existence result and some a priori estimates in the next theorem.

THEOREM 3.4. *Let* $\mathbf{S}$ *fulfill Assumption 2.4 with* $\delta \in [0,1]$, $p \in [\frac{2d}{d+2}, \infty)$ *and let Assumption 2.22 be fulfilled. Suppose* $\mathbf{f} \in W^{1,2}(I, L^2(\Omega))$ *and* $\mathbf{u}_0 \in W^{1,p}_{0,\mathrm{div}}(\Omega)$ *and let* $M_1 > 0$ *be such that*

$$\|\mathbf{f}\|_{W^{1,2}(I,L^2(\Omega))} + \|\mathbf{u}_0\|_{W^{1,p}(\Omega)} \le M_1. \tag{3.5}$$

Let $\mathbf{u}_0^h \in V_h$ *be an approximation of* $\mathbf{u}_0$ *such that there exists* $M_2 > 0$ *with*

$$\|\mathbf{u}_0^h\|_{W^{1,p}(\Omega)} \le M_2. \tag{3.6}$$

Then there exists a unique solution $\mathbf{u}_h \in C^1(\overline{I}, V_h)$ *of (3.3).Furthermore, we have the estimates*

$$\|\mathbf{u}_h\|_{L^\infty(I,L^2(\Omega))} + \|\mathbf{F}(\mathbf{D}\mathbf{u}_h)\|_{L^2(I,L^2(\Omega))} \le c(M_1, M_2), \tag{3.7}$$

$$\|\partial_t \mathbf{u}_h\|_{L^2(I,L^2(\Omega))} + \|\mathbf{F}(\mathbf{D}\mathbf{u}_h)\|_{L^\infty(I,L^2(\Omega))} \le c(M_1, M_2), \tag{3.8}$$

where the constants are independent of h.

Proof : The identity (3.3) is a system of ordinary differential equation for $\mathbf{u}_h$ which can be solved by standard methods. Let $\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_N$ be a basis of V_h and let $\alpha_1^0, \dots, \alpha_N^0 \in \mathbb{R}$ be such that

$$\mathbf{u}_0^h(x) = \sum_{i=1}^N \alpha_i^0 \boldsymbol{\xi}_i(x).$$

Since the Gram matrix $G = ((\boldsymbol{\xi}_i, \boldsymbol{\xi}_j))_{i,j=1,\dots,N}$ is invertible, Péano's theorem yields the existence of a solution $\boldsymbol{\alpha}^N(t) := (\alpha_1^N(t), \dots, \alpha_N^N(t))$ on an interval $[0, T^*]$, $T^* \le T$, of

$$\begin{aligned} \sum_{i=1}^N \partial_t \alpha_i^N(t)(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j) &= (\mathbf{f}(t), \boldsymbol{\xi}_j) - (\mathbf{S}(\sum_{i=1}^N \alpha_i^N(t)\mathbf{D}\boldsymbol{\xi}_i), \mathbf{D}\boldsymbol{\xi}_j) \qquad \forall\, t \in [0, T^*], \\ \alpha_j(0) &= \alpha_j^0, \end{aligned} \tag{3.9}$$

for all $j = 1, ..., N$. This gives the solution of (3.3) by defining

$$\mathbf{u}_h(t,x) := \sum_{i=1}^N \alpha_i^N(t) \boldsymbol{\xi}_i(x).$$

By choosing $\boldsymbol{\xi}_h = \mathbf{u}_h(t)$, using Lemma 2.12 and the continuous Gronwall inequality, we get the a priori estimate

$$\begin{aligned} \|\mathbf{u}_h\|_{C(I^*, L^2(\Omega))} + \|\mathbf{F}(\mathbf{D}\mathbf{u}_h)\|_{L^2(I^*, L^2(\Omega))} &\le c(\|\mathbf{u}_0^h\|_{L^2(\Omega)}, \|\mathbf{f}\|_{L^2(I, L^2(\Omega))}) \\ &\le c(M_1, M_2), \end{aligned} \tag{3.10}$$

where we used the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ for $p \ge \frac{2d}{d+2}$. Since the right-hand side of (3.10) is independent of T^*, we can extend the solution to the whole interval I, which also yields (3.7). The uniqueness of $\mathbf{u}_h$ is easily shown using strict monotonicity.

It remains to show the a priori estimate (3.8). Since $\mathbf{f} \in W^{1,2}(I, L^2(\Omega))$ we deduce that $\mathbf{u}_h \in W^{2,2}(I, V_h) \hookrightarrow C^1(\overline{I}, V_h)$. Thus,

$$\partial_t \mathbf{u}_h = \sum_{i=1}^N \partial_t \alpha_i^N \boldsymbol{\xi}_i \in C^0(\overline{I}, V_h).$$

We choose $\boldsymbol{\xi}_h = \partial_t \mathbf{u}_h(s)$ in (3.3) and integrate over $(0,t)$ to get

$$\int_0^t \|\partial_t \mathbf{u}_h(s)\|_2^2 \, ds + \int_0^t (\mathbf{S}(\mathbf{D}\mathbf{u}_h(s)), \mathbf{D}\partial_t \mathbf{u}_h(s)) \, ds \le c \int_0^T \|\mathbf{f}(s)\|_2^2 \, ds,$$

where we also used Young's inequality. Due to our definition of $\mathbf{F}$ and $\mathbf{S}$ we have

$$(\mathbf{S}(\mathbf{D}\mathbf{u}_h), \mathbf{D}\partial_t \mathbf{u}_h) = (\mathbf{S}(\mathbf{D}\mathbf{u}_h), \partial_t \mathbf{D}\mathbf{u}_h) \sim \frac{d}{dt} \|\mathbf{F}(\mathbf{D}\mathbf{u}_h)\|_{L^2(\Omega)}^2$$

and thus we deduce

$$\begin{aligned} \int_0^t \|\partial_t \mathbf{u}_h(s)\|_{L^2(\Omega)}^2 \, ds + \|\mathbf{F}(\mathbf{D}\mathbf{u}_h(t))\|_{L^2(\Omega)}^2 &\le c\|\mathbf{f}\|_{L^2(I, L^2(\Omega))}^2 + c\|\mathbf{F}(\mathbf{D}\mathbf{u}_0^h)\|_{L^2(\Omega)}^2 \\ &\le c(M_1, M_2) \end{aligned}$$

for every $t \in (0,T)$. In the last step, we also used Corollary 2.19 to bound $\mathbf{F}(\mathbf{D}\mathbf{u}_0^h)$. This proves (3.8). ■

REMARK 3.11. *Once the existence of a solution $\mathbf{u}_h \in C^1(\overline{I}, V_h)$ of (3.3) is ensured, Lemma 2.42 yields the existence of a discrete pressure $q_h \in C^0(\overline{I}, Q_h)$ such that*

$$(q_h(t), \operatorname{div} \boldsymbol{\xi}_h) = (\mathbf{f}(t), \boldsymbol{\xi}_h) + (\partial_t \mathbf{u}_h(t), \boldsymbol{\xi}_h) + (\mathbf{S}(\mathbf{D}\mathbf{u}_h(t)), \mathbf{D}\boldsymbol{\xi}_h) \qquad \forall\, \boldsymbol{\xi}_h \in X_h.$$

In order to prove error estimates for the time discretization, we need higher time regularity for the solution $\mathbf{u}_h$ of (3.3). To this end, we specify the initial value $\mathbf{u}_0^h$. For given $\mathbf{u}_0 \in W^{1,p}_{0,\operatorname{div}}(\Omega)$, let $\mathbf{u}_0^h \in V_h$ be the unique solution of

$$(\mathbf{S}(\mathbf{D}\mathbf{u}_0^h), \mathbf{D}\boldsymbol{\xi}_h) = (\mathbf{S}(\mathbf{D}\mathbf{u}_0), \mathbf{D}\boldsymbol{\xi}_h) \qquad \text{for all } \boldsymbol{\xi}_h \in V_h. \tag{3.12}$$

We have

LEMMA 3.13. *For given $\mathbf{u}_0 \in W^{1,p}_{0,\operatorname{div}}(\Omega)$ with $\|\mathbf{u}_0\|_{W^{1,p}(\Omega)} \le M_1$, there exists a unique solution $\mathbf{u}_0^h \in V_h$ of (3.12) such that*

$$\|\mathbf{F}(\mathbf{D}\mathbf{u}_0^h)\|_{L^2(\Omega)} \le c\|\mathbf{F}(\mathbf{D}\mathbf{u}_0)\|_{L^2(\Omega)} \le c(M_1). \tag{3.14}$$

Furthermore, for $p \in [\frac{2d}{d+2}, \infty)$ we have

$$\|\mathbf{u}_0^h\|_{W^{1,p}(\Omega)} \le c(M_1). \tag{3.15}$$

PROOF : The existence of a function $\mathbf{u}_0^h \in V_h$ that satisfies (3.12) follows from Brouwer's fixed point theorem. By setting $\boldsymbol{\xi}_h = \mathbf{u}_0^h$ in (3.12) and using (2.13) and Lemma 2.15 we get

$$\begin{aligned} \|\mathbf{F}(\mathbf{D}\mathbf{u}_0^h)\|^2_{L^2(\Omega)} &\le c(\mathbf{S}(\mathbf{D}\mathbf{u}_0^h), \mathbf{D}\mathbf{u}_0^h) = c(\mathbf{S}(\mathbf{D}\mathbf{u}_0), \mathbf{D}\mathbf{u}_0^h) \\ &\le \varepsilon\|\mathbf{F}(\mathbf{D}\mathbf{u}_0^h)\|^2_{L^2(\Omega)} + c_\varepsilon\|\mathbf{F}(\mathbf{D}\mathbf{u}_0)\|^2_{L^2(\Omega)}, \end{aligned}$$

which yields the first inequality of (3.14) by choosing ε sufficiently small. The second part of (3.14) follows from Corollary 2.19.

For $p \in [\frac{2d}{d+2}, 2)$ we use estimate (2.17) with $\mathbf{v} = \mathbf{0}$ and Young's inequality to get

$$\begin{aligned} \|\mathbf{D}\mathbf{u}_0^h\|^2_{L^p(\Omega)} &\le c\big(\|\mathbf{D}\mathbf{u}_0^h\|_{L^p(\Omega)} + K\big)^{2-p}\|\mathbf{F}(\mathbf{D}\mathbf{u}_0^h)\|^2_{L^2(\Omega)} \\ &\le cK^{2-p}\|\mathbf{F}(\mathbf{D}\mathbf{u}_0^h)\|^2_{L^2(\Omega)} + c\|\mathbf{D}\mathbf{u}_0^h\|^{2-p}_{L^p(\Omega)}\|\mathbf{F}(\mathbf{D}\mathbf{u}_0^h)\|^2_{L^2(\Omega)} \\ &\le cK^{2-p}\|\mathbf{F}(\mathbf{D}\mathbf{u}_0^h)\|^2_{L^2(\Omega)} + \varepsilon\|\mathbf{D}\mathbf{u}_0^h\|^2_{L^p(\Omega)} + c_\varepsilon\|\mathbf{F}(\mathbf{D}\mathbf{u}_0^h)\|^{\frac{4}{p}}_{L^2(\Omega)} \end{aligned}$$

where $K = \delta|\Omega|^{\frac{1}{p}}$. Choosing ε sufficiently small we get together with (3.14)

$$\|\mathbf{D}\mathbf{u}_0^h\|^2_{L^p(\Omega)} \le cK^{2-p}\|\mathbf{F}(\mathbf{D}\mathbf{u}_0)\|^2_{L^2(\Omega)} + c\|\mathbf{F}(\mathbf{D}\mathbf{u}_0)\|^{\frac{4}{p}}_{L^2(\Omega)}. \tag{3.16}$$

Now, Korn's inequality and (3.14) yield (3.15).

For the case $p \in [2, \infty)$, we use Korn's inequality and inequality (2.18) to get

$$\|\mathbf{D}\mathbf{u}_0^h\|^p_{L^p(\Omega)} \le c\|\mathbf{F}(\mathbf{D}\mathbf{u}_0^h)\|^2_{L^2(\Omega)} \le c\|\mathbf{F}(\mathbf{D}\mathbf{u}_0)\|^2_{L^2(\Omega)},$$

where we also used (3.14). ■

Using (3.12) as the definition for the initial value $\mathbf{u}_0^h$, we can improve the regularity results for $\mathbf{u}_h$.

THEOREM 3.17. *Let* $\mathbf{S}$ *fulfill* Assumption 2.4 *with* $\delta \in [0,1]$, $p \in [\frac{2d}{d+2}, \infty)$ *and let* Assumption 2.22 *be fulfilled. Assume* $\mathbf{f} \in W^{1,2}(I, L^2(\Omega))$ *and* $\mathbf{u}_0 \in W^{1,p}_{0,\operatorname{div}}(\Omega)$ *and let* M_1, $M_3 > 0$ *be such that*

$$\|\mathbf{f}\|_{W^{1,2}(I,L^2(\Omega))} + \|\mathbf{u}_0\|_{W^{1,p}(\Omega)} \le M_1 \tag{3.18}$$

$$\|\nabla \mathbf{S}(\mathbf{D}\mathbf{u}_0)\|_{L^2(\Omega)} \le M_3. \tag{3.19}$$

Also, let $\mathbf{u}_0^h \in V_h$ *be given as the solution of* (3.12)*. Then there exists a unique solution* $\mathbf{u}_h \in C^1(\overline{I}, V_h)$ *of* (3.3)*. Furthermore,* $\mathbf{u}_h$ *satisfies the estimates*

$$\|\mathbf{u}_h\|_{L^\infty(I,L^2(\Omega))} + \|\mathbf{F}(\mathbf{D}\mathbf{u}_h)\|_{L^2(I,L^2(\Omega))} \le c(M_1), \tag{3.20}$$

$$\|\partial_t \mathbf{u}_h\|_{L^2(I,L^2(\Omega))} + \|\mathbf{F}(\mathbf{D}\mathbf{u}_h)\|_{L^\infty(I,L^2(\Omega))} \le c(M_1), \tag{3.21}$$

$$\|\partial_t \mathbf{u}_h\|_{L^\infty(I,L^2(\Omega))} + \|\partial_t \mathbf{F}(\mathbf{D}\mathbf{u}_h)\|_{L^2(I,L^2(\Omega))} \le c(M_1, M_3), \tag{3.22}$$

where the constants are independent of the parameter h.

PROOF : The existence of a solution as well as the a priori estimates (3.20) and (3.21) are shown in Theorem 3.4 and Lemma 3.13. It remains to show (3.22). To this end, we are using the othogonal projection $P_h : L^2(\Omega) \to V_h$ defined by $(P_h \mathbf{v}, \boldsymbol{\xi}_h) = (\mathbf{v}, \boldsymbol{\xi}_h)$ for all $\boldsymbol{\xi}_h \in V_h$, $\mathbf{v} \in L^2(\Omega)$. It is clear that P_h is a self-adjoint, continuous projection and fulfills $\|P_h \mathbf{v}\|_2 \le \|\mathbf{v}\|_2$ for all $\mathbf{v} \in L^2(\Omega)$. At first, we prove that $\|\partial_t \mathbf{u}_h(0)\|_{L^2(\Omega)}$ is uniformly bounded. Since $\partial_t \mathbf{u}_h(0) \in V_h$ it follows that $P_h(\partial_t \mathbf{u}_h(0)) = \partial_t \mathbf{u}_h(0)$. Thus we get

$$\begin{aligned} \|\partial_t \mathbf{u}_h(0)\|_{L^2(\Omega)} &= \sup_{\substack{\boldsymbol{\xi} \in L^2(\Omega) \\ \|\boldsymbol{\xi}\|_{L^2(\Omega)} \le 1}} (\partial_t \mathbf{u}_h(0), \boldsymbol{\xi}) = \sup_{\substack{\boldsymbol{\xi} \in L^2(\Omega) \\ \|\boldsymbol{\xi}\|_{L^2(\Omega)} \le 1}} (\partial_t \mathbf{u}_h(0), P_h \boldsymbol{\xi}) \\ &\le \sup_{\substack{\boldsymbol{\xi}_h \in V_h \\ \|\boldsymbol{\xi}_h\|_{L^2(\Omega)} \le 1}} (\partial_t \mathbf{u}_h(0), \boldsymbol{\xi}_h). \end{aligned} \tag{3.23}$$

Next, we use (3.3) at time $t = 0$ and the definition of $\mathbf{u}_0^h$, (3.12), to get from (3.23)

$$\begin{aligned} \|\partial_t \mathbf{u}_h(0)\|_{L^2(\Omega)} &\le \sup_{\substack{\boldsymbol{\xi}_h \in V_h \\ \|\boldsymbol{\xi}_h\|_{L^2(\Omega)} \le 1}} (\mathbf{f}(0), \boldsymbol{\xi}_h) - (\mathbf{S}(\mathbf{D}\mathbf{u}_0^h), \mathbf{D}\boldsymbol{\xi}_h) \\ &\le \sup_{\substack{\boldsymbol{\xi}_h \in V_h \\ \|\boldsymbol{\xi}_h\|_{L^2(\Omega)} \le 1}} (\mathbf{f}(0), \boldsymbol{\xi}_h) - (\mathbf{S}(\mathbf{D}\mathbf{u}_0), \mathbf{D}\boldsymbol{\xi}_h) \\ &\le \sup_{\substack{\boldsymbol{\xi}_h \in V_h \\ \|\boldsymbol{\xi}_h\|_{L^2(\Omega)} \le 1}} \|\mathbf{f}\|_{L^\infty(I,L^2(\Omega))} \|\boldsymbol{\xi}_h\|_{L^2(\Omega)} + \|\nabla \mathbf{S}(\mathbf{D}\mathbf{u}_0)\|_{L^2(\Omega)} \|\boldsymbol{\xi}_h\|_{L^2(\Omega)} \\ &\le c(M_1, M_3). \end{aligned} \tag{3.24}$$

Now we use difference quotients in order to prove (3.22). For $t \in (0, T')$, where $T' < T$ and $0 < \tau < T - T'$, we take (3.3) at time $t + \tau$, subtract (3.3) at time t, divide by τ and choose the difference quotient $\boldsymbol{\xi}_h = d_\tau \mathbf{u}_h(t) \in V_h$ as a test function in the emerging equation to get

$$\begin{aligned} &(\partial_t d_\tau \mathbf{u}_h(t), d_\tau \mathbf{u}_h(t)) + \frac{1}{\tau^2}\big(\mathbf{S}(\mathbf{D}\mathbf{u}_h(t+\tau)) - \mathbf{S}(\mathbf{D}\mathbf{u}_h(t)), \mathbf{D}\mathbf{u}_h(t+\tau) - \mathbf{D}\mathbf{u}_h(t)\big) \\ &= (d_\tau \mathbf{f}(t), d_\tau \mathbf{u}_h(t)). \end{aligned}$$

Lemma 2.12, Young's and Hölder's inequality yield

$$\frac{d}{dt}\|d_\tau \mathbf{u}_h(t)\|^2_{L^2(\Omega)} + \|d_\tau \mathbf{F}(\mathbf{D}\mathbf{u}_h(t))\|^2_{L^2(\Omega)} \le c\|d_\tau \mathbf{f}(t)\|^2_{L^2(\Omega)} + \|d_\tau \mathbf{u}_h(t)\|^2_{L^2(\Omega)}.$$

Hence, the continuous Gronwall inequality, the boundedness of $\partial_t \mathbf{u}_h(0)$ and Theorem 2.20 imply the estimate

$$\|\partial_t \mathbf{u}_h\|^2_{L^\infty(I,L^2(\Omega))} + \|\partial_t \mathbf{F}(\mathbf{D}\mathbf{u}_h)\|^2_{L^2(I,L^2(\Omega))} \le c\|\partial_t \mathbf{f}\|^2_{L^2(I,L^2(\Omega))} + c\|\partial_t \mathbf{u}_h(0)\|^2_{L^2(\Omega)}.$$

This finally proves the a priori estimate (3.22). ∎

3.3. Error Estimates for the Spatial Error. The goal of this section is to finally prove error estimates between the continuous solution $\mathbf{u}$ of (1.1) and its finite element approximation $\mathbf{u}_h$. Motivated by the regularity results of [8] and [14], [6] (in the space-periodic setting) we make the following assumption.

Assumption 3.25. *Let $\mathbf{S}$ fulfill Assumption 2.4 with $\delta \in [0,1]$, $p \in [\frac{2d}{d+2},\infty)$. Assume $\mathbf{f} \in W^{1,2}(I,L^2(\Omega))$, $\mathbf{u}_0 \in W^{1,p}_{0,\operatorname{div}}(\Omega)$, and suppose*

$$\|\mathbf{f}\|_{W^{1,2}(I,L^2(\Omega))} + \|\mathbf{u}_0\|_{W^{1,p}_0(\Omega)} + \|\nabla \mathbf{F}(\mathbf{D}\mathbf{u}_0)\|_{L^2(\Omega)} + \|\nabla \mathbf{S}(\mathbf{D}\mathbf{u}_0)\|_{L^2(\Omega)} \le K_1.$$

Let the weak solution $(\mathbf{u},q) \in L^p(I,W^{1,p}_{0,\operatorname{div}}(\Omega)) \times L^{p'}(I,L^{p'}_0(\Omega))$ of (1.1) be such that

$$\begin{aligned}\|\mathbf{u}\|_{W^{1,2}(I,L^2(\Omega))} + \|\mathbf{u}\|_{L^2(I,W^{2,2}(\Omega))} + \|\mathbf{u}\|_{L^p(I,W^{1,p}(\Omega))} + \|\mathbf{F}(\mathbf{D}\mathbf{u})\|_{W^{1,2}(I\times\Omega)} &\le K_2,\\ \|q\|_{L^{p'}(I,W^{1,p'}(\Omega))} &\le K_3.\end{aligned}$$

Remark 3.26. *If we additionally assume $\mathbf{f} \in L^r(I\times\Omega)$, $\mathbf{u}_0 \in W^{r,2-\frac{2}{r}}(\Omega)$ for some $r > \max\{p,p',d+2\}$, and $\delta \in (0,1]$, then in [8] it is shown there exists a solution $(\mathbf{u},\, q)$ of (1.1) with the property that*

$$\|\mathbf{u}\|_{L^r(I,W^{2,r}(\Omega))} + \|\mathbf{u}\|_{W^{1,r}(I,L^r(\Omega))} + \|q\|_{L^r(I,W^{1,r}(\Omega))} \le c,$$

see Theorem 3.1. Similarly to Corollary 2.19 we can show that $\mathbf{u} \in L^p(I,W^{1,p}(\Omega))$ implies $\mathbf{F}(\mathbf{D}\mathbf{u}) \in L^2(I,L^2(\Omega))$. For $p \ge 2$, we can show that $\nabla\mathbf{F}(\mathbf{D}\mathbf{u})$ is bounded in $L^2(I\times\Omega)$, which means that in this case, the existence of a solution $(\mathbf{u},\, q)$ fulfilling Assumption 3.25 is ensured. For $p<2$, we have the estimate $\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u})\|_{L^2(I,L^2(\Omega))} \le \delta^{\frac{p-2}{2}}\|\mathbf{u}\|_{L^2(I,W^{2,2}(\Omega))}$. Together with the maximal regularity result from [8], this also provides a solution $(\mathbf{u},q)$ fulfilling Assumption 3.25 with a constant $K_2(\delta)$ that tends to infinity for $\delta \to 0$.

Remark 3.27. *Let Assumption 3.25 be fulfilled and let $\mathbf{u}^h_0 \in V_h$ be given by (3.12). Recall that in Theorem 3.17 we have shown that for the finite element solution $\mathbf{u}_h$ the following estimates hold true*

$$\begin{aligned}\|\mathbf{u}_h\|_{L^\infty(I,L^2(\Omega))} + \|\mathbf{F}(\mathbf{D}\mathbf{u}_h)\|_{L^2(I,L^2(\Omega))} &\le c(K_1),\\ \|\partial_t\mathbf{u}_h\|_{L^2(I,L^2(\Omega))} + \|\mathbf{F}(\mathbf{D}\mathbf{u}_h)\|_{L^\infty(I,L^2(\Omega))} &\le c(K_1),\\ \|\partial_t\mathbf{u}_h\|_{L^\infty(I,L^2(\Omega))} + \|\partial_t\mathbf{F}(\mathbf{D}\mathbf{u}_h)\|_{L^2(I,L^2(\Omega))} &\le c(K_1).\end{aligned} \tag{3.28}$$

Let us now start estimating the error between $\mathbf{u}$ and $\mathbf{u}_h$. As a first step, we prove a best approximation result.

PROPOSITION 3.29. *Let Assumption 3.25 and Assumption 2.22 be fulfilled. Moreover, assume that $\mathbf{u}_0^h \in V_h$ is given by (3.12) and let $\mathbf{u}_h$ be the corresponding finite element solution ensured by Theorem 3.17. Then we have*[2]

$$\begin{aligned}\frac{d}{dt}&\|\mathbf{u}-\mathbf{u}_h\|^2_{L^2(\Omega)} + \|\mathbf{F}(\mathbf{Du})-\mathbf{F}(\mathbf{Du}_h)\|^2_{L^2(\Omega)}\\ &\le c\inf_{\boldsymbol{\zeta}_h\in V_h}\Big(\|\partial_t(\mathbf{u}-\mathbf{u}_h)\|_{L^2(\Omega)}\|\mathbf{u}-\boldsymbol{\zeta}_h\|_{L^2(\Omega)} + \|\mathbf{F}(\mathbf{Du})-\mathbf{F}(\mathbf{D}\boldsymbol{\zeta}_h)\|^2_{L^2(\Omega)}\Big)\\ &\quad + c\inf_{\mu_h\in Y_h}\int_\Omega \big(\varphi_{|\mathbf{Du}|}\big)^*(|q-\mu_h|)\,dx\end{aligned}\tag{3.30}$$

for almost every $t\in I$.

PROOF : By subtracting (3.3) from the weak formulation of (1.1) and choosing $\boldsymbol{\xi}_h = \boldsymbol{\zeta}_h - \mathbf{u}_h(t)\in V_h$ for arbitrary $\boldsymbol{\zeta}_h\in V_h$ as a test function, we get the error equation

$$\begin{aligned}&(\partial_t(\mathbf{u}-\mathbf{u}_h)(t),\boldsymbol{\zeta}_h-\mathbf{u}_h) + (\mathbf{S}(\mathbf{Du}(t))-\mathbf{S}(\mathbf{Du}_h(t)),\mathbf{D}(\boldsymbol{\zeta}_h-\mathbf{u}_h))\\ &\quad= (q(t),\operatorname{div}(\boldsymbol{\zeta}_h-\mathbf{u}_h)) = (q(t)-\mu_h,\operatorname{div}(\boldsymbol{\zeta}_h-\mathbf{u}_h))\end{aligned}$$

for all $\boldsymbol{\zeta}_h\in V_h,\ \mu_h\in Y_h$ and almost every $t\in I$. In the last step we used that due to the definition of V_h we have $(\mu_h,\operatorname{div}\boldsymbol{\xi}_h)=0$ for all $\boldsymbol{\xi}_h\in V_h,\ \mu_h\in Y_h$. After rearranging the terms and using Lemma 2.12, we obtain

$$\begin{aligned}\frac{d}{dt}&\|\mathbf{u}-\mathbf{u}_h\|^2_{L^2(\Omega)} + \|\mathbf{F}(\mathbf{Du})-\mathbf{F}(\mathbf{Du}_h)\|^2_{L^2(\Omega)}\\ &\le c\big|(\partial_t(\mathbf{u}-\mathbf{u}_h),\mathbf{u}-\boldsymbol{\zeta}_h)\big| + c\big|(\mathbf{S}(\mathbf{Du})-\mathbf{S}(\mathbf{Du}_h),\mathbf{Du}-\mathbf{D}\boldsymbol{\zeta}_h)\big|\\ &\quad + c\big|(q-\mu_h,\operatorname{div}(\boldsymbol{\zeta}_h-\mathbf{u}))\big| + c\big|(q-\mu_h,\operatorname{div}(\mathbf{u}-\mathbf{u}_h))\big|\\ &=: I_1+I_2+I_3+I_4.\end{aligned}\tag{3.31}$$

I_1 is estimated by Hölder's inequality yielding the first term on the right-hand side of (3.30). For I_2 we use Lemma 2.15 to get

$$I_2 \le \varepsilon\|\mathbf{F}(\mathbf{Du})-\mathbf{F}(\mathbf{Du}_h)\|^2_{L^2(\Omega)} + c_\varepsilon\|\mathbf{F}(\mathbf{Du})-\mathbf{F}(\mathbf{D}\boldsymbol{\zeta}_h)\|^2_{L^2(\Omega)}.$$

For I_3 and I_4 we note that for a vector field $\mathbf{v}$ there holds $|\operatorname{div}\mathbf{v}| = |\operatorname{tr}(\nabla\mathbf{v})| = |\operatorname{tr}(\mathbf{Dv})|\le|\mathbf{Dv}|$. We use Young's inequality (2.9) for $\varphi_{|\mathbf{Du}|}$ to get

$$\begin{aligned}I_3+I_4 \le& c_\varepsilon\int_\Omega\big(\varphi_{|\mathbf{Du}|}\big)^*(|q-\mu_h|)\,dx + c\int_\Omega\varphi_{|\mathbf{Du}|}(|\mathbf{Du}-\mathbf{D}\boldsymbol{\zeta}_h|)\,dx\\ &+\varepsilon\int_\Omega\varphi_{|\mathbf{Du}|}(|\mathbf{Du}-\mathbf{Du}_h|)\,dx\\ \le& c_\varepsilon\int_\Omega\big(\varphi_{|\mathbf{Du}|}\big)^*(|q-\mu_h|)\,dx + c\|\mathbf{F}(\mathbf{Du})-\mathbf{F}(\mathbf{D}\boldsymbol{\zeta}_h)\|^2_{L^2(I,L^2(\Omega))}\\ &+\varepsilon\|\mathbf{F}(\mathbf{Du})-\mathbf{F}(\mathbf{Du}_h)\|^2_{L^2(I,L^2(\Omega))}.\end{aligned}$$

Here, we used Lemma 2.12 in the last step. Choosing ε sufficiently small, the assertion follows. A more elaborated version of this proof for the stationary case can be found in [4, Lemma 3.1]. ∎

[2]For the sake of readability, we omit the dependence on t in (3.30).

The terms on the right-hand side of (3.30) can be estimated using the following result:

LEMMA 3.32. *Let Assumption 3.25 and Assumption 2.22 be fulfilled. Moreover, assume that $\mathbf{u}_0^h \in V_h$ is given by (3.12) and let $\mathbf{u}_h$ be the corresponding finite element solution. Then we have*

$$\|\mathbf{u} - \Pi_h^{\mathrm{div}} \mathbf{u}\|_{L^2(I,L^2(\Omega))} + \|\mathbf{F}(\mathbf{Du}) - \mathbf{F}(\mathbf{D}\Pi_h^{\mathrm{div}} \mathbf{u})\|^2_{L^2(I,L^2(\Omega))} \le c(K_2) h^2.$$

For the pressure term, there holds

$$\int\limits_I \int\limits_\Omega \big(\varphi_{|\mathbf{Du}|}\big)^* (|q - \Pi_h^Y q|)\, dx\, dt \le c(K_2, K_3) h^{\min\{2,p'\}}.$$

PROOF : See [4] and Lemma 2.35. ■

Integrating (3.30) in time produces the term $\|\mathbf{u}_0 - \mathbf{u}_0^h\|_{L^2(\Omega)}$, which has to be estimated. Recall that $\mathbf{u}_0^h$ is given as the solution of (3.12).

LEMMA 3.33. *Let Assumption 3.25 and Assumption 2.22 be fulfilled. Moreover, assume that $\mathbf{u}_0^h \in V_h$ is given by (3.12) and let $\mathbf{u}_h$ be the corresponding finite element solution ensured by Theorem 3.17. For $p \in [\frac{2d}{d+2}, \infty)$ we have*

$$\|\mathbf{u}_0 - \mathbf{u}_0^h\|_{L^2(\Omega)} \le c(K_1) h^{\min\{1,\frac{2}{p}\}}. \tag{3.34}$$

PROOF : Equation (3.12) implies the orthogonality

$$(\mathbf{S}(\mathbf{Du}_0^h) - \mathbf{S}(\mathbf{Du}_0), \mathbf{D}\boldsymbol{\xi}_h) = 0 \qquad \text{for all } \boldsymbol{\xi}_h \in V_h.$$

Since $\Pi_h^{\mathrm{div}} : W^{1,p}_{0,\mathrm{div}}(\Omega) \to V_h$, we get from Lemma 2.12

$$\begin{aligned}\|\mathbf{F}(\mathbf{Du}_0) - \mathbf{F}(\mathbf{Du}_0^h)\|^2_{L^2(\Omega)} &\le c\big|(\mathbf{S}(\mathbf{Du}_0) - \mathbf{S}(\mathbf{Du}_0^h), \mathbf{Du}_0 - \mathbf{Du}_0^h)\big| \\ &= c\big|(\mathbf{S}(\mathbf{Du}_0) - \mathbf{S}(\mathbf{Du}_0^h), \mathbf{Du}_0 - \mathbf{D}\Pi_h^{\mathrm{div}}\mathbf{u}_0)\big|.\end{aligned}$$

By Lemma 2.15, this implies

$$\|\mathbf{F}(\mathbf{Du}_0) - \mathbf{F}(\mathbf{Du}_0^h)\|^2_{L^2(\Omega)} \le c\|\mathbf{F}(\mathbf{Du}_0) - \mathbf{F}(\mathbf{D}\Pi_h^{\mathrm{div}}\mathbf{u}_0)\|^2_{L^2(\Omega)}.$$

Using the properties of Π_h^{div}, see (2.39), we get

$$\|\mathbf{F}(\mathbf{Du}_0) - \mathbf{F}(\mathbf{Du}_0^h)\|^2_{L^2(\Omega)} \le ch^2 \|\nabla \mathbf{F}(\mathbf{Du}_0)\|^2_{L^2(\Omega)} \le c(K_1) h^2. \tag{3.35}$$

For the case $p \in [\frac{2d}{d+2}, 2)$, we may use the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$, Poincaré's and Korn's inequality, as well as (2.17), (3.16) and (3.35) to get

$$\begin{aligned}\|\mathbf{u}_0 - \mathbf{u}_0^h\|^2_{L^2(\Omega)} &\le c\|\mathbf{Du}_0 - \mathbf{Du}_0^h\|^2_{L^p(\Omega)} \\ &\le c(K + \|\mathbf{Du}_0\|_{L^p(\Omega)} + \|\mathbf{Du}_0^h\|_{L^p(\Omega)})^{2-p} \|\mathbf{F}(\mathbf{Du}_0) - \mathbf{F}(\mathbf{Du}_0^h)\|^2_{L^2(\Omega)} \\ &\le c(K_1) \|\mathbf{F}(\mathbf{Du}_0) - \mathbf{F}(\mathbf{Du}_0^h)\|^2_{L^2(\Omega)} \\ &\le c(K_1) h^2.\end{aligned}$$

For $p \in [2, \infty)$, we get in a similar manner, this time using (2.18),

$$\|\mathbf{u}_0 - \mathbf{u}_0^h\|^p_{L^2(\Omega)} \le c\|\mathbf{Du}_0 - \mathbf{Du}_0^h\|^p_{L^p(\Omega)} \le c\|\mathbf{F}(\mathbf{Du}_0) - \mathbf{F}(\mathbf{Du}_0^h)\|^2_{L^2(\Omega)} \le c(K_1) h^2,$$

which yields the assertion. ■

Now we are able to prove the main result of this section.

THEOREM 3.36. *Let Assumption 3.25 and Assumption 2.22 be fulfilled. Moreover, assume that $\mathbf{u}_0^h \in V_h$ is given by (3.12) and let $\mathbf{u}_h$ be the corresponding finite element solution ensured by Theorem 3.17. Then, for $p \in [\frac{2d}{d+2}, 2)$ we have*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(I,L^2(\Omega))} + \|\mathbf{F}(\mathbf{Du}) - \mathbf{F}(\mathbf{Du}_h)\|_{L^2(I,L^2(\Omega))} \le ch \tag{3.37}$$

and for $p \in [2, \infty)$ we obtain

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(I,L^2(\Omega))} + \|\mathbf{F}(\mathbf{Du}) - \mathbf{F}(\mathbf{Du}_h)\|_{L^2(I,L^2(\Omega))} \le ch^{\frac{2}{p}} \tag{3.38}$$

with constants depending on K_1, K_2, and K_3.

PROOF : We choose $\boldsymbol{\zeta}_h = \Pi_h^{\mathrm{div}}\mathbf{u}(t)$ and $\mu_h = \Pi_h^Y q(t)$ in Proposition 3.29 and integrate over $t \in I$ to get

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}_h\|^2_{L^\infty(I,L^2(\Omega))} + \|\mathbf{F}(\mathbf{Du}) - \mathbf{F}(\mathbf{Du}_h)\|^2_{L^2(I,L^2(\Omega))} \\
&\qquad \le\ c(K_1, K_2)\|\mathbf{u} - \Pi_h^{\mathrm{div}}\mathbf{u}\|_{L^2(I,L^2(\Omega))} + c\|\mathbf{F}(\mathbf{Du}) - \mathbf{F}(\mathbf{D}\Pi_h^{\mathrm{div}}\mathbf{u})\|^2_{L^2(I,L^2(\Omega))} \\
&\qquad\quad + c\int\limits_I\int\limits_\Omega \left(\varphi_{|\mathbf{Du}|}\right)^*(|q - \Pi_h^Y q|)\,dx\,dt + c\|\mathbf{u}_0 - \mathbf{u}_0^h\|^2_{L^2(\Omega)},
\end{aligned}$$

where we used Assumption 3.25 and (3.28) to bound $\|\partial_t(\mathbf{u} - \mathbf{u}_h)\|_{L^2(I,L^2(\Omega))}$. Now Lemma 3.32, Lemma 3.33 and Assumption 3.25 yield

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}_h\|^2_{L^\infty(I,L^2(\Omega))} + \|\mathbf{F}(\mathbf{Du}) - \mathbf{F}(\mathbf{Du}_h)\|^2_{L^2(I,L^2(\Omega))} \\
&\qquad \le c(K_1, K_2)h^2 + c(K_2, K_3)h^{\min\{2,p'\}} + c(K_1)h^{\min\{2,\frac{4}{p}\}}.
\end{aligned}$$

Since

$$h^{\min\{2,p'\}} \le h^{\min\{2,\frac{4}{p}\}},$$

the theorem is proven. ∎

4. The Fully Discrete Solution. In order to numerically compute an approximate solution for the generalized Stokes equations, we still need to get rid of the continuity in the time variable. Therefore, we start by dividing the time interval $I = (0, T)$ in M equidistant intervals $I_n := (t_{n-1}, t_n)$, $n = 1, ..., M$, where $t_0 = 0$, $t_n = n\Delta t$, and $\Delta t = \frac{T}{M}$. For technical reasons, we assume $\Delta t < 1$. The discrete time derivative is given by

$$d_t g^n := \frac{g^n - g^{n-1}}{\Delta t}, \qquad n = 1, ..., M$$

for a sequence $(g^n)_{n=0,...,M}$ of functions $g^n \in L^1(\Omega)$. Let $\mathbf{f}^n \in L^2(\Omega)$ be a suitable approximation of $\mathbf{f}(t_n)$ to be specified later. The implicit scheme for the fully discrete problem reads as follows: Given $\mathbf{U}^0 \in V_h$ and $\mathbf{f}^n \in L^2(\Omega)$ find $\mathbf{U}^n \in V_h$, $n = 1, ..., M$, as the solution of

$$\left(\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{\Delta t}, \boldsymbol{\xi}_h\right) + (\mathbf{S}(\mathbf{DU}^n), \mathbf{D}\boldsymbol{\xi}_h) = (\mathbf{f}^n, \boldsymbol{\xi}_h) \qquad \text{for all } \boldsymbol{\xi}_h \in V_h. \tag{4.1}$$

4.1. Existence and Regularity for the Fully Discretized Solution. At first we show existence of the fully discrete solution $\mathbf{U}^n$.

THEOREM 4.2. *Suppose* $\mathbf{f}^n \in L^2(\Omega)$, $n = 1, ..., M$, *and* $\mathbf{U}^0 \in V_h$ *satisfy*

$$\Delta t \sum_{n=1}^{M} \|\mathbf{f}^n\|^2_{L^2(\Omega)} + \|\mathbf{U}^0\|^2_{L^2(\Omega)} + \|\mathbf{F}(\mathbf{D}\mathbf{U}^0)\|^2_{L^2(\Omega)} \le K. \tag{4.3}$$

Then for every $n \in \{1, ..., M\}$, *there exists a unique solution* $\mathbf{U}^n \in V_h$ *of* (4.1). *If* $\Delta t \le \alpha < 1$ *we obtain*

$$\sup_{n\in\{1,...,M\}} \|\mathbf{U}^n\|^2_{L^2(\Omega)} + \Delta t \sum_{n=1}^{M} \|\mathbf{F}(\mathbf{D}\mathbf{U}^n)\|^2_{L^2(\Omega)} \le c(K, \alpha) \tag{4.4}$$

uniformly in M *and* Δt.

PROOF : The existence of $\mathbf{U}^n$ follows from Brouwer's fixed point theorem. Setting $\boldsymbol{\xi}_h = \mathbf{U}^n$ in (4.1), we get, using also the definitions of $\mathbf{S}$ and $\mathbf{F}$,

$$\frac{1}{2\Delta t}\|\mathbf{U}^n\|^2_{L^2(\Omega)} - \frac{1}{2\Delta t}\|\mathbf{U}^{n-1}\|^2_{L^2(\Omega)} + \|\mathbf{F}(\mathbf{D}\mathbf{U}^n)\|^2_{L^2(\Omega)} \le |(\mathbf{f}^n, \mathbf{U}^n)|,$$

since $-(\mathbf{U}^n, \mathbf{U}^{n-1}) = \frac{1}{2}\|\mathbf{U}^n - \mathbf{U}^{n-1}\|^2_{L^2(\Omega)} - \frac{1}{2}\|\mathbf{U}^n\|^2_{L^2(\Omega)} - \frac{1}{2}\|\mathbf{U}^{n-1}\|^2_{L^2(\Omega)}$. Summation from $n = 1, ..., l$, $l \in \{1, ..., M\}$, gives

$$\begin{aligned} &\|\mathbf{U}^l\|^2_{L^2(\Omega)} + 2\Delta t \sum_{n=1}^{l} \|\mathbf{F}(\mathbf{D}\mathbf{U}^n)\|^2_{L^2(\Omega)} \\ &\qquad \le \|\mathbf{U}^0\|^2_{L^2(\Omega)} + \Delta t \sum_{n=1}^{l} \|\mathbf{f}^n\|^2_{L^2(\Omega)} + \Delta t \sum_{n=1}^{l} \|\mathbf{U}^n\|^2_{L^2(\Omega)}. \end{aligned}$$

Now, the discrete Gronwall lemma, see [24], shows

$$\begin{aligned} &\|\mathbf{U}^l\|^2_{L^2(\Omega)} + 2\Delta t \sum_{n=1}^{l} \|\mathbf{F}(\mathbf{D}\mathbf{U}^n)\|^2_{L^2(\Omega)} \\ &\qquad \le \exp\Big(\frac{M\Delta t}{1-\Delta t}\Big)\Big(\Delta t \sum_{n=1}^{M} \|\mathbf{f}^n\|^2_{L^2(\Omega)} + \|\mathbf{U}^0\|^2_{L^2(\Omega)}\Big). \end{aligned}$$

Since $M = \frac{T}{\Delta t}$ and $\frac{1}{1-\Delta t} \le \frac{1}{1-\alpha}$, (4.4) is proven with $c(K, \alpha) = 2\exp(\frac{T}{1-\alpha})K$. Uniqueness of $\mathbf{U}^n$ can be proven by a similar calculation. ■

4.2. Error Estimates. Now we show error estimates between the semi-discrete solution $\mathbf{u}_h(t_n)$ and the fully discrete solution $\mathbf{U}^n$. To this end, we first need to establish an error equation. For $g \in L^1(I)$, we define its mean-value on I_n by

$$\overline{g}^n := \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} g(s)\, ds.$$

In (4.1) we choose

$$\mathbf{f}^n := \overline{\mathbf{f}}^n, \qquad n = 1, ..., M$$

as an approximate for $\mathbf{f}(t_n)$. This leads to

PROPOSITION 4.5. *Let Assumption 3.25 and Assumption 2.22 be fulfilled. Moreover, assume that $\mathbf{u}_0^h \in V_h$ is given by (3.12). Set $\mathbf{f}^n = \overline{\mathbf{f}}^n$ and $\mathbf{U}^0 = \mathbf{u}_0^h$. Then for every $n \in \{1, ..., M\}$, there exists a unique solution $\mathbf{U}^n \in V_h$ of (4.1) that satisfies*

$$\sup_{n\in\{1,\dots,M\}} \|\mathbf{U}^n\|_{L^2(\Omega)}^2 + \Delta t \sum_{n=1}^{M} \|\mathbf{F}(\mathbf{D}\mathbf{U}^n)\|_{L^2(\Omega)}^2 \le c(K_1, \alpha)$$

provided $\Delta t \le \alpha < 1$.

PROOF : By definition of $\overline{\mathbf{f}}^n$, it is clear that

$$\Delta t \sum_{n=1}^{M} \|\mathbf{f}^n\|_{L^2(\Omega)}^2 = \Delta t \sum_{n=1}^{M} \int_\Omega \left| \frac{1}{\Delta t} \int_{I_n} \mathbf{f}(s)\, ds \right|^2 dx \le \sum_{n=1}^{M} \int_\Omega \int_{I_n} |\mathbf{f}(s)|^2 \, ds\, dx \le c(K_1).$$

Lemma 3.13 yields

$$\|\mathbf{u}_0^h\|_{L^2(\Omega)} + \|\mathbf{F}(\mathbf{D}\mathbf{u}_0^h)\|_{L^2(\Omega)} \le c(K_1),$$

where we used the embedding $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$, which holds for $p \in [\frac{2d}{d+2}, \infty)$. Therefore, the requirements of Theorem 4.2 are fulfilled and the proposition is proven. ■

For the error equation, we take the mean-value of (3.3) on I_n and subtract (4.1) to get

$$(d_t(\mathbf{u}_h^n - \mathbf{U}^n), \boldsymbol{\xi}_h) + (\overline{\mathbf{S}(\mathbf{D}\mathbf{u}_h)}^n - \mathbf{S}(\mathbf{D}\mathbf{U}^n), \mathbf{D}\boldsymbol{\xi}_h) = 0 \tag{4.6}$$

for all $\boldsymbol{\xi}_h \in V_h$, $n = 1, ..., M$. Here we use the notation

$$\mathbf{u}_h^n := \mathbf{u}_h(t_n), \qquad n = 1, ..., M.$$

As a first step, we state a preliminary result

LEMMA 4.7. *For $n = 1, ..., M$ there holds*

$$\|\mathbf{F}(\mathbf{D}\mathbf{u}_h^n) - \overline{\mathbf{F}(\mathbf{D}\mathbf{u}_h)}^n\|_{L^2(\Omega)}^2 \le \Delta t \|\partial_t \mathbf{F}(\mathbf{D}\mathbf{u}_h)\|_{L^2(I_n, L^2(\Omega))}^2.$$

PROOF : The fundamental theorem of calculus yields

$$\begin{aligned}
\|\mathbf{F}(\mathbf{D}\mathbf{u}_h^n) - \overline{\mathbf{F}(\mathbf{D}\mathbf{u}_h)}^n\|_{L^2(\Omega)}^2 &= \int_\Omega \Big| \fint_{I_n} \int_s^{t_n} \partial_\tau \mathbf{F}(\mathbf{D}\mathbf{u}_h(\tau)) d\tau \, ds \Big|^2 dx \\
&\le \int_\Omega \Big| \int_{t_{n-1}}^{t_n} \partial_\tau \mathbf{F}(\mathbf{D}\mathbf{u}_h(\tau)) d\tau \Big|^2 dx \\
&\le (t_n - t_{n-1}) \|\partial_t \mathbf{F}(\mathbf{D}\mathbf{u}_h)\|_{L^2(I_n, L^2(\Omega))}^2,
\end{aligned}$$

where we used Hölder's inequality in the last step. This proves the lemma. ■

Now we are ready to prove the error estimate between the semi discretized solution $\mathbf{u}_h^n = \mathbf{u}_h(t_n)$ and the fully discrete solution $\mathbf{U}^n$.

PROPOSITION 4.8. *Let Assumption 3.25 and Assumption 2.22 be fulfilled. Moreover, assume that $\mathbf{u}_0^h \in V_h$ is given by (3.12) and let $\mathbf{u}_h$ be the corresponding finite element solution ensured by Theorem 3.17. Set $\mathbf{f}^n = \overline{\mathbf{f}}^n$ and $\mathbf{U}^0 = \mathbf{u}_0^h$. Then we have*

$$\sup_{n\in\{1,\dots,M\}} \|\mathbf{u}_h^n - \mathbf{U}^n\|_{L^2(\Omega)}^2 + \Delta t \sum_{n=1}^{M} \|\mathbf{F}(\mathbf{D}\mathbf{u}_h^n) - \mathbf{F}(\mathbf{D}\mathbf{U}^n)\|_{L^2(\Omega)}^2 \le c(K_1)(\Delta t)^2$$

provided $\Delta t \le \alpha < 1$.

Proof : We choose $\boldsymbol{\xi}_h = \mathbf{u}_h^n - \mathbf{U}^n$ as a test function in the error equation (4.6). After rearranging the terms and using Lemma 2.12 and Lemma 2.15, we get

$$\begin{aligned}
&(d_t(\mathbf{u}_h^n - \mathbf{U}^n), \mathbf{u}_h^n - \mathbf{U}^n) + \|\mathbf{F}(\mathbf{D}\mathbf{u}_h^n) - \mathbf{F}(\mathbf{D}\mathbf{U}^n)\|_{L^2(\Omega)}^2 \\
&\quad \le c\big|(\mathbf{S}(\mathbf{D}\mathbf{u}_h^n) - \overline{\mathbf{S}(\mathbf{D}\mathbf{u}_h)}^n, \mathbf{D}\mathbf{u}_h^n - \mathbf{D}\mathbf{U}^n)\big| \\
&\quad = c\Big|\fint_{I_n} (\mathbf{S}(\mathbf{D}\mathbf{u}_h(t_n)) - \mathbf{S}(\mathbf{D}\mathbf{u}_h(s)), \mathbf{D}\mathbf{u}_h(t_n) - \mathbf{D}\mathbf{U}^n)\, ds\Big| \qquad (4.9)\\
&\quad \le c_\varepsilon \fint_{I_n} \|\mathbf{F}(\mathbf{D}\mathbf{u}_h(t_n) - \mathbf{F}(\mathbf{D}\mathbf{u}_h(s))\|_{L^2(\Omega)}^2\, ds + \varepsilon\|\mathbf{F}(\mathbf{D}\mathbf{u}_h^n) - \mathbf{F}(\mathbf{D}\mathbf{U}^n)\|_{L^2(\Omega)}^2 \\
&\quad \le c_\varepsilon \|\mathbf{F}(\mathbf{D}\mathbf{u}_h^n) - \overline{\mathbf{F}(\mathbf{D}\mathbf{u}_h)}^n\|_{L^2(\Omega)}^2 + \varepsilon\|\mathbf{F}(\mathbf{D}\mathbf{u}_h^n) - \mathbf{F}(\mathbf{D}\mathbf{U}^n)\|_{L^2(\Omega)}^2.
\end{aligned}$$

In the first term on the left-hand side of (4.9), a simple calculation yields

$$(d_t(\mathbf{u}_h^n - \mathbf{U}^n), \mathbf{u}_h^n - \mathbf{U}^n) \ge \frac{1}{2\Delta t}\big(\|\mathbf{u}_h^n - \mathbf{U}^n\|_{L^2(\Omega)}^2 - \|\mathbf{u}_h^{n-1} - \mathbf{U}^{n-1}\|_{L^2(\Omega)}^2\big).$$

Taking this into account and choosing ε sufficiently small, we get from (4.9)

$$\begin{aligned}
&\|\mathbf{u}_h^n - \mathbf{U}^n\|_{L^2(\Omega)}^2 - \|\mathbf{u}_h^{n-1} - \mathbf{U}^{n-1}\|_{L^2(\Omega)}^2 + \Delta t\|\mathbf{F}(\mathbf{D}\mathbf{u}_h^n) - \mathbf{F}(\mathbf{D}\mathbf{U}^n)\|_{L^2(\Omega)}^2 \\
&\quad \le c\Delta t\|\mathbf{F}(\mathbf{D}\mathbf{u}_h^n) - \overline{\mathbf{F}(\mathbf{D}\mathbf{u}_h)}^n\|_{L^2(\Omega)}^2 \le c(\Delta t)^2\|\partial_t\mathbf{F}(\mathbf{D}\mathbf{u}_h)\|_{L^2(I_n,L^2(\Omega))}^2,
\end{aligned}$$

where we also used Lemma 4.7 in the last step. Now summation from $n = 1, ..., l$, taking the supremum over $l \in \{1, ..., M\}$ and taking (3.28) into account yields the assertion. ■

In order to link the continuous function $\mathbf{u}$ to the fully discrete function $\mathbf{U}^n$, $n = 1, ..., M$, we define the piecewise-constant-in-time function

$$\hat{\mathbf{U}}(t) := \begin{cases} \mathbf{U}^0, & t = 0 \\ \mathbf{U}^n, & t \in I_n,\ n = 1, ..., M. \end{cases}$$

Together with the results from the previous section, we get our main error estimate

Theorem 4.10. *Let Assumption 3.25 and Assumption 2.22 be fulfilled. Moreover, assume that* $\mathbf{u}_0^h \in V_h$ *is given by (3.12) and let* $\mathbf{U}^n$ *be the corresponding fully discrete solution ensured by Proposition 4.5. For* $p \in [\frac{2d}{d+2}, 2]$, *we have*

$$\|\mathbf{u} - \hat{\mathbf{U}}\|_{L^\infty(I,L^2(\Omega))} + \|\mathbf{F}(\mathbf{D}\mathbf{u}) - \mathbf{F}(\mathbf{D}\hat{\mathbf{U}})\|_{L^2(I,L^2(\Omega))} \le c(K_1, K_2, K_3)h + c(K_1)\Delta t,$$

and for $p \in (2, \infty)$, *we have*

$$\|\mathbf{u} - \hat{\mathbf{U}}\|_{L^\infty(I,L^2(\Omega))} + \|\mathbf{F}(\mathbf{D}\mathbf{u}) - \mathbf{F}(\mathbf{D}\hat{\mathbf{U}})\|_{L^2(I,L^2(\Omega))} \le c(K_1, K_2, K_3)h^{\frac{2}{p}} + c(K_1)\Delta t,$$

provided $\Delta t \le \alpha < 1$.

PROOF : Define $\beta := 1$, if $p \in [\frac{2d}{d+2}, 2]$ and $\beta := \frac{2}{p}$ if $p \in (2, \infty)$. Let $\mathbf{u}_h$ be the finite element solution ensured by Theorem 3.17. Note that the fundamental theorem of calculus yields for $t \in I_n$

$$\|\mathbf{u}_h(t) - \mathbf{u}_h(t_n)\|^2_{L^2(\Omega)} \leq \int_\Omega \Big| \int_{t_{n-1}}^{t_n} \partial_\tau \mathbf{u}_h(\tau, x) d\tau \Big|^2 dx \leq (\Delta t)^2 \|\partial_t \mathbf{u}_h\|^2_{L^\infty(I_n, L^2(\Omega))}.$$

This, Theorem 3.36, and Proposition 4.8 yield that

$$\begin{aligned}\sup_{t\in I_n} \|\mathbf{u} - \hat{\mathbf{U}}\|^2_{L^2(\Omega)} &\leq \sup_{t\in I_n} \|\mathbf{u} - \mathbf{u}_h\|^2_{L^2(\Omega)} + \sup_{t\in I_n} \|\mathbf{u}_h - \mathbf{u}_h^n\|^2_{L^2(\Omega)} + \sup_{t\in I_n} \|\mathbf{u}_h^n - \mathbf{U}^n\|^2_{L^2(\Omega)} \\ &\leq c(K_1, K_2, K_3) h^{2\beta} + (\Delta t)^2 \|\partial_t \mathbf{u}_h\|^2_{L^\infty(I_n, L^2(\Omega))} + c(K_1)(\Delta t)^2.\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\|\mathbf{F}(\mathbf{D}\mathbf{u}) - \mathbf{F}(\mathbf{D}\hat{\mathbf{U}})\|^2_{L^2(I, L^2(\Omega))} \\ &\quad \leq \|\mathbf{F}(\mathbf{D}\mathbf{u}) - \mathbf{F}(\mathbf{D}\mathbf{u}_h)\|^2_{L^2(I, L^2(\Omega))} + \sum_{n=1}^M \|\mathbf{F}(\mathbf{D}\mathbf{u}_h) - \mathbf{F}(\mathbf{D}\mathbf{u}_h^n)\|^2_{L^2(I_n, L^2(\Omega))} \\ &\quad\quad + \sum_{n=1}^M \|\mathbf{F}(\mathbf{D}\mathbf{u}_h^n) - \mathbf{F}(\mathbf{D}\mathbf{U}^n)\|^2_{L^2(I_n, L^2(\Omega))} \\ &\quad =: I_1 + I_2 + I_3.\end{aligned}$$

From Theorem 3.36, we have

$$I_1 \leq c(K_1, K_2, K_3) h^{2\beta}$$

and Proposition 4.8 gives

$$I_3 = \sum_{n=1}^M (t_n - t_{n-1}) \|\mathbf{F}(\mathbf{D}\mathbf{u}_h^n) - \mathbf{F}(\mathbf{D}\mathbf{U}^n)\|^2_{L^2(\Omega))} \leq c(K_1)(\Delta t)^2.$$

A similar argument as in Lemma 4.7 shows

$$I_2 = \sum_{n=1}^M \int_{I_n} \int_\Omega |\mathbf{F}(\mathbf{D}\mathbf{u}_h(s)) - \mathbf{F}(\mathbf{D}\mathbf{u}_h(t_n))|^2 \, dx \, ds \leq (\Delta t)^2 \|\partial_t \mathbf{F}(\mathbf{D}\mathbf{u}_h)\|^2_{L^2(I, L^2(\Omega))}.$$

Altogether we obtain, also using Remark 3.27, the assertion of the theorem. ∎

REFERENCES

[1] J. W. BARRETT AND W. B. LIU, *Finite element approximation of the p-Laplacian*, Math. Comp., 61 (1993), pp. 523–537.

[2] J. W. BARRETT AND W. B. LIU, *Finite element approximation of the parabolic p-Laplacian*, SIAM J. Numer. Anal., 31 (1994), pp. 413–428.

[3] J. W. Barrett and W. B. Liu, *Quasi-norm error bounds for the finite element approximation of a non-Newtonian flow*, Numer. Math., 68 (1994), pp. 437–456.

[4] L. Belenki, L. C. Berselli, L. Diening, and M. Růžička, *On the finite element approximation of p-Stokes systems*, SIAM J. Numer. Anal., 50 (2012), pp. 373–397.

[5] L. C. Berselli, L. Diening, and M. Růžička, *Optimal error estimates for a semi-implicit Euler scheme for incompressible fluids with shear dependent viscosities*, SIAM J. Numer. Anal., 47 (2009), pp. 2177–2202.

[6] L. C. Berselli, L. Diening, and M. Růžička, *Existence of strong solutions for incompressible fluids with shear dependent viscosities*, J. Math. Fluid Mech., 12 (2010), pp. 101–132.

[7] L. C. Berselli, L. Diening, and M. Růžička, *Optimal error estimate for semi-implicit space-time discretization for the equations describing incompressible generalized Newtonian fluids*, IMA J. Numer. Anal., 35 (2015), pp. 680–697.

[8] D. Bothe and J. Prüss, *L_P-theory for a class of non-Newtonian fluids*, SIAM J. Math. Anal., 39 (2007), pp. 379–421.

[9] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, vol. 15 of Springer Series in Computational Mathematics, Springer-Verlag, New York, 1991.

[10] L. Diening, C. Ebmeyer, and M. Růžička, *Optimal convergence for the implicit space-time discretization of parabolic systems with p-structure*, SIAM J. Numer. Anal., 45 (2007), pp. 457–472 (electronic).

[11] L. Diening, P. Nägele, and M. Růžička, *Monotone operator theory for unsteady problems in variable exponent spaces*, Complex Var. Elliptic Equ., 57 (2012), pp. 1209–1231.

[12] L. Diening, A. Prohl, and M. Růžička, *On time-discretizations for generalized Newtonian fluids*, in Nonlinear problems in mathematical physics and related topics, II, vol. 2 of Int. Math. Ser. (N. Y.), Kluwer/Plenum, New York, 2002, pp. 89–118.

[13] L. Diening, A. Prohl, and M. Růžička, *Semi-implicit Euler scheme for generalized Newtonian fluids*, SIAM J. Numer. Anal., 44 (2006), pp. 1172–1190 (electronic).

[14] L. Diening and M. Růžička, *Strong solutions for generalized Newtonian fluids*, J. Math. Fluid Mech., 7 (2005), pp. 413–450.

[15] L. Diening and M. Růžička, *Interpolation operators in Orlicz-Sobolev spaces*, Numer. Math., 107 (2007), pp. 107–129.

[16] S. Eckstein, *On the full space-time discretization of the generalized Stokes systems: The Dirichlet case*, PhD thesis, Albert-Ludwigs-Universität Freiburg im Breisgau, 2016.

[17] H. Gajewski, K. Gröger, and K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Akademie-Verlag, Berlin, 1974. Mathematische Lehrbücher und Monographien, II. Abteilung, Mathematische Monographien, Band 38.

[18] V. Girault and J.-L. Lions, *Two-grid finite-element schemes for the steady Navier-Stokes problem in polyhedra*, Port. Math. (N.S.), 58 (2001), pp. 25–57.

[19] V. Girault and P.-A. Raviart, *Finite element methods for Navier-Stokes equations*, vol. 5 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1986. Theory and algorithms.

[20] V. Girault and L. R. Scott, *A quasi-local interpolation operator preserving the discrete divergence*, Calcolo, 40 (2003), pp. 1–19.

[21] J. G. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization*, SIAM J. Numer. Anal., 19 (1982), pp. 275–311.

[22] J. G. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem. II. Stability of solutions and error estimates uniform in time*, SIAM J. Numer. Anal., 23 (1986), pp. 750–777.

[23] J. G. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem. III. Smoothing property and higher order error estimates for spa-*

tial discretization, SIAM J. Numer. Anal., 25 (1988), pp. 489–512.

[24] J. G. Heywood and R. Rannacher, *Finite-element approximation of the nonstationary Navier-Stokes problem. IV. Error analysis for second-order time discretization*, SIAM J. Numer. Anal., 27 (1990), pp. 353–384.

[25] M. A. Krasnosel′skiĭ and J. B. Rutickiĭ, *Convex functions and Orlicz spaces*, P. Noordhoff Ltd., Groningen, 1961.

[26] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Second English edition, revised and enlarged. Mathematics and its Applications, Vol. 2, Gordon and Breach, Science Publishers, New York-London-Paris, 1969.

[27] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod; Gauthier-Villars, Paris, 1969.

[28] J. Málek, J. Nečas, M. Rokyta, and M. Růžička, *Weak and measure-valued solutions to evolutionary PDEs*, vol. 13 of Applied Mathematics and Mathematical Computation, Chapman & Hall, London, 1996.

[29] J. Málek and K. R. Rajagopal, *Mathematical issues concerning the Navier-Stokes equations and some of its generalizations*, in Evolutionary equations. Vol. II, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2005, pp. 371–459.

[30] A. Prohl and M. Růžička, *On fully implicit space-time discretization for motions of incompressible fluids with shear-dependent viscosities: the case* $p \leq 2$, SIAM J. Numer. Anal., 39 (2001), pp. 214–249 (electronic).

[31] M. Růžička and L. Diening, *Non-Newtonian fluids and function spaces*, in NAFSA 8—Nonlinear analysis, function spaces and applications. Vol. 8, Czech. Acad. Sci., Prague, 2007, pp. 94–143.

[32] L. R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54 (1990), pp. 483–493.

[33] E. Zeidler, *Nonlinear functional analysis and its applications. II/B*, Springer-Verlag, New York, 1990. Nonlinear monotone operators.