SPECTRAL ANALYSIS OF A COMPLEX SCHRÖDINGER OPERATOR IN THE SEMICLASSICAL LIMIT

YANIV ALMOG AND RAPHAËL HENRY

Abstract. We consider the Dirichlet realization of the operator $-h^2\Delta + iV$ in the semi-classical limit $h \to 0$, where V is a smooth real potential with no critical points. For a one dimensional setting, we obtain the complete asymptotic expansion, in powers of h, of each eigenvalue. In two dimensions we obtain the left margin of the spectrum, under some additional assumptions.

1. Introduction

We consider the operator

$$\mathcal{A}_h = -h^2\Delta + iV\,, \tag{1.1a}$$

defined on

$$D(\mathcal{A}_h) = H_0^1(\Omega,\mathbb{C}) \cap H^2(\Omega,\mathbb{C})\,, \tag{1.1b}$$

where Ω is a bounded domain in $\mathbb{R}^2$.

We seek an approximation for $\inf \operatorname{Re}\sigma(\mathcal{A}_h)$ in the limit $h \to 0$. The domain Ω is smooth, i.e., $\partial\Omega \subset C^3$ and the potential V is at least in $C^3(\bar{\Omega},\mathbb{R})$. Let $\partial\Omega_\perp$ denote a subset of $\partial\Omega$ where $\nabla V \perp \partial\Omega$. Note that in view of the continuity of V on $\partial\Omega$, we must have $\partial\Omega_\perp \neq \emptyset$. Let $x_0 \in \partial\Omega_\perp$ satisfy

$$c_m = |\nabla V(x_0)| = \min_{x\in\partial\Omega_\perp} |\nabla V(x)|\,.$$

Denote by $\mathcal{S}$ the set

$$\mathcal{S} = \{x \in \partial\Omega_\perp \,:\, |\nabla V(x)| = |\nabla V(x_0)|\,,\ V(x) = V(x_0)\}\,. \tag{1.2}$$

(Note that in the case where x_0 is not unique, $\mathcal{S}$ depends on the choice of x_0.) For every $x \in \mathcal{S}$ set

$$c(x) = \nabla V(x)\cdot\nu(x) = \pm c_m\,,$$

and

$$\alpha(x) = t\cdot D^2V(x)t - \kappa(x)\frac{\partial V}{\partial\nu}(x) \quad t\cdot\nu(x) = 0\,,\ |t| = 1\,,$$

where ν is the outward normal and κ denotes the local curvature. (Note that $\alpha = \partial^2V/\partial s^2$ where s is the arclength on $\partial\Omega$.) We now assume that

$$\alpha(x)c(x) > 0 \quad \forall x \in \mathcal{S}\,. \tag{1.3}$$

Without any loss of generality we may then assume $\alpha(x) > 0$ in $\mathcal{S}$, otherwise we may consider $\bar{\mathcal{A}}_h$ instead of $\mathcal{A}_h$.

The spectral analysis of (1.1) has several applications in Mathematical Physics, among them are the Orr-Sommerfeld equations in fluid dynamics [12], the Ginzburg-Landau equation in the presence of electric current (when magnetic field effects are neglected), and the null controllability of Kolomogorov type equations [6]. In [3, 10] it has been established that

$$\liminf_{h\to 0} h^{-2/3} \inf \operatorname{Re} \sigma(\mathcal{A}_h) \geq \frac{|\mu_1|}{2} c_m^{2/3} \,, \tag{1.4}$$

where μ_1 is the rightmost zero of Airy's function [1].

We note that (1.4) has been obtained without the need to assume (1.3). In the present contribution we seek an upper bound for $\inf \operatorname{Re} \sigma(\mathcal{A}_h)$. It is to this end that we make that additional assumption. Our main result is the following

Theorem 1.1. *Let $\mathcal{A}_h$ denote the Dirichlet realization of a Schrödinger operator with a purely imaginary potential $V \in C^3(\Omega, \mathbb{R})$, satisfying $\nabla V \neq 0$ in $\bar{\Omega}$, given by* (1.1)*. Suppose that V satisfies* (1.3)*. Then, there exists $\lambda(h) \in \sigma(\mathcal{A}_h)$ satisfying*

$$\Big|\lambda - iV(x_0) - e^{i\pi/3}|\mu_1|(c_m h)^{2/3} - \sqrt{2\alpha} e^{i\frac{\pi}{4}} h\Big| \sim o(h) \quad \text{as } h \to 0 \,, \tag{1.5}$$

where $\alpha = \alpha(x_0)$.

An immediate corollary follows

Corollary 1.2. *Under the above assumptions we have that*

$$\lim_{h\to 0} h^{-2/3} \inf \operatorname{Re} \sigma(\mathcal{A}_h) = \frac{|\mu_1|}{2} c_m^{2/3} \,, \tag{1.6}$$

Remark 1.3. *While we do not prove that here, it appears that* (1.6) *can be extended to higher dimensions. Let $D^2_{\|}V$ denote the Hessian matrix of V with respect to a local curvilinear coordinate system defined on $\partial\Omega$ (including, of course, curvature effects). Suppose that $D^2_{\|}V(x)$ is either positive or negative. Then, we set α in the following manner*

$$\alpha(x) = \operatorname{sign}\left(D^2_{\|}V(x)\right) \inf_{\substack{t\cdot\nu(x)=0\\ |t|=1}} |t \cdot D^2_{\|}V(x) t| \,,$$

and assume (1.3) *once again.*

Remark 1.4. *Let $\mathcal{A}_h^N$ denote the Neumann realization of $\mathcal{A}_h$. By using the same techniques as in the sequel, one can obtain an upper bound for* $\inf \operatorname{Re} \sigma(\mathcal{A}_h^N)$*. In this case, μ_1 will be replaced by the rightmost critical point Airy's function.*

Finally, we note that it has been established in [10] that for all $\epsilon > 0$ there exist positive M_ϵ and h_ϵ such that for all $h \in (0, h_\epsilon)$ we have the following upper bound for the semigroup assciated with $-\mathcal{A}_h$,

$$\|e^{-t\mathcal{A}_h}\| \leq M_\epsilon \exp\{-(c_m^{2/3}|\mu_1|/2 - \epsilon) h^{2/3} t\} \,.$$

From (1.5) we can now establish that for some positive M, C and h_0 the following lower bound for the semigroup holds for all $h \in (0, h_0)$

$$\|e^{-t\mathcal{A}_h}\| \geq M \exp\Big\{ -c_m^{2/3} \frac{|\mu_1|}{2} h^{2/3}(1 + Ch^{1/3})t \Big\}\,.$$

The rest of this contribution is arranged as follows: in the next section we consider a one-dimensional version of (1.1). Assuming that $V \in C^\infty([0,a],\mathbb{R})$ we obtain the complete asymptotic expansion, as $h \to 0$, of any eigenvalue $\lambda_k \in \sigma(\mathcal{A}_h)$ (k is fixed in the limit). In Section 3 we construct the quasimode associated with the eigenvalue given in (1.5), and in the last section provide a rigorous proof of Theorem 1.1.

2. The one-dimensional case

2.1. **Statement of the results.** Let $a > 0$ and $V \in \mathcal{C}^\infty\big([0,a];\mathbb{R}\big)$ such that V has no critical point in $[0,a]$. Consider then the one-dimensional Schrödinger operator $\mathcal{A}_h$ defined on $(0,a)$ by

$$\mathcal{A}_h = -h^2 \frac{d^2}{dx^2} + i\big(V - V(0)\big)\,,$$

with domain

$$D(\mathcal{A}_h) = H_0^1([0,a],\mathbb{C}) \cap H^2([0,a],\mathbb{C})\,.$$

The main result we prove in this section is the following:

Theorem 2.1. *Assume that, for all $x \in [0,a]$, $V'(x) \neq 0$. Then, for all $n \geq 1$, there exists a complex sequence $(\alpha_{j,n})_{j\geq 1}$ and an eigenvalue $\lambda_n(h) \in \sigma(\mathcal{A}_h)$ such that, as $h \to 0$,*

$$h^{-2/3}\lambda_n(h) \underset{h\to 0}{\sim} e^{\sigma i\pi/3}|V'(0)|^{2/3}|\mu_n| + \sum_{j=1}^{+\infty} \alpha_{j,n} h^{2j/3} + \mathcal{O}(h^\infty)\,, \tag{2.1}$$

where σ is the (constant) sign of the function V'.

Similarly, one could also prove the existence of another sequence $(\nu_n(h))_{n\geq 1}$ of eigenvalues satisfying an asymptotic expansion of the form

$$\nu_n(h) \underset{h\to 0}{\sim} i\big(V(a) - V(0) - a\big) + e^{\sigma i\pi/3}|V'(a)|^{2/3}|\mu_n| h^{2/3} + \sum_{j=1}^{+\infty} \beta_{j,n} h^{2(j+1)/3} + \mathcal{O}(h^\infty) \tag{2.2}$$

by applying the transformation $x \to a - x$. Similar results have previously been obtained in the particular cases $V(x) = x$ and $V(x) = x^2$, see [12] and [6].

Remark 2.2. *Theorem 2.1 esablishes existence of two sequences of eigenvalues of $\mathcal{A}_h$, respectively obeying* (2.1) *and* (2.2)*. The fact that these sequences constitute the entire spectrum of $\mathcal{A}_h$ for* $\mathrm{Re}\,\lambda \leq Mh^{2/3}$ *for any positive M follows from* [10, Proposition 6.1].

Let $\varepsilon = h^{2/3}$. It is more convenient to obtain the spectrum of $\mathcal{A}_h$ by first applying the dilation operator $U : L^2(0,a) \to L^2(0,a/\varepsilon)$ defined by

$$(Uu)(\cdot/\varepsilon) = u(\cdot)\,.$$

Let

$$V_\varepsilon(x) = \frac{V(\varepsilon x)}{\varepsilon}\,.$$

Then by applying the above dilation we obtain

$$\frac{1}{\varepsilon}U^{-1}\mathcal{A}_h U = \mathcal{A}_\varepsilon = -\frac{d^2}{dx^2} + i\left(V_\varepsilon - \frac{V(0)}{\varepsilon}\right)\,, \tag{2.3}$$

defined on

$$D(\mathcal{A}_\varepsilon) = (H_0^1 \cap H^2)\big((0,a/\varepsilon),\mathbb{C}\big)\,.$$

2.2. **Quasimode construction.** In the following we construct quasimodes and approximate eigenvalues for $\mathcal{A}_\varepsilon$ in the neighborhood of the boundary point $x = 0$. In particular, we obtain the asymptotic expansion (2.1) for each approximate eigenvalue.

Proposition 2.3. *Assume that, for all $x \in [0,a]$, $V'(x) \neq 0$. Let $n \geq 1$ and σ denote the sign of V'. Then there exists $\psi_\varepsilon \in \mathcal{D}(\mathcal{A}_\varepsilon)$ and a complex sequence $(\nu_j)_{j\geq 2}$ such that*

$$\big\|(\mathcal{A}_\varepsilon - \nu(\varepsilon))\psi_\varepsilon\big\| = \mathcal{O}(\varepsilon^\infty)\|\psi_\varepsilon\|\,, \tag{2.4}$$

where

$$\nu(\varepsilon) = e^{\sigma i\pi/3}|V'(0)|^{2/3}|\mu_n| + \sum_{j=1}^{+\infty}\nu_j\varepsilon^j + \mathcal{O}(\varepsilon^\infty) \tag{2.5}$$

as $\varepsilon \to 0$.

Proof. We approximate $\mathcal{A}_\varepsilon$ at any order N by the operator

$$A_N(\varepsilon) = A_0 + \sum_{j=1}^N V_j\varepsilon^j \quad \text{on} \quad (0,+\infty)\,,$$

where

$$A_0 = -\frac{d^2}{dx^2} + i\beta_0 x\,, \quad \beta_0 = V'(0)\,,$$
$$V_j = i\beta_j x^{j+1}\,, \quad \beta_j = \frac{V^{(j+1)}(0)}{(j+1)!}\,, \quad j \in \mathbb{N}\,.$$

Then, for all $N \geq 1$, we look for a quasimode $u^N(x,\varepsilon)$ and an approximate eigenvalue $\lambda^N(\varepsilon)$ in the form

$$u^N(x,\varepsilon) = \sum_{j=0}^N u_j(x)\varepsilon^j\,,\ \lambda^N(\varepsilon) = \sum_{j=0}^N \lambda_j\varepsilon^j\,, \tag{2.6}$$

satisfying

$$\Big(A_0 + \sum_{j=1}^{N} V_j \varepsilon^j\Big) u^N(x,\varepsilon) = \lambda^N(\varepsilon) u^N(x,\varepsilon) + \mathcal{O}(\varepsilon^{N+1})\,.$$

To this end, we need to successively solve the following equations:

$$\begin{aligned}
(A_0-\lambda_0)u_0 &= 0\,,\\
(A_0-\lambda_0)u_1 &= -(V_1-\lambda_1)u_0\,,\\
&\vdots \qquad\qquad (2.7)\\
(A_0-\lambda_0)u_k &= -\sum_{j=1}^{k}(V_j-\lambda_j)u_{k-j}\,,\ k=1,\dots,N\,. \qquad (2.8)
\end{aligned}$$

Consider the first equation. If $\beta_0 > 0$, we can use the scale change $x \mapsto \beta_0^{1/3}x$ and the well-known properties of the complex Airy operator [3] to obtain

$$\sigma(A_0) = \big\{\beta_0^{1/3}\mu_n e^{-2i\pi/3} : n\in\mathbb{N}\big\}\,,$$

where μ_n denotes the n-th zero of the Airy function Ai. The associated eigenfunctions are

$$x \mapsto Ai(\beta_0^{1/3}e^{i\pi/6}x + \mu_n)\,.$$

If $\beta_0 < 0$, then the operator A_0 is the adjoint of $-\frac{d^2}{dx^2} + i|\beta_0|x$. Hence,

$$\sigma(A_0) = \big\{|\beta_0|^{1/3}\mu_n e^{+2i\pi/3} : n\in\mathbb{N}\big\}\,,$$

and the eigenfunctions are given by

$$x \mapsto \overline{Ai(\beta_0^{1/3}e^{i\pi/6}x + \mu_n)}\,.$$

Therefore, for any fixed $n\in\mathbb{N}$, we choose

$$\lambda_0 = \lambda_{0,n} = |\beta_0|^{1/3}\mu_n e^{\sigma 2i\pi/3}\,, \qquad (2.9)$$

and $u_0 = u_{0,n}$ to be a corresponding eigenfunction.

Next, consider the second equation. To ensure the existence of a u_1, we first select λ_1 such that

$$(V_1-\lambda_1)u_0 \in \mathrm{Im}\,(A_0-\lambda_0) = \ker(A_0^*-\bar\lambda_0)^\perp\,.$$

Since $\ker(A_0^*-\bar\lambda_0) = \langle \bar u_0\rangle$ we may conclude that

$$\lambda_1 \int_{\mathbb{R}_+} u_0(x)^2dx = i\beta_1 \int_{\mathbb{R}_+} x^2 u_0(x)^2dx\,. \qquad (2.10)$$

Furthermore, as $u_0(x) = Ai(\beta_0^{1/3}e^{i\pi/6}x+\mu_n)$ (respectively $u_0(x) = \overline{Ai(\beta_0^{1/3}e^{i\pi/6}x+\mu_n)}$) for $\beta_0>0$ (respectively $\beta_0<0$), Cauchy Theorem and the decay of Ai in the sector $\{|\arg z|\leq \pi/3\}$ immediately yields

$$\int_{\mathbb{R}_+} u_0(x)^2dx = e^{-i\pi/6}\int_{\mathbb{R}_+} Ai^2(\beta_0^{1/3}x+\mu_n)dx \neq 0\,.$$

Thus, we may select

$$\lambda_1 = i\beta_1 \frac{\int_{\mathbb{R}_+} x^2 u_0(x)^2 dx}{\int_{\mathbb{R}_+} u_0(x)^2 dx} = i\beta_1 e^{-i\pi/3} \frac{\int_{\mathbb{R}_+} x^2 Ai^2(\beta_0^{1/3} x + \mu_n) dx}{\int_{\mathbb{R}_+} Ai^2(\beta_0^{1/3} x + \mu_n) dx}\,, \tag{2.11}$$

and there exists $u_1 \in D(A_0)$ such that

$$(A_0 - \lambda_0) u_1 = -V_1 u_0\,.$$

Assuming that the first k equations are solved by $\lambda_0, \dots, \lambda_{k-1}$, $u_0, \dots, u_{k-1}$, we have to choose such λ_k so that a solution u_k to the $(k+1)$-th equation exists. It easily follows that the solvability condition is

$$-\sum_{j=1}^{k} (V_j - \lambda_j) u_{k-j} \in \ker(A_0^* - \bar{\lambda}_0)^\perp\,,$$

yielding

(2.12)

$$\lambda_k = \frac{1}{\int_{\mathbb{R}_+} u_0(x)^2 dx} \left(\sum_{j=1}^{k-1} \int_{\mathbb{R}_+} \big(i\beta_j x^{j+1} - \lambda_j\big) u_{k-j}(x) u_0(x) dx + i\beta_k \int_{\mathbb{R}_+} x^{k+1} u_0(x)^2 dx \right).$$

For this value of λ_k, there exists $u_k \in \mathcal{D}(A_0)$ satisfying (2.8). Invoking inductive arguments, we assume that each function $u_0, \dots, u_{k-1}$ is in $\mathcal{S}(\mathbb{R}_+)$. Then, it easily follows that $u_k \in \mathcal{S}(\mathbb{R}_+)$. We can then set $u(x,\varepsilon)$ and $\lambda(\varepsilon)$ to be some appropriate Borel sums of the formal series $\sum u_j(x)\varepsilon^j$ and $\sum \lambda_j \varepsilon^j$, respectively.

We now construct from $u(\cdot,\varepsilon)$ a quasimode satisfying the desired boundary conditions. Let $c_0 > 0$ and $\chi \in \mathcal{C}_0^\infty\big((-c_0, c_0); [0,1]\big)$ be such that $\chi(y) = 1$ for all $y \in [-c_0/2, c_0/2]$, and such that χ', χ'' are bounded. We set

$$\chi_\varepsilon(x) = \chi(\varepsilon^{1-\rho} x)\,.$$

Then, for $p = 1, 2$, we have

$$\mathbb{R}_+ \cap \operatorname{Supp} \chi_\varepsilon^{(p)} \subset [c_0 \varepsilon^{\rho-1}/2, c_0 \varepsilon^{\rho-1}]\,, \tag{2.13}$$

and

$$\sup_{x\in\mathbb{R}} \big|\chi_\varepsilon^{(p)}(x)\big| = \mathcal{O}\big(\varepsilon^{p(1-\rho)}\big)\,. \tag{2.14}$$

We next define

$$\psi_\varepsilon(x) = \mathbf{1}_{\mathbb{R}_+}(x) \chi_\varepsilon(x) u(x,\varepsilon)\,.$$

Then, we write

$$\mathcal{A}_\varepsilon = A_0 + \sum_{j=1}^{N} V_j(x) \varepsilon^j + \frac{1}{\varepsilon} R_{N+1}(\varepsilon, x)\,,$$

where R_{N+1} denotes the remainder term in the $(N+1)$-th order Taylor expansion of V near $x=0$ (so that $\varepsilon^{-1}R_{N+1}(\varepsilon x)$ is of order $\mathcal{O}(\varepsilon^{N+1})$).
Then, we have

$$\big(\mathcal{A}_\varepsilon - \lambda(\varepsilon)\big)\psi_\varepsilon = \chi_\varepsilon\big(\mathcal{A}_\varepsilon - \lambda(\varepsilon)\big)u(\cdot,\varepsilon) + [\mathcal{A}_\varepsilon,\chi_\varepsilon]u(\cdot,\varepsilon)\,. \tag{2.15}$$

We seek an estimate for both terms on the right-hand side. Consider the first term, for which we have

$$\big\|\chi_\varepsilon\big(\mathcal{A}_\varepsilon - \lambda(\varepsilon)\big)u(\cdot,\varepsilon)\big\| \leq \left\|\left(A_0 + \sum_{j=1}^N V_j\varepsilon^j - \lambda(\varepsilon)\right)u(\cdot,\varepsilon)\right\| + \big\|\varepsilon^{-1}R_{N+1}(\varepsilon,\cdot)u(\cdot,\varepsilon)\big\|\,. \tag{2.16}$$

By the construction of u and λ, the first term on the right-hand side is of order $\mathcal{O}(\varepsilon^{N+1})$. Furthermore, there exists $c_N>0$ such that

$$\big\|\varepsilon^{-1}R_{N+1}(\varepsilon\cdot)u(\cdot,\varepsilon)\big\| \leq c_N\varepsilon^{N+1}\|x^{N+2}u(\cdot,\varepsilon)\| = \mathcal{O}(\varepsilon^{N+1})\,, \tag{2.17}$$

where we made use of the fact that $u(\cdot,\varepsilon)\in\mathcal{S}(\mathbb{R})$. Therefore, there exists $K_N>0$ such that

$$\big\|\chi_\varepsilon\big(\mathcal{A}_\varepsilon - \lambda(\varepsilon)\big)u(\cdot,\varepsilon)\big\| \leq K_N\varepsilon^{N+1}\,. \tag{2.18}$$

Consider, next, the commutator term in (2.15). Since $u(\cdot,\varepsilon)\in\mathcal{S}(\mathbb{R})$, (2.13) and (2.14) yield

$$\|[\mathcal{A}_\varepsilon,\chi_\varepsilon]u(\cdot,\varepsilon)\| \leq 2\|\chi_\varepsilon'\partial_x u(\cdot,\varepsilon)\| + \|\chi_\varepsilon'' u(\cdot,x)\| = \mathcal{O}(\varepsilon^\infty)\|\psi_\varepsilon\|\,. \tag{2.19}$$

Finally, by (2.15), (2.18) and (2.19), we have

$$\big\|\big(\mathcal{A}_\varepsilon - \lambda(\varepsilon)\big)\psi_\varepsilon\big\| = \mathcal{O}(\varepsilon^\infty)\|\psi_\varepsilon\|\,.$$

∎

2.3. **Proof of Theorem 2.1.** Once the quasimodes associated with the approximate eigenvalues (2.1) have been found, it remains necessary to prove that such eigenvalues indeed exist in $\sigma(\mathcal{A}_h)$.

Lemma 2.4. *Let $n\in\mathbb{N}$ and λ_n be given by the expansion* (2.1)*. Let $\lambda=\lambda_n+re^{i\theta}$ where $\theta\in[0,2\pi)$. Then for $\alpha\in(1,4/3)$, there exist $\delta>0$, $\varepsilon_0>0$ and $C>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$ and r satisfying $\varepsilon^\alpha<r<\delta$, we have*

$$\|(\mathcal{A}_\varepsilon-\lambda)^{-1}\| \leq \frac{C}{r}\,. \tag{2.20}$$

Proof. Let $f\in L^2(0,a/\varepsilon)$ and $u\in D(\mathcal{A}_\varepsilon)$ satisfy

$$(\mathcal{A}_\varepsilon-\lambda)u = f\,. \tag{2.21}$$

Let $\tilde\chi_\varepsilon$ satisfy

$$\chi_\varepsilon^2+\tilde\chi_\varepsilon^2 = 1$$

and

$$\sup_{x\in\mathbb{R}}\big|\nabla\tilde{\chi}_\varepsilon(x)\big| = \mathcal{O}\big(\varepsilon^{(1-\rho)}\big)\,. \tag{2.22}$$

Taking the inner product in $L^2(0,a/\varepsilon)$ of (2.21) with $\tilde{\chi}_\varepsilon^2u$ we obtain from the real part

$$\|\nabla(\tilde{\chi}_\varepsilon u)\|_2^2 = \mathrm{Re}\,\langle\tilde{\chi}_\varepsilon u,\tilde{\chi}_\varepsilon f\rangle + \|u\nabla\tilde{\chi}_\varepsilon\|_2^2 + \mathrm{Re}\,\lambda\|\tilde{\chi}_\varepsilon u\|_2^2\,.$$

Hence,

$$\|\nabla(\tilde{\chi}_\varepsilon u)\|_2 \leq C\big(\varepsilon^{-(1-\rho)}\|\tilde{\chi}_\varepsilon f\|_2 + \|\tilde{\chi}_\varepsilon u\|_2 + \varepsilon^{1-\rho}\|u\|_2\big)\,. \tag{2.23}$$

From the imaginary part of the above inner product we obtain that

$$\langle\tilde{\chi}_\varepsilon(V_\varepsilon-\varepsilon^{-1}V(0))u,\tilde{\chi}_\varepsilon u\rangle = \mathrm{Im}\,\langle\tilde{\chi}_\varepsilon u,\tilde{\chi}_\varepsilon f\rangle + \mathrm{Im}\,\langle\nabla(\tilde{\chi}_\varepsilon u),u\nabla\tilde{\chi}_\varepsilon\rangle + \mathrm{Im}\,\lambda\|\tilde{\chi}_\varepsilon u\|_2^2\,.$$

Since

$$\min_{x\in(0,a/\varepsilon)}|\tilde{\chi}_\varepsilon(V_\varepsilon-\varepsilon^{-1}V(0))| \geq C\varepsilon^{\rho-1}\,,$$

We obtain that

$$\|\tilde{\chi}_\varepsilon u\|_2^2 \leq C\varepsilon^{1-\rho}\big[\|\tilde{\chi}_\varepsilon u\|_2^2 + \|\tilde{\chi}_\varepsilon f\|_2^2 + \varepsilon^{2(1-\rho)}\|\nabla(\tilde{\chi}_\varepsilon u)\|_2^2 + \|u\|_2^2\big]\,.$$

With the aid of (2.23) we then obtain

$$\|\tilde{\chi}_\varepsilon u\|_2 \leq C\varepsilon^{(1-\rho)/2}(\|u\|_2+\|f\|_2)\,. \tag{2.24}$$

We next seek an estimate for $\|\chi_\varepsilon u\|_2$. To this end we write

$$(A_0-\lambda)(\chi_\varepsilon u) = \chi_\varepsilon f - i\Big(V_\varepsilon - \frac{V(0)}{\varepsilon} - \beta_0x\Big)\chi_\varepsilon u + [\mathcal{A}_\varepsilon,\chi_\varepsilon]u\,. \tag{2.25}$$

Denote by v_n the eigenfunction of A_0 associated with the eigenvalue $e^{i\pi/3}\beta_0^{1/3}\mu_n$. For any $g\in L^2(0,a/\varepsilon)$ let

$$\Pi_ng = \langle\bar{v}_n,g\rangle v_n\,.$$

Let further

$$w_n = (I-\Pi_n)(\chi_\varepsilon u)\,.$$

By (2.25) we easily obtain that

$$(A_0-\lambda)w_n = (I-\Pi_n)\Big(\chi_\varepsilon f - i\Big(V_\varepsilon - \frac{V(0)}{\varepsilon} - \beta_0x\Big)\chi_\varepsilon u + [\mathcal{A}_\varepsilon,\chi_\varepsilon]u\Big)\,.$$

By the Riesz-Schauder theory for compact operators (cf. [2] for instance) we have that

$$(A_0-\lambda)^{-1} = \frac{\Pi_n}{\lambda-\lambda_{0,n}} + T_n(\lambda)\,,$$

where $T_n(\lambda)$ is holomorphic, and hence bounded, in some fixed neighborhood of $\lambda_{0,n}$. Consequently, there exists $C(n,\beta_0)$ such that $\|(A_0-\lambda)^{-1}(I-\Pi_n)\|\leq C$, and therefore,

$$\begin{aligned}\|w_n\|_2 &\leq C\Big\|\Big(\chi_\varepsilon f - i\Big(V_\varepsilon - \frac{V(0)}{\varepsilon} - \beta_0x\Big)\chi_\varepsilon u + [\mathcal{A}_\varepsilon,\chi_\varepsilon]u\Big)\Big\|_2 \\ &\leq C\Big(\|f\|_2 + \Big\|\Big(V_\varepsilon - \frac{V(0)}{\varepsilon} - \beta_0x\Big)\chi_\varepsilon u\Big\|_2 + \|[\mathcal{A}_\varepsilon,\chi_\varepsilon]u\|_2\Big)\,.\end{aligned}$$

Hence,

$$\|w_n\|_2 \leq C\big(\|f\|_2 + [\varepsilon^{2\rho-1} + \varepsilon^{2(1-\rho)}]\|u\|_2 + \varepsilon^{1-\rho}\|\nabla u\|_2\big)\,,$$

and since

$$\|\nabla u\|_2^2 = \operatorname{Re}\langle u, f\rangle + \operatorname{Re}\lambda\|u\|_2^2\,, \tag{2.26}$$

we obtain that

$$\|w_n\|_2 \leq C\big(\|f\|_2 + [\varepsilon^{2\rho-1} + \varepsilon^{1-\rho}]\|u\|_2\big)\,. \tag{2.27}$$

To complete the proof, we seek an estimate for $\Pi_n(\chi_\varepsilon u)$. Taking the inner product of (2.25) with $\chi_\varepsilon \bar{v}_n$ yields

$$\begin{aligned}(e^{i\pi/3}\beta_0^{1/3}\mu_n - \lambda)\gamma_n = {}& \langle \bar{v}_n, f\rangle + \langle [A_0,\chi_\varepsilon]\bar{v}_n, \chi_\varepsilon u\rangle - \langle \tilde{\chi}_\varepsilon \bar{v}_n, \tilde{\chi}_\varepsilon f\rangle + \\ & i\Big\langle \bar{v}_n, \Big(V_\varepsilon - \frac{V(0)}{\varepsilon} - \beta_0 x\Big)\chi_\varepsilon u\Big\rangle + \langle \chi_\varepsilon \bar{v}_n, [A_0,\chi_\varepsilon]u\rangle + \\ & (e^{i\pi/3}\beta_0^{1/3}\mu_n - \lambda)\langle \tilde{\chi}_\varepsilon v_n, \tilde{\chi}_\varepsilon u\rangle - i\Big\langle (1-\chi_\varepsilon)\bar{v}_n, \Big(V_\varepsilon - \frac{V(0)}{\varepsilon} - \beta_0 x\Big)\chi_\varepsilon u\Big\rangle\,,\end{aligned} \tag{2.28}$$

where

$$\gamma_n = \langle \bar{v}_n, \chi_\varepsilon u\rangle\,.$$

By the exponential decay of v_n and (2.26) we have that

$$\begin{aligned}\Big|&\langle [A_0,\chi_\varepsilon]\bar{v}_n, \chi_\varepsilon u\rangle - \langle \tilde{\chi}_\varepsilon \bar{v}_n, \tilde{\chi}_\varepsilon f\rangle + (e^{i\pi/3}\beta_0^{1/3}\mu_n - \lambda)\langle \tilde{\chi}_\varepsilon v_n, \tilde{\chi}_\varepsilon u\rangle - \\ & i\Big\langle (1-\chi_\varepsilon)\bar{v}_n, \Big(V_\varepsilon - \frac{V(0)}{\varepsilon} - \beta_0 x\Big)\chi_\varepsilon u\Big\rangle\Big| \leq Ce^{-\varepsilon^{-3(1-\rho)/2}}(\|u\|_2 + \|f\|_2)\,.\end{aligned} \tag{2.29}$$

We next write

$$\begin{aligned}\Big\langle \bar{v}_n, \Big(V_\varepsilon - \frac{V(0)}{\varepsilon} - \beta_0 x\Big)\chi_\varepsilon u\Big\rangle &= \varepsilon\gamma_n\langle \bar{v}_n, \beta_1 x^2 v_n\rangle \\ &+ \Big\langle \bar{v}_n, \Big(V_\varepsilon - \frac{V(0)}{\varepsilon} - \beta_0 x\Big)w_n\Big\rangle + \gamma_n\Big\langle \bar{v}_n, \Big(V_\varepsilon - \frac{V(0)}{\varepsilon} - \beta_0 x - \varepsilon\beta_1 x^2\Big)v_n\Big\rangle\,.\end{aligned}$$

We now observe that

$$\Big\|\bar{v}_n\Big(V_\varepsilon - \frac{V(0)}{\varepsilon} - \beta_0 x\Big)\Big\|_2 \leq C\varepsilon\,,$$

and that

$$\Big|\Big\langle \bar{v}_n, \Big(V_\varepsilon - \frac{V(0)}{\varepsilon} - \beta_0 x - \varepsilon\beta_1 x^2\Big)v_n\Big\rangle\Big| \leq C\varepsilon^2\,.$$

As $|\gamma_n| \leq \|u\|_2$, we obtain with the aid of (2.27) that

$$\Big|\Big\langle \bar{v}_n, \Big(V_\varepsilon - \frac{V(0)}{\varepsilon} - \beta_0 x\Big)\chi_\varepsilon u\Big\rangle - \varepsilon\gamma_n\langle \bar{v}_n, \beta_1 x^2 v_n\rangle\Big| \leq C\varepsilon(\|f\|_2 + [\varepsilon^{2\rho-1} + \varepsilon^{1-\rho}]\|u\|_2)\,.$$

Substituting the above, together with (2.29) into (2.28) yields

$$|(e^{i\pi/3}\beta_0^{1/3}\mu_n + i\varepsilon\gamma_n\langle \bar{v}_n, \beta_1 x^2 v_n\rangle - \lambda)\gamma_n| \leq C(\|f\|_2 + [\varepsilon^{2\rho} + \varepsilon^{2-\rho}]\|u\|_2)$$

Consequently, we must have

$$|\gamma_n| \leq \frac{C}{r}(\|f\|_2 + [\varepsilon^{2\rho} + \varepsilon^{2-\rho}]\|u\|_2)\,. \tag{2.30}$$

We now choose $\rho = 2/3$. Since

$$\|u\|_2 \leq C(|\gamma_n| + \|w_n\|_2 + \|\tilde{\chi}_\varepsilon u\|_2)\,,$$

(2.20) easily follows from (2.24), (2.27), and (2.30). ∎

Lemma 2.5. *Let* $1 < \alpha < 4/3$*. Let further*

$$\Lambda_{n,N}(\varepsilon) = e^{\sigma i\pi/3}|\beta_0|^{2/3}|\mu_n| + \sum_{j=1}^{N} \alpha_{j,n}\varepsilon^j\,. \tag{2.31}$$

Then, for sufficiently small ε *there exists* $\lambda_n(\varepsilon)$ *such that*

$$\sigma(\mathcal{A}_\varepsilon) \cap B(\Lambda_{n,1}, 2\varepsilon^\alpha) = \{\lambda_n(\varepsilon)\}\,. \tag{2.32}$$

Furthermore, the eigenspace associated with $\lambda_n(\varepsilon)$ *is of dimension* 1.

Proof. We follow the same procedure used in [5, 4] to prove existence of eigenvalues. Let $u_{n,N}$ be given by (2.6) and set $\psi_{n,N} = \chi_\varepsilon u_{n,N}$. Let $\varepsilon^\alpha < r < 2\varepsilon^\alpha$ be such that $\partial B(\Lambda_{n,N}, r) \in \rho(\mathcal{A}_\varepsilon)$. Let further $\lambda \in \partial B(\Lambda_{n,N}, r)$. Then, by (2.4) we have

$$(\mathcal{A}_\varepsilon - \lambda)\psi_{n,N} = (\Lambda_{n,N} - \lambda)\psi_{n,N} + \varepsilon^{N+1} f\,,$$

where $\|f\|_2 \leq C$, for some $C > 0$ which is independent of ε. Applying $(\mathcal{A}_\varepsilon - \lambda)^{-1}$ to both sides of the above equation yields

$$(\mathcal{A}_\varepsilon - \lambda)^{-1}\psi_{n,N} = \frac{1}{\Lambda_{n,N} - \lambda}\left[\psi_{n,N} - \varepsilon^{N+1}(\mathcal{A}_\varepsilon - \lambda)^{-1} f\right].$$

Integrating the above identity with respect to λ along $\partial B(\Lambda_{n,N}, r)$ yields

$$P_n\psi_{n,N} = \psi_{n,N} - \oint_{\partial B(\Lambda_{n,N}, r)} \frac{\varepsilon^{N+1}(\mathcal{A}_\varepsilon - \lambda)^{-1} f}{2\pi i(\Lambda_{n,N} - \lambda)}\, d\lambda\,,$$

where P_n is the spectral projection

$$P_n = \frac{1}{2\pi i}\oint_{\partial B(\Lambda_{n,N}, r)} (\mathcal{A}_\varepsilon - \lambda)^{-1}\, d\lambda\,. \tag{2.33}$$

With the aid of (2.20) we then obtain that

$$\|(I - P_n)\psi_{n,N}\|_2 \leq C\varepsilon^{N+1-\alpha}\,. \tag{2.34}$$

By Cauchy Theorem we now readily obtain that

$$\sigma(\mathcal{A}_\varepsilon) \cap B(\Lambda_{n,1}, 2\varepsilon^\alpha) \neq \emptyset\,.$$

We now prove that $P_n L^2(0, a/\varepsilon)$ is one dimensional. To this end suppose that for some $\nu_1, \nu_2 \in B(\Lambda_{n,1}, 2\varepsilon^\alpha)$ (which can be equal or not) and $w_1, w_2 \in D(\mathcal{A}_\varepsilon)$ we have

$$(\mathcal{A}_\varepsilon - \nu_j)w_j = 0 \quad j = 1, 2 \tag{2.35}$$

such that $\|w_1\|_2 = \|w_2\|_2 = 1$ and

$$\langle \bar{w}_1, w_2\rangle = 0\,. \tag{2.36}$$

Let further

$$f_j = (A_0 - \Lambda_{n,0})(\chi_\varepsilon w_j) \quad j = 1, 2\,. \tag{2.37}$$

A simple calculation yields

$$f_j = \chi_\varepsilon(\nu_j - \Lambda_{n,0})w_j - i(V_\varepsilon - \varepsilon^{-1}V(0) - \beta_0 x)\chi_\varepsilon w_j + [A_0, \chi_\varepsilon]w_j \quad j = 1, 2\,. \tag{2.38}$$

We now turn to estimate the various terms on the right-hand-side of (2.38). Let $j \in \{1, 2\}$. For the first term we easily obtain, since $\nu_j \in B(\Lambda_{n,1}, 2\varepsilon^\alpha)$ that

$$\|\chi_\varepsilon(\nu_j - \Lambda_{n,0})w_j\|_2 \leq C\varepsilon\,. \tag{2.39}$$

For the second term we have that

$$\|(V_\varepsilon - \varepsilon^{-1}V(0) - \beta_0 x)\chi_\varepsilon w_j\|_2 \leq C\varepsilon^{1-2\rho}\,. \tag{2.40}$$

To estimate the last term we take the inner product of (2.35) with w_j to obtain from the real part that

$$\|\nabla w_j\|_2 \leq C\,.$$

Consequently, we have that

$$\|[A_0, \chi_\varepsilon]w_j\|_2 \leq \|\Delta\chi_\varepsilon w_j\|_2 + 2\|\nabla\chi_\varepsilon \cdot \nabla w_j\|_2 \leq C\varepsilon^{1-\rho}\,.$$

Substituting the above, together with (2.39) and (2.40) into (2.38) then yields

$$\|f_j\|_2 \leq C\varepsilon^{1-2\rho}\,. \tag{2.41}$$

We now write

$$\chi_\varepsilon w_j = (\chi_\varepsilon w_j)_\| + (\chi_\varepsilon w_j)_\perp\,,$$

where

$$(\chi_\varepsilon w_j)_\| = \langle \bar{u}_0, \chi_\varepsilon w_j\rangle u_0\,.$$

Applying Riesz-Schauder theory to A_0 yields, by (2.37) and (2.38),

$$\|(\chi_\varepsilon w_j)_\perp\| \leq C\varepsilon^{1-2\rho}\,.$$

Consequently,

$$|\langle \chi_\varepsilon \bar{w}_1, \chi_\varepsilon w_2\rangle| \geq 1 - C\varepsilon^{1-2\rho}\,.$$

Hence, by (2.36) we have that

$$|\langle \tilde{\chi}_\varepsilon \bar{w}_1, \tilde{\chi}_\varepsilon w_2\rangle| \geq 1 - C\varepsilon^{1-2\rho}\,. \tag{2.42}$$

To complete the proof we take again the inner product of (2.35) with w_j to obtain, this time from the imaginary part, that

$$\|(V_\varepsilon - \varepsilon^{-1}V(0))w_j\|_2 \leq C\,.$$

Hence,

$$\|w_j\|_{L^2(\varepsilon^{\rho-1}, a/\varepsilon)} \leq C\varepsilon^{1-\rho}\,,$$

from which we easily conclude that

$$|\langle \tilde{\chi}_\varepsilon \bar{w}_1, \tilde{\chi}_\varepsilon w_2\rangle| \leq \|w_1\|_{L^2(\varepsilon^{\rho-1}, a/\varepsilon)}\|w_2\|_{L^2(\varepsilon^{\rho-1}, a/\varepsilon)} \leq C\varepsilon^{2(1-\rho)}\,,$$

contradicting (2.42) and therefore (2.36). ∎

Proof of Theorem 2.1 . Recall that by (2.4) we have

$$(\mathcal{A}_\varepsilon - \Lambda_{n,N})\psi_{n,N} = \varepsilon^{N+1} f\,,$$

where $\|f\|_2$ is uniformly bounded as $\varepsilon \to 0$. We now apply the spectral projection P_n, defined in (2.33) to both side of the above equations. It can be easily verified that $[P_n, \mathcal{A}_\varepsilon] = 0$. Consequently

$$(\mathcal{A}_\varepsilon - \Lambda_{n,N})P_n\psi_{n,N} = \varepsilon^{N+1} P_n f\,. \tag{2.43}$$

By (2.32) we have that

$$(\mathcal{A}_\varepsilon - \Lambda_{n,N})P_n\psi_{n,N} = (\lambda_n - \Lambda_{n,N})P_n\psi_{n,N}\,. \tag{2.44}$$

By (2.34) we have that

$$\|P_n\psi_{n,N}\|_2 \geq 1 - C\varepsilon^{N+1}\,.$$

Substituting the above, together with (2.44) into (2.43) then yields

$$|\lambda_n - \Lambda_{n,N}| \leq C\varepsilon^{N+1}$$

Theorem 2.1 now easily follows from (2.3) ∎

3. Two dimensions: Quasimode construction

Let $\Omega \subset\subset \mathbb{R}^2$ be a C^3 domain and $V \in C^3(\bar{\Omega})$. Let $\partial\Omega_\perp$ denote the portion of the boundary $\partial\Omega$ where ∇V is orthogonal to $\partial\Omega$. (Note that $\partial\Omega_\perp$ may be finite, but is never empty by the continuity of V on $\partial\Omega$.) Let $x_0 \in \partial\Omega_\perp$ such that

$$|\nabla V(x_0)| = \min_{x\in\partial\Omega_\perp} |\nabla V(x)|\,,$$

and let $V_0 = V(x_0)$. We look for an approximation of the principal eigenvalue and the corresponding eigenfunction of the operator

$$\mathcal{A}_h = -h^2\Delta + i(V - V_0)\,, \tag{3.1}$$

defined over

$$D(\mathcal{A}_h) = H^1_0(\Omega, \mathbb{C}) \cap H^2(\Omega, \mathbb{C})\,.$$

Define in a vicinity of $\partial\Omega$ a curvilinear coordinate system (t, s) such that $t = d(x, \partial\Omega)$ and $s(x)$ denotes the distance (or arclength) along $\partial\Omega$ connecting x_0 and the projection of x on $\partial\Omega$. We have

$$\Delta = \Big(\frac{1}{g}\frac{\partial}{\partial s}\Big)^2 + \frac{1}{g}\frac{\partial}{\partial t}\Big(g\frac{\partial}{\partial t}\Big)\,, \tag{3.2}$$

where

$$g = 1 - t\kappa(s)\,, \tag{3.3}$$

and $\kappa(s)$ is the curvature at s on $\partial\Omega$. Expanding Δ near x_0 ($t^2+s^2 \ll 1$) yields for some $u \in D(\mathcal{A}_h)$

$$\Delta u = u_{tt} + u_{ss} + \Upsilon u \,, \tag{3.4}$$

where

$$\Upsilon u = \Big(\frac{1}{g^2}-1\Big)u_{ss} + \frac{t\kappa'}{g^3}u_s - \frac{\kappa}{g}u_t \,. \tag{3.5}$$

We next expand V near x_0

$$V(s,t) - V_0 = ct + \frac{1}{2}(\alpha s^2 + \beta t^2 + 2\sigma st) + \mathcal{O}((s^2+t^2)^{3/2}) \,, \tag{3.6}$$

where

$$c = V_t(x_0) \quad ; \quad \alpha = V_{ss}(x_0) \quad ; \quad \beta = V_{tt}(x_0) \quad ; \quad \sigma = V_{st}(x_0) \,.$$

We note that $V_s(x_0)=0$ since $x_0 \in \partial\Omega_\perp$. We confine the discussion, in view of (1.3) to the case where $\alpha c > 0$. Without any loss of generality we may assume $c > 0$ (and hence $\alpha > 0$ as well), otherwise we can consider the spectrum of the complex conjugate of $\mathcal{A}_h$.

We search for an approximate eigenpair (u,λ) of $\mathcal{A}_h$. Previous works [3, 10] suggest that one should look for such u which is localized near x_0. Applying the transformation

$$\tau = \Big(\frac{c}{h^2}\Big)^{1/3} t \quad ; \quad \xi = \Big(\frac{\alpha}{h^2}\Big)^{1/4} s \tag{3.7}$$

to (3.6) and (3.4) leads to the following approximation for every $u \in D(\mathcal{A}_h)$

$$\frac{\alpha}{\varepsilon c^2}\mathcal{A}_h u = -u_{\tau\tau} + i\tau u + \varepsilon^{1/2}\Big(-u_{\xi\xi} + \frac{i}{2}\xi^2 u\Big) + \Big(\frac{\varepsilon}{\alpha}\Big)^{3/4} i\sigma\xi\tau u + Ru \,, \tag{3.8}$$

where

$$\varepsilon = \alpha (h^2/c^4)^{1/3} \,, \tag{3.9}$$

$\|u\|_2 = 1$, and the operator R satisfies, for all $u \in D(\mathcal{A}_h)$

$$\begin{aligned} Ru = c^{2/3}\Big(\frac{\varepsilon}{\alpha}\Big)^{1/2}\Big(\frac{1}{g^2}-1\Big)u_{\xi\xi} + c^{2/3}\Big(\frac{\varepsilon}{\alpha}\Big)^{9/4}\frac{\tau c^{1/3}\kappa'}{g^3}u_\xi - \Big(\frac{\varepsilon}{\alpha}\Big)\frac{c^{1/3}\kappa}{g}u_\tau + \\ i\frac{\alpha}{\varepsilon c^2}\Big(V(\xi,\tau) - V_0 - \frac{\varepsilon}{\alpha}c^2\tau - \frac{c^2\varepsilon^{3/2}}{\alpha}\frac{1}{2}\xi^2 - \Big(\frac{\varepsilon}{\alpha}\Big)^{7/4}c^2\sigma\xi\tau\Big) \,. \end{aligned} \tag{3.10}$$

It can be easily verified that for any $0 < \gamma < 1$ we have

$$\begin{aligned} \|Ru\|_{L^2(B_+(0,\varepsilon^{-\gamma}))} \leq C\varepsilon\Big[\|\varepsilon^{1/2}|\tau u_{\xi\xi}| + \varepsilon^{5/4}|\tau u_\xi| + |u_\tau|\|_{L^2(B_+(0,\varepsilon^{-\gamma}))} + \\ C\varepsilon\Big[\|\tau^2 u\|_{L^2(B_+(0,\varepsilon^{-\gamma}))} + \varepsilon^{1/4}\|\xi^3 u\|_{L^2(B_+(0,\varepsilon^{-\gamma}))}\Big] \,. \end{aligned} \tag{3.11}$$

We seek an approximate solution for $\mathcal{A}_h u = \lambda u$. To this end, we introduce the expansion

$$u \cong u_0 + \varepsilon^{1/4}u_1 + \varepsilon^{1/2}u_2 + \varepsilon^{3/4}u_3 + \mathcal{O}(\varepsilon) \quad ; \quad \frac{\alpha}{\varepsilon c^2}\lambda = \lambda_0 + \varepsilon^{1/4}\lambda_1 + \varepsilon^{1/2}\lambda_2 + \varepsilon^{3/4}\lambda_3 + \mathcal{O}(\varepsilon) \,.$$

Substituting into (3.8) leads to the following $\mathcal{O}(1)$ balance

$$\mathcal{L}_\tau u_0 \overset{def}{=} -\frac{\partial^2 u_0}{\partial \tau^2} + i\tau u_0 = \lambda_0 u_0 \quad ; \quad u_0(0,\xi) = 0\,, \tag{3.12a}$$

where the operator $\mathcal{L}_\tau$ is defined over

$$D(\mathcal{L}_\tau) = \{u \in H^2(\mathbb{R}_+,\mathbb{C}) \cap H^1_0(\mathbb{R}_+,\mathbb{C}) \,|\, \tau u \in L^2(\mathbb{R},\mathbb{C})\}\,. \tag{3.12b}$$

The solution to (3.12) associated with the energy λ_0 having the smallest real part is given by

$$u_0(\tau,\xi) = v_0(\tau) w_0(\xi) \quad \text{where} \quad v_0(\tau) = A_i(e^{i\pi/6}\tau + \mu_1)\,, \tag{3.13}$$

and

$$\lambda_0 = e^{-i2\pi/3}\mu_1\,, \tag{3.14}$$

where A_i is Airy's function and $\mu_1 < 0$ is its rightmost zero. The function $w_0(\xi)$ will be determined from the $\mathcal{O}(\varepsilon^{1/2})$ balance.

The next order, or $\mathcal{O}(\varepsilon^{1/4})$, balance in (3.8) assumes the form

$$(\mathcal{L}_\tau - \lambda_0) u_1 = \lambda_1 u_0 \quad ; \quad u_1(0,\xi) = 0\,, \tag{3.15}$$

Taking the inner product of (3.15) with $\bar{v}_0$ yields $\lambda_1 = 0$. Hence, $u_1 = v_0(\tau) w_1(\xi)$.

The next order, or $\mathcal{O}(\varepsilon^{1/2})$, balance in (3.8) assumes the form

$$(\mathcal{L}_\tau - \lambda_0) u_2 = -\Big(\mathcal{L}_\xi - \lambda_2\Big) u_0 \quad ; \quad u_2(0,\xi) = 0\,, \tag{3.16}$$

where

$$\mathcal{L}_\xi = -\frac{\partial^2}{\partial \xi^2} + \frac{i}{2}\xi^2\,, \tag{3.17}$$

is defined over

$$D(\mathcal{L}_\xi) = \{u \in H^2(\mathbb{R},\mathbb{C}) \,|\, \xi^2 u \in L^2(\mathbb{R},\mathbb{C})\}$$

For fixed ξ we now take the inner product of the above equation with $\bar{v}_0$, in $L^2(\mathbb{R}_+)$. After noticing that by Cauchy's Theorem

$$\int_0^\infty v_0^2(\tau)\, d\tau = e^{-i\pi/6} \int_0^\infty A_i^2(x + \mu_1)\, dx \neq 0\,, \tag{3.18}$$

we obtain

$$(\mathcal{L}_\xi - \lambda_2) w_0 = 0\,.$$

The solution of the above problem corresponding to the λ_2 with smallest real part is given by

$$w_0(\xi) = C_0 \exp\Big\{ -\frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \xi^2 \Big\} \quad ; \quad \lambda_2 = \sqrt{2} e^{i\frac{\pi}{4}}\,. \tag{3.19}$$

The constant C_0 should be obtain, up to a product by -1, from the normalization condition $\|u\|_2 = 1$. We allow dependence of C_0 on ε (see below). Substituting into (3.16) yields

$$u_2(\tau,\xi) = v_0(\tau) w_2(\xi)\,.$$

For the $\mathcal{O}(\varepsilon^{3/4})$ balance in (3.8) we have

$$(\mathcal{L}_\tau - \lambda_0)u_3 = -v_0(\mathcal{L}_\xi - \lambda_2)w_1 - \Big(i\sigma\xi\tau - \lambda_3\Big)v_0w_0 \quad ; \quad u_2(0,\xi) = 0\,.$$

We take once again the inner product of the above balance with $\bar{v}_0$ to obtain

$$(\mathcal{L}_\xi - \lambda_2)w_1 + \Big(i\gamma\xi - \lambda_3\Big)w_0 = 0\,, \tag{3.20}$$

where

$$\gamma = \sigma\frac{\int_0^\infty \tau v_0^2(\tau)\,d\tau}{\int_0^\infty v_0^2(\tau)\,d\tau}\,.$$

Note that this expression is well-defined due to (3.18). Taking the inner product, this time in $L^2(\mathbb{R},\mathbb{C})$, of (3.20) with w_0, which is even, yields

$$\lambda_3 = 0\,.$$

Furthermore, w_1 is the unique solution of

$$(\mathcal{L}_\xi - \lambda_2)w_1 = -i\gamma\xi w_0 \quad ; \quad \int_{\mathbb{R}} w_1(\xi)w_0(\xi)\,d\xi = 0\,,$$

and

$$u_3 = v_3(\xi,\tau) + v_0(\tau)w_3(\xi)\,,$$

where v_3 is the unique solution of the problem

$$\begin{cases} (\mathcal{L}_\tau - \lambda_0)v_3 = -i\xi(\tau-\gamma)v_0w_0 & \tau > 0 \\ v_3(0,\xi) = 0 \\ \int_0^\infty v_2(\tau,\xi)v_0(\tau)d\tau = 0\,. \end{cases} \tag{3.21}$$

Notice that, if $\mathcal{S}(\mathbb{R}^2_+)$ denotes the Schwartz space of rapidly decaying functions along with all their derivatives, then the right-hand side in (3.21) belongs to $\mathcal{S}(\mathbb{R}^2_+)$. As the operator $-\partial^2/\partial\tau^2 + i\tau - \lambda_0$ is globally elliptic with respect to τ, in the sense of [8, Definition 1.5.6], we have that

$$v_3 \in \mathcal{S}(\mathbb{R}^2_+)\,, \tag{3.22}$$

(see [8, Theorem 1.6.4]). For the same reason, the $\mathcal{O}(\varepsilon)$ balance would yield $w_3 \in \mathcal{S}(\mathbb{R})$.

We have thus obtained the quasimode

$$U = \Big(C_0(\varepsilon)\exp\Big\{-\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\xi^2\Big\} + \varepsilon^{1/2}w_1(\xi)\Big)A_i(e^{i\pi/6}\tau + \mu_1) + \varepsilon^{3/4}v_3(\xi,\tau) + \varepsilon^{3/4}w_3(\xi)A_i(e^{i\pi/6}\tau + \mu_1)\,. \tag{3.23}$$

We obtain the various constants by requiring that

$$\|U\|_2 = 1\,.$$

We now conclude this section by the following proposition

Proposition 3.1. *Let $\mathcal{A}_h$ be given by* (3.1) *and U by* (3.23). *Let further*

$$\Lambda = \lambda_0 + \varepsilon^{1/2}\lambda_2 \,. \tag{3.24}$$

Let $\eta_r = \eta_r^0(\tau)\eta_r^1(\xi)$, where $\eta_r^0 \in C^\infty(\mathbb{R}_+,[0,1])$ and $\eta_r^1 \in C^\infty(\mathbb{R},[0,1])$ are chosen so that

$$\eta_r = \begin{cases} 1 & |x-x_0| < r \\ 0 & |x-x_0| > 2r \,, \end{cases} \quad |\nabla \eta_r| \leq \frac{C}{r} \,. \tag{3.25}$$

Then,

$$\Big\| \Big(\frac{\alpha}{\varepsilon c^2}\mathcal{A}_h - \Lambda \Big)(\eta_{\varepsilon^{-1/2}}U) \Big\|_2 \leq C\varepsilon \|\eta_{\varepsilon^{-1/2}}U\|_2 \,. \tag{3.26}$$

Proof. We first write

$$\begin{aligned} \frac{\alpha}{\varepsilon c^2}\mathcal{A}_h(\eta_{\varepsilon^{-1/2}}U) &= \big(\mathcal{L}_\tau + \varepsilon^{1/2}\mathcal{L}_\xi + \varepsilon^{3/4} i\sigma\xi\tau\big)(\eta_{\varepsilon^{-1/2}}U) + R\eta_{\varepsilon^{-1/2}}U \\ &= \Lambda \eta_{\varepsilon^{-1/2}}U + \Big[\mathcal{L}_\tau + \varepsilon^{1/2}\mathcal{L}_\xi, \eta_{\varepsilon^{-1/2}}\Big]U + R\eta_{\varepsilon^{-1/2}}U \,, \end{aligned} \tag{3.27}$$

where the operator R is defined by (3.10). We next seek an estimate for the commutator term in (3.27), given by

$$[\mathcal{L}_\tau, \eta_{\varepsilon^{-1/2}}]U = -\partial_\tau^2(\eta_{\varepsilon^{-1/2}})U - 2\partial_\tau \eta_{\varepsilon^{-1/2}}\partial_\tau U \,. \tag{3.28}$$

In order to estimate the norm of U and $\partial_\tau U$ on the support of $\partial_\tau^2\eta_{\varepsilon^{-1/2}}$ and $\partial_\tau \eta_{\varepsilon^{-1/2}}$, we recall the well-known asymptotic behavior of the Airy function [1]:

$$Ai(z) = \frac{1}{2\sqrt{\pi} z^{1/4}} e^{-\frac{2}{3}z^{3/2}} \big(1 + \mathcal{O}(z^{-3/2})\big) \tag{3.29}$$

as $|z| \to +\infty$ in any sector of the form $|\arg z| \leq \pi - \delta$, $\delta > 0$. By (3.23), and since for all $(\tau,\xi) \in \operatorname{Supp} \partial_\tau \eta_{\varepsilon^{-1/2}}$ we have $\varepsilon^{-1/2} \leq \tau \leq 2\varepsilon^{-1/2}$, (3.22) and (3.29) yield

$$\|(\partial_\tau^2 \eta_{\varepsilon^{-1/2}})U\|_2 \leq C_1 \varepsilon \,,$$

for some positive constant C_1.

Since the asymptotic behaviour of Ai', as $|z| \to \infty$ is not substantially different from (3.29) (cf. [1]), we easily obtain that

$$\|\partial_\tau \eta_{\varepsilon^{-1/2}} \partial_\tau U\|_2 \leq C_2 \varepsilon \,, \ C_2 > 0 \,.$$

Thus (3.28) yields, for some $C > 0$,

$$\|[\mathcal{L}_\tau, \eta_{\varepsilon^{-1/2}}]U\|_2 \leq C\varepsilon \,. \tag{3.30}$$

Due to the decay of the U and $\partial_\xi U$ as $|\xi| \to +\infty$ (recall that $w_3 \in \mathcal{S}(\mathbb{R})$), we similarly obtain

$$\|[\varepsilon^{1/2}\mathcal{L}_\xi, \eta_{\varepsilon^{-1/2}}]U\|_2 \leq K\varepsilon \,, \tag{3.31}$$

for some $K > 0$ .can be estimated as follows. Using

To estimate the remaining term $R\eta_{\varepsilon^{-1/2}}U$ we use (3.11) to obtain, for $\alpha \in (1/2, 1)$,

$$\|R\eta_{\varepsilon^{-1/2}}U\|_2 \leq \|RU\|_{L^2(B_+(0,\varepsilon^{-\alpha}))} \leq C'\varepsilon \tag{3.32}$$

for some $C' > 0$. Finally (3.27), (3.30), (3.31) and (3.32) yield, for some positive $\tilde{C}$ and C,

$$\begin{aligned}\Big\|\Big(\frac{\alpha}{\varepsilon c^2}\mathcal{A}_h-\Lambda\Big)(\eta_{\varepsilon^{-1/2}}U)\Big\|_2 &\leq C'\varepsilon\\ &\leq C\varepsilon\|\eta_{\varepsilon^{-1/2}}U\|_2\,,\end{aligned}$$

where we have used the that for some $C'' > 0$, $\|\eta_{\varepsilon^{-1/2}}U\|_2 \geq 1/C''$. ∎

4. Eigenvalue existence

Let $\mathcal{L}_\tau$ and $\mathcal{L}_\xi$ be respectively defined by (3.12) and (3.17). Then let

$$\mathcal{B}_\varepsilon = \mathcal{L}_\tau + \varepsilon^{1/2}\mathcal{L}_\xi \tag{4.1}$$

be the closed operator associated with the quadratic form

$$\langle\nabla u,\nabla v\rangle + i\langle u,(\tau+\varepsilon^{1/2}\xi^2)v\rangle$$

whose domain is given by $\tilde{V}\times\tilde{V}$ where

$$\tilde{V} = \{u\in H^1_0(\mathbb{R}^2_+,\mathbb{C})\,|\,|(\tau^{1/2}+|\xi|)u\in L^2(\mathbb{R}^2_+,\mathbb{C})\}\,.$$

It can be easily verified that

$$D(\mathcal{B}_\varepsilon) = \{u\in H^2(\mathbb{R}^2_+,\mathbb{C})\cap H^1_0(\mathbb{R}^2_+)\,|\,(\tau+\xi^2)u\in L^2(\mathbb{R}^2_+),\}\,.$$

We begin by the following straightforward observation

Lemma 4.1. *We have*

$$\sigma(\mathcal{B}_\varepsilon) = \{c^{2/3}\mu_n e^{-i2\pi/3} + (2k-1)\varepsilon^{1/2}\sqrt{2}e^{i\frac{\pi}{4}}\}_{n,k=1}^\infty\,. \tag{4.2}$$

Proof. After the scale changes $\tau\mapsto c^{1/3}\tau$ and $\xi\mapsto\big(|\alpha|/2\big)^{1/4}\xi$, we obtain from [3] and [7, Section 14.5] the following expressions for the eigenvalues of the complex Airy operator $\mathcal{L}_\tau$ and the complex harmonic oscillator $\mathcal{L}_\xi$:

$$\sigma(\mathcal{L}_\tau) = \Big\{c^{2/3}\mu_n e^{-i2\pi/3} : n\geq 1\Big\}\,,$$

μ_n being the n-th (negative) zero of the Airy function Ai, and

$$\sigma(\mathcal{L}_\xi) = \Big\{(2k-1)\sqrt{2}\,e^{i\frac{\pi}{4}} : k\geq 1\Big\}\,.$$

Denote by $\mathcal{L}_\tau \dotplus \varepsilon^{1/2}\mathcal{L}_\xi$ the closure of the operator $\mathcal{L}_\tau\otimes I + I\otimes(\varepsilon^{1/2}\mathcal{L}_\xi)$ whose domain is $D(\mathcal{L}_\tau)\otimes D(\mathcal{L}_\xi)$. We first need to verify that the domains of $\mathcal{B}_\varepsilon$ and $\mathcal{L}_\tau \dotplus \varepsilon^{1/2}\mathcal{L}_\xi$ coincide. Let $e^{-t\mathcal{B}_\varepsilon}$ denote the contraction semigroup generated by $\mathcal{B}_\varepsilon$, and let $\varphi\in D(\mathcal{L}_\tau)$, $\psi\in D(\mathcal{L}_\xi)$. Clearly,

$$e^{-t\mathcal{B}_\varepsilon}(\varphi\otimes\psi) = e^{-t\mathcal{L}_\tau}\varphi\otimes e^{-t(\varepsilon^{1/2}\mathcal{L}_\xi)}\psi\,,$$

where $e^{-t\mathcal{L}_\tau}$ and $e^{-t(\varepsilon^{1/2}\mathcal{L}_\xi)}$ denote respectively the contraction semigroups generated by $\mathcal{L}_\tau$ and $\varepsilon^{1/2}\mathcal{L}_\xi$.Thus,

$$e^{-t\mathcal{B}_\varepsilon}\big(D(\mathcal{L}_\tau)\otimes D(\mathcal{L}_\xi)\big)\subset D(\mathcal{L}_\tau)\otimes D(\mathcal{L}_\xi)\,.$$

Consequently, due to [11, Theorem X.49] we have $\mathcal{B}_\varepsilon = \overline{(\mathcal{B}_\varepsilon)_{|D(\mathcal{L}_\tau)\otimes D(\mathcal{L}_\xi)}}$, and $\mathcal{B}_\varepsilon$ clearly coincides with $\mathcal{L}_\tau \otimes I + I \otimes (\varepsilon^{1/2}\mathcal{L}_\xi)$ on $D(\mathcal{L}_\tau)\otimes D(\mathcal{L}_\xi)$, and hence $\mathcal{B}_\varepsilon = \mathcal{L}_\tau \dotplus \varepsilon^{1/2}\mathcal{L}_\xi$.

Noticing that $\mathcal{L}_\tau$ and $\mathcal{L}_\xi$ are both sectorial with respect to the same sector $\mathcal{S} = \{z \in \mathbb{C} : 0 \leq \arg z \leq \pi/2\}$, we can then apply the so-called Ichinose Lemma (see [11, Theorem XIII.35, Corollary 2]) which yields

$$\sigma\big(\mathcal{L}_\tau \dotplus \varepsilon^{1/2}\mathcal{L}_\xi\big) = \sigma(\mathcal{L}_\tau) + \sigma(\varepsilon^{1/2}\mathcal{L}_\xi)\,,$$

and (4.2) follows. ∎

The following auxiliary lemma will be necessary in the sequel

Lemma 4.2. *Let v_n denote the (unique up to multiplication by a complex number of modulus 1) unity norm eigenfunction associated with the eigenvalue*

$$\nu_{n-1} = \mu_n e^{-i2\pi/3} \quad n \in \mathbb{N} \tag{4.3}$$

of $\mathcal{L}_\tau$. Let further $\mathcal{V}$ denote the form domain of $\mathcal{L}_\tau$, i.e,

$$\mathcal{V} = \{u \in H_0^1(\mathbb{R}_+,\mathbb{C}) \,|\, \tau^{1/2}u \in L^2(\mathbb{R}_+,\mathbb{C})\}\,,$$

and $\mathcal{V}_n = \mathrm{span}\{v_n\}_{n=k+1}^\infty \cap \mathcal{V}$. Set

$$\beta_k = \inf_{\substack{u\in\mathcal{V}_n\\ \|u\|=1}} \|u_\tau\|_2^2 + \|\tau^{1/2}u\|_2^2\,. \tag{4.4a}$$

Then,

$$\beta_k \to \infty\,. \tag{4.4b}$$

Proof. Let us assume by contradiction that there exists a subsequence (k_n) and a positive constant C such that

$$\sup_{n\in\mathbb{N}} \beta_{k_n} \leq C\,.$$

Then there exists a sequence (u_n) of functions in $H_0^1(\mathbb{R}_+,\mathbb{C})$, $\tau^{1/2}u_n \in L^2(\mathbb{R}_+,\mathbb{C})$ such that, for all $n \in \mathbb{N}$, $u_n \in \mathrm{span}\{v_j\}_{j=k_n+1}^\infty$, $\|u_n\|_2 = 1$ and

$$\sup_{n\in\mathbb{N}} \big(\|\partial_\tau u_n\|_2^2 + \|\tau^{1/2}u_n\|_2^2\big) \leq 2C\,. \tag{4.5}$$

Since for any $r > 0$ we have

$$\int_r^\infty |u_n|^2 \leq \frac{1}{r}\int_r^\infty \tau|u_n|^2 \leq \frac{2C}{r}\,,$$

we can choose such r for which

$$\int_0^r |u_n|^2 \geq \frac{1}{2}\,.$$

Since by (4.5) the $H^1(\mathbb{R}_+,\mathbb{C})$ norms of $\{u_n\}_{n=1}^\infty$ are bounded, we can extract a subsequence $(u_{\varphi(n)})$ such that $u_{\varphi(n)} \to u_\infty$ in $L^2(\mathbb{R}_+,\mathbb{C})$ weakly, and in $L^2([0,r],\mathbb{C})$ strongly, for some limit function $u_\infty \in L^2(\mathbb{R}_+,\mathbb{C})$. We note that

$$\int_0^r |u_\infty|^2 \geq \frac{1}{2}\,. \tag{4.6}$$

Now let $k \in \mathbb{N}$ be fixed. Then for all n such that $k_{\varphi(n)} \geq k$ we have

$$u_{\varphi(n)} \in \operatorname{span}\{v_j\}_{j\geq k+1} = \big(\operatorname{span}\{\bar{v}_n\}_{n=1}^k\big)_\perp\,,$$

hence, by the weak convergence in $L^2(\mathbb{R}_+,\mathbb{C})$.

$$0 = \langle u_{\varphi(n)}, \bar{v}_k\rangle \longrightarrow \langle u_\infty, \bar{v}_k\rangle = 0\,.$$

Consequently $u_\infty \in \big(\operatorname{span}\{\bar{v}_j\}_{j=1}^{+\infty}\big)_\perp$, thus $u_\infty = 0$ since the eigenfunctions $\{\bar{v}_j\}_{j\geq 1}$ of $\mathcal{L}_\tau^*$ form a complete family of $L^2(\mathbb{R}_+,\mathbb{C})$ (see [3]). A contradiction, in view of (4.6). ∎

We next claim the following

Lemma 4.3. *There exist $r_0 > 0$, $\varepsilon_0 > 0$ and $C > 0$, such that if $r \in (0, r_0)$, then*

$$|\lambda - \lambda_0 - \varepsilon^{1/2}\lambda_2| = r\varepsilon^{1/2} \Rightarrow \|(\mathcal{B}_\varepsilon - \lambda)^{-1}\| \leq \frac{C}{r}\varepsilon^{-1/2} \quad \forall 0 < \varepsilon < \varepsilon_0\,. \tag{4.7}$$

Proof. Suppose that r is so chosen such that $\partial B(\lambda_0 + \varepsilon^{1/2}\lambda_2, r\varepsilon^{1/2}) \in \rho(\mathcal{B}_\varepsilon)$. Let $g \in \operatorname{span}\{v_n w_m\}_{n,m=0}^\infty$ and w denote the solution of

$$(\mathcal{B}_\varepsilon - \lambda)w = g\,. \tag{4.8}$$

Let further

$$\lambda - \lambda_0 - \varepsilon^{1/2}\lambda_2 = \varepsilon^{1/2} r e^{i\alpha}\,,$$

where $\alpha \in [0, 2\pi)$. By the Riesz-Schauder Theory (cf. [2, Eq. (16.4)] for instance) we have that

$$(\mathcal{L}_\tau - \lambda)^{-1} = \frac{\Pi_0}{\lambda - \nu_0} + \sum_{k=1}^K \frac{\Pi_k}{\lambda - \nu_k} + T_k(\lambda)\,, \tag{4.9}$$

where $\{\nu_n\}_{n=0}^\infty$ are given by (4.3), and $\|T_K\| \leq C_K$ in $B(\nu_0, \tilde{r})$ for some fixed $\tilde{r} > 0$. In the above Π_k is the projection operator on $\operatorname{span}\{v_k\}$, which is explicitly given, for any $u \in \operatorname{span}\{v_n\}_{n=0}^\infty$, by

$$\Pi_k(u) = \langle \bar{v_k}, u\rangle_\tau v_k(\tau)\,,$$

where $\langle\cdot,\cdot\rangle_\tau$ denotes the standard $L^2(\mathbb{R}_+,\mathbb{C})$ inner product.

Let $u_k = \Pi_k(w)$. It can be easily verified that

$$u_k = \varepsilon^{-1/2}(\mathcal{L}_\xi - \lambda_2 - re^{i\alpha} + \varepsilon^{-1/2}(\nu_k - \nu_0))^{-1}\Pi_k(g)\,.$$

It easily follows from here that

$$\|u_0\|_2 \leq \frac{C}{r\varepsilon^{1/2}}\|\Pi_0(g)\|_2 \leq \frac{C}{r\varepsilon^{1/2}}\|g\|_2\,, \tag{4.10}$$

whereas

$$\|u_k\|_2 \leq C_k\|g\|_2\,, \tag{4.11}$$

where C_k is independent of r and ε. For every $K\geq 1$ we have

$$\|w\|_2 \leq \Big(\frac{C}{r\varepsilon^{1/2}}+\sum_{k=1}^K C_k\Big)\|g\|_2 + \|P_K(w)\|_2\,, \tag{4.12}$$

where

$$P_K = I - \sum_{k=0}^K \Pi_k\,. \tag{4.13}$$

To complete the proof we need an estimate for $\|P_K(w)\|_2$. Let then $u_K = P_K(w)$. Clearly,

$$(\mathcal{B}_\varepsilon-\lambda)u_K = P_K(g)\,.$$

Taking the inner product of the above equation by u_K yields

$$\begin{aligned}\Big\|\frac{\partial u_K}{\partial\tau}\Big\|_2^2+\varepsilon^{1/2}\Big\|\frac{\partial u_K}{\partial\xi}\Big\|_2^2-\operatorname{Re}\lambda\|u_K\|_2^2 &= \operatorname{Re}\langle u_K,P_K(g)\rangle\\ \|\tau^{1/2}u_K\|_2^2+\varepsilon^{1/2}\|\xi u_K\|_2^2-\operatorname{Im}\lambda\|u_K\|_2^2 &= \operatorname{Im}\langle u_K,P_K(g)\rangle\,.\end{aligned}$$

Combining the above equations yields

$$\Big\|\frac{\partial u_K}{\partial\tau}\Big\|_2^2+\|\tau^{1/2}u_K\|_2^2-(\operatorname{Im}\lambda+\operatorname{Re}\lambda)\|u_K\|_2^2 \leq 2\|u_K\|_2\|P_K(g)\|_2\,. \tag{4.14}$$

As

$$\|P_K(g)\|_2 \leq C_K\|g\|_2\,, \tag{4.15}$$

we obtain by (4.4) and (4.14) that for sufficiently large K (but independent of ε)

$$\|u_K\|_2 \leq C_K\|g\|_2\,.$$

The lemma is now proved by the above and (4.12) for any $g\in\operatorname{span}\{v_nw_m\}_{n,m=0}^\infty$, and hence for any $g\in L^2(\mathbb{R}^2_+,\mathbb{C})$ via a density argument. ∎

Note that r may depend on ε. As a matter of fact (4.7) remains valid independently of the pace at which $r\to 0$ as $\varepsilon\to 0$.

Corollary 4.4. *Under the conditions of 4.3 we have that*

$$\|(\mathcal{B}_\varepsilon-\lambda)^{-1}P_1\| \leq C\,, \tag{4.16}$$

where C is independent of ε.

The corollary follows immediately from (4.11) and (4.15).

Recall now the definition of $\mathcal{S}$ from the introduction

$$\mathcal{S} = \{x\in\partial\Omega_\perp\,:\,|\nabla V(x)| = |\nabla V(x_0)|\,,\ V(x)=V(x_0)\}\,.$$

By (1.3), $\mathcal{S}$ is a finite set of isolated points $\{x_j\}_{j\in J_{\mathcal{S}}}$. Recall the definition of the curvilinear coordinate system (s,t) from the previous section, and then let $x_j = (s_j,0)$. Let further $f \in L^\infty(\Omega,\mathbb{C})$ be supported on $\Omega \cap \bigcup_{j\in J_{\mathcal{S}}} B(x_j,\delta)$ and satisfy

$$|f| \leq C\|f\|_2 \varepsilon^{7/8} e^{-\gamma_1\varepsilon^{-3/2}[(s-s_j)^2+t^{3/2}]} \quad \text{in } B(x_j,\delta)\cap\Omega \quad \forall j \in J_{\mathcal{S}}\,, \tag{4.17}$$

for some fixed and positive γ_1 and C.

We seek an estimate for the resolvent of $\mathcal{A}_h$. To this end a few auxiliary estimates, beyond (4.7), are necessary. Set then

$$\Omega_+ = \{x\in\Omega \,|\, V(x) > V(x_0)\,\} \quad ; \quad \Omega_- = \{x\in\Omega \,|\, V(x) < V(x_0)\,\}\,,$$

and

$$\Gamma = \{x\in\Omega \,|\, V(x) = V(x_0)\,\}\,.$$

Define then the cutoff function $\chi^+_{\varepsilon,n} \in C^\infty(\Omega,[0,1])$, where $n\in\mathbb{N}$, in the following manner

$$\chi^+_{\varepsilon,n}(x) = \begin{cases} 1 & x\in\Omega_- \\ 1 & x\in\Omega_+\cap\{V(x)-V(x_0)\leq 2^{n-1}\varepsilon^\rho\} \\ 0 & x\in\Omega_+\cap\{V(x)-V(x_0)\geq 2^n\varepsilon^\rho\}\,, \end{cases} \qquad \|\nabla\chi^+_{\varepsilon,n}\|_\infty \leq \frac{C_n}{\varepsilon^\rho} \tag{4.18}$$

where $0<\rho<1$. We further set

$$(\tilde\chi^+_{\varepsilon,n})^2 + (\chi^+_{\varepsilon,n})^2 = 1\,. \tag{4.19}$$

In a similar manner we then define $\chi^-_{\Gamma,\varepsilon,n}$:

$$\chi^-_{\varepsilon,n}(x) = \begin{cases} 1 & x\in\Omega_+ \\ 1 & x\in\Omega_-\cap\{V(x_0)-V(x)\leq 2^{n-1}\varepsilon^\rho\} \\ 0 & x\in\Omega_-\cap\{V(x_0)-V(x)\geq 2^n\varepsilon^\rho\}\,. \end{cases}$$

The complementary cutoff function $\tilde\chi^-_{\varepsilon,n}$ is then given by

$$(\tilde\chi^-_{\varepsilon,n})^2 = 1 - (\chi^-_{\varepsilon,n})^2$$

We begin with the following estimate

Lemma 4.5. *Let f satisfy* (4.17) *and*

$$(\mathcal{A}_h - \lambda^*)w = f\,, \tag{4.20}$$

where

$$|\lambda^*| \leq C\varepsilon$$

Then, for any $n\in\mathbb{N}$ there exists $C_n>0$ and $\gamma_2>0$ such that for sufficiently small ε we have

$$\|\tilde\chi^-_{\varepsilon,n}w\|_2 + \|\tilde\chi^+_{\varepsilon,n}w\|_2 \leq C_n(\varepsilon^{n\rho-1}\|w\|_2 + e^{-\gamma_2\varepsilon^{-\frac{3}{2}(1-\rho)}}\|f\|_2)\,. \tag{4.21a}$$

Furthermore, we have that

(4.21b)
$$\|\nabla(\tilde\chi^+_{\varepsilon,n}w)\|_2+\|\nabla(\tilde\chi^-_{\varepsilon,n}w)\|_2+\varepsilon^2(\|D^2(\tilde\chi^+_{\varepsilon,n}w)\|_2+\|D^2(\tilde\chi^-_{\varepsilon,n}w)\|_2)\leq C_n\varepsilon^{n\rho-1}(\|w\|_2+\|f\|_2)\,.$$

Proof. In the following the constants C and γ_2 depend on n. Taking the inner product of (4.20) with $(\tilde\chi^+_{\varepsilon,n})^2w$ yields

$$\begin{cases} \|\nabla(\tilde\chi^+_{\varepsilon,n}w)\|_2^2-\|w\nabla\tilde\chi^+_{\varepsilon,n}\|_2^2=\dfrac{\alpha}{\varepsilon^3c^4}\big(\mathrm{Re}\,\lambda^*\|\tilde\chi^+_{\varepsilon,n}w\|_2^2+\mathrm{Re}\,\langle\tilde\chi^+_{\varepsilon,n}w,\tilde\chi^+_{\varepsilon,n}f\rangle\big) & (4.22a)\\ \dfrac{\alpha}{\varepsilon^3c^4}\|\tilde\chi^+_{\varepsilon,n}|V-V(x_0)|^{1/2}w\|_2^2+\mathrm{Im}\,\langle w\nabla\tilde\chi^+_{\varepsilon,n},\nabla(\tilde\chi^+_{\varepsilon,n}w)\rangle \\ \qquad =\dfrac{\alpha}{\varepsilon^3c^4}\big(\mathrm{Im}\,\lambda^*\|\tilde\chi^+_{\varepsilon,n}w\|_2^2+\mathrm{Im}\,\langle\tilde\chi^+_{\varepsilon,n}w,\tilde\chi^+_{\varepsilon,n}f\rangle\big)\,. & (4.22b)\end{cases}$$

From the definition of $\tilde\chi^+_{\varepsilon,n}$ and (4.22b) we get

$$\|\tilde\chi^+_{\varepsilon,n}w\|_2^2\leq C\varepsilon^{3-\rho}\Big(\|\nabla(\tilde\chi^+_{\varepsilon,n}w)\|_2^2+\|w\nabla\tilde\chi^+_{\varepsilon,n}\|_2^2+\varepsilon^{-4}\|\tilde\chi^+_{\varepsilon,n}f\|_2^2+\varepsilon^{-2}\|\tilde\chi^+_{\varepsilon,n}w\|_2^2\Big)\,. \tag{4.23}$$

By (4.22a) we have

$$\|\nabla(\tilde\chi^+_{\varepsilon,n}w)\|_2^2\leq C\Big(\|w\nabla\tilde\chi^+_{\varepsilon,n}\|_2^2+\varepsilon^{-4}\|\tilde\chi^+_{\varepsilon,n}f\|_2^2+\varepsilon^{-2}\|\tilde\chi^+_{\varepsilon,n}w\|_2^2\Big)\,. \tag{4.24}$$

Substituting the above into (4.23) then yields

$$\|\tilde\chi^+_{\varepsilon,n}w\|_2^2\leq C\varepsilon^{3-\rho}\Big(\|w\nabla\tilde\chi^+_{\varepsilon,n}\|_2^2+\varepsilon^{-4}\|\tilde\chi^+_{\varepsilon,n}f\|_2^2+\varepsilon^{-2}\|\tilde\chi^+_{\varepsilon,n}w\|_2^2\Big)\,,$$

from which we easily obtain, for sufficiently small ε,

$$\|\tilde\chi^+_{\varepsilon,n}w\|_2^2\leq C\varepsilon^{3-\rho}\Big(\|w\nabla\tilde\chi^+_{\varepsilon,n}\|_2^2+\varepsilon^{-4}\|\tilde\chi^+_{\varepsilon,n}f\|_2^2\Big)\,. \tag{4.25}$$

By (4.17) we have that for sufficiently small γ_2 and ε,

$$\|\tilde\chi^+_{\varepsilon,n}f\|_2\leq Ce^{-\gamma_2\varepsilon^{-\frac{3}{2}(1-\rho)}}\|f\|_2\,. \tag{4.26}$$

Furthermore, by (4.18) and (4.19) we have that

$$\|w\nabla\tilde\chi^+_{\varepsilon,n}\|_2\leq\frac{C}{\varepsilon^\rho}\|\tilde\chi^+_{\varepsilon,n-1}w\|_2\,.$$

Combining the above, (4.26), and (4.25) then yields

$$\|\tilde\chi^+_{\varepsilon,n}w\|_2\leq C\big(\varepsilon^\rho\|\tilde\chi^+_{\varepsilon,n-1}w\|_2+e^{-\gamma_2\varepsilon^{-\frac{3}{2}(1-\rho)}}\|f\|_2\big)\,.$$

Similarly we obtain that

$$\|\tilde\chi^-_{\varepsilon,n}w\|_2\leq C\big(\varepsilon^\rho\|\tilde\chi^+_{\varepsilon,n-1}w\|_2+e^{-\gamma_2\varepsilon^{-\frac{3}{2}(1-\rho)}}\|f\|_2\big)\,.$$

The above pair of inequalities, when recursively applied, readily yield (4.21a).

We begin the proof of (4.21b) by combining (4.24) and (4.21a) to obtain

$$\|\nabla(\tilde\chi^+_{\varepsilon,n}w)\|_2\leq C_n(\varepsilon^{n\rho-1}\|w\|_2+e^{-\gamma_2\varepsilon^{-\frac{3}{2}(1-\rho)}}\|f\|_2)\,. \tag{4.27}$$

Furthermore, we have that

$$\|\tilde{\chi}_{\varepsilon,n}^{+}\Delta w\|_2 \leq \frac{C}{\varepsilon^3}\|(V-V(x_0))\tilde{\chi}_{\varepsilon,n}^{+}w\|_2 + \frac{C}{\varepsilon^2}\|\tilde{\chi}_{\varepsilon,n}^{+}w\|_2 + \frac{C}{\varepsilon^3}\|\tilde{\chi}_{\varepsilon,n}^{+}f\|_2 \leq C_n(\varepsilon^{n\rho-3}\|w\|_2 + e^{-\gamma_2\varepsilon^{-\frac{3}{2}(1-\rho)}}\|f\|_2)\,.$$

As,

$$\|\Delta(\tilde{\chi}_{\varepsilon,n}^{+}w)\|_2 \leq \frac{C}{\varepsilon^\rho}\|\nabla(\tilde{\chi}_{\varepsilon,n-1}^{+}w)\|_2 + \frac{C}{\varepsilon^{2\rho}}\|\tilde{\chi}_{\varepsilon,n-1}^{+}w\|_2 + \|\tilde{\chi}_{\varepsilon,n}^{+}\Delta w\|_2\,,$$

we readily conclude that

$$\|\Delta(\tilde{\chi}_{\varepsilon,n}^{+}w)\|_2 \leq C_n(\varepsilon^{n\rho-3}\|w\|_2 + e^{-\gamma_2\varepsilon^{-\frac{3}{2}(1-\rho)}}\|f\|_2)\,.$$

Standard elliptic estimates, together with (4.27) then yield (4.21b), after repeating the same argument for $\tilde{\chi}_{\varepsilon,n}^{-}w$. ∎

Before we attempt to estimate $(\mathcal{A}_h-\lambda^*)^{-1}f$ we need yet the following auxiliary estimate.

Lemma 4.6. *Under the same conditions of Lemma 4.5 we have that*

$$\begin{cases} \|\nabla w\|_2 \leq \dfrac{C}{\varepsilon}\|w\|_2 + \dfrac{C}{\varepsilon^2}\|f\|_2\,, & \text{(4.28a)} \\ \|D^2 w\|_2 \leq \dfrac{C}{\varepsilon^{3-\rho}}\|w\|_2 + \dfrac{C}{\varepsilon^3}\|f\|_2\,, & \text{(4.28b)} \end{cases}$$

where $w=(\mathcal{A}_h-\lambda^*)^{-1}f$ *and* $0<\rho<1$.

Proof. As

$$\|\nabla w\|_2^2 = \frac{\alpha}{\varepsilon^3 c^4}(\lambda^*\|w\|_2^2 + \mathrm{Re}\,\langle w,f\rangle)\,,$$

we readily obtain (4.28a). To prove (4.28b) we first note that

$$\|\Delta w\|_2 \leq \frac{C}{\varepsilon^3}(\|(V-V(x_0))w\|_2 + \lambda^*\|w\|_2 + \|f\|_2) \tag{4.29}$$

Let

$$\zeta^2 = 1-(\tilde{\chi}_{\varepsilon,n}^{-})^2-(\tilde{\chi}_{\varepsilon,n}^{+})^2\,.$$

By (4.21) we have, for sufficiently large n,

$$\|(V-V(x_0))w\|_2 \leq C(\|\tilde{\chi}_{\varepsilon,n}^{-}w\|_2 + \|\tilde{\chi}_{\varepsilon,n}^{+}w\|_2) + \|\zeta(V-V(x_0))w\|_2 \leq C(\varepsilon^{n\rho-1}\|w\|_2 + e^{-\gamma_2\varepsilon^{-\frac{3}{2}(1-\rho)}}\|f\|_2 + \varepsilon^\rho\|w\|_2) \leq C(\varepsilon^\rho\|w\|_2 + e^{-\gamma_2\varepsilon^{-\frac{3}{2}(1-\rho)}}\|f\|_2)\,,$$

which, when substituted into (4.29), yields (4.28) with the aid of standard elliptic estimates. ∎

Lemmas 4.3 and 4.5 can now be used to estimate $(\mathcal{A}_h-\lambda^*)^{-1}f$ in the close vicinity of x_0 where $\lambda^*\in\partial B(\Lambda_0,(c^2r\varepsilon^{3/2}/\alpha))$, $r\in(0,1)$ being chosen so that $\partial B(\Lambda_0,(c^2r\varepsilon^{3/2}/\alpha))\subset\rho(\mathcal{A}_h)$, where

$$\Lambda_0=\frac{\varepsilon c^2}{\alpha}(\lambda_0+\varepsilon^{1/2}\lambda_2)\,. \tag{4.30}$$

Lemma 4.7. *Let $f\in L^\infty(\Omega,\mathbb{C})$ satisfy (4.17), and $7/8<\rho<1$. Let $w=(\mathcal{A}_h-\lambda^*)^{-1}f^*$ and ζ_0 be given by*

$$\zeta_0^*(\varepsilon,\rho)=[1-(\tilde{\chi}_{\varepsilon,n}^-)^2-(\tilde{\chi}_{\varepsilon,n}^+)^2]\mathbf{1}_{B(x_0,\delta)\cap\Omega}\,, \tag{4.31}$$

where $\delta>0$ is so chosen so that $B(x_0,\delta)\cap\Gamma=\{x_0\}$. Then,

$$\|\zeta_0^*w^*\|_2\leq\frac{C}{r}(\varepsilon^{-3/2}\|f\|_2+\varepsilon^{1/8}\|w^*\|_2)\,. \tag{4.32}$$

Proof. Clearly,

$$(\mathcal{A}_h-\lambda^*)(\zeta_0^*w^*)=\zeta_0^*f^*+[\mathcal{A}_h,\zeta_0^*]w^*$$

We next write

$$\mathcal{A}_h=\mathcal{A}_0+\mathcal{D}^*\,,$$

where A_0 is given by

$$\mathcal{A}_0=-\frac{\varepsilon^3c^4}{\alpha^3}(\partial_{tt}+\partial_{ss})+i(ct+\alpha s^2)\,,$$

and

$$\mathcal{D}^*=-\frac{\varepsilon^3c^4}{\alpha^3}\Upsilon+i(V-V(x_0)-ct-\frac{1}{2}\alpha s^2)\,,$$

where Υ is given by (3.5). Then,

$$(\mathcal{A}_0-\lambda^*)(\zeta_0^*w^*)=\zeta_0f^*-\mathcal{D}^*(\zeta_0^*w^*)+[\mathcal{A}_h,\zeta_0^*]w^*\,.$$

Applying the transformation (3.7) yields

$$(\mathcal{B}_\varepsilon-\lambda)(\zeta_0w)=\frac{\alpha}{\varepsilon c^2}\zeta_0f+[\mathcal{B}_\varepsilon,\zeta_0]w-R(\zeta_0w)\,. \tag{4.33}$$

where f, ζ_0, and w are respectively obtained from f^*, ζ_0^*, and w^* via the dilation $\cdot(\xi,\tau)=\cdot^*(s,t)$, in which (ξ,τ) are given by (3.7), R is given by (3.10) and $\lambda=\frac{\alpha}{\varepsilon c^2}\lambda^*$.

We next apply to (4.33) the operator P_1 defined in (4.13). Since $\mathcal{B}_\varepsilon$ and P_1 commute, we easily obtain from (4.16) that

$$\|P_1(\zeta_0w)\|_2\leq C(\varepsilon^{-1}\|f\|_2+\|[\mathcal{B}_\varepsilon,\zeta_0]w\|_2+\|R(\zeta_0w)\|_2)\,. \tag{4.34}$$

We now attempt to estimate $\|R(\zeta_0w)\|_2$. We first note that R is given by (3.10). We then observe that

$$\Big|\frac{\alpha}{\varepsilon c^2}[V-V(x_0)]-\tau-\varepsilon^{1/2}\frac{1}{2}\xi^2\Big|\leq C(\varepsilon^{5/4}\xi^3+\varepsilon^{3/4}\tau\xi+\varepsilon\tau^2)\quad\forall x\in B(x_0,\delta)\,, \tag{4.35}$$

Since

$$\frac{1}{2}\Big(\tau+\frac{\varepsilon^{1/2}}{2}\xi^2\Big)\leq\frac{\alpha}{\varepsilon c^2}|V(x)-V(x_0)|\leq2\varepsilon^{-(1-\rho)}\quad\forall x\in supp(\zeta_0)\,,$$

we obtain that for some $C>0$

$$\operatorname{supp}\zeta_0\subset\{(\xi,\tau)\,|\,|\xi|\le C\varepsilon^{-3/4+\rho/2}\,,\;0\le\tau<C\varepsilon^{-(1-\rho)}\}\,. \tag{4.36}$$

Consequently, by (4.35) we have that

$$\zeta_0\Big|\frac{\alpha}{\varepsilon c^2}[V-V(x_0)]-\tau-\varepsilon^{1/2}\frac{1}{2}\xi^2\Big|\le C\varepsilon^{\frac{3\rho}{2}-1}\,.$$

Hence,

$$\Big\|\Big(\frac{\alpha}{\varepsilon c^2}[V-V(x_0)]-\tau-\varepsilon^{1/2}\frac{1}{2}\xi^2\Big)\zeta_0 w\Big\|_2\le C\varepsilon^{\frac{3\rho}{2}-1}\|\zeta_0 w\|_2\,. \tag{4.37}$$

To complete the estimation of $R(\zeta_0 w)$, it is necessary to bound

$$\tilde R(\zeta_0 w)=\varepsilon^{3/2}\Big\|\tau(\zeta_0 w)_{\xi\xi}\Big\|_2+\varepsilon^{9/4}\|\tau(\zeta_0 w)_\xi\|_2+\varepsilon\|(\zeta_0 w)_\tau\|_2\,. \tag{4.38}$$

Since by (4.36) we have that

$$\|\zeta_0\|_{C^{2,0}}\le C\,,$$

we have by (3.7), (4.28), and (4.36) that

$$\Big\|\tau(\zeta_0 w)_{\xi\xi}\Big\|_2\le C\Big(\frac{1}{\varepsilon^{3/2-\rho}}\|w\|_2+\frac{1}{\varepsilon^{5/2-\rho}}\|f\|_2\Big)\,. \tag{4.39}$$

Furthermore,

$$\|\tau(\zeta_0 w)_\xi\|_2\le C\Big(\frac{1}{\varepsilon^{1/4}}\|w\|_2+\frac{1}{\varepsilon^{9/4-\rho}}\|f\|_2\Big)\,, \tag{4.40}$$

and

$$\|(\zeta_0 w)_\tau\|_2\le C(\|w\|_2+\varepsilon^{-1}\|f\|_2)\,.$$

Substituting the above together with (4.40) and (4.39) into (4.38) then yields

$$\tilde R(\zeta_0 w)\le C(\varepsilon^\rho\|w\|_2+\|f\|_2)\,. \tag{4.41}$$

Combining the above with (4.37) yields

$$\|R(\zeta_0 w)\|_2\le C(\varepsilon^{\frac{3\rho}{2}-1}\|w\|_2+\|f\|_2)\,. \tag{4.42}$$

We now turn to estimate $[\mathcal{B}_\varepsilon,\zeta_0]w$. From (4.21) we learn that, for any $n\in\mathbb{N}$, there exists some $\varepsilon_0(n)$, such that for all $\varepsilon<\varepsilon_0(n)$ we have

$$\begin{aligned}\|[\mathcal{B}_\varepsilon,\zeta_0]w\|_2&=\frac{\alpha}{c}\varepsilon^{-7/8}\Big\|\frac{\alpha}{\varepsilon c^2}[\mathcal{A}_h,\zeta_0^*]w^*\Big\|_2\le\\ &C\varepsilon^{9/8}[\varepsilon^{-2\rho}(\|\tilde\chi^-_{\varepsilon,n-1}w^*\|_2+\|\tilde\chi^+_{\varepsilon,n-1}w^*\|_2)+\varepsilon^{-\tilde\rho}(\|\nabla(\tilde\chi^-_{\varepsilon,n-1}w^*)\|_2\\ &+\|\nabla(\tilde\chi^+_{\varepsilon,n-1}w^*)\|_2)]\le C_n(\varepsilon^{n\rho-15/8}\|w^*\|_2+e^{-\gamma_2\varepsilon^{-\frac{3}{2}(1-\rho)}}\|f^*\|_2)\\ &\le C_n(\varepsilon^{n\rho-1}\|w\|_2+e^{-\gamma_2\varepsilon^{-\frac{3}{2}(1-\rho)}}\|f\|_2)\,.\end{aligned} \tag{4.43}$$

Substituting the above together with (4.42) into (4.34) yields

$$\|P_1(\zeta_0 w)\|_2\le C(\varepsilon^{\frac{3\rho}{2}-1}\|w\|_2+\|f\|_2)\,. \tag{4.44}$$

We now turn to estimate $\Pi_0(w)$. Taking the inner product of (4.33) in $L^2(\mathbb{R}_+,\mathbb{C})$ with $\bar{v}_0$ yields

$$(\mathcal{L}_\xi-\tilde{\lambda})w_0=\varepsilon^{-1/2}\Big\langle \bar{v}_0,\frac{\alpha}{\varepsilon c^2}\zeta_0 f-R(\zeta_0w)+[\mathcal{B}_\varepsilon,\zeta_0]w\Big\rangle_{\mathbb{R}_+}\,, \tag{4.45}$$

where $w_0=\langle\bar{v}_0,\zeta_0w\rangle$, and $\tilde{\lambda}=\varepsilon^{-1/2}(\lambda-\lambda_0)$. (Note that $\Pi_0(\zeta_0w)=w_0(\xi)v_0(\tau)$.) Multiplying (4.45) by $\bar{w}_0$ and integrating by parts yields, from the imaginary part

$$\|\xi w_0\|^2_{L^2(\mathbb{R})}\leq C\big(\|w_0\|^2_{L^2(\mathbb{R})}+\varepsilon^{-1/2}|\langle\bar{v}_0w_0,\varepsilon^{-1}\zeta_0f-R(\zeta_0w)+[\mathcal{B}_\varepsilon,\zeta_0]w\rangle|\big)\,.$$

We now use (4.41), (4.43), and (4.35) to obtain that

$$\|\xi w_0\|_{L^2(\mathbb{R})}\leq C(\|w_0\|_{L^2(\mathbb{R})}+\varepsilon^{-3/2}\|f\|_2+\varepsilon^{\rho-1/2}\|w\|_2+ \\ \varepsilon^{3/4}\|\xi^3\zeta_0w\|_2+\varepsilon^{1/4}\|\tau\xi\zeta_0w\|_2+\varepsilon^{1/2}\|\tau^2\zeta_0w\|_2)$$

In view of (4.36) we then have

$$\|\xi w_0\|_{L^2(\mathbb{R})}\leq C(\|w_0\|_{L^2(\mathbb{R})}+\varepsilon^{-3/2}\|f\|_2+\varepsilon^{\rho-1/2}\|w\|_2+\varepsilon^{1/4}\|\xi\zeta_0w\|_2)\,. \tag{4.46}$$

We now use (4.44) to obtain

$$\|\xi\zeta_0w\|_2\leq\|\xi P_1(\zeta_0w)\|_2+\|\xi w_0\|_{L^2(\mathbb{R})}\leq C(\varepsilon^{2\rho-7/4}\|w\|_2+\varepsilon^{-3/2}\|f\|_2)+\|\xi w_0\|_{L^2(\mathbb{R})}\,.$$

Substituting the above into (4.46) then yields

$$\|\xi w_0\|_{L^2(\mathbb{R})}\leq C(\|w_0\|_{L^2(\mathbb{R})}+\varepsilon^{2\rho-\frac{3}{2}}\|w\|_2+\varepsilon^{-3/2}\|f\|_2)\,,$$

and hence,

$$\|\xi\zeta_0w\|_2\leq C(\|w_0\|_{L^2(\mathbb{R})}+\varepsilon^{2\rho-\frac{7}{4}}\|w\|_2+\varepsilon^{-3/2}\|f\|_2)\,.$$

From the above and (4.44) once again we can conclude that

$$\|\xi^3\zeta_0w\|_2\leq C\varepsilon^{-3/2+\rho}\|\xi\zeta_0w\|_2\leq C\varepsilon^{-3/2+\rho}(\|w_0\|_{L^2(\mathbb{R})}+\varepsilon^{2\rho-\frac{7}{4}}\|w\|_2+\varepsilon^{-3/2}\|f\|_2)\,. \tag{4.47}$$

Similarly, we obtain

$$\|\xi\tau\zeta_0w\|_2\leq C\varepsilon^{-(1-\rho)}(\|w_0\|_{L^2(\mathbb{R})}+\varepsilon^{2\rho-\frac{7}{4}}\|w\|_2+\varepsilon^{3/2}\|f\|_2)\,.$$

The above, together with (4.47), (4.35), and (4.36) yield the following improvement of (4.37) (recall that $\|\Pi_0(w)\|_2\leq C\|w\|_2$)

$$\Big\|\big[\frac{\alpha}{\varepsilon c^2}[V-V(x_0)]-\tau-\varepsilon^{1/2}\frac{1}{2}\xi^2\big]\zeta_0w\Big\|_2\leq C\varepsilon^{\rho-1/4}(\|w\|_2+\varepsilon^{-3/2}\|f\|_2)\,.$$

We now combine the above inequality with (4.41) to obtain an improved version of (4.42)

$$\|R(\zeta_0w)\|_2\leq C\varepsilon^{\rho-1/4}(\varepsilon^{\tilde{\rho}}\|w\|_2+\varepsilon^{3/2}\|f\|_2)\,. \tag{4.48}$$

Returning to (4.33) we obtain from (4.7) that

$$\|\zeta_0w\|_2\leq\frac{C}{r\varepsilon^{1/2}}(\varepsilon^2\|f\|_2+\|[\mathcal{B}_\varepsilon,\zeta_0]w\|_2+\|R(\zeta_0w)\|_2)\,.$$

With the aid of (4.43) and (4.48) we then obtain

$$\|\zeta_0w\|_2\leq\frac{C}{r\varepsilon^{1/2}}(\varepsilon^{-1}\|f\|_2+\varepsilon^{5/8}\|w\|_2)\,,$$

from which (4.32) easily follows. ∎

Remark 4.8. *Clearly,* (4.32) *can be extended to the neighborhood of each point in* $\mathcal{S}$*. Thus, if we set for any* $x_j \in \mathcal{S}$

$$\zeta_j^*(\varepsilon,\rho) = [1 - (\tilde{\chi}_{\varepsilon,n}^-)^2 - (\tilde{\chi}_{\varepsilon,n}^+)^2]\mathbf{1}_{B(x_j,\delta)\cap\Omega}\,, \tag{4.49}$$

where $\delta > 0$ *is so chosen so that* $B(x_j,\delta)\cap\Gamma = \{x_j\}$ *for all* $j \in J_{\mathcal{S}}$*. Then,*

$$\|\zeta_j^* w^*\|_2 \leq \frac{C}{r}(\varepsilon^{3/2}\|f\|_2 + \varepsilon^{1/8}\|w^*\|_2)\,. \tag{4.50}$$

We can now estimate $\|(\mathcal{A}_h - \lambda^*)^{-1}f\|$ in the simplest possible case where $\Gamma = \{x_0\}$.

Corollary 4.9. *Let* $f \in L^\infty(\Omega,\mathbb{C})$ *satisfy* (4.17)*. Let* $\lambda^* \in \partial B(\Lambda_0, r\varepsilon^{-1/2}) \subset \rho(\mathcal{A}_h)$*, where* Λ_0 *is given by* (4.30)*, for some* $\varepsilon^{1/8} \ll r < 1$*. Then, there exists* $C > 0$ *such that for sufficiently small* ε *we have*

$$\|(\mathcal{A}_h - \lambda^*)^{-1}f\|_2 \leq \frac{C}{\varepsilon^{3/2}r}\|f\|_2\,. \tag{4.51}$$

Proof. Since $\Gamma = \{x_0\}$ we may set with any loss of generality $\Omega = \Omega_+$. Hence, we have that $\chi_{\varepsilon,n}^+ = \zeta_0^*$, where ζ_0^* is defined by (4.31). Let $w = (\mathcal{A}_h - \lambda)^{-1}f$. Then,

$$\|w\|_2^2 = \|\chi_{\varepsilon,n}^+ w\|_2^2 + \|\tilde{\chi}_{\varepsilon,n}^+ w\|_2^2 = \|\zeta_0^* w\|_2^2 + \|\tilde{\chi}_{\varepsilon,n}^+ w\|_2^2\,.$$

The corollary now easily follows from (4.21a) and (4.32). ∎

Consider next the general case where $\Gamma \setminus \{x_0\} \neq \emptyset$. We begin by defining some local approximations of the operator $\tilde{\mathcal{A}}_h$. Let $\rho \in (7/8, 1)$, and then define two sets of indices $J_{\partial\Omega} = J_{\partial\Omega}(\varepsilon)$ and $J_\Omega = J_\Omega(\varepsilon)$. Set then $J = J_{\partial\Omega} \cup J_\Omega$ and let $\delta > 0$ be the same as in (4.31). Next, choose a sequence of points $(x_j)_{j\in J} = (x_j(\varepsilon))_{j\in J} \subset \bar{\Omega}\setminus \bigcup_{j\in J_{\mathcal{S}}} B(x_j,\delta)$, where $x_j \in \partial\Omega$ (respectively $x_j \in \Omega$) if $j \in J_{\partial\Omega}$ (respectively $j \in \Omega$), such that

$$\bar{\Omega} \setminus \bigcup_{j\in J_{\mathcal{S}}} B(x_j,\delta) \subset \bigcup_{j\in J} B(x_j,\varepsilon^\rho)\,.$$

Let $(\eta_j)_{j\in J}$ be a family of cutoff functions associated with the partition above, namely $\eta_j(x) = 1$ if $x \in B(x_j,\varepsilon^\rho/2)$, $\operatorname{Supp}\eta_j \subset B(x_j,\varepsilon^\rho)$, and

$$\forall x \in \bar{\Omega} \setminus \bigcup_{j\in J_{\mathcal{S}}} B(x_j,\delta)\,,\ \sum_{j\in J}\eta_j(x)^2 = 1\,.$$

We further assume that for all $j \in J$, $\|\nabla\eta_j\|_\infty = \mathcal{O}(\varepsilon^{-\rho})$ and $\|\Delta\eta_j\|_\infty = \mathcal{O}(\varepsilon^{-2\rho})$. Finally we set, for all $j \in J$,

$$\chi_j = \eta_j \mathbf{1}_{\bar{\Omega}}\,.$$

In the neighborhood of each point x_j, $j \in J_\Omega$, we shall approximate $\mathcal{A}_h$ by the following operator:

$$\mathcal{A}_{j,h} := -\frac{\varepsilon^3 c^4}{\alpha^3}\Delta + i(\mathbf{c}_j.x + V(x_j) - V(x_0))\,,\ \mathbf{c}_j = (c_j^1, c_j^2) = \nabla V(x_j)\,, \tag{4.52a}$$

whose domain is given by

$$D(\mathcal{A}_{j,h}) = H^2(\mathbb{R}^2;\mathbb{C}) \cap L^2(\mathbb{R}^2, |x|^2 dx;\mathbb{C})\,. \tag{4.52b}$$

In the neighborhood of the boundary points x_j, $j \in J_{\partial\Omega}$, we use different approximate operators, depending on the local behaviour of V. To this end, denote by $J^1_{\partial\Omega} \subset J_{\partial\Omega}$ the set of indices j such that $x_j \in \partial\Omega_\perp$ and

$$|\nabla V(x_j)| = |\nabla V(x_0)| = \min_{x\in\partial\Omega_\perp} |\nabla V(x)|\,.$$

Notice that $J^1_{\partial\Omega}$ may be an empty set, since $x_0 \notin \bar\Omega \setminus B(x_0,\delta)$. We then let $J^2_{\partial\Omega} = J_{\partial\Omega} \setminus J^1_{\partial\Omega}$ and $J^3_{\partial\Omega} = J^1_{\partial\Omega} \setminus J_{\mathcal{S}}$. In the neighborhood of the boundary points x_j for $j \in J^2_{\partial\Omega}$, we use the following approximation of $\mathcal{A}_h$. Let (t,s) be the same curvilinear coordinate system as defined in Section 3, centered at x_j. In these coordinates the leading order approximation of $\mathcal{A}_h$ reads

$$\mathcal{A}_{j,h} = -\frac{\varepsilon^3 c^4}{\alpha^3}\Delta + i(\mathbf{c}_j.(t,s) + V(x_j) - V(x_0))\,,\ \mathbf{c}_j = (c^1_j, c^2_j) = \nabla V(x_j)\,, \tag{4.53a}$$

with the following domain

$$D(\mathcal{A}_{j,h}) = H^1_0(\mathbb{R}^2_+;\mathbb{C}) \cap H^2(\mathbb{R}^2_+;\mathbb{C}) \cap L^2(\mathbb{R}^2_+, (t^2+s^2)dtds;\mathbb{C})\,. \tag{4.53b}$$

In the following we provide resolvent estimates on the approximate operators $\mathcal{A}_{j,h}$ introduced above. These estimates are stated in the following lemma

Lemma 4.10. *There exists $r_0 > 0$ such that, for all $r \in (0, r_0)$ and $j \in J$, $\partial B(\Lambda_0, r\varepsilon^{-1/2}) \subset \rho(\mathcal{A}_{j,h})$, where Λ_0 is given by (4.30). Moreover, there exists $C > 0$ such that for all $\lambda^* \in \partial B(\Lambda_0, r\varepsilon^{-1/2})$ and for all $j \in J_\Omega \cup J^2_{\partial\Omega}$,*

$$\|(\mathcal{A}_{j,h} - \lambda^*)^{-1}\|_2 \le \frac{C}{\varepsilon}\,. \tag{4.54}$$

Proof. Let $j \in J_\Omega$. Recall that the operator $\mathcal{A}_{j,h}$ is given in this case by (4.53). It has been established in [3, 9] that $\mathcal{A}_{j,h}$ has empty spectrum, and for all $\omega \in \mathbb{R}$ there exists $C_\omega > 0$ such that

$$\sup_{\operatorname{Re} z \le \omega} \left\|(-\Delta + i\mathbf{c}_j.x - z)^{-1}\right\| \le C_\omega\,. \tag{4.55}$$

Since the scale change $x \mapsto \alpha/(\varepsilon c^{4/3})x$ gives

$$\|(\mathcal{A}_{j,h} - \lambda^*)^{-1}\| = \frac{\alpha}{\varepsilon c^{4/3}} \left\|\Big(-\Delta + i\Big[\frac{\alpha}{\varepsilon c^{4/3}}\big(V(x_j) - V(x_0)\big) + \mathbf{c}_j.x\Big] - \frac{\alpha}{\varepsilon c^{4/3}}\lambda^*\Big)^{-1}\right\|\,. \tag{4.56}$$

and since $\alpha/(\varepsilon c^{4/3})\lambda^*$ remains bounded as $\varepsilon \to 0$, (4.55) and (4.56) easily yield (4.54) for any $j \in J_\Omega$.

The same argument can be used in the case where $j \in J^2_{\partial\Omega}$ with $x_j \notin \partial\Omega_\perp$, since the operator $-\Delta + ic^1_j t + ic^2_j s$ on $\mathbb{R}^2_+$ has empty spectrum and satisfies (4.55) as well as soon as $c^2_j \ne 0$, see Theorem A.3.

We next consider the case where $j \in J^2_{\partial\Omega}$ and $x_j \in \partial\Omega_\perp$. Then,

$$\mathcal{A}_{j,h} = -\frac{\varepsilon^3 c^4}{\alpha^3}\Delta + i\big(c_j t + V(x_j) - V(x_0)\big)$$

where $c_j := c_j^1$. The domain $D(\mathcal{A}_{j,h})$ is given by ((4.53)b). Suppose that $c_j > 0$ (otherwise apply the same argument to the operator $\mathcal{A}_{j,h}^*$). Denote by $\mathcal{A}_0^\perp$ the Dirichlet realization on $\mathbb{R}^2_+$ of the operator $-\Delta + it$. Then, the scale change

$$(t,s) \longmapsto \frac{\alpha c_j^{1/3}}{\varepsilon c^{4/3}}\,(t,s)$$

gives

$$\|(\mathcal{A}_{j,h}-\lambda^*)^{-1}\| = \frac{\alpha c_j^{1/3}}{\varepsilon c^{4/3}}\left\|\left(\mathcal{A}_0^\perp + i\frac{\alpha c_j^{1/3}}{\varepsilon c^{4/3}}\big(V(x_j)-V(x_0)\big) - \frac{\alpha c_j^{1/3}}{\varepsilon c^{4/3}}\lambda^*\right)^{-1}\right\|. \tag{4.57}$$

By the definition of $J^2_{\partial\Omega}$, we have $c_j < c$. Hence for any fixed $\delta_0 \in (0,1)$ we have

$$\frac{\alpha c_j^{1/3}}{\varepsilon c^{4/3}}\lambda^* = \left(\frac{c}{c_j}\right)^{2/3}\lambda_0 + \mathcal{O}(\varepsilon^{1/2}) \leq (1-\delta_0)\lambda_0$$

for all sufficiently small ε. It has been established in [9] that

$$\sup_{\operatorname{Re} z \leq (1-\delta_0)\lambda_0} \|(\mathcal{A}_0^\perp - z)^{-1}\| < +\infty\,.$$

Consequently, (4.54) follows from (4.57) and the above estimate. ∎

We now extend (4.51) to the general case

Proposition 4.11. *Let $\varepsilon^{1/8} \ll r < 1$. Under the assumptions of Theorem 1.1, (4.51) holds for any $f \in L^\infty(\Omega,\mathbb{C})$ satisfying (4.17), and $\lambda^* \in \partial B(\Lambda_0, r\varepsilon^{-1/2})$.*

Proof. Let $w = (\mathcal{A}_h - \lambda^*)^{-1} f$. Let $j \in J^2_{\partial\Omega} \cup J_\Omega$. Clearly

$$(\mathcal{A}_{j,h}-\lambda^*)(\chi_j w) = [\mathcal{A}_h, \chi_j]w - (\mathcal{A}_h - \mathcal{A}_{j,h})(\chi_j w)\,. \tag{4.58}$$

We now attempt to estimate the right-hand-side of (4.58). Clearly,

$$\|[\mathcal{A}_h,\chi_j]w\|_2 \leq C\varepsilon^{-2\rho}\|w\|_{L^2(B(x_j,\varepsilon^\rho))} + C\varepsilon^{-\rho}\|\nabla(\chi_j w)\|_2\,. \tag{4.59}$$

As

$$\operatorname{Re}\langle \chi_j^2 w, (\mathcal{A}_h-\lambda^*)w\rangle = \|\nabla(\chi_j w)\|_2^2 - \lambda^*\|\chi_j w\|_2^2 - \|w\nabla\chi_j\|_2^2 = 0\,,$$

we obtain that

$$\|\nabla(\chi_j w)\|_2 \leq C\varepsilon^{-1}\|w\|_{L^2(B(x_j,\varepsilon^\rho))}\,, \tag{4.60}$$

which, when substituted into (4.59) yields

$$\|[\mathcal{A}_h,\chi_j]w\|_2 \leq C\varepsilon^{-(1+\rho)}\|w\|_{L^2(B(x_j,\varepsilon^\rho))}\,. \tag{4.61}$$

We now attempt to estimate $(\mathcal{A}_h - \mathcal{A}_{j,h})(\chi_j w)$. By (4.53) and (4.52) we have that

$$\mathcal{A}_h - \mathcal{A}_{j,h} = i\frac{\alpha^3}{\varepsilon^3 c^4}\big(V(x) - V(x_j) - \mathbf{c}_j.(x-x_j)\big)\,.$$

Consequently,

$$\|(\mathcal{A}_h - \mathcal{A}_{j,h})(\chi_j w)\|_2 \leq C\varepsilon^{-3+2\rho}\|w\|_{L^2(B(x_j,\varepsilon^\rho))}\,.$$

Combining the above with (4.61), (4.58), and (4.54) yields

$$\|\chi_j w\|_2 \leq C\varepsilon^{2\rho-1}\|w\|_{L^2(B(x_j,\varepsilon^\rho))}\,. \tag{4.62}$$

Consider next the case where $j\in J^3_{\partial\Omega}$. Here we have

$$\mathrm{Im}\,\langle\chi_j^2 w,(\mathcal{A}_h-\lambda^*)w\rangle = \frac{\alpha^3 c_j}{\varepsilon^3 c^4}\||V(\cdot)-V(x_0)|^{1/2}\chi_j w\|_2^2 - \mathrm{Im}\,\lambda^*\|\chi_j w\|_2^2 + 2\mathrm{Im}\,\langle w\nabla\chi_j,\chi_j\nabla w\rangle = 0\,.$$

By (1.3), there exists $\delta_1>0$ such that $|V(x_j)-V(x_0)|>\delta_1$. Consequently,

$$\|\chi_j w\|_2^2 \leq C[\varepsilon\|\chi_j w\|_2^2 + \varepsilon^3\|w\nabla\chi_j\|_2\|\chi_j\nabla w\|_2]\,.$$

With the aid of (4.60), which is valid for every $j\in J$, we then obtain

$$\|\chi_j w\|_2 \leq C\varepsilon^{1-\rho/2}\|w\|_{L^2(B(x_j,\varepsilon^\rho))}\,. \tag{4.63}$$

Combining (4.63) and (4.62) then yields

$$\|w\|_{L^2\left(\Omega\setminus\bigcup\limits_{j\in J_{\mathcal{S}}}B(x_j,\delta)\right)} \leq C\varepsilon^{1-\rho/2}\sum_{j\in J_\Omega\cup J^2_{\partial\Omega}}\|w\|_{L^2(B(x_j,\varepsilon^\rho))} \leq C\varepsilon^{1-\rho/2}\|w\|_2\,. \tag{4.64}$$

We conclude the proof by recalling that for all $j\in J_{\mathcal{S}}$ we have, by (4.50)

$$\|\zeta_j^* w\|_2 \leq \frac{C}{r}(\varepsilon^{3/2}\|f\|_2+\varepsilon^{1/8}\|w\|_2)\,. \tag{4.65}$$

Furthermore, let

$$\tilde{\zeta_j^*}^2 + (\zeta_j^*)^2 = \mathbf{1}_{B(x_j,\delta)}\,.$$

Then, by (4.21a)

$$\|\tilde{\zeta_j^*}w\|_2^2 \leq \|\tilde{\chi}^+_{\varepsilon,n}w\|_2^2 + \|\tilde{\chi}^-_{\varepsilon,n}w\|_2^2 \leq C_n(\varepsilon^{n\rho-1}\|w\|_2 + e^{-c\varepsilon^{-\frac{3}{2}(1-\rho)}}\|f\|_2)\,.$$

which, together with (4.65) and (4.64) yields (4.17). ∎

Proof of Theorem 1.1. Let U be given by (3.23) and Λ_0 be given by (4.30). Let $f=(\mathcal{A}_h-\Lambda_0)(\eta_{\varepsilon^{1/2}}U)$. Then, for $\lambda^*\in\partial B(\Lambda_0,r\varepsilon^{-1/2})\subset\rho(\mathcal{A}_h)$ where $\varepsilon^{1/8}\ll r<1$,

$$(\mathcal{A}_h-\lambda^*)(\eta_{\varepsilon^{1/2}}U) = f + (\Lambda_0-\lambda)\eta_{\varepsilon^{1/2}}U\,.$$

Hence

$$\langle\eta_{\varepsilon^{1/2}}U,(\mathcal{A}_h-\lambda^*)^{-1}(\eta_{\varepsilon^{1/2}}U)\rangle = -\frac{1}{\lambda-\Lambda_0}[1-\langle\eta_{\varepsilon^{1/2}}U,(\mathcal{A}_h-\lambda)^{-1}f\rangle]$$

By (4.51) and (3.26) we then obtain that

$$\|(\mathcal{A}_h-\lambda)^{-1}f\|_2 \leq C\frac{\varepsilon^{-3/2}}{r}\|f\|_2 \leq C\frac{\varepsilon^{1/2}}{r} \leq C\varepsilon^{1/4}\,.$$

Consequently

$$\frac{1}{2\pi i}\oint_{\partial B(\Lambda_0,r\varepsilon^{-3/2})}\langle\eta_{\varepsilon^{1/2}}U,(\mathcal{A}_h-\lambda)^{-1}(\eta_{\varepsilon^{1/2}}U)\rangle \leq -1+C\varepsilon^{1/4}\,.$$

Hence $(\mathcal{A}_h-\lambda)^{-1}$ is not holomorphic in $B(\Lambda_0,r\varepsilon^{-3/2})$ and the Theorem is proved via (3.9). ∎

Appendix A. Spectral analysis of (4.53))

In the following we provide the spectrum, semigroup estimates, and resolvent estimates for the operator $\mathcal{A}_{j,h}$ given by (4.53). This operator has already been investigated in [3, 9], but since resolvent estimates have not been obtained there we derive them here.

Let $\mathbf{c} = (c^1, c^2) \in \mathbb{R}^2$ such that $c^2 \neq 0$. We study here the spectrum and the resolvent of the Dirichlet realization in $\mathbb{R}^2_+ = \{(t,s) \in \mathbb{R}^2 : t > 0\}$ of $-\Delta + i(c^1 t + c^2 s)$, whose domain is given by (4.53b). The imaginary part of the potential

$$\ell(t,s) = \mathbf{c} \cdot (t,s)$$

does not have a constant sign, hence we are unable to use the variational approach to define the operator. We shall instead define the operator by separation of variables.

Let

$$\mathcal{A}_s = -\partial_s^2 + ic^2 s\,, \tag{A.1}$$

and let $\mathcal{A}_t^+$ be the Dirichlet realization in $\mathbb{R}_+$ of the complex Airy operator

$$-\frac{d^2}{dt^2} + ic^1 t\,. \tag{A.2}$$

Both $\mathcal{A}_s$ and $\mathcal{A}_t^+$ are maximally accretive and hence they serve as generators of contraction semigroups $(e^{-t\mathcal{A}_s})_{t>0}$ and $(e^{-t\mathcal{A}_t^+})_{t>0}$ respectively. One can easily verify that the family $(e^{-t\mathcal{A}_s} \otimes e^{-t\mathcal{A}_t^+})_{t>0}$ is a contraction semigroup on $L^2(\mathbb{R}^2_+)$. Thus, we can define the desired operator as follows:

Definition A.1. *$\mathcal{A}_+$ is the generator of the semigroup $(e^{-t\mathcal{A}_s} \otimes e^{-t\mathcal{A}_t^+})_{t>0}$.*

Let $D = D(\mathcal{A}_s) \otimes D(\mathcal{A}_t^+)$ be the set of all finite linear combinations of functions of the form $f \otimes g = f(s)g(t)$, where $f \in D(\mathcal{A}_s)$ and $g \in D(\mathcal{A}_t^+)$. Then it is clear that D satisfies the conditions of [11, Theorem X.49], hence $\mathcal{A}_+ = \overline{\mathcal{A}_{+|D}}$. Consequently, we may chacterize $D(\mathcal{A}_+)$ as follows:

$$\begin{aligned} D(\mathcal{A}_+) \;=\; \{ & u \in L^2(\mathbb{R}^2_+) : \exists (u_j)_{j\geq 1} \subset D\,,\ u_j \xrightarrow[j\to+\infty]{L^2} u\,, \\ & (\mathcal{A}_+ u_j)_{j\geq 1} \text{ is a Cauchy sequence } \}\,. \end{aligned} \tag{A.3}$$

In the following lemma we give a more constructive description of $D(\mathcal{A}_+)$.

Lemma A.2. *We have*

$$D(\mathcal{A}_+) = H_0^1(\mathbb{R}^2_+) \cap H^2(\mathbb{R}^2_+) \cap L^2(\mathbb{R}^2_+; |\ell(t,s)|^2 dtds)\,, \tag{A.4}$$

and there exists $C > 0$ such that, for all $u \in D(\mathcal{A}_+)$,

$$\|\Delta u\|^2_{L^2(\mathbb{R}^2_+)} + \|\ell u\|^2_{L^2(\mathbb{R}^2_+)} \leq \|\mathcal{A}_+ u\|^2_{L^2(\mathbb{R}^2_+)} + C\|\nabla u\|_{L^2(\mathbb{R}^2_+)} \|u\|_{L^2(\mathbb{R}^2_+)}\,. \tag{A.5}$$

Proof: Let $u \in D(\mathcal{A}_+)$ and $(u_j)_{j\geq 1} \subset D$ such that $u_j \xrightarrow[j\to+\infty]{L^2} u$ and $(\mathcal{A}_+ u_j)_{j\geq 1}$ is a Cauchy sequence. Then, using the identity

$$\operatorname{Re}\langle \mathcal{A}_+ u_j, u_j \rangle = \|\nabla u_j\|^2_{L^2(\mathbb{R}^2_+)}\,,$$

which holds for every $j \in \mathbb{N}$, we obtain that $(\nabla u_j)_{j\geq 1}$ is a Cauchy sequence in $L^2(\mathbb{R}^2_+)$ and hence

$$u_j \xrightarrow[j\to+\infty]{H^1} u\,, \tag{A.6}$$

and $u \in H^1_0(\mathbb{R}^2_+)$.
To prove (A.5), we write (hereafter $\|\cdot\|$ denotes the $L^2(\mathbb{R}^2_+,\mathbb{C})$ norm)

$$\begin{aligned} \|\mathcal{A}_+u_j\|^2 &= \langle(-\Delta+i\ell)u_j,(-\Delta+i\ell)u_j\rangle \\ &= \|\Delta u_j\|^2+\|\ell u_j\|^2+2\mathrm{Im}\,\langle-\Delta u_j,\ell u_j\rangle\,. \end{aligned} \tag{A.7}$$

As

$$\begin{aligned} \mathrm{Im}\,\langle-\Delta u_j,\ell u_j\rangle &= \mathrm{Im}\int_{\mathbb{R}^2_+}\nabla u_j(t,s)\cdot\overline{\nabla(\ell u_j)(t,s)}dtds \\ &= \mathrm{Im}\left(\int_{\mathbb{R}^2_+}\ell(t,s)|\nabla u_j(t,s)|^2dtds+\int_{\mathbb{R}^2_+}\nabla u_j(t,s)\cdot\overline{\nabla\ell(t,s)u_j(t,s)}dtds\right) \\ &= \mathrm{Im}\int_{\mathbb{R}^2_+}\mathbf{c}\cdot\nabla u_j(t,s)\overline{u_j(t,s)}dtds\,, \end{aligned}$$

it follows that for some $C>0$,

$$|\mathrm{Im}\,\langle-\Delta u_j,\ell u_j\rangle|\leq C\,\|\nabla u_j\|\,\|u_j\|\,.$$

Thus, by (A.7), (A.5) holds for u_j for all $j\in\mathbb{N}$. Consequently, $(u_j)_{j\geq1}$ is a Cauchy sequence in $H^2(\mathbb{R}^2_+)$ and in $L^2(\mathbb{R}^2_+;|\ell(t,s)|^2dtds)$. Hence, (A.4) follows, and so does (A.5) for every $u\in D(\mathcal{A}_+)$. ⊟

We now obtain the spectrum of $\mathcal{A}_+$. Since $\mathcal{A}_s$ has an empty spectrum (see [3, 9]), we expect $\sigma(\mathcal{A}_+)$ to be empty as well [3]. To establish this fact we employ semigroup estimates.

Theorem A.3. *We have $\sigma(\mathcal{A}_+)=\emptyset$. Moreover, for every $\omega\in\mathbb{R}$, there exists $C_\omega>0$ such that*

$$\sup_{\mathrm{Re}\,z\leq\omega}\|(\mathcal{A}_+-z)^{-1}\|\leq C_\omega\,. \tag{A.8}$$

Finally, the semigroup generated by $\mathcal{A}_+$ satisfies

$$\forall t>0,\ \|e^{-t\mathcal{A}_+}\|\leq e^{-t^3/12}\,. \tag{A.9}$$

Proof: Recall that $e^{-t\mathcal{A}_+}=e^{-t\mathcal{A}_s}\otimes e^{-t\mathcal{A}_t^+}$, where $\mathcal{A}_s$ and $\mathcal{A}_t^+$ are respectively defined by (A.1) and (A.2). Recall further the following estimates (see [9]):

$$\forall t>0\,,\ \|e^{-t\mathcal{A}_s}\|=e^{-t^3/12}\,, \tag{A.10}$$

and for all $\omega<|\mu_1|/2$ (μ_1 being the rightmost zero of Airy's function), there exists $M_\omega>0$ such that

$$\forall t>0\,,\ \|e^{-t\mathcal{A}_t^+}\|\leq M_\omega\,e^{-\omega t}\,. \tag{A.11}$$

Thus, (A.9) follows, and the formula

$$(\mathcal{A}_+ - z)^{-1} = \int_0^{+\infty} e^{-t(\mathcal{A}_+ - z)} dt \,, \tag{A.12}$$

which holds *a priori* for $\mathrm{Re}\, z < 0$, can be extended to the entire complex plane. Hence the resolvent of $\mathcal{A}_+$ is an entire function, and we must have $\sigma(\mathcal{A}_+) = \emptyset$ together with (A.8). □

Acknowledgements. The authors are grateful to Bernard Helffer for his valuable comments. Y. Almog was partially supported by NSF grant DMS-1109030. R. Henry acknowledges the support of the ANR project NOSEVOL.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, 1972.

[2] S. Agmon, *Lectures on elliptic boundary value problems*, Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr. Van Nostrand Mathematical Studies, No. 2, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965.

[3] Y. Almog, *The Stability of the Normal State of Superconductors in the Presence of Electric Currents*, SIAM Journal on Mathematical Analysis, 40 (2008), pp. 824–850.

[4] Y. Almog, B. Helffer, and X.-B. Pan, *Superconductivity near the normal state in a half-plane under the action of a perpendicular electric current and an induced magnetic field, Part II: The large conductivity limit*, SIAM J. Math. Anal., 44 (2012), pp. 3671–3733.

[5] ——, *Superconductivity near the normal state in a half-plane under the action of a perpendicular electric current and an induced magnetic field*, Trans. Amer. Math. Soc., 365 (2013), pp. 1183–1217.

[6] K. Beauchard, B. Helffer, R. Henry, and L. Robbiano, *Degenerate parabolic operators of Kolmogorov type with a geometric control condition*, To appear in ESAIM: COCV, (2014).

[7] E. B. Davies, *Linear operators and their spectra*, vol. 106 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2007.

[8] B. Helffer, *Thorie spectrale pour des oprateurs globalement elliptiques*, Astrisque, 1984.

[9] B. Helffer, *On spectral problems related to a time dependent model in superconductivity with electric current*, in Proceedings of Evian conference., June 2009.

[10] R. Henry, *On the semiclassical analysis of Schrdinger operators with purely imaginary electric potentials in a bounded domain*, arXiv preprint arXiv:1405.6183.

[11] M. Reed and B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.

[12] A. A. Shkalikov, *Spectral portraits of the Orr-Sommerfeld operator at large Reynolds numbers*, Sovrem. Mat. Fundam. Napravl., 3 (2003), pp. 89–112.

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803

E-mail address, Y. Almog: almog@math.lsu.edu

Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, France.

E-mail address, R. Henry: raphael.henry@math.u-psud.fr