

ON INTERPOLATION APPROXIMATION: CONVERGENCE RATES FOR POLYNOMIAL INTERPOLATION FOR FUNCTIONS OF LIMITED REGULARITY*

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Abstract. The convergence rates on polynomial interpolation in most cases are estimated by Lebesgue constants. These estimates may be overestimated for some special points of sets for functions of limited regularities. In this paper, by applying the Peano kernel theorem and Wainerman's lemma, new formulas on the convergence rates are considered. Based upon these new estimates, it shows that the interpolation at strongly normal pointsystems can achieve the optimal convergence rate, the same as the best polynomial approximation. Furthermore, by using the asymptotics on Jacobi polynomials, the convergence rates are established for Gauss-Jacobi, Jacobi-Gauss-Lobatto or Jacobi-Gauss-Radau pointsystems. From these results, we see that the interpolations at the Gauss-Legendre, Legendre-Gauss-Lobatto pointsystem, or at strongly normal pointsystems, has essentially the same approximation accuracy compared with those at the two Chebyshev pointsystems, which also illustrates the equally accuracy of the Gauss and Clenshaw-Curtis quadrature. In addition, numerical examples illustrate the perfect coincidence with the estimates, which means the convergence rates are optimal.

Key words. polynomial interpolation, Peano kernel, convergence rate, limited regularity, strongly normal pointsystem, Gauss-Jacobi point, Jacobi-Gauss-Lobatto point, Chebyshev point.

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1. Introduction. A central problem in approximation theory is the construction of simple functions that are easily implemented on computers and approximate well a given set of functions.

There exist many investigations for the behavior of continuous functions approximated by polynomials. Weierstrass [73] in 1885 proved the well known result that every continuous function $f(x)$ in $[-1, 1]$ can be uniformly approximated as closely as desired by a polynomial function. This result has both practical and theoretical relevance, especially in polynomial interpolation.

Polynomial interpolation is a fundamental tool in many areas of scientific computing. Lagrange interpolation is a classical technique for approximation of continuous functions. Let us denote by

$$(1.1) \quad -1 \leq x_n^{(n)} < x_{n-1}^{(n)} < \cdots < x_2^{(n)} < x_1^{(n)} \leq 1$$

the n distinct points in the interval $[-1, 1]$ and let $f(x)$ be a function defined in the same interval. The n th Lagrange interpolation polynomial of $f(x)$ is unique and given by the formula

$$(1.2) \quad L_n[f] = \sum_{k=1}^n f(x_k^{(n)}) \ell_k^{(n)}(x), \quad \ell_k^{(n)}(x) = \frac{\omega_n(x)}{\omega_n'(x_k^{(n)})(x - x_k^{(n)})},$$

where $\omega_n(x) = (x - x_1^{(n)})(x - x_2^{(n)}) \cdots (x - x_n^{(n)})$.

There is a well developed theory that quantifies the convergence or divergence of the Lagrange interpolation polynomials (Brutman [7, 8] and Trefethen [59]). Two key notions for interpolation in a given set of points are that of the *Lebesgue function*

$$(1.3) \quad \lambda_n(x) = \sum_{k=1}^n |\ell_k^{(n)}(x)|$$

and *Lebesgue constant*

$$(1.4) \quad \Lambda_n = \max_{x \in [-1, 1]} \lambda_n(x),$$

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which are of fundamental importance (Cheney [9], Davis [12] and Szegő [55]). The Lebesgue constant can also be interpreted as the ∞ -norm of the projection operator $L_n : C([-1, 1]) \rightarrow \mathcal{P}_{n-1}$

$$\Lambda_n = \sup_f \frac{\|L_n[f]\|_\infty}{\|f\|_\infty},$$

where \mathcal{P}_{n-1} is the set of polynomials of degree less than or equal to $n - 1$.

Based upon the Lebesgue constant, the interpolation error can be estimated by

$$(1.5) \quad \|L_n[f] - f\|_\infty \leq (1 + \Lambda_n) \|p_{n-1}^* - f\|_\infty,$$

where p_{n-1}^* is the best polynomial approximation of degree $n - 1$. Thus, the Lebesgue constant Λ_n indicates how good the interpolant $L_n[f]$ is in comparison with the best polynomial approximation p_{n-1}^* .

The study of the Lebesgue constant Λ_n originated more than 100 years ago. Comprehensive reviews can be found in Brutman [8], Lubinsky [41], Trefethen [59, Chapter 15], etc. For an arbitrarily given system of points $\{x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}\}_{n=1}^\infty$, Bernstein [2] and Faber [18] in 1914 obtained that

$$\Lambda_n \geq \frac{1}{12} \log n,$$

which, together with the boundedness principle, implies that there exists a continuous function $f(x)$ in $[-1, 1]$ for which the sequence $L_n[f]$ ($n = 1, 2, \dots$) is not uniformly convergent to f in $[-1, 1]^1$. More precisely, Erdős [15] and Brutman [7] proved that

$$(1.6) \quad \Lambda_n \geq \frac{2}{\pi} \log n + C \text{ for some constant } C \text{ ([15]); } \quad \Lambda_n \geq \frac{2}{\pi} \left(\gamma_0 + \log \frac{4}{\pi} \right) + \frac{2}{\pi} \log n \text{ ([7]),}$$

where $\gamma_0 = 0.577 \dots$ is the Euler's constant. In particular, for equidistant pointsystem

$$\left\{ x_k^{(n)} = -1 + \frac{2k}{n-1} \right\}_{k=0}^{n-1},$$

Schönhage [52] showed that

$$\Lambda_n \sim \frac{2^n}{e(\log(n-1) + \gamma_0)(n-1)}, \quad n \rightarrow \infty.$$

Additionally, Trefethen and Weideman [61] established that

$$\frac{2^{n-3}}{(n-1)^2} \leq \Lambda_n \leq \frac{2^{n+2}}{n-1}, \quad n \geq 0.$$

Then generally, the set of equally spaced points is a bad choice for Lagrange interpolation (see Runge [49]).

Whereas, for well chosen sets of points, the growth of Λ_n may be extremely slow as $n \rightarrow \infty$:

- Chebyshev pointsystem of first kind $T_n = \left\{ x_k^{(n)} = \cos \left(\frac{2k-1}{2n} \pi \right) \right\}_{k=1}^n$: An asymptotic estimate of $\Lambda_n(T_n)$ was given by Bernstein [1] as

$$(1.7) \quad \Lambda_n(T_n) \sim \frac{2}{\pi} \log n, \quad n \rightarrow \infty,$$

¹Grünwald [24] in 1935 and Marcinkiewicz [43] in 1937, independently, showed that even for the Chebyshev points of first kind

$$x_k^{(n)} = \cos \left(\frac{2k-1}{2n} \pi \right), \quad k = 1, 2, \dots, n, \quad n = 1, 2, \dots,$$

there is a continuous function $f(x)$ in $[-1, 1]$ for which the sequence $L_n[f]$ is divergent everywhere in $[-1, 1]$.

which is improved by Ehlich and Zeller [14], Rivlin [47] and Brutman [7] as

$$\frac{2}{\pi} \left(\gamma_0 + \log \frac{4}{\pi} \right) + \frac{2}{\pi} \log n < \Lambda_n(T_n) \leq 1 + \frac{2}{\pi} \log n, \quad n = 1, 2, \dots$$

- Chebyshev pointsystem of second kind $U_n = \left\{ x_k^{(n)} = \cos \left(\frac{k}{n-1} \pi \right) \right\}_{k=0}^{n-1}$ (also called Chebyshev extreme or Clenshaw-Curtis points [58]): Ehlich and Zeller [14] proved that

$$(1.8) \quad \Lambda_n(U_n) = \begin{cases} \Lambda_{n-1}(T_{n-1}), & n = 2, 4, 6, \dots \\ \Lambda_{n-1}(T_{n-1}) - \alpha_n, & 0 \leq \alpha_n < \frac{1}{(n-1)^2}, \quad n = 3, 5, 7, \dots \end{cases}$$

- The roots of Jacobi polynomial $P_n^{(\alpha, \beta)}(x)$ ($\alpha, \beta > -1$): The asymptotic estimate of $\Lambda_n(J_n)$ was found by Szegö [55] as

$$(1.9) \quad \Lambda_n(J_n) = \begin{cases} O(n^{\gamma+\frac{1}{2}}), & \gamma > -\frac{1}{2} \\ O(\log n), & \gamma \leq -\frac{1}{2} \end{cases}, \quad \gamma = \max\{\alpha, \beta\}.$$

Comparing Equations (1.7), (1.8) and (1.9) with (1.6), we see that the two Chebyshev pointsystems and the Jacobi pointsystem with $\gamma \leq -\frac{1}{2}$ are nearly optimal and of order $O(\log n)$.

Nevertheless, it is worth noting that if $f(x)$ has an absolutely continuous $(k-1)$ st derivative $f^{(k-1)}$ on $[-1, 1]$ for some $k \geq 1$ and its k -th derivative $f^{(k)}$ is of bounded variation $\text{Var}(f^{(k)}) < \infty$, Mastroianni and Szabados [42], Trefethen [59] and Xiang et al. [76] proved that

$$(1.10) \quad \|f - L_n[f]\|_\infty = O(n^{-k}),$$

where $L_n[f]$ is at the n Chebyshev points of first or second kind, which has the same asymptotic order as $\|f - p_{n-1}^*\|_\infty$ for the best approximation p_{n-1}^* , following de la Vallée Poussin [62]. In particular, for $f(x) = |x|$, the error on the $L_n[f]$ at the above two Chebyshev pointsystems satisfies

$$\|f - L_n[f]\|_\infty \leq \frac{4}{\pi(n-1)}$$

(see [59, 76]), while

$$\|f - p_{n-1}^*\|_\infty \sim \frac{\beta}{n}, \quad 0.2801685 < \beta < 0.2801734$$

(see Bernstein [3] and Varga and Capenter [63]). Thus, the error estimate (1.5) by using the Lebesgue constant may be overestimated for some special points of sets for functions of limited regularities.

Moreover, it has been observed, by Clenshaw-Curtis [10] and O'Hara and Smith [29], that n -point Gauss quadrature and n -point Clenshaw-Curtis quadrature have essentially the same accuracy, which has been showed recently by Trefethen [58, 59], Brass and Petras [6] and Xiang and Bornemann [75]. Both of these two quadrature are derived from the interpolation polynomial $L_n[f]$ by

$$Q_n[f] = \int_{-1}^1 L_n[f](x) dx,$$

based on the n Gauss-Legendre and Clenshaw-Curtis points, respectively. From this observation, we may conclude that the corresponding interpolation $L_n[f]$ based on these two pointsystems may have the same convergence rate. However, it can not be derived from (1.5).

In this paper, we present new convergence rates of the interpolation polynomials for functions of limited regularities, based upon the famous Peano kernel theorem [45] and applying an interesting Wainerman's lemma [72]. Suppose $f(x)$ has an absolutely continuous $(r-1)$ st

derivative $f^{(r-1)}$ on $[-1, 1]$, and its r -th derivative $f^{(r)}$ is of bounded variation $\text{Var}(f^{(r)}) < \infty$. We will show that

$$(1.11) \quad \|f - L_n[f]\|_\infty \leq \frac{\pi^r \text{Var}(f^{(r)})}{(n-1)(n-2)\cdots(n-r)} \max_{1 \leq j \leq n} \|\ell_j^{(n)}\|_\infty,$$

which leads to

$$(1.12) \quad \|f - L_n[f]\|_\infty = O(n^{-r} \max_{1 \leq j \leq n} \|\ell_j^{(n)}\|_\infty).$$

The Lebesgue constant $\Lambda_n = \max_{x \in [-1, 1]} \sum_{k=1}^n |\ell_k^{(n)}(x)|$ is replaced by $\max_{1 \leq j \leq n} \|\ell_j^{(n)}\|_\infty$ in some sense since $\|f - p_{n-1}^*\|_\infty = O(n^{-r})$ [62].

Particularly, from (1.12), it directly follows that the interpolation $L_n[f]$ at a strongly normal pointsystem (see Fejér [19]) can achieve the optimal convergence rate as $O(\|f - p_{n-1}^*\|_\infty)$.

Furthermore, $\|\ell_j\|_\infty$ can be explicitly estimated for Gauss-Jacobi, Jacobi-Gauss-Lobatto or Jacobi-Gauss-Radau pointsystems, by using the asymptotics on Jacobi polynomials given by Szegő [55] and some results given in Kelzon [34, 35], Vértesi [66, 68], Sun [54], Prestin [46], Kvernadze [38], Vecchia et al. [69], etc., as follows

- For the n Gauss-Jacobi points:

$$\max_{1 \leq j \leq n} \|\ell_j^{(n)}\|_\infty = O(n^{\max\{\gamma - \frac{1}{2}, 0\}}), \quad \gamma = \max\{\alpha, \beta\}.$$

- For the n Jacobi-Gauss-Lobatto points (the roots of $(1-x^2)P_{n-2}^{(\alpha, \beta)}(x) = 0$):

$$\max_{1 \leq j \leq n} \|\ell_j^{(n)}\|_\infty = \begin{cases} O\left(n^{-\min\{0, \alpha + \frac{1}{2}, \beta + \frac{1}{2}\}}\right), & -1 < \alpha, \beta \leq \frac{3}{2} \\ O\left(n^{-\min\{0, \alpha + \frac{1}{2}, 2 + \alpha - \beta, \frac{5}{2} - \beta\}}\right), & -1 < \alpha \leq \frac{3}{2}, \beta > \frac{3}{2} \\ O\left(n^{-\min\{0, \beta + \frac{1}{2}, 2 + \beta - \alpha, \frac{5}{2} - \alpha\}}\right), & \alpha > \frac{3}{2}, -1 < \beta \leq \frac{3}{2} \\ O\left(n^{-\min\{0, 2 + \alpha - \beta, 2 + \beta - \alpha, \frac{5}{2} - \alpha, \frac{5}{2} - \beta\}}\right), & \alpha, \beta > \frac{3}{2} \end{cases}.$$

- For the n Jacobi-Gauss-Radau points $(1-x)P_{n-1}^{(\alpha, \beta)}(x)$

$$\max_{0 \leq j \leq n-1} \|\ell_j^{(n)}\|_\infty = \begin{cases} O\left(n^{-\min\{0, \alpha + \frac{1}{2}, \alpha - \beta\}}\right), & -1 < \alpha \leq \frac{1}{2} \\ O\left(n^{-\min\{0, \frac{1}{2} - \beta, \frac{5}{2} - \alpha, \alpha - \beta\}}\right), & \alpha > \frac{1}{2} \end{cases}.$$

- For the n Jacobi-Gauss-Radau points $(1+x)P_{n-1}^{(\alpha, \beta)}(x)$

$$\max_{1 \leq j \leq n} \|\ell_j^{(n)}\|_\infty = \begin{cases} O\left(n^{-\min\{0, \beta + \frac{1}{2}, \beta - \alpha\}}\right), & -1 < \beta \leq \frac{1}{2} \\ O\left(n^{-\min\{0, \frac{1}{2} - \alpha, \frac{5}{2} - \beta, \beta - \alpha\}}\right), & \beta > \frac{1}{2} \end{cases}.$$

From the above estimates, we see that the interpolation at the Gauss-Legendre or at the Legendre-Gauss-Lobatto pointsystem, has essentially the same approximation accuracy compared with those at the two Chebyshev pointsystems. All of them satisfy that $\max_{1 \leq j \leq n} \|\ell_j^{(n)}\|_\infty = O(1)$ (for more general cases see FIG. 1.1). In addition, the convergence rate is attainable illustrated by some functions of limited regularities.

Thus, the best approximation polynomial is challenged by the interpolation polynomials at the special pointsystems showed in FIG. 1.1. Furthermore, we will see that the interpolation polynomials at the special pointsystems perform much better than the best approximation polynomial for approximation the derivatives f' and f'' by $L_n'[f]$, $L_n''[f]$, $[p_{n-1}^*]'$ and $[p_{n-1}^*]''$, respectively, illustrated by numerical examples in the final section.

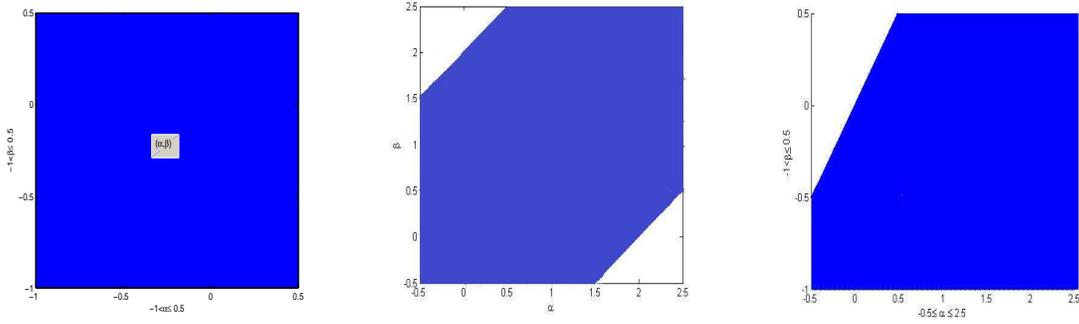


FIG. 1.1. The neighbourhood on (α, β) such that $\|f - L_n[f]\|_\infty = O(n^{-r})$ for the Gauss-Jacobi pointsystems (left), Jacobi-Gauss-Lobatto pointsystems (middle) and Jacobi-Gauss-Radau $((1-x)P_{n-1}^{(\alpha, \beta)}(x) = 0)$ pointsystems (right), respectively: $\text{Var}(f^{(r)}) < \infty$.

It is worthy of special mention that the interpolation polynomial $L_n[f]$, at the Gauss-Jacobi, Jacobi-Gauss-Lobatto or Gauss-Jacobi-Radau pointsystem, can be efficiently evaluated by applying the second barycentric formula

$$L_n[f](x) = \frac{\sum_{j=1}^n \frac{\lambda_j}{x-x_j} f(x_j)}{\sum_{j=1}^n \frac{\lambda_j}{x-x_j}},$$

which is robust in the presence of rounding errors [33] and costs overall computational complexity $O(n)$ [4], where the nodes x_j and the barycentric weights λ_j are computed by **jacpts** and the formulas given in [28, 70, 71], respectively. A MATLAB routine **jacpts**, which uses the algorithm in [26] for the computation of these nodes and weights, can be found in CHEBFUN system [60]. For more details on this topic, see Salzer [50], Henrich [30], Berrut and Trefethen [4], Higham [32, 33], Glaser et al. [23], Wang and Xiang [70], Bogaert et al. [5], Hale and Trefethen [28], Hale and Townsend [26], Trefethen [59], Wang et al. [71], etc. MATLAB routines can be found in CHEBFUN system [60] and Xiang and He [77].

The paper is organized as follows: In section 2, we present the error of $f(x) - L_n[f](x)$ for each fixed $x \in [-1, 1]$ by using the Peano representation and the bounded variation. In section 3, we introduce the interesting Wainerman's lemma and deduce the error bound on $\|f - L_n[f]\|_\infty$ by $\max_{1 \leq j \leq n} \|\ell_j^{(n)}\|_\infty$. We consider, in section 4, the estimates of $\|\ell_j^{(n)}\|_\infty$ and derive the convergence rates for the interpolation polynomial at strongly normal pointsystems, Gauss-Jacobi, Jacobi-Gauss-Lobatto and Jacobi-Gauss-Radau pointsystems, respectively, where the convergence rates and attainability are illustrated by numerical experiments.

Throughout this paper, $A \sim B$ means that there exist positive constants C_1 and C_2 such that

$$C_1 B \leq A \leq C_2 B.$$

For simplicity, in the following we abbreviate $x_k^{(n)}$ as x_k and $\ell_k^{(n)}(x)$ as $\ell_k(x)$.

All the numerical results in this paper are carried out by using MATLAB R2012a on a desktop (2.8 GB RAM, 2 Core2 (32 bit) processors at 2.80 GHz) with Windows XP operating system.

2. The Peano kernel theorem. There are two general methods for deriving strict error bounds (Dahlquist and Björck [11]). One applies the norms and distance formula together with the Lebesgue constants, which often overestimates the error. The other is due to the Peano kernel theorem.

Suppose \mathcal{L} a continuously linear functional that maps functions $f \in C([-1, 1])$ to R satisfying $\mathcal{L}(f_1 + f_2) = \mathcal{L}f_1 + \mathcal{L}f_2$ for any $f_1, f_2 \in C([-1, 1])$ and $\mathcal{L}(\alpha f) = \alpha \mathcal{L}f$ for any scalar α .

In addition, we assume $\mathcal{L}[\mathcal{P}_{r-1}] = \{0\}$ for some $r \in \{1, 2, \dots\}$, where \mathcal{P}_{r-1} denotes the set of polynomials with degree less than or equal to $r - 1$.

The Peano kernel theorem (Peano [45], see also Kowalewski [36], Schmidt [51] and Mises [44]) is the identity

$$(2.1) \quad \mathcal{L}[f] = \int_{-1}^1 f^{(r)}(t) K_r(t) dt$$

holding for all such functions $f \in C^r([-1, 1])$, where $K_r(t) = \frac{1}{(r-1)!} \mathcal{L}[(x-t)_+^{r-1}]$ and

$$(2.2) \quad (x-t)_+^{r-1} = \begin{cases} (x-t)^{r-1}, & x \geq t \\ 0, & x < t \end{cases} \quad (r \geq 2), \quad (x-t)_+^0 = \begin{cases} 1, & x \geq t \\ 0, & x < t. \end{cases} \quad (r = 1).$$

For each fixed $x \in [-1, 1]$, we consider the special functional $\mathcal{L} = E_n$, where $E_n[f](x)$ is defined for $\forall f \in C([-1, 1])$ by

$$E_n[f](x) = f(x) - \sum_{j=1}^n f(x_j) \ell_j(x) = f(x) - L_n[f](x)$$

with $-1 \leq x_n < x_{n-1} < \dots < x_2 < x_1 \leq 1$. $E_n[f]$ is a continuously linear functional since $|E_n[f](x) - E_n[g](x)| \leq (1 + \Lambda_n) \|f - g\|_\infty$ for arbitrary $f, g \in C([-1, 1])$, and then by the Peano theorem [45] $E_n[f]$ can be represented if $f \in C^r([-1, 1])$ for $n \geq r$ as

$$(2.3) \quad E_n[f](x) = \int_{-1}^1 f^{(r)}(t) K_r(t) dt$$

with

$$(2.4) \quad K_r(t) = \frac{1}{(r-1)!} (x-t)_+^{r-1} - \frac{1}{(r-1)!} \sum_{j=1}^n (x_j - t)_+^{r-1} \ell_j(x).$$

Particularly, from (2.3) it implies

$$|E_n[f](x)| \leq \|f^{(r)}\|_\infty \int_{-1}^1 |K_r(t)| dt \leq 2 \|f^{(r)}\|_\infty \|K_r\|_\infty.$$

Similar to the Peano kernel for quadrature [6], the kernel for interpolation satisfies the following proposition.

PROPOSITION 2.1. (*Peano representation*) Let

$$(2.5) \quad K_s(t) = \frac{1}{(s-1)!} (x-t)_+^{s-1} - \frac{1}{(s-1)!} \sum_{j=1}^n (x_j - t)_+^{s-1} \ell_j(x), \quad s = 1, 2, \dots$$

Then for $s \geq 2$, the Peano kernel satisfies $K_s(-1) = K_s(1) = 0$ and can be rewritten as

$$(2.6) \quad K_s(u) = \int_u^1 K_{s-1}(t) dt, \quad s = 2, 3, \dots$$

Proof. From the definition of K_s in (2.5), it is easy to verify that $K_s(-1) = K_s(1) = 0$ by using $\sum_{j=1}^n \ell_j(t) \equiv 1$ for $t \in [-1, 1]$. Furthermore, we find that

$$(2.7) \quad \begin{aligned} \frac{1}{(s-2)!} \int_u^1 (x-t)_+^{s-1} dt &= \begin{cases} 0, & u > x \\ \frac{1}{(s-2)!} \int_u^x (x-t)^{s-1} dt = \frac{1}{(s-1)!} (x-u)^{s-1}, & u \leq x \end{cases} \\ &= \frac{1}{(s-1)!} (x-u)_+^{s-1}. \end{aligned}$$

Define $x_0 = 1$ and $x_{n+1} = -1$ and suppose $x_{m+1} < u \leq x_m$ for some nonnegative integer m . By (2.7), similarly we have

$$(2.8) \quad \begin{aligned} \frac{1}{(s-2)!} \sum_{j=1}^n \int_u^1 (x_j - t)_+^{s-2} \ell_j(x) dt &= \frac{1}{(s-2)!} \sum_{j=1}^m \ell_j(x) \begin{cases} 0, & u > x_j \\ \int_u^{x_j} (x_j - t)^{s-2} dt, & u \leq x_j \end{cases} \\ &= \frac{1}{(s-1)!} \sum_{j=1}^n (x_j - u)_+^{s-1} \ell_j(x). \end{aligned}$$

Then from

$$\int_u^1 K_{s-1}(t) dt = \frac{1}{(s-2)!} \int_u^1 (x-t)_+^{s-2} dt - \frac{1}{(s-2)!} \sum_{j=1}^n \int_u^1 (x_j - t)_+^{s-2} \ell_j(x) dt,$$

we get $\int_u^1 K_{s-1}(t) dt = K_s(u)$ by (2.7) and (2.8). \square

In the following, we consider functions of limited regularities as

$$(2.9) \quad \begin{aligned} &\text{Suppose that } f(t) \text{ has an absolutely continuous } (r-1)\text{st derivative } f^{(r-1)} \text{ on } [-1, 1] \\ &\text{for some } r \geq 1 \text{ with } f^{(r-1)}(t) = f^{(r-1)}(-1) + \int_{-1}^t g(y) dy, \text{ where } g \text{ is absolutely} \\ &\text{integrable and of bounded variation } \text{Var}(g) < \infty \text{ on } [-1, 1]. \end{aligned}$$

From Stein and Shakarchi [53, p. 130] and Tao [56, pp. 143-145], we see that a function $G : [-1, 1] \rightarrow R$ is absolutely continuous if and only if it takes the form $G(t) = \int_{-1}^t g(y) dy + C$ for some absolutely integrable $g : [-1, 1] \rightarrow R$ and a constant C . It is obvious that such g is not unique. Then in this paper, we suppose $f(t)$ satisfies (2.9) and define

$$V_r = \inf \left\{ \text{Var}(g) \mid \begin{array}{l} f^{(r-1)}(t) = f^{(r-1)}(-1) + \int_{-1}^t g(y) dy \text{ for all } t \in [-1, 1] \text{ with } g \text{ being} \\ \text{absolutely integrable and of bounded variation} \end{array} \right\}.$$

REMARK 1. Here, we use the condition “ $f^{(r-1)}(t) = f^{(r-1)}(-1) + \int_{-1}^t g(y) dy$, where g is absolutely integrable and of bounded variation $\text{Var}(g) < \infty$ ” instead of “ $f^{(r)}$ is of bounded variation $V_r = \text{Var}(f^{(r)}) < \infty$ ” in [58, 59]. If $f^{(r)}$ is of bounded variation, then $f^{(r+1)}$ exists almost everywhere and $f^{(r+1)} \in L^1([-1, 1])$ (see Lang [39] and Rudin [48]). Whereas, $f^{(r)}$ in [58, 59] denotes an equivalent representation in the sense of almost everywhere. An example for $f(x) = |x|$ is given in [58, 59], where $f(t)$ is not differentiable at $t = 0$, but f' can be chosen as

$$f'(t) = \begin{cases} 1, & t > 0 \\ c, & t = 0 \\ -1, & t < 0 \end{cases},$$

then $\text{Var}(f') = \begin{cases} 2, & |c| \leq 1 \\ |1+c| + |1-c|, & \text{otherwise} \end{cases}$. Using the new condition, we see that $|t|$ can

be represented as $|t| = 1 + \int_{-1}^t g(y) dy$ with $g(y) = \begin{cases} 1, & y > 0 \\ c, & y = 0 \\ -1, & y < 0 \end{cases}$ and $V_1 = 2$ is unique.

THEOREM 2.2. *Suppose $f(t)$ satisfies (2.9), then for $n \geq r$, we have*

$$(2.10) \quad \|E_n[f]\|_\infty \leq V_r \|K_{r+1}\|_\infty.$$

Proof. Applying the Peano theorem implies that for each fixed $x \in [-1, 1]$,

$$E_n[f](x) = \int_{-1}^1 f^{(s)}(t) K_s(t) dt, \quad s = 1, 2, \dots, r-1.$$

Then, directly following Brass and Petras [6], integrating by parts and using $K_r(-1) = K_r(1) = 0$ yields

$$E_n[f](x) = \int_{-1}^1 f^{(r-1)}(t)K_{r-1}(t)dt = \int_{-1}^1 g(t)K_r(t)dt.$$

Since g can be written as $g = g_1 - g_2$ with g_1 and g_2 are monotonically increasing, and $\text{Var}(g) = \text{Var}(g_1) + \text{Var}(g_2)$ (see Lang [39, pp. 280-281]). Without loss of generality, assume g is monotonically increasing. Then by the second mean value theorem of integral calculus, it follows from $K_{r+1}(-1) = \int_{-1}^1 K_r(t)dt = 0$ that there exists a $\xi \in [-1, 1]$ such that

$$E_n[f](x) = g(-1) \int_{-1}^{\xi} K_r(t)dt + g(1) \int_{\xi}^1 K_r(t)dt = (g(1) - g(-1))K_{r+1}(\xi) = \text{Var}(g)K_{r+1}(\xi),$$

which leads to the desired result. \square

LEMMA 2.3. [6, Lemma 5.7.1] Assume that

$$\sup_{-1 \leq t \leq 1} w(t)\sqrt{1-t^2} < \infty, \quad t_u(y) = \begin{cases} 0, & y < u \\ 1, & y \geq u \end{cases}.$$

Then, for every positive integer ℓ and every $u \in [-1, 1]$, there is a $q_u \in \mathcal{P}_\ell$ satisfying

$$q_u(y) \geq t_u(y) \quad \text{for all } y \in [-1, 1]$$

and

$$\int_{-1}^1 [t_u(y) - q_u(y)] w(y)dy \geq -\frac{\pi}{\ell+1} \sup_{-1 \leq t \leq 1} w(t)\sqrt{1-t^2}.$$

LEMMA 2.4.

$$(2.11) \quad |K_{s+1}(u)| \leq \frac{\pi}{n-s+1} \sup_{-1 \leq t \leq 1} |K_s(t)|.$$

Proof. In Lemma 2.3, letting $\ell = n - s - 1$, $w(t) \equiv 1$, representing q_u as $q_u(t) = p_{n-1}^{(s)}(t)$, and noting that $E_n[\mathcal{P}_{n-1}] = 0$, by Theorem 2.2 we have

$$0 = E_n[p_{n-1}] = \int_{-1}^1 p_{n-1}^{(s)}(t)K_s(t)dt = \int_{-1}^1 q_u(t)K_s(t)dt.$$

Consequently, by Lemma 2.3 we get that

$$\begin{aligned} |K_{s+1}(u)| &= \left| \int_u^1 K_s(t)dt \right| = \left| \int_{-1}^1 K_s(t)t_u(t)dt \right| = \left| \int_{-1}^1 K_s(t)[t_u(t) - q_u(t)]dt \right| \\ &\leq \frac{\pi}{n-s} \sup_{-1 \leq t \leq 1} |K_s(t)|. \end{aligned}$$

\square

From Theorem 2.2 and Lemma 2.4 we obtain that

THEOREM 2.5. Suppose $f(t)$ satisfies (2.9), then for $n \geq r + 1$

$$(2.12) \quad \|E_n[f]\|_\infty \leq \frac{\pi^r V_r}{(n-1)(n-2)\cdots(n-r)} \|K_1\|_\infty.$$

3. Wainerman's lemma. In the following, we shall focus on the estimate of $\|K_1\|_\infty$. Notice that $\sum_{j=1}^n \ell_j(t) \equiv 1$ for $t \in [-1, 1]$ and

$$(3.1) \quad K_1(u) = (x - u)_+^0 - \sum_{j=1}^n (x_j - u)_+^0 \ell_j(x).$$

If $x_1 < u \leq 1$, we have $K_1(u) = 1$ for $u \leq x$, and $K_1(u) = 0$ for $u > x$. While for $-1 < u \leq x_n$, we have $K_1(u) = 0$ for $u \leq x$, and $K_1(u) = -1$ for $u > x$. Thus, in these cases we obtain

$$(3.2) \quad |K_1(u)| \leq 1 \leq \max_{1 \leq j \leq n} \|\ell_j\|_\infty$$

since $\ell_j(x_j) = 1$ for $j = 1, 2, \dots, n$.

Suppose that $x_{m+1} < u \leq x_m$ for some positive integer m , then for $u \leq x$ we get

$$(3.3) \quad K_1(u) = 1 - \sum_{j=1}^n (x_j - u)_+^0 \ell_j(x) = 1 - \sum_{j=1}^m \ell_j(x) = \sum_{j=m+1}^n \ell_j(x),$$

while for $u > x$ we have

$$(3.4) \quad K_1(u) = - \sum_{j=1}^n (x_j - u)_+^0 \ell_j(x) = - \sum_{j=1}^m \ell_j(x).$$

LEMMA 3.1. (*Wainerman's lemma [72]*) Suppose $x_{m+1} < u \leq x_m$ for some positive integer m , and let

$$a_k(u) = \begin{cases} \sum_{j=1}^k \ell_j(u), & k = 1, 2, \dots, m \\ \sum_{j=k}^n \ell_j(u), & k = m+1, m+2, \dots, n \end{cases}$$

and $a_0(u) = a_{n+1}(u) \equiv 0$. Then it follows for $x_{m+1} < u < x_m$ that

$$(3.5) \quad \text{sgn}(a_k(u)) = \text{sgn}(\ell_k(u)) = \begin{cases} (-1)^{m-k}, & k = 1, 2, \dots, m \\ (-1)^{k-m-1}, & k = m+1, m+2, \dots, n \end{cases}$$

and for $x_{m+1} < u \leq x_m$ that

$$(3.6) \quad |a_k(u)| \leq |\ell_k(u)|, \quad k = 1, 2, \dots, n,$$

where sgn denotes the sign function.

Proof. The interesting result and its proof is published in Russian in [72]. For convenience and completeness, we present the proof here.

For $x_{m+1} < u < x_m$, from the definition of $\ell_k(t)$ we see that

$$\begin{aligned} \text{sgn}(\ell_k(u)) &= \text{sgn} \left(\frac{(u - x_1) \cdots (u - x_{k-1})(u - x_{k+1}) \cdots (u - x_n)}{(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)} \right) \\ &= (-1)^{1-k} \text{sgn}((u - x_1) \cdots (u - x_{k-1})(u - x_{k+1}) \cdots (u - x_n)), \end{aligned}$$

which directly leads to the desired result (3.5) for $\text{sgn}(\ell_k(u))$ based on $k \leq m$ or $k > m$, respectively.

In the following, we will show that $\text{sgn}(a_k(u))$ also satisfies (3.5).

In the case $k \leq m$: Since

$$a_k(x_j) = \sum_{i=1}^k \ell_i(x_j) = \begin{cases} 1, & j = 1, 2, \dots, k \\ 0, & j = k+1, k+2, \dots, n \end{cases},$$

then by the Rolle's theorem it follows

$$a'_k(y_j) = 0$$

for some y_j satisfying $x_{j+1} < y_j < x_j$ for $j = 1, \dots, k-1, k+1, \dots, n-1$.

$$a_k(x_k) = 1 \underset{*}{a_k(x_{k-1})} = 1 \underset{*}{a_k(x_2)} = 1 \underset{*}{a_k(x_1)} = 1$$

Note that $a_k(t)$ is a polynomial of degree $n - 1$, then $a'_k(t)$ is a polynomial of degree $n - 2$, which implies that y_j are the exact zeros of $a'_k(t)$ and then $a'_k(t)$ has the form of

$$(3.7) \quad a'_k(t) = C(t - y_1) \cdots (t - y_{k-1})(t - y_{k+1}) \cdots (t - y_{n-1})$$

for some non-zero constant C . In addition, from (3.7) $a'_k(t)$ has alternative sign between these roots. Then, by $a_k(x_{k+1}) = 0$ and $a_k(x_k) = 1$, it yields

$$a'_k(t) > 0, \quad t \in (y_{k+1}, y_{k-1})$$

and

$$\text{sgn}(a_k(t)) = 1, \quad t \in (x_{k+1}, x_k) \subset (y_{k+1}, y_{k-1})$$

since $a(t)$ is strictly increasing in (x_{k+1}, x_k) and $a_k(x_{k+1}) = 0$.

By the alternative property of $a'_k(t)$ between these roots, it deduces that $\text{sgn}(a'_k(t)) = (-1)^{j-k}$ for $t \in (y_{j+1}, y_j)$ and $j > k$, particularly,

$$\text{sgn}(a'_k(t)) = 1, \quad t \in (y_{k+1}, x_{k+1}) \subset (y_{k+1}, y_{k-1})$$

and

$$\text{sgn}(a'_k(t)) = -1, \quad t \in (x_{k+2}, y_{k+1}) \subset (y_{k+2}, y_{k+1}),$$

which, together with $a_k(x_{k+1}) = a_k(x_{k+2}) = 0$, derives $\text{sgn}(a_k(t)) = -1$ for $t \in (x_{k+2}, x_{k+1})$. Similarly, applying

$$\text{sgn}(a'_k(t)) = (-1)^{j-k}, \quad t \in (y_{j+1}, x_{j+1}); \quad \text{sgn}(a'_k(t)) = (-1)^{j-k+1}, \quad t \in (x_{j+2}, y_{j+1}),$$

together with $a_k(x_{j+2}) = a_k(x_{j+1}) = 0$ for $j > k$, derives $\text{sgn}(a_k(t)) = (-1)^{j-k+1}$ for $j > k$ and $t \in (x_{j+2}, x_{j+1})$ by induction. So we get $a_k(u) = (-1)^{m-k}$.

In the case $k > m$: By

$$a_k(x_j) = \sum_{i=k}^n \ell_i(x_j) = \begin{cases} 0, & j = 1, 2, \dots, k-1 \\ 1, & j = k, k+1, \dots, n \end{cases},$$

applying similar arguments derives $a_k(u) = (-1)^{k-m-1}$ for $k > m$.

Furthermore, from (3.5) and the definition of $a_k(t)$, we see that immediately: for $k \leq m$ and $x_{m+1} < u < x_m$,

$$|a_k(u)| = |\ell_k(u) + a_{k-1}(u)| = |\ell_k(u)| - |a_{k-1}(u)| \leq |\ell_k(u)|,$$

and for $k > m$

$$|a_k(u)| = |\ell_k(u) + a_{k+1}(u)| = |\ell_k(u)| - |a_{k+1}(u)| \leq |\ell_k(u)|.$$

The special case of (3.6) for $u = x_m$ directly follows from $|a_k(x_m)| = |\ell_k(x_m)|$ by the definitions of $a_k(u)$ and $\ell_k(u)$. \square

Theorem 2.5 together with (3.2), (3.3), (3.4) and (3.6) leads to the following estimate.

THEOREM 3.2. *Suppose $f(t)$ satisfies (2.9), then for $n \geq r + 1$*

$$(3.8) \quad \|E_n[f]\|_\infty \leq \frac{\pi^r V_r}{(n-1)(n-2) \cdots (n-r)} \max_{1 \leq j \leq n} \|\ell_j\|_\infty.$$

In the next section, we shall focus on estimates of $\max_{1 \leq j \leq n} \|\ell_j\|_\infty$ for special points of sets.

4. Estimates of $\|l_j\|_\infty$ and convergence rates on $\|f - L_n[f]\|_\infty$. For any convergent quadrature derived from polynomial interpolation at the grid points (1.1) for

$$\int_{-1}^1 f(x)w(x)dx = \int_{-1}^1 f(x)d\sigma(x)$$

for each $\sigma(x)$ of bounded variation and any analytic function $f(x)$ on $[-1, 1]$, the clustering of the n points has a limiting Chebyshev distribution

$$\mu(t) = \frac{1}{\pi} \int_{-1}^t \frac{1}{\sqrt{1-x^2}} dx$$

(see Krylov [37, Theorem 7, p. 263]); that is, the clustering will be asymptotically the same: on $[-1, 1]$, n points will be distributed with density

$$\frac{n}{\pi\sqrt{1-x^2}}$$

as n tends to infinity (see Hale and Trefethen [27] and Trefethen [58]).

Moreover, the clustering of optimal pointsystems for polynomial interpolation implies near endpoints ± 1 (see Z. Ditzian and V. Totik [13] and [58]). (The Gauss-Jacobi type pointsystems have this proposition.) The density of the zeros of orthogonal polynomials has been extensively studied in Erdős and Turán [16, 17], Gatteschi [22] and Szegő [55].

4.1. Strongly normal pointsystems. One of the proofs of Weierstrass approximation theorem using interpolation polynomials was first presented by Fejér [19] in 1916 based on the Chebyshev pointsystem of first kind $\{x_k = \cos(\frac{2k-1}{2n}\pi)\}_{k=1}^n$: If $f \in C([-1, 1])$, then there is a unique polynomial $H_{2n-1}(f, t)$ of degree at most $2n-1$ such that $\lim_{n \rightarrow \infty} \|H_{2n-1}(f) - f\|_\infty = 0$, where $H_{2n-1}(f, t)$ is determined by

$$(4.1) \quad H_{2n-1}(f, x_k) = f(x_k), \quad H'_{2n-1}(f, x_k) = 0, \quad k = 1, 2, \dots, n.$$

This polynomial is known as the Hermite-Fejér interpolation polynomial.

The convergence result has been extended to general Hermite-Fejér interpolation of $f(x)$ at nodes (1.1) by Grünwald [25] in 1942, upon strongly normal pointsystems introduced in Fejér [20]: Given, respectively, the function values $f(x_1), f(x_2), \dots, f(x_n)$ and derivatives d_1, d_2, \dots, d_n at these grids, the general Hermite-Fejér interpolation polynomial $H_{2n-1}(f)$ has the form of

$$(4.2) \quad H_{2n-1}(f, t) = \sum_{k=1}^n f(x_k)h_k(t) + \sum_{k=1}^n d_k b_k(t),$$

where $h_k(t) = v_k(t) (\ell_k(t))^2$, $b_k(t) = (t - x_k) (\ell_k(t))^2$ and

$$(4.3) \quad v_k(t) = 1 - (t - x_k) \frac{\omega_n''(x_k)}{\omega_n'(x_k)} \quad (\text{see Fejér [21]}).$$

The pointsystem (1.1) is called strongly normal if for all n

$$(4.4) \quad v_k(t) \geq c > 0, \quad k = 1, 2, \dots, n, \quad t \in [-1, 1]$$

for some positive constant c . The pointsystem (1.1) is called normal if for all n

$$(4.5) \quad v_k(t) \geq 0, \quad k = 1, 2, \dots, n, \quad t \in [-1, 1].$$

Fejér [20] (also see Szegő [55, p. 339]) showed that for the zeros of Jacobi polynomial $P_n^{(\alpha, \beta)}(t)$ of degree n ($\alpha > -1, \beta > -1$)

$$(4.6) \quad v_k(t) \geq \min\{-\alpha, -\beta\} \quad \text{for } -1 < \alpha \leq 0, -1 < \beta \leq 0, k = 1, 2, \dots, n \text{ and } t \in [-1, 1].$$

While for the Legendre-Gauss-Lobatto pointsystem (the roots of $(1-t^2)P_{n-2}^{(1,1)}(t) = 0$),

$$(4.7) \quad v_k(t) \geq 1, \quad k = 1, 2, \dots, n, \quad t \in [-1, 1] \text{ (Fejér [21])}.$$

These results have been extended to Jacobi-Gauss-Lobatto pointsystem (the roots of $(1-t^2)P_{n-2}^{(\alpha,\beta)}(t) = 0$) and Jacobi-Gauss-Radau pointsystem (the roots of $(1-t)P_{n-1}^{(\alpha,\beta)}(t) = 0$ or $(1+t)P_{n-1}^{(\alpha,\beta)}(t) = 0$) by Vértési [64, 65]: for all k and $t \in [-1, 1]$,

$$(4.8) \quad v_k(t) \geq \min\{2 - \alpha, 2 - \beta\} \quad \text{for } \{x_k\} \cup \{-1, 1\} \text{ with } 1 \leq \alpha \leq 2 \text{ and } 1 \leq \beta \leq 2,$$

$$(4.9) \quad v_k(t) \geq \min\{2 - \alpha, -\beta\} \quad \text{for } \{x_k\} \cup \{1\} \text{ with } 1 \leq \alpha \leq 2 \text{ and } -1 < \beta \leq 0,$$

$$(4.10) \quad v_k(t) \geq \min\{-\alpha, 2 - \beta\} \quad \text{for } \{x_k\} \cup \{-1\} \text{ with } -1 < \alpha \leq 0 \text{ and } 1 \leq \beta \leq 2.$$

PROPOSITION 4.1. (i) [20, 55] *The Gauss-Jacobi pointsystem is strongly normal if and only if $\max\{\alpha, \beta\} < 0$.*

(ii) [64, 65] *The Jacobi-Gauss-Lobatto pointsystem is strongly normal if and only if $1 \leq \alpha < 2$ and $1 \leq \beta < 2$.*

(iii) [64, 65] *The Jacobi-Gauss-Radau pointsystem including $x_1 = 1$ is strongly normal if and only if $1 \leq \alpha < 2$ and $-1 \leq \beta < 0$, and the Jacobi-Gauss-Radau pointsystem including $x_n = -1$ is strongly normal if and only if $-1 < \alpha < 0$ and $1 \leq \beta < 2$.*

It is worth noticing that if the pointsystem is strongly normal, then it implies $v_i(t) \geq c > 0$ for all $i = 1, 2, \dots, n$ and $t \in [-1, 1]$, and

$$(4.11) \quad 1 \equiv \sum_{i=1}^n h_i(t) = \sum_{i=1}^n v_i(t) \ell_i^2(t) \geq c \sum_{i=1}^n \ell_i^2(t)$$

(see [20]) and then

$$\|\ell_i\|_\infty \leq \frac{1}{\sqrt{c}}, \quad i = 1, 2, \dots, n.$$

THEOREM 4.2. *Suppose $f(t)$ satisfies (2.9) and $\{x_j\}_{j=1}^n$ is a strongly normal pointsystem, then for $n \geq r + 1$*

$$(4.12) \quad \|E_n[f]\|_\infty \leq \frac{\pi^r V_r}{\sqrt{c}(n-1)(n-2)\cdots(n-r)}.$$

Following de la Vallée Poussin [62], the error bound indicates that $\|f - L_n[f]\|_\infty$ has the same asymptotic order as the estimate of $\|f - p_{n-1}^*\|_\infty$ for the interpolant at a strongly normal pointsystem for a functions of limited regularity with $V_r < \infty$ for some $r \geq 1$.

To check the error bounds in Theorem 4.2 numerically, we consider two limited regularity functions: $f(x) = |x|$ ($V_1 < \infty$) and $f(x) = |x|^3$ ($V_3 < \infty$). All (α, β) are generated by `rand(1, 2)`² except for $(\alpha, \beta) = (-0.5, -0.5)$, $(\alpha, \beta) = (0, 0)$, $(\alpha, \beta) = (1, 1)$ or $(\alpha, \beta) = (1.5, 1.5)$. Particularly, we used `-rand(1, 2)` in FIGS. 4.1-4.2 for strongly normal Gauss-Jacobi pointsystems, while `rand(1, 2) + 1` in FIGS. 4.3-4.4 for strongly normal Jacobi-Gauss-Lobatto pointsystems. In FIGS. 4.5-4.6, we used `(rand(1) + 1, -rand(1))` (1st row) and `(-rand(1), rand(1) + 1)` (2nd row) for strongly normal Jacobi-Gauss-Radau pointsystems, respectively.

From FIGS. 4.1-4.6, we see that these convergence rates are in conformity to the estimates and attainable.

²`rand(m, n)` returns an m-by-n matrix containing pseudorandom values drawn from the standard uniform distribution on the open interval (0, 1).

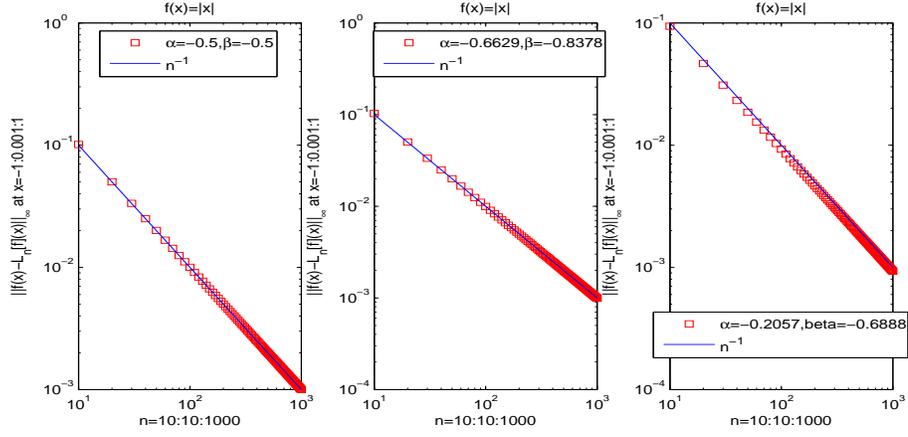


FIG. 4.1. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ with $n = 10 : 10 : 1000$ at the strongly normal Gauss-Jacobi pointsystems for $f(x) = |x|$, respectively.

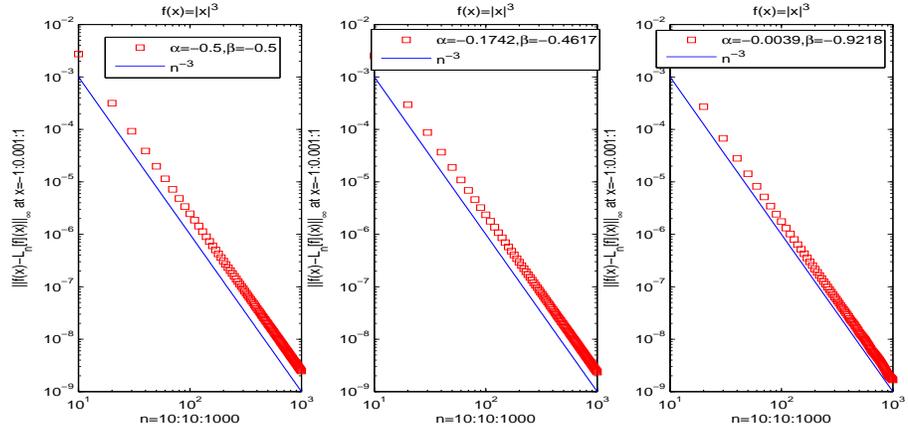


FIG. 4.2. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ with $n = 10 : 10 : 1000$ at the strongly normal Gauss-Jacobi pointsystems for $f(x) = |x|^3$, respectively.

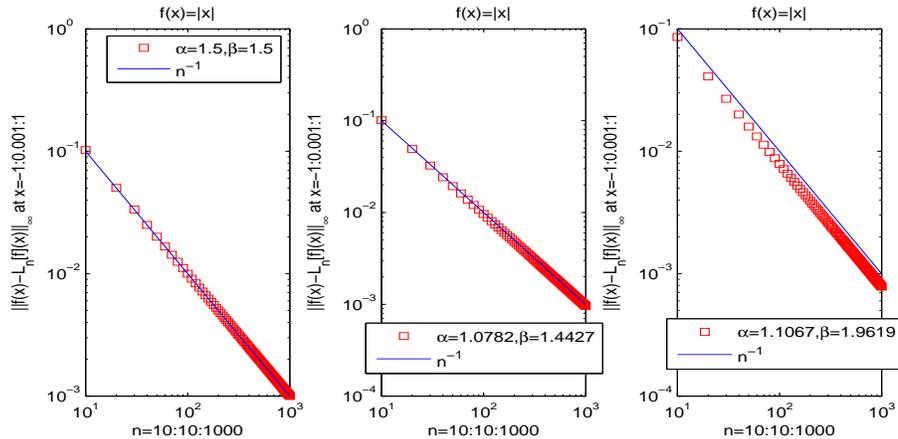


FIG. 4.3. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ with $n = 10 : 10 : 1000$ at the strongly normal Jacobi-Gauss-Lobatto pointsystems for $f(x) = |x|$, respectively.

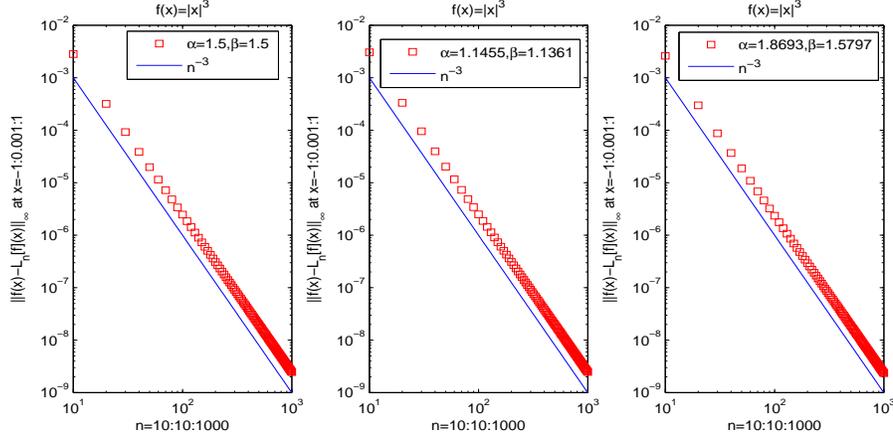


FIG. 4.4. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ with $n = 10 : 10 : 1000$ at the strongly normal Jacobi-Gauss-Lobatto pointsystems for $f(x) = |x|^3$, respectively.

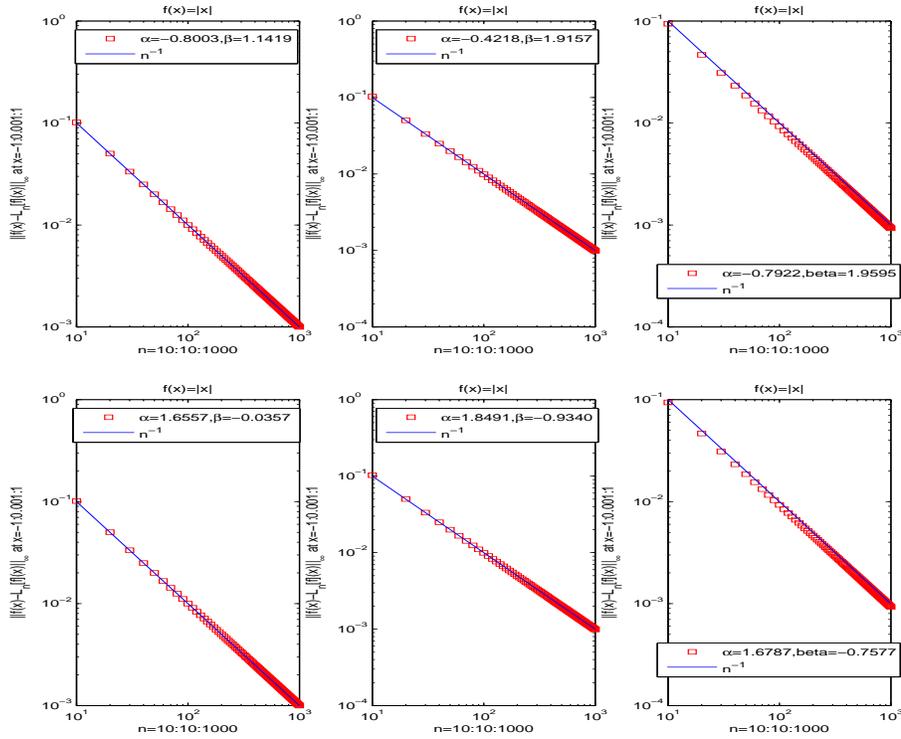


FIG. 4.5. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ with $n = 10 : 10 : 1000$ at the strongly normal Jacobi-Gauss-Radau pointsystems including -1 (1st row) and 1 (2nd row) for $f(x) = |x|$, respectively.

4.2. General Gauss-Jacobi pointsystems. In this subsection, we will consider convergence rates for general Gauss-Jacobi pointsystems, which includes the corresponding strongly normal pointsystems ($-1 < \alpha, \beta < 0$) as special cases.

Let $\{x_k\}_{k=1}^n$ be the roots of the Jacobi polynomial $P_n^{(\alpha, \beta)}(t)$ ($\alpha, \beta > -1$) and $x_k = \cos(\theta_k)$. Then from Szegő [55], it follows

$$(4.13) \quad P_n^{(\alpha, \beta)}(t) = (-1)^n P_n^{(\beta, \alpha)}(-t) \quad ([55, (4.1.3)])$$

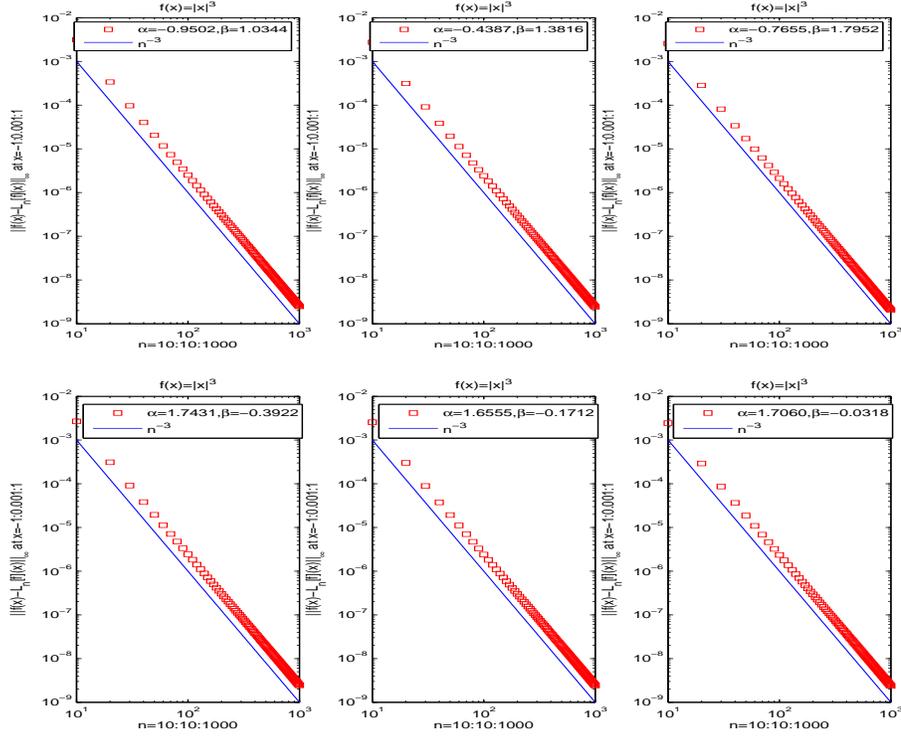


FIG. 4.6. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ with $n = 10 : 10 : 1000$ at the strongly normal Jacobi-Gauss-Radau pointsystems including -1 (1st row) and 1 (2nd row) for $f(x) = |x|^3$, respectively.

$$(4.14) \quad \max_{-1 \leq t \leq 1} |P_n^{(\alpha, \beta)}(t)| = \begin{cases} \binom{n+q}{n} \sim n^q, & q = \max\{\alpha, \beta\} \geq -\frac{1}{2} \\ |P_n^{(\alpha, \beta)}(t)| \sim n^{-\frac{1}{2}}, & q = \max\{\alpha, \beta\} < -\frac{1}{2} \end{cases} \quad ([55, (7.32.2)])$$

where x' is one of the two maximum points, and for $t = \cos(\theta)$ and any fixed constant c with $0 < c < 1$,

$$(4.15) \quad P_n^{(\alpha, \beta)}(\cos(\theta)) = \begin{cases} O(n^\alpha), & 0 \leq \theta \leq cn^{-1} \\ \theta^{-\alpha-\frac{1}{2}} O(n^{-\frac{1}{2}}), & cn^{-1} \leq \theta \leq \frac{\pi}{2} \end{cases} \quad ([55, \text{Theorem 7.32.2}]),$$

$$(4.16) \quad \theta_k = n^{-1} [k\pi + O(1)] \quad ([55, (8.9.1)]),$$

$$(4.17) \quad |P_n^{(\alpha, \beta)' }(\cos(\theta_k))| \sim k^{-\alpha-\frac{3}{2}} n^{\alpha+2}, \quad 0 < \theta_k \leq \frac{\pi}{2} \quad ([55, (8.9.2)]).$$

Moreover, expression (4.17) can be extended to

$$(4.18) \quad |P_n^{(\alpha, \beta)' }(\cos(\theta_k))| \sim k^{-\alpha-\frac{3}{2}} n^{\alpha+2}, \quad 0 < \theta_k \leq c_1\pi$$

for any fixed c_1 with $0 < c_1 < 1$ ([67, (4.6)]).

Based on these identities, the estimates on $\ell_k(t) = \frac{P_n^{(\alpha, \beta)}(t)}{P_n^{(\alpha, \beta)' }(x_k)(t - x_k)}$ have been extensively studied in Kelzon [34, 35], Vértési [66, 68], Sun [54], Prestin [46], Kvernadze [38], Vecchia et al. [69], etc.

LEMMA 4.3. [54] (also see [38]) For $t \in [-1, 1]$, let x_m be the root of the Jacobi polynomial

$P_n^{(\alpha, \beta)}$ which is closest to t . Then we have

$$(4.19) \quad \ell_k(t) = \begin{cases} O\left(|k-m|^{-1} + |k-m|^{\gamma-\frac{1}{2}}\right), & k \neq m \\ O(1) & k = m \end{cases}, \quad \gamma = \max\{\alpha, \beta\},$$

for $k = 1, 2, \dots, n$.

Proof. In [54], the proof of Lemma 4.3 is given only for $0 \leq \theta_k \leq \frac{\pi}{2}$ or $k = m$. That proof can be readily extended to $0 \leq \theta_k \leq \frac{2\pi}{3}$ due to (4.18). We complement the proof for $\frac{2\pi}{3} < \theta_k < \pi$ and $k \neq m$ next.

From (4.13) and (4.18), we see that

$$(4.20) \quad |P_n^{(\alpha, \beta)'(\cos(\theta_k))}| \sim (n-k+1)^{-\beta-\frac{3}{2}} n^{\beta+2}, \quad \frac{2\pi}{3} < \theta_k < \pi \quad ([46, (9)]).$$

Then for $0 \leq t = \cos(\theta) \leq 1$ with $0 \leq \theta \leq cn^{-1}$ and $\frac{2\pi}{3} < \theta_k < \pi$, it follows by (4.15) and (4.20) that

$$\ell_k(t) = O\left(\frac{n^\alpha}{(n-k+1)^{-\beta-\frac{3}{2}} n^{\beta+2}}\right) = O\left(\frac{(n-k+1)^{\beta+\frac{3}{2}}}{n^{\beta+2-\alpha}}\right) = O\left(n^{\alpha-\frac{1}{2}}\right).$$

While for $cn^{-1} \leq \theta \leq \frac{\pi}{2}$ and $\frac{2\pi}{3} < \theta_k < \pi$, it follows by (4.15)-(4.18) and (4.20) that

$$\ell_k(t) = O\left(\frac{(m\pi/n)^{-\alpha-\frac{1}{2}} n^{-\frac{1}{2}}}{(n-k+1)^{-\beta-\frac{3}{2}} n^{\beta+2}}\right) = O\left(\frac{1}{m^{\frac{1}{2}+\alpha} n^{\frac{1}{2}-\alpha}}\right) = \begin{cases} O\left(n^{\alpha-\frac{1}{2}}\right), & \alpha > -\frac{1}{2} \\ O\left(n^{-1}\right), & -1 < \alpha \leq -\frac{1}{2}. \end{cases}$$

Thus for $0 \leq t \leq 1$, we have $\ell_k(t) = O\left(n^{-1} + n^{\alpha-\frac{1}{2}}\right)$ for $k \neq m$, which leads to the desired result due to that $k-m \sim n$ in the case $\frac{2\pi}{3} < \theta_k < \pi$.

Similarly, by (4.13) together with the above analysis, we get for $-1 \leq t \leq 0$ that

$$\ell_k(t) = O\left(|k-m|^{-1} + |k-m|^{\beta-\frac{1}{2}}\right), \quad k \neq m.$$

These together lead to the desired result (4.19) for $k \neq m$. \square

THEOREM 4.4. *Suppose $f(t)$ satisfies (2.9) and $\{x_j\}_{j=1}^n$ are the roots of the Jacobi polynomial $P_n^{(\alpha, \beta)}(t)$, then for $n \geq r+1$*

$$(4.21) \quad \|E_n[f]\|_\infty = O\left(n^{-r+\max\{0, \gamma-\frac{1}{2}\}}\right), \quad \gamma = \max\{\alpha, \beta\}$$

Proof. From Lemma 4.3, we see that $\max_{1 \leq j \leq n} \|\ell_j\|_\infty = O\left(n^{\max\{0, \gamma-\frac{1}{2}\}}\right)$, which together with Theorem 3.2 yields the desired result. \square

REMARK 2. Theorem 4.4 implies that $\|f - L_n[f]\|_\infty$ has the same asymptotic order as $\|f - p_{n-1}^*\|_\infty$ [62] at the roots of the Jacobi polynomial $P_n^{(\alpha, \beta)}(t)$ for $-1 < \alpha, \beta \leq \frac{1}{2}$. Then the interpolations at the n -point Gauss-Legendre points and at the n -point Chebyshev points of first kind or second kind have essentially the same accuracy. All of them can achieve the optimal convergence rate $O(\|f - p_{n-1}^*\|_\infty)$. Consequently, the corresponding quadrature Gauss, Clenshaw-Curtis and Fejér first rule have essentially the same accuracy [74].

Here, we used FIGS. 4.7-4.8 to illustrate the convergence rates for general Gauss-Jacobi pointsystems, where (α, β) are obtained by $rand(1, 2)$ (1st row) and $mrand(1, 2)$ with $m\|rand(1, 2)\|_\infty > m-1$ for $m = 2, 3, 4$ (2nd row), respectively. From these figures, we see that the convergence rates are attainable too, which are in accordance with the estimates. Then the convergence rates at the Gauss-Jacobi pointsystems are optimal.

REMARK 3. It is of particular relevance from FIGS. 4.7-4.8 in the cases that the polynomial interpolations are divergent if $r - \max\{0, \gamma - \frac{1}{2}\} \leq 0$, the divergence rate is also controlled by the order $O\left(n^{-r+\max\{0, \gamma-\frac{1}{2}\}}\right)$.

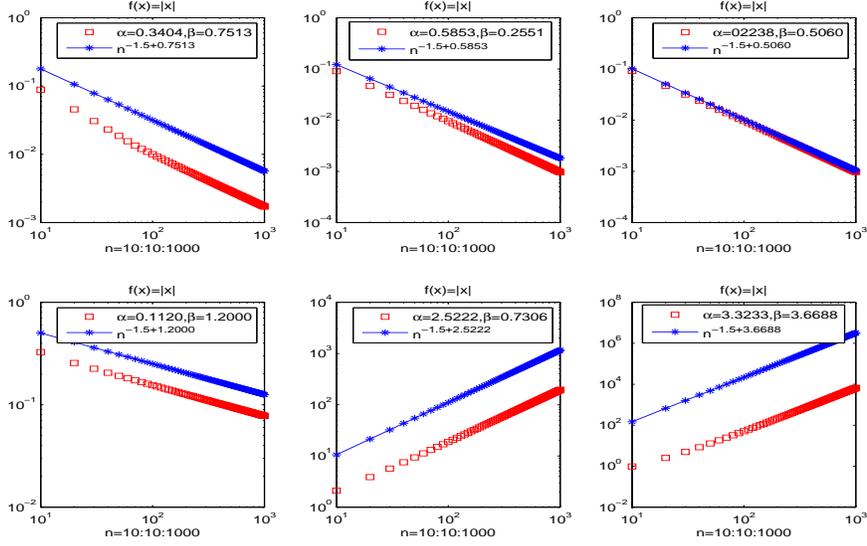


FIG. 4.7. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ with $n = 10 : 10 : 1000$ at the Gauss-Jacobi pointsystems for $f(x) = |x|$, respectively.

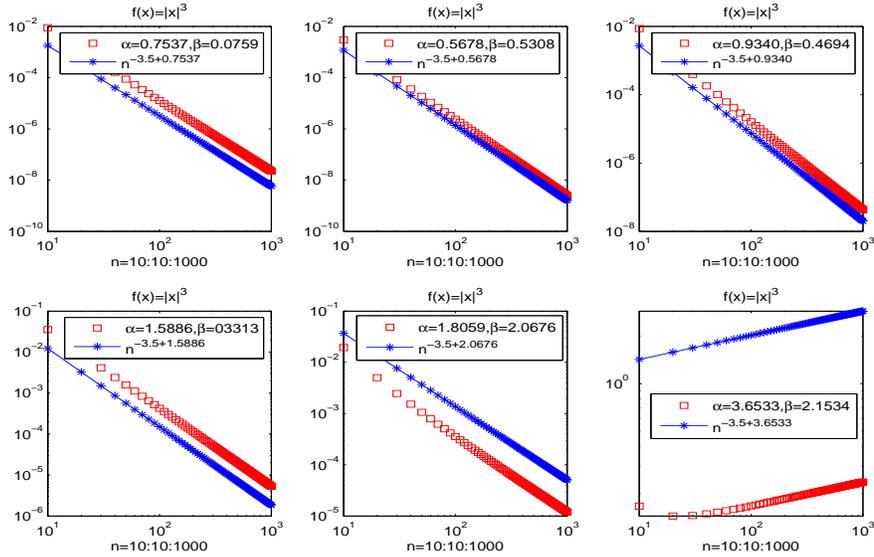


FIG. 4.8. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ with $n = 10 : 10 : 1000$ at the Gauss-Jacobi pointsystems for $f(x) = |x|^3$, respectively.

4.3. General Jacobi-Gauss-Lobatto pointsystems. Let

$$(4.22) \quad -1 = x_{n+1} < x_n < x_{n-1} < \cdots < x_2 < x_1 < x_0 = 1$$

be the roots of $(1-t^2)P_n^{(\alpha,\beta)}(t) = 0$ ($\alpha, \beta > -1$), $x_k = \cos(\theta_k)$ and

$$\omega(t) = (t-x_0)(t-x_1)\cdots(t-x_n)(t-x_{n+1}), \quad \ell_k(t) = \frac{\omega(t)}{(t-x_k)\omega'(x_k)}.$$

Then

$$(4.23) \quad \ell_0(t) = \frac{1+t}{2} \cdot \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(1)}, \quad \ell_{n+1}(t) = \frac{1-t}{2} \cdot \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(-1)},$$

and

$$(4.24) \quad \ell_k(t) = \frac{(1-t^2)P_n^{(\alpha,\beta)}(t)}{(t-x_k)(1-x_k^2)P_n^{(\alpha,\beta)'}(x_k)}, \quad k = 1, 2, \dots, n.$$

In the next, we shall concentrate on estimates of $\ell_k(t)$ for $k = 0, 1, 2, \dots, n+1$.

- On the estimate of $\ell_0(t)$: (i) In the case $0 \leq t \leq 1$, setting $t = \cos \theta$ for $0 \leq \theta \leq \frac{\pi}{2}$, and using

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} \sim n^\alpha \quad ([55, (4.1.1), (7.32.2)]),$$

we find that from (4.15) and (4.23) for $0 \leq \theta \leq \frac{\pi}{2}$,

$$\ell_0(t) = \begin{cases} O(1), & 0 \leq \theta \leq cn^{-1} \\ O\left(\theta^{-\alpha-\frac{1}{2}}n^{-\frac{1}{2}}n^{-\alpha}\right) = O\left((n\theta)^{-\alpha-\frac{1}{2}}\right) = O\left(n^{-\min\{0, \alpha+\frac{1}{2}\}}\right), & cn^{-1} \leq \theta \leq \frac{\pi}{2} \end{cases}.$$

- (ii) In the case $-1 \leq t \leq 0$, letting $t = -\cos(\theta)$ for $0 \leq \theta \leq \frac{\pi}{2}$ and applying $P_n^{(\alpha,\beta)}(-\cos(\theta)) = (-1)^n P_n^{(\beta,\alpha)}(\cos(\theta))$ and $1 - \cos(\theta) = 2 \sin^2\left(\frac{\theta}{2}\right)$ and $\frac{2}{\pi}(\theta) \leq \sin(\theta) \leq \theta$, together with (4.15) and (4.23), we have

$$\begin{aligned} \ell_0(t) &= O\left(\theta^2 P_n^{(\beta,\alpha)}(\cos(\theta))n^{-\alpha}\right) \\ &= \begin{cases} O\left(\frac{1}{n^{2+\alpha-\beta}}\right), & 0 \leq \theta \leq cn^{-1} \\ O\left(\theta^{-\beta+\frac{3}{2}}n^{-\frac{1}{2}}n^{-\alpha}\right) = O\left(n^{-\min\{2+\alpha-\beta, \alpha+\frac{1}{2}\}}\right), & cn^{-1} \leq \theta \leq \frac{\pi}{2} \end{cases}. \end{aligned}$$

These together yield

$$(4.25) \quad \|\ell_0\|_\infty = O\left(\frac{1}{n^{\min\{0, 2+\alpha-\beta, \alpha+\frac{1}{2}\}}}\right).$$

- Similarly, we have

$$(4.26) \quad \|\ell_{n+1}\|_\infty = O\left(\frac{1}{n^{\min\{0, 2+\beta-\alpha, \beta+\frac{1}{2}\}}}\right).$$

- For $k = 1, 2, \dots, n$, let x_m be the nearest to $t \in [0, 1]$ and $t = \cos(\theta)$. From (4.24), we have for $k \neq m$ that

$$(4.27) \quad \ell_k(t) = \frac{\sin^2 \theta P_n^{(\alpha,\beta)}(\cos \theta)}{(\cos \theta - \cos \theta_k) \sin^2 \theta_k P_n^{(\alpha,\beta)' }(\cos \theta_k)} = \frac{-\sin^2 \theta P_n^{(\alpha,\beta)}(\cos \theta)}{2 \sin\left(\frac{\theta-\theta_k}{2}\right) \sin\left(\frac{\theta+\theta_k}{2}\right) \sin^2 \theta_k P_n^{(\alpha,\beta)' }(\cos \theta_k)}.$$

In the case $0 \leq \theta \leq cn^{-1}$ **and** $0 \leq \theta_k \leq \frac{2\pi}{3}$: From (4.15)-(4.18), it follows

$$(4.28) \quad \ell_k(\cos \theta) = O\left(\frac{n^{-2}n^\alpha}{|k-m||k+m|n^{-2}k^2n^{-2}k^{-\alpha-\frac{3}{2}}n^{\alpha+2}}\right) = O\left(\frac{k^{\alpha-\frac{1}{2}}}{|k-m||k+m|}\right).$$

Define

$$h_1(u) = \frac{u^{\alpha-\frac{1}{2}}}{u^2-m^2} \quad \text{for } m+1 \leq u \leq n; \quad h_2(u) = -\frac{u^{\alpha-\frac{1}{2}}}{u^2-m^2} \quad \text{for } 1 \leq u \leq m-1.$$

Then by an elementary proof and noting that $m \leq c_1 n$ for $0 < c_1 < 1$, we get

$$\max_{m+1 \leq u \leq n} h_1(u) = \begin{cases} h_1(m+1) = O\left(m^{\alpha-\frac{3}{2}}\right), & -1 < \alpha \leq \frac{5}{2} \\ \max\{h_1(m+1), h_1(n)\} = O\left(\max\left\{m^{\alpha-\frac{3}{2}}, n^{\alpha-\frac{5}{2}}\right\}\right), & \alpha > \frac{5}{2} \end{cases}$$

and

$$\max_{1 \leq u \leq m-1} h_2(u) = \begin{cases} \max \{h_2(1), h_2(m-1)\} = O\left(\max \left\{m^{-2}, m^{\alpha-\frac{5}{2}}\right\}\right), & -1 < \alpha \leq \frac{1}{2} \\ h_2(m-1) = O\left(m^{\alpha-\frac{3}{2}}\right), & \alpha > \frac{1}{2} \end{cases},$$

which, together with $m \sim 1$ under the assumption, establishes that

$$(4.29) \quad \ell_k(\cos \theta) = \begin{cases} O(1), & -1 < \alpha \leq \frac{5}{2} \\ O\left(n^{\alpha-\frac{5}{2}}\right), & \alpha > \frac{5}{2} \end{cases}.$$

In the case $0 \leq \theta \leq cn^{-1}$ **and** $\frac{2\pi}{3} < \theta_k < \pi$: Similarly, from (4.15) and (4.20) we have

$$(4.30) \quad \begin{aligned} \ell_k(\cos \theta) &= O\left(\frac{n^{-2}n^\alpha}{(n-k+1)^2 n^{-2}(n-k+1)^{-\beta-\frac{3}{2}} n^{\beta+2}}\right) \\ &= O\left(\frac{(n-k+1)^{\beta-\frac{1}{2}}}{n^{2+\beta-\alpha}}\right) \\ &= O\left(n^{-\min\{2+\beta-\alpha, \frac{5}{2}-\alpha\}}\right). \end{aligned}$$

In the case $cn^{-1} \leq \theta \leq \frac{\pi}{2}$ **and** $0 \leq \theta_k \leq \frac{2\pi}{3}$: By (4.27), together with (4.15)-(4.18), we obtain

$$(4.31) \quad \ell_k(\cos \theta) = O\left(\frac{m^{\frac{3}{2}-\alpha} k^{\alpha-\frac{1}{2}}}{|k-m||k+m|}\right) = m^{\frac{3}{2}-\alpha} O\left(\frac{k^{\alpha-\frac{1}{2}}}{|k-m||k+m|}\right)$$

which establishes that by applying the estimates to $h_1(u)$ and $h_2(u)$

$$(4.32) \quad \ell_k(\cos \theta) = \begin{cases} O\left(n^{-\alpha-\frac{1}{2}}\right), & -1 < \alpha < -\frac{1}{2} \\ O(1), & -\frac{1}{2} \leq \alpha \leq \frac{5}{2} \\ O\left(n^{\alpha-\frac{5}{2}}\right), & \alpha > \frac{5}{2} \end{cases}.$$

In the case $cn^{-1} \leq \theta \leq \frac{\pi}{2}$ **and** $\frac{2\pi}{3} < \theta_k < \pi$: From (4.15)-(4.18), (4.20) and (4.27), we find that

$$(4.33) \quad \ell_k(\cos \theta) = m^{\frac{3}{2}-\alpha} O\left(\frac{(n-k+1)^{\beta-\frac{1}{2}}}{n^{2+\beta-\alpha}}\right) = \begin{cases} O\left(n^{-\min\{0, \beta+\frac{1}{2}\}}\right), & -1 < \alpha \leq \frac{3}{2} \\ O\left(n^{-\min\{2+\beta-\alpha, \frac{5}{2}-\alpha\}}\right), & \alpha > \frac{3}{2} \end{cases}.$$

Thus for $t \in [0, 1]$, we get

$$(4.34) \quad \|\ell_k\|_\infty = \begin{cases} O\left(n^{-\min\{0, \beta+\frac{1}{2}, \alpha+\frac{1}{2}\}}\right), & -1 < \alpha \leq \frac{3}{2} \\ O\left(n^{-\min\{0, 2+\beta-\alpha, \frac{5}{2}-\alpha\}}\right), & \alpha > \frac{3}{2} \end{cases}.$$

For $t \in [-1, 0]$, by $P_n^{(\beta, \alpha)}(-t) = (-1)^n P_n^{(\alpha, \beta)}(t)$, setting $t = -\cos \theta$ and $y_k = -x_{n-k+1} = \cos \bar{\theta}_k$ for $k = 1, 2, \dots, n$, we see that y_k are the roots of $P_n^{(\alpha, \beta)}(-t) = (-1)^n P_n^{(\beta, \alpha)}(t)$, then (4.24) can be represented as for $k = 1, 2, \dots, n$

$$\ell_{n-k+1}(t) = \frac{(1 - \cos^2 \theta)(-1)^n P_n^{(\beta, \alpha)}(\cos \theta)}{-(\cos \theta - y_k)(1 - y_k^2) P_n^{(\alpha, \beta)' }(-y_k)} = \frac{\sin^2 \theta P_n^{(\beta, \alpha)}(\cos \theta)}{(\cos \theta - y_k)(1 - y_k^2) P_n^{(\beta, \alpha)' } (y_k)}$$

followed

$$\left[P_n^{(\alpha, \beta)}(t) \right]' = \frac{1}{2}(n + \alpha + \beta + 1) P_{n-1}^{(\alpha+1, \beta+1)}(t) \text{ [55, (4.21.7)].}$$

Similarly, we get that

$$(4.35) \quad \|\ell_k\|_\infty = \begin{cases} O\left(n^{-\min\{0, \alpha + \frac{1}{2}, \beta + \frac{1}{2}\}}\right), & -1 < \beta \leq \frac{3}{2} \\ O\left(n^{-\min\{0, 2 + \alpha - \beta, \frac{5}{2} - \beta\}}\right), & \beta > \frac{3}{2} \end{cases},$$

which together with (4.34) leads to that for $t \in [-1, 1]$

$$(4.36) \quad \|\ell_k\|_\infty = \begin{cases} O\left(n^{-\min\{0, \alpha + \frac{1}{2}, \beta + \frac{1}{2}\}}\right), & -1 < \alpha, \beta \leq \frac{3}{2} \\ O\left(n^{-\min\{0, \alpha + \frac{1}{2}, 2 + \alpha - \beta, \frac{5}{2} - \beta\}}\right), & -1 < \alpha \leq \frac{3}{2}, \beta > \frac{3}{2} \\ O\left(n^{-\min\{0, \beta + \frac{1}{2}, 2 + \beta - \alpha, \frac{5}{2} - \alpha\}}\right), & \alpha > \frac{3}{2}, -1 < \beta \leq \frac{3}{2} \\ O\left(n^{-\min\{0, 2 + \alpha - \beta, 2 + \beta - \alpha, \frac{5}{2} - \alpha, \frac{5}{2} - \beta\}}\right), & \alpha, \beta > \frac{3}{2} \end{cases}.$$

THEOREM 4.5. *Suppose $f(t)$ satisfies (2.9) and $\{x_j\}_{j=0}^{n+1}$ are the roots of $(1-t^2)P_n^{(\alpha, \beta)}(t)$, then for $n \geq r+1$*

$$(4.37) \quad E_n[f] = n^{-r} \cdot \begin{cases} O\left(n^{-\min\{0, \alpha + \frac{1}{2}, \beta + \frac{1}{2}\}}\right), & -1 < \alpha, \beta \leq \frac{3}{2} \\ O\left(n^{-\min\{0, \alpha + \frac{1}{2}, 2 + \alpha - \beta, \frac{5}{2} - \beta\}}\right), & -1 < \alpha \leq \frac{3}{2}, \beta > \frac{3}{2} \\ O\left(n^{-\min\{0, \beta + \frac{1}{2}, 2 + \beta - \alpha, \frac{5}{2} - \alpha\}}\right), & \alpha > \frac{3}{2}, -1 < \beta \leq \frac{3}{2} \\ O\left(n^{-\min\{0, 2 + \alpha - \beta, 2 + \beta - \alpha, \frac{5}{2} - \alpha, \frac{5}{2} - \beta\}}\right), & \alpha, \beta > \frac{3}{2} \end{cases}.$$

Particularly, we have for $(\alpha, \beta) \in S$

$$(4.38) \quad \|E_n[f]\|_\infty = O\left(\frac{1}{n^r}\right),$$

where $S := [-\frac{1}{2}, \frac{5}{2}] \times [-\frac{1}{2}, \frac{5}{2}] - \{(\alpha, \beta) : -\frac{1}{2} \leq \alpha \leq \frac{1}{2}, 2 + \alpha < \beta \leq \frac{5}{2}\} \cup \{(\alpha, \beta) : \frac{3}{2} \leq \alpha \leq \frac{5}{2}, -\frac{1}{2} \leq \beta < 2 - \alpha\}$.

REMARK 3. Theorem 4.5 implies $\|f - L_n[f]\|_\infty$ has the same asymptotic order as $\|f - p_{n-1}^*\|_\infty$ [62] at the roots of the Jacobi polynomial $(1-t^2)P_n^{(\alpha, \beta)}(t)$ for $(\alpha, \beta) \in S$, which includes the corresponding strongly normal pointsystems as special cases.

FIGS. 4.9-4.10 show the convergence rates for $f(x) = |x|$ or $f(x) = |x|^3$ at the Jacobi-Gauss-Lobatto pointsystems, respectively, where each (α, β) is generated by $2\text{rand}(1, 2) - 0.5$.

4.4. General Jacobi-Gauss-Radau pointsystems. Let

$$(4.39) \quad -1 < x_n < x_{n-1} < \cdots < x_2 < x_1 < x_0 = 1$$

be the roots of $(1-t)P_n^{(\alpha, \beta)}(t) = 0$ ($\alpha, \beta > -1$), $x_k = \cos(\theta_k)$ and

$$\omega(t) = (t - x_0)(t - x_1) \cdots (t - x_n), \quad \ell_k(t) = \frac{\omega(t)}{(t - x_k)\omega'(x_k)}.$$

Then

$$(4.40) \quad \ell_0(t) = \frac{P_n^{(\alpha, \beta)}(t)}{P_n^{(\alpha, \beta)}(1)}, \quad \ell_k(t) = \frac{(1-t)P_n^{(\alpha, \beta)}(t)}{(t - x_k)(1 - x_k)P_n^{(\alpha, \beta)'}(x_k)}, \quad k = 1, 2, \dots, n.$$

Additionally, for $t = \cos \theta \in [0, 1]$, we have

$$\ell_0(t) = \begin{cases} O\left(\frac{n^\alpha}{n^\alpha}\right), & 0 \leq \theta \leq cn^{-1} \\ O\left(\theta^{-\alpha - \frac{1}{2}} n^{-\alpha - \frac{1}{2}}\right), & cn^{-1} \leq \theta < \frac{\pi}{2} \end{cases} = O\left(n^{-\min\{0, \alpha + \frac{1}{2}\}}\right),$$

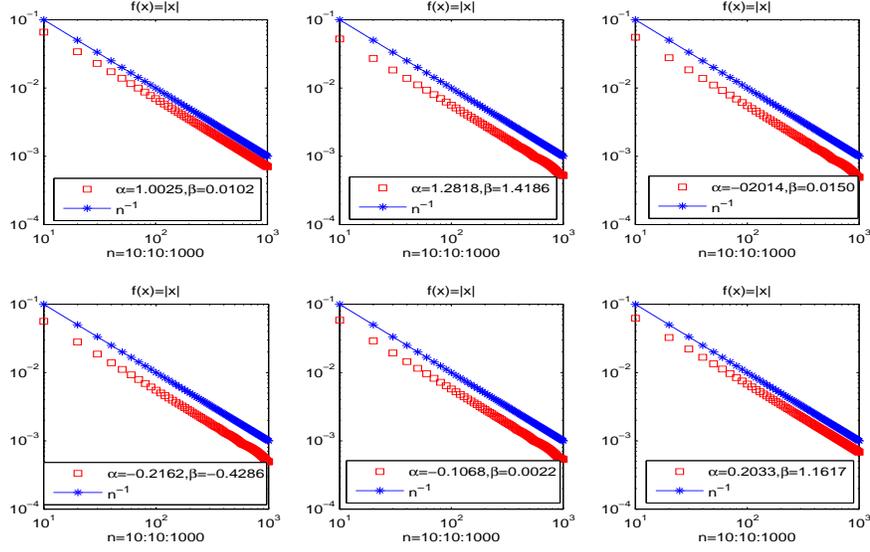


FIG. 4.9. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ with $n = 10 : 10 : 1000$ at the Jacobi-Gauss-Lobatto pointsystems for $f(x) = |x|$, respectively.

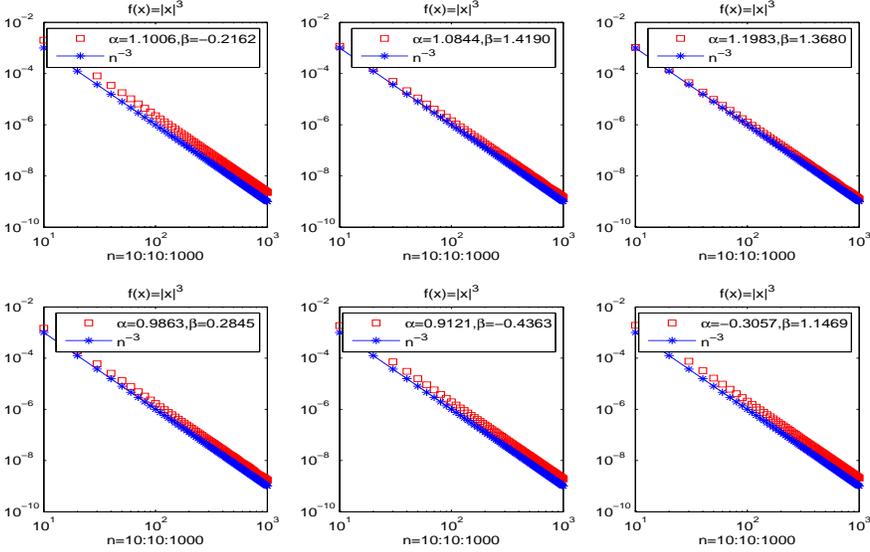


FIG. 4.10. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ with $n = 10 : 10 : 1000$ at the Jacobi-Gauss-Lobatto pointsystems for $f(x) = |x|^3$, respectively.

and

$$\begin{aligned}
 \ell_k(t) &= - \frac{2 \sin^2(\theta/2) P_n^{(\alpha, \beta)}(\cos \theta)}{2 \sin^2(\theta_k/2) P_n^{(\alpha, \beta)' }(\cos \theta_k) 2 \sin((\theta - \theta_k)/2) \sin((\theta + \theta_k)/2)} \\
 (4.41) \quad &= \begin{cases} O\left(n^{-\min\{0, \alpha + \frac{1}{2}\}}\right), & -1 < \alpha \leq -\frac{1}{2} \\ O\left(n^{-\min\{0, \frac{5}{2} - \alpha\}}\right), & \alpha > -\frac{1}{2} \end{cases}
 \end{aligned}$$

following (4.27) similarly.

Similarly, for $t \in [-1, 0]$, by $P_n^{(\beta, \alpha)}(-t) = (-1)^n P_n^{(\alpha, \beta)}(t)$, setting $t = -\cos \theta$ and $y_k = -x_{n-k+1} = \cos \bar{\theta}_k$ for $k = 1, 2, \dots, n$, we obtain $\ell_0(t) = O\left(n^{-\min\{\alpha+\frac{1}{2}, \alpha-\beta\}}\right)$ and

$$(4.42) \quad \ell_k(t) = \begin{cases} O\left(n^{-\min\{0, \alpha+\frac{1}{2}, \alpha-\beta\}}\right), & -1 < \alpha < \frac{1}{2} \\ O\left(n^{-\min\{0, \frac{1}{2}-\beta\}}\right), & \alpha \geq \frac{1}{2} \end{cases}.$$

Thus for $t \in [-1, 1]$, we get

$$(4.43) \quad \|\ell_k\|_\infty = \begin{cases} O\left(n^{-\min\{0, \alpha+\frac{1}{2}, \alpha-\beta\}}\right), & -1 < \alpha \leq \frac{1}{2} \\ O\left(n^{-\min\{0, \frac{1}{2}-\beta, \frac{5}{2}-\alpha, \alpha-\beta\}}\right), & \alpha > \frac{1}{2} \end{cases}$$

for $k = 0, 1, 2, \dots, n$.

THEOREM 4.6. *Suppose $f(t)$ satisfies (2.9) and $\{x_j\}_{j=0}^n$ are the roots of $(1-t)P_n^{(\alpha, \beta)}(t)$, then for $n \geq r+1$*

$$(4.44) \quad E_n[f] = n^{-r} \cdot \begin{cases} O\left(n^{-\min\{0, \alpha+\frac{1}{2}, \alpha-\beta\}}\right), & -1 < \alpha \leq \frac{1}{2} \\ O\left(n^{-\min\{0, \frac{1}{2}-\beta, \frac{5}{2}-\alpha, \alpha-\beta\}}\right), & \alpha > \frac{1}{2} \end{cases}$$

Particularly, we have for $(\alpha, \beta) \in \bar{S}$

$$(4.45) \quad \|E_n[f]\|_\infty = O\left(\frac{1}{n^r}\right),$$

where $\bar{S} := [-\frac{1}{2}, \frac{5}{2}] \times (-1, \frac{1}{2}] - \{(\alpha, \beta) : -\frac{1}{2} \leq \alpha \leq \frac{1}{2}, \alpha < \beta \leq \frac{1}{2}\}$.

Similarly we have

THEOREM 4.7. *Suppose $f(t)$ satisfies (2.9) and $\{x_j\}_{j=0}^n$ are the roots of $(1+t)P_n^{(\alpha, \beta)}(t)$, then for $n \geq r+1$*

$$(4.46) \quad E_n[f] = n^{-r} \cdot \begin{cases} O\left(n^{-\min\{0, \beta+\frac{1}{2}, \beta-\alpha\}}\right), & -1 < \beta \leq \frac{1}{2} \\ O\left(n^{-\min\{0, \frac{1}{2}-\alpha, \frac{5}{2}-\beta, \beta-\alpha\}}\right), & \beta > \frac{1}{2} \end{cases}$$

Particularly, we have for $(\alpha, \beta) \in \hat{S}$

$$(4.47) \quad \|E_n[f]\|_\infty = O\left(\frac{1}{n^r}\right),$$

where $\hat{S} := (-1, \frac{1}{2}] \times [-\frac{1}{2}, \frac{5}{2}] - \{(\alpha, \beta) : -\frac{1}{2} \leq \beta \leq \frac{1}{2}, \beta < \alpha \leq \frac{1}{2}\}$.

FIG. 4.11 shows the convergence rates for $f(x) = |x|$ at the Jacobi-Gauss-Radau pointsystems, where each $(\alpha, \beta) \in \bar{S}$ or $(\alpha, \beta) \in \hat{S}$.

5. Final remarks. The results in section 4 indicate the fact that the interpolations, for functions of limited regularities, at strongly normal pointsystems, Gauss-Jacobi pointsystems with $-1 < \alpha, \beta \leq \frac{1}{2}$, Jacobi-Gauss-Lobatto pointsystems with $(\alpha, \beta) \in S$, Gauss-Jacob-Radau pointsystems \bar{S} or \hat{S} , have the same convergence order compared with the best polynomial approximation of the same degree. Numerical experiments give in line with the estimates.

In addition, numerical experiments also show that the same occurs for analytic or smooth functions. Here we illustrate the phenomenons by entire function $f(x) = e^x$, i.e., analytic throughout the complex plane, $f(x) = 1/(1+25x^2)$, which is analytic in a neighborhood of $[-1, 1]$ but not throughout the complex plane, and $f(x) = e^{-1/x^2}$, which is not analytic in a neighborhood of $[-1, 1]$ but is infinitely differentiable in $[-1, 1]$.

In FIGS. 5.1-5.3, the left columns are computed by zeros of Gauss-Jacobi polynomial $P_n^{(\alpha, \beta)}(x)$, the middles by Jacobi-Gauss-Lobatto $(1-x^2)P_{n-2}^{(\alpha, \beta)}(x)$, while the rights by Jacobi-Gauss-Radau $(1-x)P_{n-1}^{(\alpha, \beta)}(x)$ (first three cases) or $(1+x)P_{n-1}^{(\alpha, \beta)}(x)$ (last three cases), respectively. From these figures, we see that the interpolations at these pointsystems including the

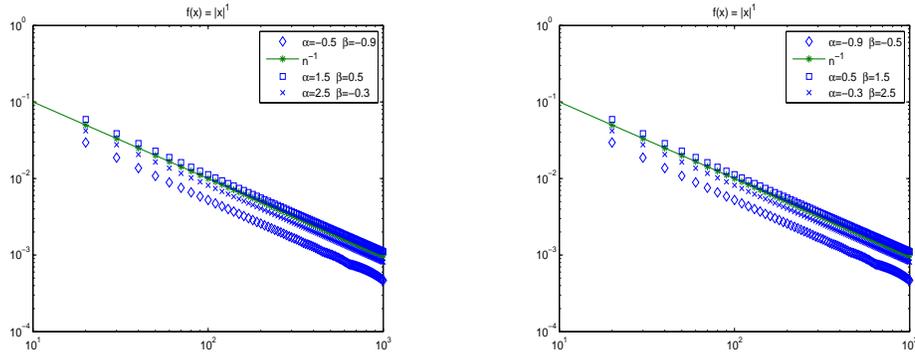


FIG. 4.11. The absolute errors of $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ for Jacobi-Gauss-Radau pointsystems: the roots of $(1-t)P_n^{(\alpha,\beta)}(t)$ (left) and the roots of $(1+t)P_n^{(\alpha,\beta)}(t)$ (right) for $f(x) = |x|$, respectively.

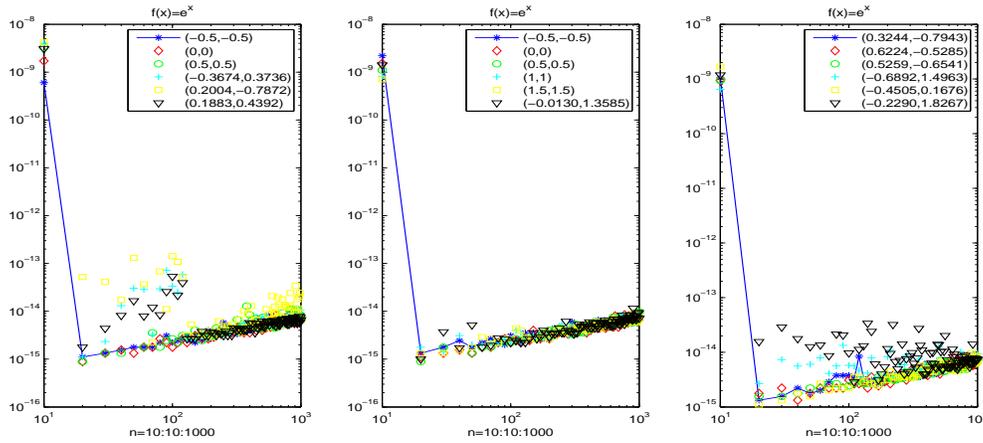


FIG. 5.1. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ with $n = 10 : 10 : 1000$ at Gauss-Jacobi pointsystems for $f(x) = e^x$, respectively.

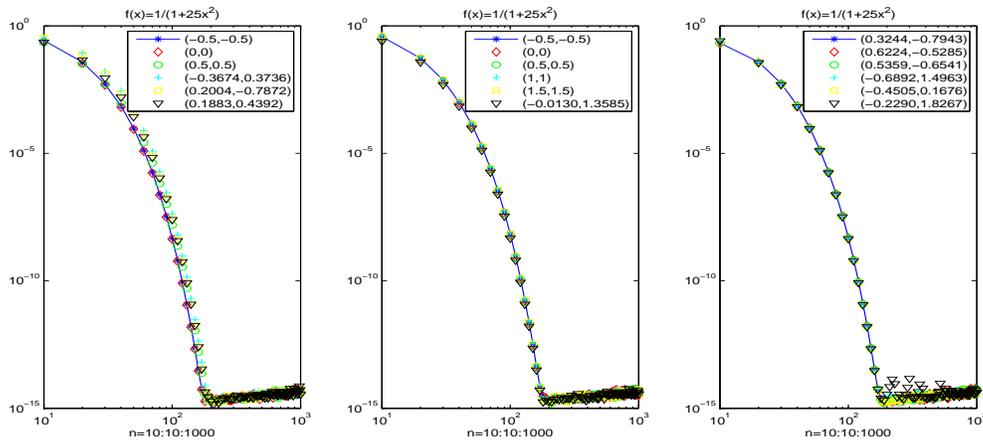


FIG. 5.2. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ with $n = 10 : 10 : 1000$ at Gauss-Jacobi pointsystems for $f(x) = \frac{1}{1+25x^2}$, respectively.

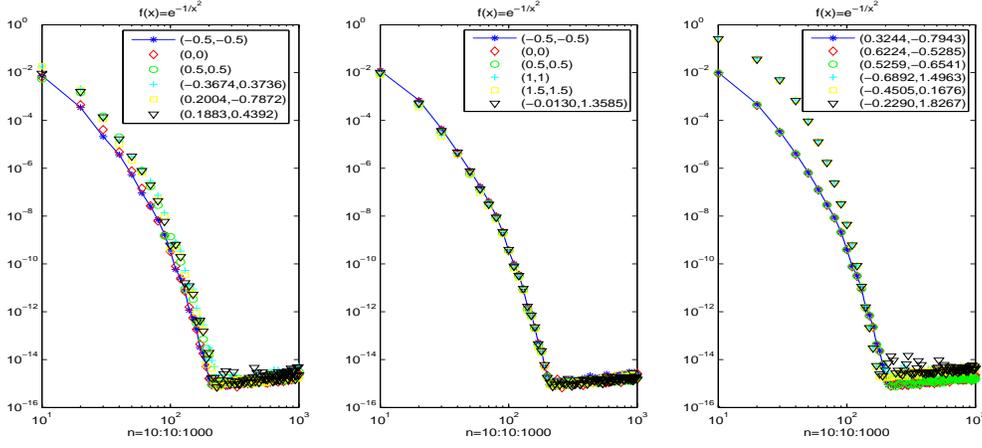


FIG. 5.3. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ with $n = 10 : 10 : 1000$ at Jacobi-Gauss-Radau pointsystems for $f(x) = e^{-1/x^2}$, respectively.

Gauss-Legendre and Legendre-Gauss-Lobatto, achieve essentially the same approximation accuracy compared with those at the two Chebyshev pointsystems too.

It is interesting to noting that the interpolation approximation polynomial $L_n[f]$ challenges the best approximation polynomial p_{n-1}^* of f at the above nice pointsystems not only on the equally asymptotic order on the convergence rate, but also on the faster convergence on the first derivative or second derivative approximation by the polynomials, which has plenty applications in spectral methods [31, 57].

FIG. 5.4 illustrates that the convergence rates $\max_{x=-1:0.001:1} |f'(x) - [p_{n-1}^*]'(x)|$ and $\max_{x=-1:0.001:1} |f''(x) - [p_{n-1}^*]''(x)|$ reduces 2-order and 3-order, respectively, compared with $\max_{x=-1:0.001:1} |f(x) - p_{n-1}^*(x)|$ for $f(x) = |x|^5$ or $|x|^7$. Here, the best approximation polynomial p_{n-1}^* is obtained by **remez** algorithm in CHEBFUN system [60].

However, the interpolation polynomial $L_n[f]$ at the above nice pointsystems performs much better than p_{n-1}^* for approximation f' and f'' by $L'_n[f]$, $L''_n[f]$, $[p_{n-1}^*]'$ and $[p_{n-1}^*]''$, respectively. Here, we use usual pointsystems to show the performs (see FIGS. 5.5-5.7).

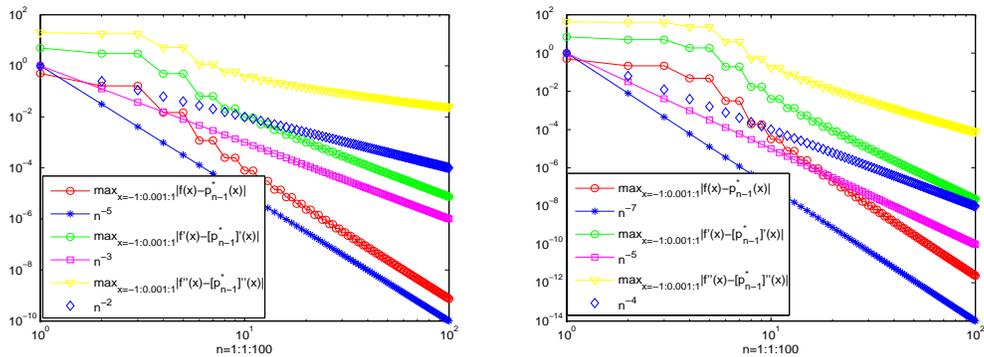


FIG. 5.4. $\max_{x=-1:0.001:1} |f^{(m)}(x) - [p_{n-1}^*]^{(m)}(x)|$ with $n = 10 : 1 : 100$ for $f(x) = |x|^5$ (left), $f(x) = |x|^7$ (right) and $m = 0, 1, 2$, respectively.

FIG. 5.8 shows that the convergence rates $\max_{x=-1:0.001:1} |f^{(m)}(x) - [p_{n-1}^*]^{(m)}(x)|$ and $\max_{x=-1:0.001:1} |f^{(m)}(x) - L_n^{(m)}[f](x)|$ for $f(x) = |x|^5$ and $|x|^7$, respectively, where $L_n[f]$ is the interpolation at Jacobi-Gauss-Lobatto points $\{x_k = \cos\left(\frac{k\pi}{n-1}\right)\}_{k=0}^{n-1}$.

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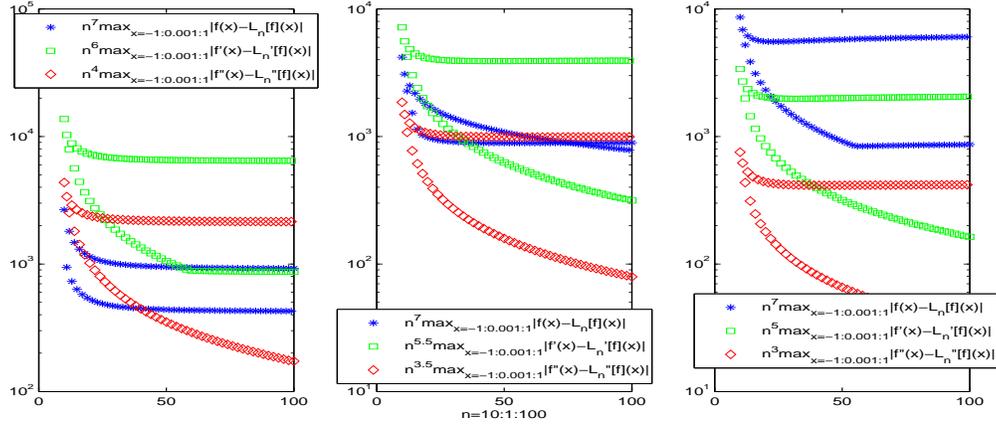


FIG. 5.5. $n^{s(m)} \max_{x=-1:0.001:1} |f^{(m)}(x) - L_n^{(m)}[f](x)|$ with $n = 10 : 1 : 100$ at Gauss-Jacobi pointsystems for $f(x) = |x|^7$ and $m = 0, 1, 2$, and $(-0.5, -0.5)$ (left), $(0, 0)$ (middle), $(0.5, 0.5)$ (right), respectively.

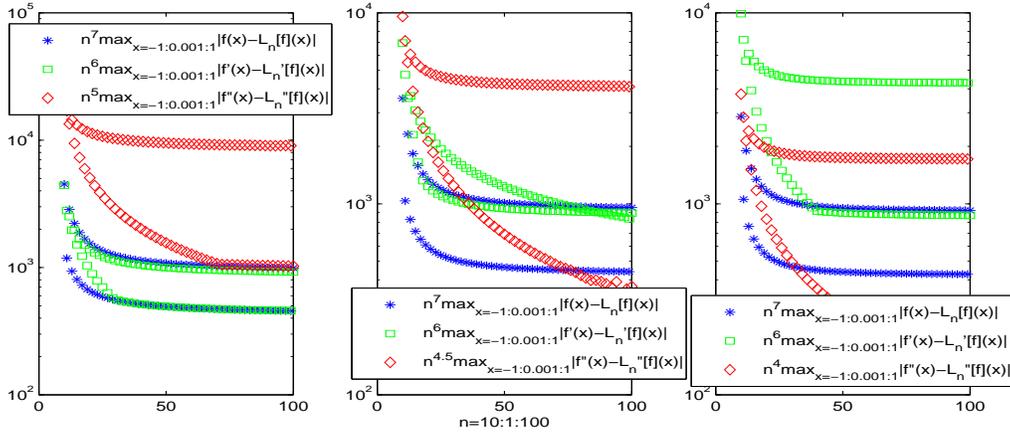


FIG. 5.6. $n^{s(m)} \max_{x=-1:0.001:1} |f^{(m)}(x) - L_n^{(m)}[f](x)|$ with $n = 10 : 1 : 100$ at Jacobi-Gauss-Lobatto pointsystems for $f(x) = |x|^7$ and $m = 0, 1, 2$, and $(0.5, 0.5)$ (left), $(1, 1)$ (middle), $(1.5, 1.5)$ (right), respectively.

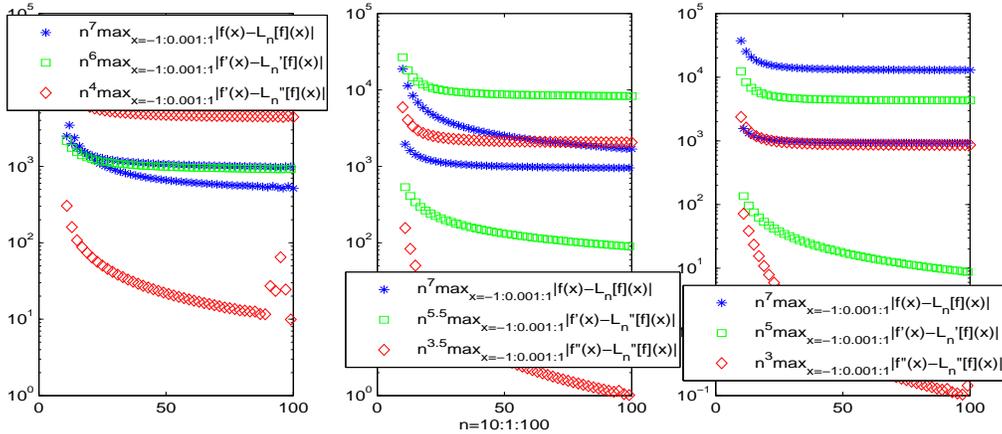


FIG. 5.7. $n^{s(m)} \max_{x=-1:0.001:1} |f^{(m)}(x) - L_n^{(m)}[f](x)|$ with $n = 10 : 1 : 100$ at Jacobi-Gauss-Radau pointsystems with $x_0 = 1$ for $f(x) = |x|^7$ and $m = 0, 1, 2$, and $(-0.5, -0.5)$ (left), $(0, 0)$ (middle), $(0.5, 0.5)$ (right), respectively.

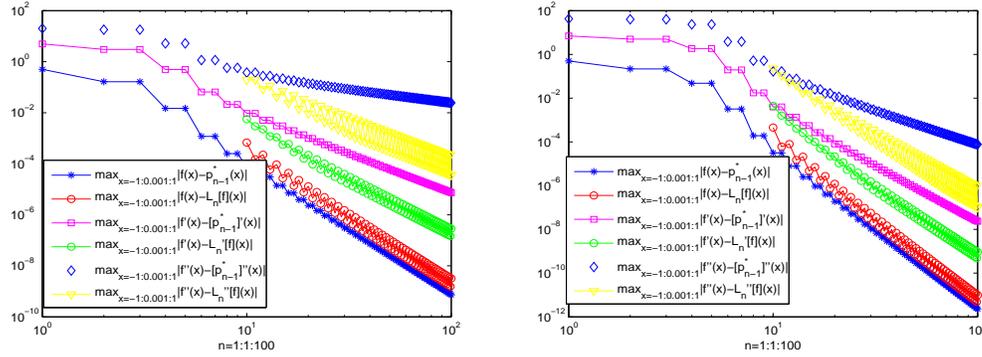


FIG. 5.8. The convergence rates $\max_{x=-1.0001:1} |f^{(m)}(x) - [p_{n-1}^*]^{(m)}(x)|$ and $\max_{x=-1.0001:1} |f^{(m)}(x) - L_n^{(m)}[f](x)|$ for $f(x) = |x|^5$ (left) and $|x|^7$ (right) with $m = 0, 1, 2$, respectively, where $L_n[f]$ is the interpolation at Jacobi-Gauss-Lobatto points $\{x_k = \cos(\frac{k\pi}{n-1})\}_{k=0}^{n-1}$.

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