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LOCALLY INERTIAL APPROXIMATIONS OF BALANCE LAWS ARISING IN (1+1)-DIMENSIONAL GENERAL RELATIVITY

LAURENT GOSSE*

Abstract. An elementary model of $1+1$-dimensional general relativity, known as "$\mathcal{R} = \mathcal{T}$" and mainly developed by Mann *et al.* [44, 45, 48, 50, 51, 56, 63], is set up in various contexts. Its formulation, mostly in isothermal coordinates, is derived and a relativistic Euler system of self-gravitating gas coupled to a Liouville equation for the metric's conformal factor is deduced. First, external field approximations are carried out: both a Klein-Gordon equation is studied along with its corresponding density, and a Dirac one inside an hydrostatic gravitational field induced by a static, piecewise constant mass repartition. Finally, the coupled Euler-Liouville system is simulated, by means of a locally inertial Godunov scheme: the gravitational collapse of a static random initial distribution of density is displayed. Well-balanced discretizations rely on the treatment of source terms at each interface of the computational grid, hence the metric remains flat in every computational cell.

Key words. 1+1 general relativity; Dirac and Klein-Gordon equations; Intrinsic finite-differences; Locally inertial scheme: Relativistic hydrodynamics; Structure-Preserving and Well-balanced schemes.

AMS subject classifications. 65M06, 35Q75, 58D30.

1. Introduction.

A major obstacle in studying Einstein's Field Equations (EFE) is their huge complexity and high dimensionality: for instance, the $(1+3)$-dimensional formalism leads to a set of 10 coupled, nonlinear, hyperbolic-elliptic partial differential equations, see *e.g.* [21, 43, 59]. Thus, for both theoretical and practical purposes (left aside a development of quantum gravity), simpler, lower-dimensional gravity models [8] were sought during a long time. The Einstein-tensor $\mathcal{G}_{\alpha\beta}$ vanishes identically for a $(1+1)$-dimensional spacetime. Hence, setting up the usual Einstein equations,

$$\mathcal{G}_{\alpha\beta} \doteq R_{\alpha\beta} - \left(\frac{\mathcal{R}}{2} - \Lambda\right) g_{\alpha\beta} = 8\pi T_{\alpha\beta}, \quad R_{\alpha\beta} \text{ the Ricci tensor of the metric } g_{\alpha\beta},$$

$\mathcal{R}$ its Ricci scalar, and $T_{\alpha\beta}$ the stress-energy tensor, is meaningless, as explained by *e.g.* Collas [13], because $\mathcal{G}_{\alpha\beta}$ vanishes identically for $\Lambda = 0$. The "cosmological constant" Λ was initially meant to cope with a static universe, nowadays, it stands for a residual energy in an otherwise empty spacetime. A reasonable alternative was proposed in $1+1$ dimensions, the "Jackiw-Tetelboim (JT) model" [29], at the price of introducing an auxiliary scalar field Φ into the Einstein-Hilbert action,

$$\mathcal{S}_{2D} = \int \Phi(x)\sqrt{-\det g}(\mathcal{R} - \Lambda)\mathrm{d}^2x, \qquad \mathcal{R} - \Lambda = 0,$$

which was studied especially by Desloge [17]. This reduced "empty universe" model was later improved by R.B. Mann *et al.* during the nineties, see *e.g.* [7, 35, 44, 45, 50, 63] in order to include matter. It was delicate to justify the inclusion of the trace $\mathcal{T} = g^{\alpha\beta}T_{\alpha\beta}$ of the stress-energy tensor, while keeping the field equation coming from a variational principle. This was solved by means of another auxiliary field ψ,

$$\tilde{\mathcal{S}}_{2D} = \int \sqrt{-\det g}\left(\psi\mathcal{R} + \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\psi\ \nabla_\beta\psi + \Lambda + 2\mathcal{L}_{matter}\right)\mathrm{d}^2x, \qquad \mathcal{R} - \Lambda = \mathcal{T},$$

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and ∇ standing for the covariant derivative with respect to the metric g (see *e.g.* [48, page 43], or [39, 56]). Viewed as a particular case of "dilatonic gravity", it appears to be the unique one where the dilaton decouples for the gravity field equation [66, 46]: it is also endowed with remarkable features like a well-defined Newtonian limit, gravitational collapse, black hole radiation, and simple cosmologies [9, 14, 11]. Covariant conservation laws for the stress-energy tensor, $\nabla_\alpha T^{\alpha\beta} = 0$, actually follow from a consideration of the behavior of the dilaton equation, as shown in [47, 48, 49].

Remark 1. *A rather natural approach to derive a gravity action in* $1+1$ *dimensions is to dimensionally reduce the* $1+3$*-dimensional Einstein-Hilbert action. Implementing spherical symmetry, by imposing the* $1+3$*-dimensional line element,*

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\mu \mathrm{d}x^\nu - \phi^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \cdot \mathrm{d}\varphi^2), \qquad \Lambda = 0,$$

with $\mu, \nu \in \{0,1\}^2$*, the auxiliary field* ϕ *being restricted to positive values, and integrating over both angle coordinates* ϑ, φ*, one derives the* $1+1$*-dimensional action,*

$$\tilde{\mathcal{S}}_{2D} = \int \sqrt{-\det g}\left(\frac{\phi^2}{4}\mathcal{R} + \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\ \nabla_\beta\phi - \frac{1}{2}\right)\mathrm{d}^2x,$$

This leads in particular to Scharzchild-type solutions, [33]. Thus, $(1+1)$*-dimensional gravity is sometimes assumed to hold, in first approximation, at Earth's surface [52].*

The paper is organized as follows: in Section 2, quantitative features of $1+1$-dimensional relativity are presented, especially we show that in the conformal gauge, the expression of $\mathcal{R}$ yields a Liouville wave equation [16, 30, 32]. This allows to derive, in Section 2.2 a model of coupled Euler-Louville self-gravitating fluid which meets with a special case of a more general one obtained assuming planar Gowdy symmetry [5, 25, 37]. We recover both an hydrostatic and a simple FRW model by simplifying it in Section 2.3. In Section 3, we first work entirely in the "external-field approximation", that is, we take the classical perturbed metric to be given and study time-evolution of remaining fields in this static background [2, 55, 53, 51, 56, 64, 70]. A Klein-Gordon equation is derived for spinless scalar fields, along with its (sign-indefinite) continuity equation acting on ρ, J. Its time-evolution is displayed numerically by means of an energy-preserving scheme, following methods of [20]. Later, a Dirac equation is considered in a curved space-time induced by the hydrostatic repartition of a piecewise constant mass. There, numerical results are obtained by means of the schemes proposed in [24]. Numerical study of Dirac equation on a curved space-time can be useful for graphene applications [57], too. Section 4 deals with the more ambitious nonlinearly coupled Euler-Liouville system formerly derived: a well-balanced [23], naturally locally inertial, Godunov scheme is carefully derived for subsonic, non-resonant, relativistic flows (see also [34, 36]). The source terms arising from the variations in both space and time of the metric's conformal factor are handled by both well-balanced and time-splitting processes, respectively, in such a manner that they don't impose supplementary time-step restrictions, other than the usual, homogeneous, CFL condition. Substantial differences exist with respect to the Godunov scheme presented in [68, 69], especially in the treatment of the geometrical source terms (here, there's no need for any staggering process). This is illustrated by means of an elementary gravitational collapse starting from random initial data with null velocity and isothermal pressure law. Finally, an Appendix contains basic facts about conservation laws on smooth curved surfaces, the link with well-balanced approximations is explained and numerical absorbing boundary conditions for a 1D Liouville equation are derived. Besides, in the spirit of [43], we skip as much as possible tensorial notations and Einstein's convention of repeated indexes summation.

2. A short synthesis on $1+1$-dimensional "$\mathcal{R}=\mathcal{T}$ relativity". Within a two-dimensional space-time, the Einstein curvature tensor vanishes identically: the general theory of relativity induces an empty space-time with no dynamical equations.

2.1. An extension of Newtonian gravity. Let us start with a smooth surface embedded in $\mathbb{R}^3$ and a general (pseudo-)Riemannian metric expressed in a system of curvilinear coordinates t, x such that the line element reads:

$$\mathrm{d}s^2 = E(t,x)\mathrm{d}t^2 + 2F(t,x)\mathrm{d}x\mathrm{d}t + G(t,x)\mathrm{d}x^2,$$

where E, F, G are smooth. By rescaling the time variable $t \to \tilde{t}(t,x)$ according to an "integrating factor" denoted $\sigma(t,x)$, [13], F can be (locally) eliminated because,

$$\frac{\partial \tilde{t}}{\partial t}\mathrm{d}t + \frac{\partial \tilde{t}}{\partial x}\mathrm{d}x \overset{def}{=} \mathrm{d}\tilde{t} = \sigma(t,x)\big(E(t,x)\mathrm{d}t + F(t,x)\mathrm{d}x\big), \qquad \frac{\partial(\sigma E)}{\partial x} = \frac{\partial(\sigma F)}{\partial t},$$

the second condition expressing that we have an exact differential form. Such an integrating factor σ exists when the differential equation $E(t,x)\mathrm{d}t + F(t,x)\mathrm{d}x = 0$ has a (local) solution, which happens as soon as $\frac{F}{E}$ (or $\frac{E}{F}$) is sufficiently smooth. Hence the line element is $\mathrm{d}s^2 = \frac{\mathrm{d}\tilde{t}^2}{E\sigma^2} + (G - \frac{F^2}{E})\mathrm{d}x^2$ in these *orthogonalized coordinates.* Accordingly we hereafter restrict ourselves to metrics written in diagonal form[1], that is, for which the 2×2 matrix involved in the first fundamental form reduces to:

$$g = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}. \tag{2.1}$$

A standard result implies that the Gaussian curvature in orthogonal coordinates is,

$$K = -\frac{1}{2\sqrt{|EG|}}\left[\partial_t\left(\frac{\partial_t G}{\sqrt{|EG|}}\right) + \partial_x\left(\frac{\partial_x E}{\sqrt{|EG|}}\right)\right] = \frac{\mathcal{R}}{2}, \tag{2.2}$$

with $\mathcal{R}$ standing for the Ricci curvature scalar. Even in such a seemingly elementary framework, we may have to face the strongly nonlinear wave equation (2.2) in order to get the space-time curvature: Collas [13] chose $E = \exp(2\nu)$, $G = \exp(2\lambda)$, so

$$K = \frac{1}{\exp(\lambda+\nu)}\Big[\partial_t\big(\exp(\lambda-\nu)\partial_t\lambda\big) - \partial_x\big(\exp(\nu-\lambda)\partial_x\nu\big)\Big].$$

In some cases, the metric may be even simpler: following Mann et al. [44, 45, 50, 63], we can select in (2.1), $E(t,x) = \alpha(t,x) = -\frac{1}{G(t,x)}$ where $\alpha \neq 0$ is a Lipschitz function:

$$g = \begin{pmatrix} -\alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}, \qquad \mathcal{R} = \partial_{tt}\left(\frac{1}{\alpha}\right) - \partial_{xx}(\alpha). \tag{2.3}$$

Let's now *admit*[2] that the trace of the stress-energy tensor is given by $\mathcal{T} = p - \rho$, where ρ, p stands for the density and pressure of the matter, respectively. A very elementary example is the static point mass M concentrated at $x = 0$, [19], yielding

$$\partial_{xx}(\alpha) = M\delta(x), \quad p = 0, \qquad \alpha(x) = |x| + C_1 x + C_2,$$

[1]In the ADM formalism, the "shift factor" is $F = 0$; E, G are called "lapse" and "metric" factors.
[2]It will be shown in §5 that this expression actually holds.

C_1, C_2 being arbitrary constants. Vanishing pressure is usually associated with dust. The metric element $\alpha(x)$ thus identifies with the classical Newtonian gravity potential. Newton's law of motion with an absolute time leads to a geodesic equation,

$$\frac{\mathrm{d}t}{\mathrm{d}\tau^2} = 0, \qquad \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} + \Gamma^x_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mathrm{d}\alpha}{\mathrm{d}x} = 0. \tag{2.4}$$

Hence "Newton spacetime" may have a metric such that $F = 0$, G is constant and $\Gamma^x_{tt} = \frac{\mathrm{d}\alpha}{\mathrm{d}x} = -\frac{\partial_x E}{2}$. The metric (2.3) is expressed in so–called *Schwarzschild gauge*.

REMARK 2. *The metric (2.3) leads to a quasi-linear wave equation, that is,*

$$\partial_t(\frac{1}{\alpha}) = \partial_x \beta, \qquad \partial_t \beta = \partial_x \alpha + \int^x \mathcal{T},$$

an inhomogeneous, isothermal p-system "à la Nishida" which (in general) admits discontinuous BV entropy solutions, "Gauge shocks" [1, 3]. Consequently, Christoffel symbols will display Dirac masses at the corresponding shock locations, bringing non-conservative products [38] into covariant matter dynamics equations.

2.2. Derivation of the $1+1$ coupled Euler-Liouville system.

2.2.1. Liouville field equation in isothermal coordinates. Thanks to the property of any 2D smooth manifold to be conformally flat, one can work in the *conformal gauge*, with t, x such that the Lorentzian metric tensor reads :

$$g = \exp(2\phi)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} := \exp(2\phi)\eta, \qquad \eta \text{ the Minkowski metric,} \tag{2.5}$$

where the function $2\phi(t,x)$ is usually called the "conformal factor". Yet 2D scalar curvature $\mathcal{R}$ (Ricci curvature tensor's trace) equals twice Gaussian curvature K, and completely characterizes a surface's curvature. In isothermal coordinates [4],

$$\mathcal{R} = 2K = -\exp(-2\phi)\left(\frac{\partial^2(2\phi)}{\partial t^2} - \frac{\partial^2(2\phi)}{\partial x^2}\right),$$

meaning that the curvature of a 1+1-Lorentzian space-time can be computed through the Liouville equation which displays a weaker nonlinearity compared to (2.2). Since the Ricci scalar equals the trace $\mathcal{T} = tr(T)$ of the stress-energy tensor, we finally get:

$$\frac{\partial^2(2\phi)}{\partial t^2} - \frac{\partial^2(2\phi)}{\partial x^2} = -\mathcal{T}\exp(2\phi).$$

For later use, Christoffel symbols (A.2) with $D = \det g = -\exp(4\phi)$ and $F = 0$ read:

$$\Gamma^t = \begin{pmatrix} \partial_t\phi & \partial_x\phi \\ \partial_x\phi & \partial_t\phi \end{pmatrix}, \qquad \Gamma^x = \begin{pmatrix} \partial_x\phi & \partial_t\phi \\ \partial_t\phi & \partial_x\phi \end{pmatrix}. \tag{2.6}$$

REMARK 3. *The property of 2D surfaces being conformally flat [7, 31] doesn't say much about curvature; pick for instance the 2-sphere $\mathbb{S}^2$, for which $K = \frac{1}{R^2}$, R being its radius. Their geodesic flow can be integrable, see [60] and Appendix D.*

2.2.2. A model of self-gravitating perfect fluid. To finalize our model, we still need 2 things: the trace of the (symmetric) stress-energy tensor T to be inserted in the Liouville equation, and then the expression of the covariant divergence of T.

- a standard form of T for a perfect fluid is, in tensorial notation, [7, p.329]

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta - p\, g^{\alpha\beta}, \qquad u^\alpha u_\alpha = (g^{\alpha\beta}\ u_\beta)u_\alpha = -1.$$

One must define a (local) Lorentz factor γ and a scalar 1D velocity v:

$$\gamma = \frac{1}{\sqrt{1-v^2}}, \qquad u := (1, v)\exp(-\phi)\gamma.$$

Component-wise, since $g^{\alpha\beta}$ is the inverse matrix of g, this means:

$$\left\{\begin{array}{rclcl} T^{tt} &=& (\rho+p)u^t u^t - p\, g^{tt} &=& \exp(-2\phi)\big[(\rho+p)\gamma^2 - p\big], \\ T^{tx} &=& (\rho+p)u^t u^x - p\, g^{tx} &=& \exp(-2\phi)(\rho+p)\gamma^2 v, \\ T^{xx} &=& (\rho+p)u^x u^x - p\, g^{xx} &=& \exp(-2\phi)\big[(\rho+p)(\gamma v)^2 + p\big]. \end{array}\right. \tag{2.7}$$

At steady-state, there is no x-component in u, we recover $v = 0$, $\gamma = 1$ and $T = \exp(-2\phi)\left(\begin{smallmatrix}\rho & 0\\ 0 & p\end{smallmatrix}\right)$ like in [7]. The trace of T reads $\mathcal{T} = g_{\alpha\beta}\ T^{\alpha\beta}$:

$$\begin{array}{rcl} \mathcal{T} &=& \begin{pmatrix} \exp(2\phi) \\ -\exp(2\phi) \end{pmatrix} \cdot \begin{pmatrix} \exp(-2\phi)\big[(\rho+p)\gamma^2 - p\big] \\ \exp(-2\phi)\big[(\rho+p)(\gamma v)^2 + p\big] \end{pmatrix} \\ &=& (\rho+p)\gamma^2(1-v^2) - 2p \ = \ \rho - p. \end{array}$$

The Liouville equation describing the curvature/conformal factor reads:

$$\frac{\partial^2(2\phi)}{\partial t^2} - \frac{\partial^2(2\phi)}{\partial x^2} = -(\rho - p)\exp(2\phi) \tag{2.8}$$

- the covariant divergence of an "order 2 tensor", in tensorial notation,

$$\begin{array}{rcl} \mathsf{div}_g(T) &=& T^{\alpha\beta}{}_{;\beta} = \nabla_\beta T^{\alpha\beta} = \frac{\partial_k\left(\sqrt{-\det g}\ T^{\alpha k}\right)}{\sqrt{-\det g}} + \Gamma^\alpha_{lm} T^{lm} \\ &=& \frac{1}{\sqrt{\exp(4\phi)}} \begin{pmatrix} \partial_t(\exp(2\phi)T^{tt}) + \partial_x(\exp(2\phi)T^{tx}) \\ \partial_t(\exp(2\phi)T^{tx}) + \partial_x(\exp(2\phi)T^{xx}) \end{pmatrix} \\ && \qquad + \begin{pmatrix} \Gamma^t_{tt}T^{tt} + 2\Gamma^t_{tx}T^{tx} + \Gamma^t_{xx}T^{xx} \\ \Gamma^x_{tt}T^{tt} + 2\Gamma^x_{tx}T^{tx} + \Gamma^x_{xx}T^{xx} \end{pmatrix}, \end{array} \tag{2.9}$$

a 2-component vector. As Euler perfect fluid equations read $T^{\alpha\beta}{}_{;\beta} = 0$, one can multiply the former expression by $\exp(2\phi)$ in order to get:

$$\boxed{\left\{\begin{array}{rcl} \partial_t\tau + \partial_x S + 2S\ \partial_x\phi + (\tau + \Sigma)\ \partial_t\phi &=& 0, \\ \partial_t S + \partial_x\Sigma + (\tau+\Sigma)\ \partial_x\phi + 2S\ \partial_t\phi &=& 0, \end{array}\right.} \tag{2.10}$$

thanks to the simple expressions (2.6) and with some easier notations,

$$\left\{\begin{array}{rcccl} \tau &=& \exp(2\phi)T^{tt} &=& (\rho+p)\gamma^2 - p, \\ S &=& \exp(2\phi)T^{tx} &=& (\rho+p)\gamma^2 v, \\ \Sigma(\tau, S) &=& \exp(2\phi)T^{xx} &=& (\rho+p)(\gamma v)^2 + p. \end{array}\right.$$

Within those notations, the field equation (2.8) rewrites as follows:

$$\boxed{\frac{\partial^2(2\phi)}{\partial t^2} - \frac{\partial^2(2\phi)}{\partial x^2} + (\tau - \Sigma)\exp(2\phi) = 0.}$$

Our 1 + 1-dimensional model of relativity in isothermal coordinates is the Euler-Liouville system (2.8)–(2.10). It is endowed with several (interesting) properties:

1. it is a particular case of the "Gowdy models" [25] studied in [5, 37]
2. our equations are time-dependent generalizations of the ones derived in [7]
3. theoretical results on time-dependent Liouville equation (alone) are available in [16, 32], its numerical analysis can be done following [20, 58]
4. it isn't clear that this model supports gravitational waves in vacuum, [15].

2.3. Two well-known $1+1$ relativistic gravity models. Static or homogeneous gravity field models will prove useful in a first set of numerical experiments, that is, for simulating dynamics of elementary particles on curved space-times.

2.3.1. Hydrostatic $1+1$ "stellar model". A first case of interest, evoked in [51, 63, 64], involves a (static) metric (2.1), in *unitary gauge*, which elements satisfy

$$\mathrm{d}s^2 = -B(x)^2\mathrm{d}t^2 + \mathrm{d}x^2, \qquad E = -B(x)^2, \quad G \equiv 1.$$

The scalar velocity $v \equiv 0$, so $\gamma \equiv 1$ and the stress-energy tensor is now,

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + p\, g^{\alpha\beta}, \qquad u^\alpha u_\alpha = -\frac{1}{B^2}|u_t|^2 = -1.$$

The field equation $\mathcal{R} = \mathcal{T}$ implies that the lapse factor $B \neq 0$ solves:

$$T = \begin{pmatrix} \rho/B^2 & 0 \\ 0 & p \end{pmatrix}, \qquad \mathcal{R} = 2K = -\frac{1}{|B|}\partial_x\left(\frac{\partial_x(|B|^2)}{|B|}\right) = \mathcal{T} = -\rho + p,$$

which gives finally that $2\partial_{xx}|B| = |B|(\rho - p)$. Yet, since matter is at steady-state, there is an hydrostatic balance law that is deduced from the second equation in (2.9). As $v \equiv 0$, the only useful Christoffel symbols are $\Gamma^x_{tt} = \frac{\partial_x(B^2)}{2}$ and $\Gamma^x_{xx} = 0$; thus,

$$0 = \frac{\partial_x\left(\sqrt{-\det g}\; T^{xx}\right)}{\sqrt{-\det g}} + \Gamma^x_{tt}T^{tt} = \frac{1}{B}\partial_x(pB) + \frac{\rho}{2}\frac{\partial_x(B^2)}{B^2}.$$

Given an equation of state $p = p(\rho)$, the field and matter laws for $\rho(x), B(x)$ rewrite:

$$2\partial_{xx}|B| = |B|(\rho - p), \qquad \partial_x p + \partial_x(\log|B|)(\rho + p) = 0. \tag{2.11}$$

An example of such a metric appears in [64], with $B(x) = \tanh x$, as the analytic continuation of a black hole (torsion-free) to a Lorentz signature, see also [66]. Other examples are either $B(x) = \sinh x$, or $B(x) = \cosh x$, the so–called "anti-de Sitter solution", which can be recast in the Schwarzschild gauge by defining $r = \sinh x$,

$$\mathrm{d}s^2 = -\cosh^2 x\mathrm{d}t^2 + \mathrm{d}x^2 = -\cosh^2 x\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\cosh^2 x} = -(1+r^2)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1+r^2}.$$

A key feature is the presence of an *horizon* in $x = 0$ for $B(x) = \sinh x$, [19, 54]. Its geodesic flow is retrieved from symbols $\Gamma^x_{tt} = \frac{\partial_x(B^2)}{2}$, $\Gamma^t_{tx} = \partial_x \log|B(x)|$, if $B \neq 0$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(|B|\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = 0, \qquad \frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} = -\Gamma^x_{tt}\left|\frac{\mathrm{d}t}{\mathrm{d}\tau}\right|^2 = -C\frac{\partial_x(B^2)}{2B^2} = -C\partial_x \log|B(x(\tau))|.$$

2.3.2. Friedmann-Robertson-Walker (FRW) homogeneous model. The 1+1-dimensional FRW metric is, with $a(t)$ the metric, or scale factor and $k \in \{\pm 1, 0\}$,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \frac{\mathrm{d}x^2}{1-kx^2}, \qquad E \equiv -1, \quad G = a(t)^2.$$

In one space dimension, one can rescale the x coordinate into $\frac{\sin^{-1}(\sqrt{k}x)}{\sqrt{k}}$ or $\frac{\sinh^{-1}(\sqrt{k}x)}{\sqrt{k}}$ in order to eventually absorb the x-dependence of the original metric. Hence only $k = 0$ is a meaningful case to study, [41, 62]. The field equation $\mathcal{R} = \mathcal{T}$ reads:

$$K = \frac{1}{2|a|}\partial_t\left(\frac{\partial_t |a|^2}{|a|}\right) = \frac{\mathcal{R}}{2} = \frac{\mathcal{T}}{2}, \qquad 2\partial_{tt}|a| = |a|\mathcal{T}.$$

From the first equation in (2.9), relevant Christoffel symbols are $\Gamma^t_{tt} = 0$ and $\Gamma^t_{xx} = \frac{\partial_t |a|^2}{2}$. The second equation in (2.9) yields $\partial_x(p/a) = 0$ because $\Gamma^x_{tt} = 0 = \Gamma^x_{xx}$, so p depends only on t. The equation of state $p = p(\rho)$ leads to $\rho = \rho(t)$. The stress-energy tensor $T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + p\, g^{\alpha\beta}$ with a comoving 2-velocity $u = (1, 0)$ gives:

$$T = \begin{pmatrix} \rho & 0 \\ 0 & p/a^2 \end{pmatrix}, \qquad 2\partial_{tt}|a| = |a|\mathcal{T} = |a|(p - \rho), \tag{2.12}$$

along with the first fluid dynamics equation in (2.9) which yields:

$$\frac{\partial_t(|a|\rho)}{|a|} + \frac{\partial_t |a|^2}{2}\frac{p}{|a|^2} = 0, \qquad \partial_t \rho + (\rho + p)\partial_t \log|a| = 0. \tag{2.13}$$

Both (2.12) and (2.13) form a coupled system of nonlinear ODE in time for the unknowns $|a(t)|, \rho(t)$. The pressure $p(t)$ is deduced thanks to the equation of state.

3. External field approximation: waves in a fixed gravity. Accordingly, one may consider as a first (and less difficult) step the treatment of an evolution equation like the Dirac equation for relativistic fermions in a space-time curved by a time-independent metric [64] (not in isothermal coordinates, though): the corresponding Christoffel symbols induce a so–called "spin connection" bringing new source terms.

3.1. Klein-Gordon model including gravitational effects. Following *e.g.* [48] (see also $1+1$ versions of Birkhoff's theorem relying on the existence of a Killing vector [41, 62]), the inclusion of microscopic particles of mass m in an existing gravitational background induces the following modification of the field equation:

$$\mathcal{R} = \mathcal{T} + \mathcal{T}^{micro}, \qquad \mathcal{T}^{micro} = m^2|\varphi|^2. \tag{3.1}$$

We shall assume that the stress-energy tensor $\mathcal{T}$ results from a "big" mass $M \gg m$, hence the perturbation term $m^2|\varphi|^2$ results as being negligible so the evolution equation for the scalar field φ safely decouples from the former gravitational equation.

3.1.1. Klein-Gordon (KG) model for scalar spinless fields. In tensorial notation and with the summation convention, the usual relativistic KG equation reads:

$$\Box_g \varphi + m^2\varphi = \underbrace{\frac{-1}{\sqrt{-\det g}}\partial_\mu\left(\sqrt{-\det g}\; g^{\mu\nu}\partial_\nu \varphi\right)}_{\textit{intrinsic box operator}} + m^2\varphi = 0,$$

where $m \geq 0$ is a parameter standing for the mass of the scalar field φ. Now, in our $1+1$-dimensional context, and for a diagonal metric of the type (2.1), it rewrites like

$$-\partial_t \left(\frac{\sqrt{|EG|}}{E} \partial_t \varphi \right) - \partial_x \left(\frac{\sqrt{|EG|}}{G} \partial_x \varphi \right) + m^2 \varphi \sqrt{|EG|} = 0.$$

As $m \ll 1$, it is reasonable to assume that the dynamics of the scalar field $\varphi(t,x)$ cannot perturb the already existing gravitational field induced by the metric g. So both E, G are considered to be static functions of x only, some simplification occurs:

$$\partial_{tt}\varphi + \frac{E(x)}{\sqrt{|EG|}} \partial_x \left(\frac{\sqrt{|EG|}}{G(x)} \partial_x \varphi \right) - Em^2 \varphi = 0.$$

Since a smooth 2D surface is conformally flat, it follows that the metric g has only one degree of freedom, which is given by the conformal factor in isothermal coordinates. We have therefore two main examples of static metrics, given by (2.3) and the stellar model (2.11) respectively. As the choice (2.3) was already studied in *e.g.* [48, 51], we switch to (2.11), and the resulting equation reads:

$$\partial_{tt}\varphi - |B(x)|\partial_x \left(|B(x)|\partial_x \varphi \right) + m^2 |B(x)|^2 \varphi = 0. \tag{3.2}$$

The KG equation doesn't generally preserve the L^2 norm: it is however endowed with a continuity equation for a quantity denoted by $\rho(t,x)$ which hasn't a definite sign. Starting from (3.2), one proceeds by subtracting the two following equations,

$$\varphi^* \big[\partial_{tt}\varphi - |B(x)|\partial_x \left(|B(x)|\partial_x \varphi \right) \big] = -m^2 |B(x)|^2 |\varphi|^2 \ ,$$
$$\varphi \big[\partial_{tt}\varphi^* - |B(x)|\partial_x \left(|B(x)|\partial_x \varphi^* \right) \big] = -m^2 |B(x)|^2 |\varphi|^2 \ ,$$

(φ^* standing here for the complex conjugate of φ), in order to produce,

$$\frac{\varphi^* \partial_{tt}\varphi - \varphi \partial_{tt}\varphi^*}{|B(x)|} + \big(\varphi \partial_x \left(|B(x)|\partial_x \varphi^* \right) - \varphi^* \partial_x \left(|B(x)|\partial_x \varphi \right) \big) = 0.$$

Now, standard computations yield a $\rho(t,\cdot)$ with no definite sign:

$$i \frac{\varphi^* \partial_{tt}\varphi - \varphi \partial_{tt}\varphi^*}{|B(x)|} = \partial_t \left(i \cdot \frac{\varphi^* \partial_t \varphi - \varphi \partial_t \varphi^*}{|B(x)|} \right) \stackrel{def}{=} \partial_t \rho. \tag{3.3}$$

It vanishes if $\varphi \in \mathbb{R}$. The mass flux is obtained by observing that:

$$\partial_x(\varphi(|B(x)|\partial_x \varphi^*)) = \partial_x \varphi \cdot |B(x)|\partial_x \varphi^* + \varphi \cdot \partial_x(|B(x)|\partial_x \varphi^*),$$
$$\partial_x(\varphi^*(|B(x)|\partial_x \varphi)) = \partial_x \varphi^* \cdot |B(x)|\partial_x \varphi + \varphi^* \cdot \partial_x(|B(x)|\partial_x \varphi),$$

so cross-products cancel in the difference. The corresponding mass flux $J(t,x)$ reads,

$$J(t,x) = -i \ |B(x)| \left(\varphi^* \partial_x \varphi - \varphi \partial_x \varphi^* \right),$$

and there holds $\partial_t \rho + \partial_x J = 0$, implying that $\frac{d}{dt} \int \rho(t,x) \mathrm{d}x \equiv 0$ formally. This conservation property should hold numerically in order to validate the results.

3.2. Structure-Preserving discretization of Klein-Gordon equation. An energy estimate for (3.2) is easily derived by multiplying it by $\partial_t\varphi$ and integrating:

$$\int_{\mathbb{R}} \partial_t\varphi \cdot \partial_{tt}\varphi - \partial_t(|B|\varphi) \cdot \partial_x(|B|\partial_x\varphi) + m^2|B|^2 \frac{\partial_t|\varphi|^2}{2}\ \mathrm{d}x = 0.$$

By observing that, if $|B(x)|\partial_x\varphi \to 0$ as $|x| \to \pm\infty$, an integration by parts yields:

$$-\int_{\mathbb{R}} \partial_t(|B|\varphi) \cdot \partial_x(|B|\partial_x\varphi)\ \mathrm{d}x = \int_{\mathbb{R}} \partial_t\big(\partial_x(|B|\partial_x\varphi)\big) \cdot \partial_x(|B|\partial_x\varphi)\ \mathrm{d}x.$$

And this leads to the property of the (non-definite) energy conservation,

$$\frac{d}{dt}\int_{\mathbb{R}} \left[|\partial_t\varphi|^2 + |B(x)|^2\left(|\partial_x\varphi|^2 + m^2|\varphi|^2\right)\right]\ \mathrm{d}x \equiv 0.$$

We intend to follow ideas of [20] in order to derive a numerical process able to maintain this property at the discrete level without stringent restrictions as time grows. Accordingly, we denote by $\mathcal{E}$ the associated "conserved energy",

$$\mathcal{E} = \int_{\mathbb{R}} |\partial_t\varphi|^2 + \mathcal{G}(x,\varphi,\partial_x\varphi)\ \mathrm{d}x, \quad \mathcal{G}(x,\varphi,\partial_x\varphi) \stackrel{def}{=} |B(x)|^2\left(|\partial_x\varphi|^2 + m^2|\varphi|^2\right),$$

and Liouville equation (2.8) rewrites in a form inspired by the "gradient flow" PDE's:

$$\frac{\partial^2\varphi}{\partial t^2} + \frac{\delta\mathcal{G}}{\delta\varphi} = 0, \qquad \frac{\delta\mathcal{G}}{\delta\varphi} = \frac{\partial\mathcal{G}}{\partial\varphi} - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{G}}{\partial(\partial_x\varphi)}\right),$$

see [20, pp. 27–31]. The procedure involves the following steps:

1. Defining a discrete $\mathbf{G}$, a discrete analogue of $\mathcal{G}$: for $j,n \in \mathbb{Z}\times\mathbb{N}$,

$$\mathbf{G}_j^n \stackrel{def}{=} \frac{|B(x_j)|^2}{2}\left(\left|\frac{\varphi_{j+1}^n - \varphi_j^n}{\Delta x}\right|^2 + \left|\frac{\varphi_j^n - \varphi_{j-1}^n}{\Delta x}\right|^2 + 2m^2|\varphi_j^n|^2\right).$$

2. Computing its discrete variation in such a way that $\mathbf{E} \simeq \mathcal{E}$ is constant,

$$\mathbf{E}^{n+\frac12} \stackrel{def}{=} \Delta x \sum_{j\in\mathbb{Z}} \left|\frac{\varphi_j^{n+1} - \varphi_j^n}{\Delta t}\right|^2 + \mathbf{G}_j^{n+\frac12}, \quad \mathbf{G}_j^{n+\frac12} = \frac12\left(\mathbf{G}_j^{n+1} + \mathbf{G}_j^n\right).$$

3. Deducing the numerical scheme as a time-marching process: namely, we aim at linearizing, for any $n \in \mathbb{N}$, the difference $\mathbf{E}^{n+\frac12} - \mathbf{E}^{n-\frac12} = 0$ in order to derive an energy-preserving algorithm. Indeed, the time-difference yields:

$$0 = \sum_{j\in\mathbb{Z}} \left(\varphi_j^{n+1} - \varphi_j^{n-1}\right)\left(\frac{\varphi_j^{n+1} - 2\varphi_j^n + \varphi_j^{n-1}}{\Delta t^2} + \underbrace{\frac{\mathbf{G}_j^{n+\frac12} - \mathbf{G}_j^{n-\frac12}}{\varphi_j^{n+1} - \varphi_j^{n-1}}}_{\delta\mathbf{G}_j^n/\delta\varphi}\right),$$

and it remains to express the quantity $\frac{\delta\mathbf{G}_j^n}{\delta\varphi}$. Toward this end, let us define the following notation:

$$\forall j,n \in \mathbb{Z}\times\mathbb{N}, \qquad \delta_{j+\frac12}(\varphi^n) \stackrel{def}{=} \frac{\varphi_{j+1}^n - \varphi_j^n}{\Delta x}.$$

The difference of the time-average values $\mathbf{G}_j^{n+\frac{1}{2}} - \mathbf{G}_j^{n-\frac{1}{2}}$ rewrites:

$$\begin{aligned}
\mathbf{G}_j^{n+\frac{1}{2}} - \mathbf{G}_j^{n-\frac{1}{2}} &= \tfrac{|B(x_j)|^2}{4} \Big[|\delta_{j+\frac{1}{2}}(\varphi^{n+1})|^2 - |\delta_{j+\frac{1}{2}}(\varphi^{n-1})|^2 \\
&\qquad + 2m^2 \left(|\varphi_j^{n+1}|^2 - |\varphi_j^{n-1}|^2 \right) \\
&\qquad + |\delta_{j-\frac{1}{2}}(\varphi^{n+1})|^2 - |\delta_{j-\frac{1}{2}}(\varphi^{n-1})|^2 \Big] \\
&= \tfrac{|B(x_j)|^2}{4} \Big[\delta_{j+\frac{1}{2}}(\varphi^{n+1} + \varphi^{n-1}) \cdot \delta_{j+\frac{1}{2}}(\varphi^{n+1} - \varphi^{n-1}) \\
&\qquad + 2m^2 \left(\varphi_j^{n+1} + \varphi_j^{n-1} \right) \cdot \left(\varphi_j^{n+1} - \varphi_j^{n-1} \right) \\
&\qquad + \delta_{j-\frac{1}{2}}(\varphi^{n+1} + \varphi^{n-1}) \cdot \delta_{j-\frac{1}{2}}(\varphi^{n+1} - \varphi^{n-1}) \Big].
\end{aligned}$$

We need to let $\varphi_j^{n+1} - \varphi_j^{n-1}$ appear hence a summation by parts is convenient:

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \frac{\mathbf{G}_j^{n+\frac{1}{2}} - \mathbf{G}_j^{n-\frac{1}{2}}}{\varphi_j^{n+1} - \varphi_j^{n-1}} &= -\sum_{j\in\mathbb{Z}} \Big[\tfrac{|B(x_{j+1})|^2 + |B(x_j)|^2}{4\Delta x} \delta_{j+\frac{1}{2}}(\varphi^{n+1} + \varphi^{n-1}) \\
&\qquad - \tfrac{m^2 |B(x_j)|^2}{2} \left(\varphi_j^{n+1} + \varphi_j^{n-1} \right) \\
&\qquad - \tfrac{|B(x_j)|^2 + |B(x_{j-1})|^2}{4\Delta x} \delta_{j-\frac{1}{2}}(\varphi^{n+1} + \varphi^{n-1}) \Big].
\end{aligned}$$

Gathering all these results, we obtain the following (implicit) numerical scheme:

$$\boxed{\begin{aligned}
\frac{\varphi_j^{n+1} - 2\varphi_j^n + \varphi_j^{n-1}}{\Delta t^2} &= \frac{1}{2} \Bigg[\frac{|B(x_{j+1})|^2 + |B(x_j)|^2}{2} \left(\frac{\varphi_{j+1}^{n+1} - \varphi_j^{n+1}}{\Delta x^2} + \frac{\varphi_{j+1}^{n-1} - \varphi_j^{n-1}}{\Delta x^2} \right) \\
&\qquad - \frac{|B(x_j)|^2 + |B(x_{j-1})|^2}{2} \left(\frac{\varphi_j^{n+1} - \varphi_{j-1}^{n+1}}{\Delta x^2} + \frac{\varphi_j^{n-1} - \varphi_{j-1}^{n-1}}{\Delta x^2} \right) \Bigg] \\
&\qquad - m^2 |B(x_j)|^2 \left(\frac{\varphi_j^{n+1} + \varphi_j^{n-1}}{2} \right).
\end{aligned}} \tag{3.4}$$

If one imposes periodic boundary conditions at the edges of the (finite) computational

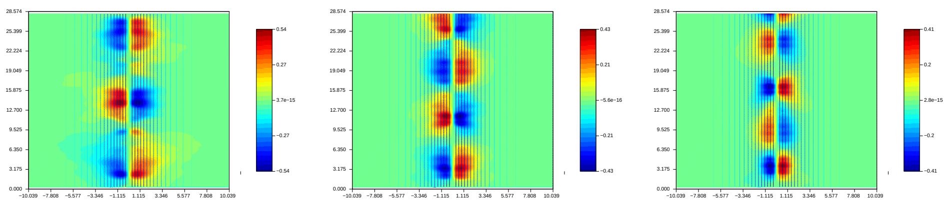

FIG. 3.1. *Position densities* ρ *in* (x,t)*-plane,* $0 \le t \le 40$ *for* $m^2 = 1, 2, 4$.

domain, the scheme (3.4) can easily put in matrix form, and if $\boldsymbol{\varphi}^n$ stands for the vector with components φ_j^n at each $t^n = n\Delta t$, one finds $A\boldsymbol{\varphi}^{n+1} = 2\boldsymbol{\varphi}^n - A\boldsymbol{\varphi}^{n-1}$, with A strictly diagonal-dominant, hence invertible. We set up a slight variation of the aforementioned example, where $B(x) = 1 + \tanh(x/5)$ in $x \in (-10, 10)$, and 2^8 grid points. By prescribing Gaussian-type initial data,

$$\varphi(t=0,x) = \exp(-ikx - (x/\sigma)^2), \qquad \partial_t \varphi(t=0,\cdot) = 0, \qquad k = 0.2,$$

with $\sigma = 3$, we get the results on Fig. 3.1 for increasing masses and $\Delta t = 0.95\Delta x$ (in all figures, color coding is always in increasing order of magnitude). For such a benchmark, the position density ρ_j^n (3.3) is defined at each step as,

$$\rho_j^n = i\left((\varphi_j^n)^* \frac{\varphi_j^n - \varphi_j^{n-1}}{\Delta t} - \varphi_j^n \frac{(\varphi_j^n - \varphi_j^{n-1})^*}{\Delta t}\right)$$

and the deviation of $\sum_j \rho_j^n$ as times grow remains of the order of 10^{-12}. The metric's isolines are superimposed onto the values of ρ. The main effect of the curvature induced by the stellar model (2.11) is to produce smooth oscillations of the quantum particle satisfying (3.2). Moreover, increasing its mass m, Fig. 3.1 reveals that their frequency grows whereas their amplitude decreases, which is the expected behavior.

3.3. Dirac equation for fermions in a curved space-time. Here again, we follow [48] in assuming that including the microscopic term $\mathcal{T}^{micro} = m|\Psi|^2$ induces only a negligible contribution thanks to the smallness of m. Hence we postulate that the Dirac equation decouples from the gravitational field one. Such a model of fermions dynamics can be seen as rendering either gravitational effects and local curvature [42, 55, 56, 64] or effects of impurities/dislocations in a graphene sheet [57].

3.3.1. Inertial coordinates and *Zweibeine*. As we did for the Klein-Gordon equation, we hereafter assume that the fermions dynamics aren't able to perturb the existing gravitational field in a sensible manner. A time-dependent 2-spinor in curved spacetime with metric (2.3) was studied in detail in [56], see equations (20)–(21).

A more challenging situation is the one where a 2-spinor moves in the gravity field resulting of a static, uniform density source, for which one substitutes the point mass concentrated in $x = 0$ by a discontinuous function $\rho(x) = M\chi(|x| < x_0)$, χ being the standard indicator function. In order to set up our framework, we need to compute hydrostatic solutions ϕ, p of the Euler-Liouville system (2.8)–(2.10) and then to derive the $1 + 1$ Dirac equation corresponding to the resulting gravity field.

- Following [7, 62], let us compute, in isothermal coordinates, the hydrostatic solution corresponding to a uniform density of limited extent. The 2-velocity reads $u^t = \exp(-\phi), u^x = 0$ because the scalar velocity $v \equiv 0$ and $\gamma \equiv 1$. The resulting stress-energy tensor is deduced by inserting these values in (2.7). By examining remaining terms in (2.10) and (2.8), the hydrostatic and field equations read, for the pressure $p(x)$ and the conformal factor $\phi(x)$:

$$\partial_x p + (\rho + p)\partial_x \phi = 0, \quad \partial_{xx}\phi = \frac{1}{2}(\rho - p)\exp(2\phi), \quad \rho(x) = M\chi(|x| < x_0).$$

Let's first deal with the interval $x \in (-x_0, x_0)$, where $\rho(x) \equiv M$: the first equation is a logarithmic derivative, which can be handled like

$$-\partial_x \phi = \frac{\partial_x p}{p + M}, \qquad \phi(x) = \phi(0) - \log\left|\frac{p(x) + M}{p(0) + M}\right|.$$

The hydrostatic pressure rewrites:

$$p(x) = (p(0) + M)\exp(-(\phi(x) - \phi(0))) - M,$$

and this leads to the following differential equation for ϕ,

$$2\partial_{xx}\phi = 2M\exp(2\phi(x)) - (p(0) + M)\exp(\phi(x) - \phi(0)).$$

As the density is even $\rho(-x) = \rho(x)$, we can normalize the pressure $p(0) = 1$ and the coformal factor like $\phi(0) = 0$, $\phi'(0) = 0$ in order to get:

$$\phi(x) = -\log\left(\frac{M(1-\cosh x) + (1+\cosh x)}{2}\right), \qquad p(0) = 1.$$

This expression easily yields $\exp(-\phi(x))$ from which one deduces the pressure

$$p(x) = \frac{1}{2}\left((1+M^2) + (1-M^2)\cosh x\right),$$

along with the value x_0, which is such that $p(x_0) = 0$, so

$$p(x_0) = 0, \qquad x_0 = \log\left|\frac{M+1}{M-1}\right|, \qquad M > 1.$$

Concerning the interval $x \notin (-x_0, x_0)$, the density vanishes thus we are in vacuum. The static field equation gives $\partial_{xx}\phi \equiv 0$, with an even solution which moreover must be continuous in x_0. So we get that for $x \notin (-x_0, x_0)$,

$$\phi(x) = A|x| + B, \qquad A|x_0| + B = \phi(x_0) = -\log\frac{M}{1+M}.$$

A second constraint comes from the hydrostatic equation posed in x_0,

$$\partial_x\phi(x_0) + \frac{\partial_x p(x_0)}{\rho(x_0)} = \partial_x\phi(x_0) + \frac{1-M^2}{M}\sinh x_0, \qquad \partial_x\phi(x_0) = 1 = A.$$

Finally, the (hydrostatic) gravitational field rewrites:

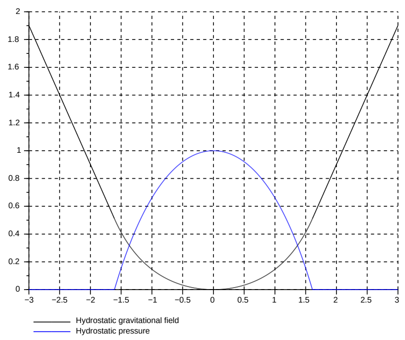

FIG. 3.2. *Illustration of hydrostatic solution for $M = 1.5$, so $x_0 = \log(5)$.*

$$\phi(x) = \begin{cases} |x| + \log\left(\frac{M-1}{M}\right), & x \notin \left(-\log\left|\frac{M+1}{M-1}\right|, \log\left|\frac{M+1}{M-1}\right|\right), \\ -\log\left(\frac{M+1-(M-1)\cosh x}{2}\right), & x \in \left(-\log\left|\frac{M+1}{M-1}\right|, \log\left|\frac{M+1}{M-1}\right|\right). \end{cases} \tag{3.5}$$

- Following for instance [55, 56], we now derive the corresponding expression of the Dirac equation for the 2-spinor $\Psi(t,x) \in \mathbb{C}^2$ in this static spacetime. In order to express properly this equation on a curved surface $\mathcal{S}$, it is necessary to introduce the so–called *Dyad* (or *Zweibein*), which, given any point $p \in \mathcal{S}$, is a linear map from the tangent space $T_p\mathcal{S}$ onto the Minkowski flat space which preserves the inner product. In tensorial notation, it comes:

$$\forall x, y \in (T_p\mathcal{S})^2, \qquad g_{\mu\nu}x^\mu y^\nu = \eta_{ab}(e^a_\alpha x^\alpha)(e^b_\beta y^\beta).$$

Since we chose to work in conformally flat (isothermal) coordinates, the metric $g = \exp(2\phi)\eta$ and the dyads read simply: $e^a_\alpha(x) = \exp(\phi)\delta^a_\alpha$, where $\delta^a_\alpha = 1$ for $\alpha = a$, and 0 if $\alpha \neq a$. For the "stellar" and the FRW metrics,

$$[e^a_\alpha]^{stellar} = \begin{pmatrix} |B(x)| & 0 \\ 0 & 1 \end{pmatrix}, \qquad [e^a_\alpha]^{FRW} = \begin{pmatrix} 1 & 0 \\ 0 & |a(t)| \end{pmatrix}.$$

Relying on [64], the Dirac equation on curved spacetime reads, in tensorial notation:

$$i\gamma^a \left[E^\mu_a \partial_\mu(\Psi) + \frac{1}{2\sqrt{-\det g}}\partial_\mu\left(\sqrt{-\det g}\ E^\mu_a\right)\Psi\right] = m\Psi, \tag{3.6}$$

wth $E^\mu_a = \exp(-\phi)\delta^\mu_a$ standing for the inverse of e^a_μ. Here, the matrices γ^0, γ^1 are:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

like in [56], but other choices are possible, see [55, 64]. We can rewrite (3.6) as:

$$\begin{aligned} -im\Psi = \gamma^0 &\left[\exp(-\phi)\partial_t\Psi + \tfrac{1}{2\exp(2\phi)}\partial_t\big(\exp(2\phi)\exp(-\phi)\big)\Psi\right] \\ +\gamma^1 &\left[\exp(-\phi)\partial_x\Psi + \tfrac{1}{2\exp(2\phi)}\partial_x\big(\exp(2\phi)\exp(-\phi)\big)\Psi\right]. \end{aligned}$$

As the static conformal factor $\phi(x)$ is independent of t, there are simplifications:

$$\gamma^0\partial_t\Psi + \gamma^1 \underbrace{\left(\partial_x\Psi + \frac{\partial_x\phi}{2}\Psi\right)}_{intrinsic\ space-derivative} + im\exp(\phi)\Psi = 0.$$

It is advantageous to introduce a rescaled spinor $\tilde\Psi = \exp(\phi/2)\Psi$, so it comes

$$\gamma^0\partial_t\tilde\Psi + \gamma^1\partial_x\tilde\Psi + im\exp(\phi)\tilde\Psi = 0. \tag{3.7}$$

By standard computations, one recovers the L^2 conservation law:

$$\partial_t\left(|\tilde\Psi_+|^2 + |\tilde\Psi_-|^2\right) + \partial_x\left(|\tilde\Psi_+|^2 - |\tilde\Psi_-|^2\right) = 0, \qquad \tilde\Psi_\pm = \exp(\phi/2)\Psi_\pm.$$

3.3.2. L^2-preserving WB numerical scheme. A recent numerical scheme was studied in [24] for the simulation of the Dirac equation within an x-dependent scalar potential. Such a framework matches the one derived in the former subsection:

1. the mass term $m\exp(\phi(x))$ isn't a constant if the spacetime is curved,
2. there is no supplementary scalar potential, $V_t(x)$, appearing in (3.7).

Thus, with preceding notations, (3.7) rewrites:

$$\partial_t\tilde\psi_\pm(t,x) \pm \partial_x\tilde\psi_\pm + im\exp(\phi)\tilde\psi_\mp = 0, \qquad \tilde\psi = (\tilde\psi_-, \tilde\psi_+) \in \mathbb{C}^2.$$

Clearly, we intend to proceed like in §B.2, that is to say by localizing the right-hand side by means of a lattice of Dirac masses adapted to the computational grid:

$$\partial_t \tilde{\psi}_\pm(t,x) \pm \partial_x \tilde{\psi}_\pm + im\Delta x \sum_{j\in\mathbb{Z}} \left(\exp(\phi)\tilde{\psi}_\mp \cdot \delta(x - x_{j-\frac{1}{2}}) \right) = 0.$$

A Godunov scheme is deduced by integrating the preceding (modified) equation on each numerical elementary domain,

$$D_j^n = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \times (t^n, t^{n+1}), \qquad j,n \in \mathbb{Z}\times\mathbb{N}.$$

A piecewise-constant approximation can be defined as follows:

$$\tilde{\psi}^{\Delta x} = \tilde{\psi}_\pm^{\Delta x} \in \mathbb{C}^2, \qquad \tilde{\psi}_\pm^{\Delta x}(t,x) = \tilde{\psi}_{j,\pm}^n, \qquad (t,x) \in D_j^n.$$

Because of the Dirac masses, "scattering states" $\tilde{\psi}^n_{j-\frac{1}{2},\pm}$ appear at the discrete level, which are obtained out of the stationary equations $\pm\partial_x\tilde{\psi}_\pm + im\exp(\phi)\tilde{\psi}_\mp = 0$:

$$\left.\begin{array}{l} \tilde{\psi}^{n+1}_{j,+} = \tilde{\psi}^n_{j,+} - \frac{\Delta t}{\Delta x}\left(\tilde{\psi}^n_{j,+} - \tilde{\psi}^n_{j-\frac{1}{2},+}\right) \\ \tilde{\psi}^{n+1}_{j,-} = \tilde{\psi}^n_{j,-} + \frac{\Delta t}{\Delta x}\left(\tilde{\psi}^n_{j+\frac{1}{2},-} - \tilde{\psi}^n_{j,-}\right) \end{array}\right\}$$

It was noticed in [24] that this scheme rewrites in the following (vectorial) form:

$$\begin{pmatrix} \tilde{\psi}^{n+1}_{j,+} \\ \tilde{\psi}^{n+1}_{j-1,-} \end{pmatrix} = \left(1 - \frac{\Delta t}{\Delta x}\right) \begin{pmatrix} \tilde{\psi}^{n}_{j,+} \\ \tilde{\psi}^{n}_{j-1,-} \end{pmatrix} + \frac{\Delta t}{\Delta x}\mathbf{S}_{j-\frac{1}{2}} \begin{pmatrix} \tilde{\psi}^{n}_{j-1,+} \\ \tilde{\psi}^{n}_{j,-} \end{pmatrix}, \tag{3.8}$$

where $\mathbf{S}_{j-\frac{1}{2}}$ is a "local 2×2 scattering matrix" which reads:

$$\mathbf{S}_{j-\frac{1}{2}} = \frac{1}{\cosh\omega} \begin{pmatrix} 1 & -i\ \sinh\omega_{j-\frac{1}{2}} \\ -i\ \sinh\omega_{j-\frac{1}{2}} & 1 \end{pmatrix}, \quad \omega_{j-\frac{1}{2}} = m\Delta x \exp(\phi(x_{j-\frac{1}{2}})).$$

As usual, it relates the scattering states to the "incoming states":

$$\begin{pmatrix} \tilde{\psi}^{n}_{j-\frac{1}{2},+} \\ \tilde{\psi}^{n}_{j-\frac{1}{2},-} \end{pmatrix} = \mathbf{S}_{j-\frac{1}{2}} \begin{pmatrix} \tilde{\psi}^{n}_{j-1,+} \\ \tilde{\psi}^{n}_{j,-} \end{pmatrix}.$$

It is clear from (3.8) that numerical dissipation is minimized when $\Delta t = \Delta x$, so that the Courant number is 1: the L^2 norm, a particular entropy, is conserved as time grows, and this is equivalent to the preservation of the probability of presence of relativistic quantum particles. The scheme (3.8) is applied to the equation (3.7) endowed with the conformal factor (3.5) in the interval $x \in (-3,3)$ and periodic boundary conditions (which hardly play any role because particles don't reach borders). There are 2^7 points in space, $\Delta t = \Delta x$, and initial data read (before normalization):

$$\psi_\pm(t=0,x) = \exp\left(-2(x\pm 0.6)^2\right).$$

Corresponding results show up on Fig. 3.3 on the (curved) x,t spacetime, namely position densities $\rho(t,\cdot) = |\psi_+(t,\cdot)|^2 + |\psi_-(t,\cdot)|^2$, $0 \le t \le 25$, for several masses $m = \frac{3}{4}, \frac{3}{2}, 3$. The scheme works in $\tilde{\Psi}$ variables, but results appear in the original ones. Metric's isolines are displayed together with the numerical values of ρ. As expected, the space-time's (static) curvature induced by (3.5) produces oscillating dynamics for the quantum particle. Moreover, increasing its mass m in (3.7) restricts noticeably the spreading of its density of presence (see Fig. 3.3, from left to right).

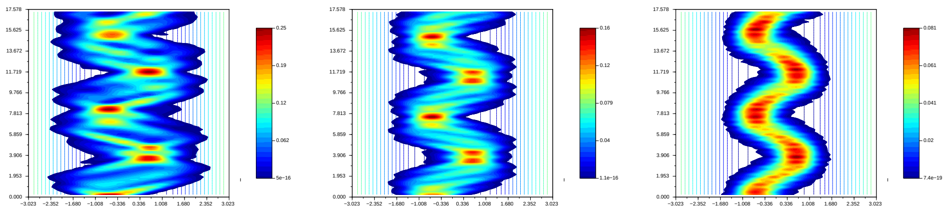

FIG. 3.3. *Spinor's position densities in (x,t)-plane for increasing masses $m = 0.75, 1.5, 3$.*

4. Dynamical coupling: a model of self-gravitating perfect fluid. Opposite to both the preceding sections where we considered microscopic particles moving into an existing gravitational field induced by a much more massive object, hence allowing for a decoupling of their dynamics from the feedback of the $\mathcal{R} = \mathcal{T}$ equation, we shall consider hereafter the numerical resolution of a fully coupled problem where a (relativistic) perfect gas moves according to the gravity it creates. In some sense, this is a very elementary $1+1$-dimensional illustration of J.A. Wheeler's famous sentence: "*Matter tells spacetime how to curve, and curvature tells matter how to move*".

4.1. Locally inertial numerical discretization. Here, we follow [26, pp. 18–20] and work on a uniform Cartesian grid with cells centered around t^n, x_j, where $n, j \in \mathbb{N} \times \mathbb{Z}$. Those computational cells are such that $\phi \equiv \phi_j^n \simeq \phi(t^n, x_j)$, *i.e.* ϕ is a constant in each cell, meaning that it is a local inertial reference frame (in the sense that the metric is flat inside each cell): the conformal factor jumps at each interface

4.2. Structure-Preserving discretization for Liouville equation. By multiplying (2.8) by $\partial_t(2\phi)$, assuming that $\partial_t(\rho - p) \equiv 0$ and integrating on $\mathbb{R}$, one gets:

$$\frac{d}{dt}\int_{\mathbb{R}} \left[|\partial_t \phi|^2 + |\partial_x \phi|^2 + \frac{(\rho - p)(x)}{2} \exp(2\phi) \right] \cdot \mathrm{d}x \equiv 0.$$

Thus it appears convenient to follow again [20] in order to develop a numerical discretization able to keep constant a discrete approximation of this energy functional. An alternative, less costly, would be to use the simpler scheme recalled in *e.g.* [58],

$$\phi_j^{n+1} = 2\phi_j^n \left(1 - \frac{\Delta t^2}{\Delta x^2}\right) - \phi_j^{n-1} + \frac{\Delta t^2}{\Delta x^2}(\phi_{j+1}^n + \phi_{j-1}^n) - \frac{\Delta t^2}{2}(\rho_j^n - p_j^n) \exp(\phi_{j+1}^n + \phi_{j-1}^n). \tag{4.1}$$

One issue is to find a correct manner to initialize (4.1): one manner is to prescribe constant initial data, another is to derive a "pseudo-static" conformal factor, *i.e.* an approximation of the solution to (with appropriate boundary/decay conditions),

$$-\partial_{xx}\phi^0(x) + \frac{\rho^0(x) - p(\rho^0(x))}{2} \exp(2\phi^0(x)) = 0. \tag{4.2}$$

4.3. Well-balanced Godunov scheme for (2.9) with $p(\rho) = \sigma^2\rho$.

4.3.1. Riemann problem for relativistic Euler system. Hereafter, we quickly recall elementary properties of the system (2.9) when the pressure law is "isothermal": according to [26, page 22], the choice $\sigma^2 = \frac{1}{3}$ is especially meaningful. The first observation is the very easy inversion relating both sets of variables $\tau, S, \Sigma \mapsto \rho, v$:

$$\rho = \frac{\tau - \Sigma}{1 - \sigma^2}, \qquad v = \frac{(1 - \sigma^2)S}{(1 + \sigma - \sigma^2)\tau - \sigma\Sigma}.$$

Moreover, the value of v can be easily deduced from $\tau, S \neq 0$ as follows:

$$s = \frac{S}{\tau}, \qquad v = \frac{(1+\sigma^2) - \sqrt{(1+\sigma^2)^2 - 4\sigma^2 s^2}}{2\sigma^2 s}.$$

Following [26], characteristic velocities are found to be,

$$\lambda^{\pm} = v \pm \sigma, \qquad \lambda^- < \lambda^+,$$

thus system (2.9) with the aforementioned pressure law is unconditionally strictly hyperbolic as soon as the sound speed $\sigma > 0$. Since $|v| < 1$, the homogeneous CFL stability restriction is simply $(1+\sigma)\Delta t \leq \Delta x$. More importantly, if one assumes that the system is to remain in subsonic regime, then the CFL lightens into $2\sigma\Delta t \leq \Delta x$. Sonic points are such that $|v| = \sigma$ and Riemann invariants read:

$$\mathcal{W}_{\pm}(\rho, v) = \frac{1}{2}\left[\ln\left(\frac{1+v}{1-v}\right) \pm \frac{\sigma}{1+\sigma^2}\ln\rho\right] = \frac{1}{2}\ln\left[\left(\frac{1+v}{1-v}\right)\rho^{\pm\frac{\sigma}{1+\sigma^2}}\right]. \tag{4.3}$$

It is convenient to set up a subsonic (approximate) Riemann solver based only on rarefaction curves by imposing that each Riemann invariant must remain constant across its corresponding simple wave: given left/right states ρ_L, v_L, and ρ_R, v_R, both not in vacuum, one seeks an intermediate state ρ_*, v_* such that,

$$\mathcal{W}_+(\rho_L, v_L) = \mathcal{W}_+(\rho_*, v_*), \qquad \mathcal{W}_-(\rho_*, v_*) = \mathcal{W}_-(\rho_R, v_R). \tag{4.4}$$

These 2 equations are straightforwardly solved and it comes:

$$\begin{aligned} \rho_* &= \exp\left(\frac{1+\sigma^2}{\sigma}(\mathcal{W}_+ - \mathcal{W}_-)\right), \\ v_* &= \frac{\exp(\mathcal{W}_+ + \mathcal{W}_-) - 1}{\exp(\mathcal{W}_+ + \mathcal{W}_-) + 1} = \tanh\left(\frac{\mathcal{W}_+ + \mathcal{W}_-}{2}\right), \end{aligned} \tag{4.5}$$

in particular, $v_* \in (-1, 1)$.The corresponding Godunov scheme reads accordingly,

$$\left.\begin{aligned} \tau_j^{n+1} &= \tau_j^n - \tfrac{\Delta t}{\Delta x}\left(S^n_{j+\frac12} - S^n_{j-\frac12}\right), \\ S_j^{n+1} &= S_j^n - \tfrac{\Delta t}{\Delta x}\left(\Sigma^n_{j+\frac12} - \Sigma^n_{j-\frac12}\right), \end{aligned}\right\} \tag{4.6}$$

where the numerical fluxes $S^n_{j\pm\frac12}, \Sigma^n_{j\pm\frac12}$ are deduced from both (4.5) and the definition of S, Σ. At each interface $x_{j+\frac12} = (j+\frac12)\Delta x$, one considers the elementary Riemann problem separating ρ_j^n, v_j^n and ρ_{j+1}^n, v_{j+1}^n, which yields the intermediate state (4.5).

4.3.2. Inclusion of zero-waves for source terms in $\partial_x\phi$. Here we shall follow the ideas of [23, 28] and choose to solve the "augmented Riemann problem" for

$$\left.\begin{aligned} \partial_t\tau + \partial_x S + 2S\partial_x\phi &= 0, \\ \partial_t S + \partial_x\Sigma + (\tau+\Sigma)\partial_x\phi &= 0, \\ \partial_t\phi &= 0, \end{aligned}\right\} \tag{4.7}$$

where the last (static) equation reflects the "locally inertial" nature of the numerical process. Indeed, in each computational cell, the metric is flat $\phi \equiv \phi_j^n$ and $\partial_t\phi \equiv 0$. The characteristic velocities for (4.7) are now $\lambda^{\pm}$ and 0, hence in order to preserve strict hyperbolicity, one must restrict v for preventing crossing of eigenvalues $\lambda^{\pm} = 0$

which induces very intricate "nonlinear resonance" phenomena [28]. Accordingly we shall again assume in the sequel that (4.7) remains in subsonic regime, $|v| < \sigma < 1$ (for $\sigma = 1$, $p = \rho$ and (2.8) gives that the conformal factor ϕ decouples from matter).

Besides, the supplementary (static) jump relations generated by the terms in $\partial_x \phi$ are integral curves of the stationary equations, so another advantage of limiting $|v|$ is their simplification. More precisely, let's assume $|v|$ small enough so that one writes:

$$\partial_x \Sigma = -(\tau + \Sigma)\partial_x \phi \simeq -(\tau_* + \Sigma)\partial_x \phi, \qquad \tau_* = \frac{\rho_*(1 + \sigma^2 v_*^2)}{1 - v_*^2},$$

and ρ_*, v_* are given by (4.5). Accordingly, stationary equations of (4.7) become

$$\partial_x S + 2S\partial_x \phi = 0, \qquad \partial_x \Sigma = -(\tau_* + \Sigma)\partial_x \phi. \tag{4.8}$$

Since it degenerates correctly when $\phi_+ - \phi_- \to 0$, the approximation involving τ_* is justified because the Liouville equation (2.8) doesn't create shocks. So, by refining the grid if necessary, the amplitude of the zero-waves decrease and so their truncation errors. Integral curves of the (approximate) static discontinuities read accordingly:

$$\begin{array}{rcl} S(x)\exp(2\phi(x)) & = & S(0)\exp(2\phi(0)), \\ \Sigma(x) & = & \Sigma(0)\exp(\phi(0) - \phi(x)) - \tau_*[1 - \exp(\phi(0) - \phi(x))], \end{array} \tag{4.9}$$

so the Riemann problem for (4.7) involves 2 genuinely nonlinear waves corresponding to the Riemann invariants $\mathcal{W}_\pm$ and the very linearly degenerate (static) discontinuity induced by the jump of ϕ. Such a Riemann solver, when inserted inside a Godunov scheme, displays many advantages as long as nonlinear resonance doesn't occur:

- the resulting scheme is truly "locally inertial" in the sense that the metric is a constant inside each computational cell. Discontinuities are resolved at each interface $x_{j+\frac{1}{2}}$ involving the jump of the conformal factor, too;
- by construction, if ϕ remains constant between two computational cells, the Godunov scheme degenerates onto (4.6);
- including source terms by means of a supplementary static jump relation doesn't have consequences on the CFL restriction, in subsonic regime stability holds for $2\sigma\Delta t \leq \Delta x$;
- an hydrostatic equilibrium, that is to say a stationary state in which pressure balances gravity with a null velocity $v \equiv 0$ is automatically preserved at the discrete level because the approximate equations (4.8) are exact, and the Riemann problem for (4.7) is solved by means of the supplementary wave only (in general, this is referred to as the "well-balanced" (WB) property).

The corresponding Godunov scheme for (4.7) reads accordingly,

$$\left.\begin{array}{l} \tau_j^{n+1} = \tau_j^n - \Delta t \overbrace{\left(\dfrac{S^n_{j+\frac{1}{2},-} - S^n_{j-\frac{1}{2},+}}{\Delta x}\right)}^{numer.\ intrinsic\ space-derivative} \\ S_j^{n+1} = S_j^n - \Delta t \underbrace{\left(\dfrac{\Sigma^n_{j+\frac{1}{2},-} - \Sigma^n_{j-\frac{1}{2},+}}{\Delta x}\right)}_{numer.\ intrinsic\ space-derivative} \end{array}\right\} \tag{4.10}$$

where now $S^n_{j+\frac{1}{2},\mp}, \Sigma^n_{j+\frac{1}{2},\mp}$ are the left/right states in the Riemann fan which are separated by the static discontinuity induced by the jump $\phi^n_{j+1} - \phi^n_j$ of the metric's conformal factor located at the interface $x_{j+\frac{1}{2}}$. These left/right states are derived from the values $\rho_\pm, v_\pm$ appearing in the Riemann problem:

$$\begin{pmatrix} \rho_L \\ v_L \end{pmatrix} \overset{\mathcal{W}_+,\phi\equiv C}{\rightarrow} \begin{pmatrix} \rho_- \\ v_- \end{pmatrix} \overset{\substack{S\exp(2\phi),\\ \Sigma[\dots]}\ \equiv C}{\rightarrow} \begin{pmatrix} \rho_+ \\ v_+ \end{pmatrix} \overset{\mathcal{W}_-,\phi\equiv C}{\rightarrow} \begin{pmatrix} \rho_- \\ v_- \end{pmatrix},$$

where the quantities constant across each simple wave are written over each arrow.

We have the following relations, for $\mathcal{W}_- = \mathcal{W}_-(\rho_L, v_L)$ and $\mathcal{W}_+ = \mathcal{W}_+(\rho_R, v_R)$:

- on the left side of the static discontinuity,

$$\rho_- = \left(\frac{(1-v_-)\exp(2\mathcal{W}_+)}{1+v_-}\right)^{\frac{1+\sigma^2}{\sigma}} \Rightarrow \begin{cases} S_- = \frac{(1+\sigma^2)v_-}{1-v_-^2}\left(\frac{(1-v_-)\exp(2\mathcal{W}_+)}{1+v_-}\right)^{\frac{1+\sigma^2}{\sigma}} \\ \Sigma_- = \frac{\sigma^2+v_-^2}{1-v_-^2}\left(\frac{(1-v_-)\exp(2\mathcal{W}_+)}{1+v_-}\right)^{\frac{1+\sigma^2}{\sigma}} \end{cases}$$

- analogously, on the right side of the static discontinuity,

$$\rho_+ = \left(\frac{1+v_+}{(1-v_+)\exp(2\mathcal{W}_-)}\right)^{\frac{1+\sigma^2}{\sigma}} \Rightarrow \begin{cases} S_+ = \frac{(1+\sigma^2)v_+}{1-v_+^2}\left(\frac{1+v_+}{(1-v_+)\exp(2\mathcal{W}_-)}\right)^{\frac{1+\sigma^2}{\sigma}} \\ \Sigma_+ = \frac{\sigma^2+v_+^2}{1-v_+^2}\left(\frac{1+v_+}{(1-v_+)\exp(2\mathcal{W}_-)}\right)^{\frac{1+\sigma^2}{\sigma}} \end{cases}$$

- and both these states must satisfy:

$$\boxed{\begin{aligned} S_+\exp(2\phi_+) &= S_-\exp(2\phi_-), \\ \Sigma_+\exp(\phi_+) &= \Sigma_-\exp(\phi_-) - \tau_*[\exp(\phi_+) - \exp(\phi_-)]. \end{aligned}} \tag{4.11}$$

The system (4.11) is a set of two nonlinear algebraic equations for $v_\pm$, which can be solved numerically, by prescribing an starting value of $v_+ = v_- = v_*$, corresponding to $\phi_+ = \phi_-$. Then, having at hand both values $v_\pm$, $\rho_\pm$ are deduced thanks to (4.4):

$$\mathcal{W}_+(\rho_L, v_L) = \mathcal{W}_+(\rho_-, v_-), \qquad \mathcal{W}_-(\rho_R, v_R) = \mathcal{W}_-(\rho_+, v_+).$$

If $\phi_+ = \phi_-$, the metric remains flat inside 2 consecutive computational cells, (4.11) implies automatically that $S_+ = S_-$, $\Sigma_+ = \Sigma_-$, so it yields the v_* value found in (4.5), along with $\rho_+ = \rho_- = \rho_*$. Inversion of this nonlinear algebraic system allows to compute all the numerical fluxes of the locally inertial Godunov scheme (4.10).

4.3.3. Inclusion of time-splitting for source terms in $\partial_t\phi$. It remains to treat the source terms depending on $\partial_t\phi$ inside (2.9); according to [22], one may choose to treat them by means of a time-splitting algorithm involving the aforementioned approximation in order to integrate the differential system, for $t \in (n\Delta t, (n+1)\Delta t)$,

$$\partial_t\tau(t) = -(\tau+\Sigma)\partial_t\phi \simeq -(\tau+\Sigma^n)\partial_t\phi, \qquad \partial_t S(t) + 2S\partial_t\phi = 0,$$

where Σ^n stands for a "frozen" value of $\Sigma(t)$ at the time $t^n = n\Delta t$. More precisely, given, for some time-index $n \in \mathbb{N}$ and any space-index $j \in \mathbb{Z}$, both approximate values $\tau_j^n \simeq \tau(t^n, x_j)$, $S_j^n \simeq S(t^n, x_j)$, this ODE step reads:

$$\boxed{\begin{aligned} \tilde{\tau}_j^{n+1} &= \tau_j^n \exp\left(-(\phi_j^{n+1} - \phi_j^n)\right) - \Sigma_j^n \left[1 - \exp\left(-(\phi_j^{n+1} - \phi_j^n)\right)\right], \\ \tilde{S}_j^{n+1} &= S_j^n \exp\left(-2(\phi_j^{n+1} - \phi_j^n)\right), \end{aligned}} \tag{4.12}$$

where $\phi_j^{n+1} - \phi_j^n$ is the time-variation of the metric's conformal factor, computed by means of the Structure-Preserving scheme or (4.1) for the Liouville equation (2.8).

REMARK 4. *Since $\rho \geq p(\rho)$, $\phi(t,\cdot)$ decreases in time by (2.8); yet $\tau + \Sigma \geq 0$, so $\tau(t,\cdot)$, hence $\rho(t,\rho)$, always grows. This may perturb gravitational self-interactions, so we imposed at each time-step that the arithmetic mean of ϕ remains constant. Inserting a "cosmological constant" $\Lambda > 0$ in (2.8), modeling a neutralizing background may appear as being necessary for global stability.*

4.4. Summary of the time-marching scheme for (2.8)–(2.9). Given initial data for $\phi_j^{n=-1}, \phi_j^{n=0}$, and $\rho_j^{n=0}, v_j^{n=0}$, $j \in \mathbb{Z}$, our numerical process consists in advancing first the conformal factor by means of *e.g.* (4.1), then compute all the jumps in space $\phi_j^{n=1} - \phi_{j-1}^{n=1}$, solve (4.11) at each interface in order to compute corresponding Godunov fluxes $S^{n=0}_{j-\frac{1}{2},\pm}, \Sigma^{n=0}_{j-\frac{1}{2},\pm}$, then update each couple $\tau_j^{n=0}, S_j^{n=0}$, and finally apply the time-splitting correction (4.12) in order to derive $\tau_j^{n=1}, S_j^{n=1}$. This sequence applies iteratively until a prescribed stopping time, t^N, $N \in \mathbb{N}$, is reached.

4.4.1. Self-gravitating relativistic isothermal gas cloud. As an illustrative example, we consider a classical problem consisting in a gravitational collapse of randomly distributed dust. This meets with the initial data,

$$\rho^0(x_j) = \texttt{random}_j \in (0,1), \qquad v_j \equiv 0, \qquad j \in \{1, 2, ..., J\},$$

and $\phi_j^{n=-1} = \phi_j^{n=0} = 0$ (the flat Minkowski metric), with transparent boundary conditions (see Appendix C). On a finite computational domain, boundary conditions must be specified at each time t^n, in $j = 0$ and $j = J + 1$: we imposed that $v = 0 = \partial_x \rho$. Clearly, this is enough to compute the Godunov fluxes at the edges of the computational domain by means of (4.5). On Fig. 4.1, we display the results in $T = 10$ of such a simulation on the domain $x \in (-5, 5)$, with 100 grid points, so $\Delta x = 0.1$ and a constant CFL number $2\sigma\Delta t \leq \Delta x$. The isothermal pressure law is fixed by $\sigma^2 = \frac{1}{3}$; gravitational effects decrease with the sound speed $0 < \sigma < 1$. On the top, left, the conformal factor $\phi(t = 10, \cdot)$ indicates that the accumulated mass in the center of the computational domain curved the spacetime's metric. Corresponding isolines of $\exp(2\phi)$ are superimposed onto numerical values of ρ (t is vertical): the picture on top, right in1dicates that densities pass from a very disordered state to a more ordered one. The final states (in $t = 10$) of ρ and v are displayed on the bottom, together with the random fluctuations of $\rho(t = 0, \cdot)$. The 1D scalar velocity v nearly vanished around $x \simeq 0$, that is where matter accumulated. Red lines indicate sonic speeds $\pm\sigma$: crossing them during the simulation makes the numerical process break down because of nonlinear resonance [28] and loss of strict hyperbolicity in (4.7).

REMARK 5. *A conformally flat metric (2.5) cannot hold a simulation of complete gravitational collapse as ϕ would blow up close to the event horizon [31]; notice that as $\sigma^2 = \frac{1}{3}$, could such a simulation be achieved, the final object would be a "black membrane" according to [65]. Instead, as the pressure law is barotropic, a polytrope likely emerged as an isothermal self-gravitating configuration of gas. Its structure is similar to the one of a collisionless system of stars, like a (low-dimensional caricature of a) globular cluster; see [27, 61] for polytropes arising in gravitational collapse.*

4.4.2. Random perturbation of a $1 + 1$ polytrope. Another interesting benchmark consists in simulating the effects of an initially Gaussian mass distribution, which is perturbed by random (scalar) velocity fluctuations:

$$\rho_0(x) = 0.15 + 4.35 \exp(-x^2), \qquad v(t = 0, x_j) = 0.9\sigma(\texttt{random}_j - 0.5).$$

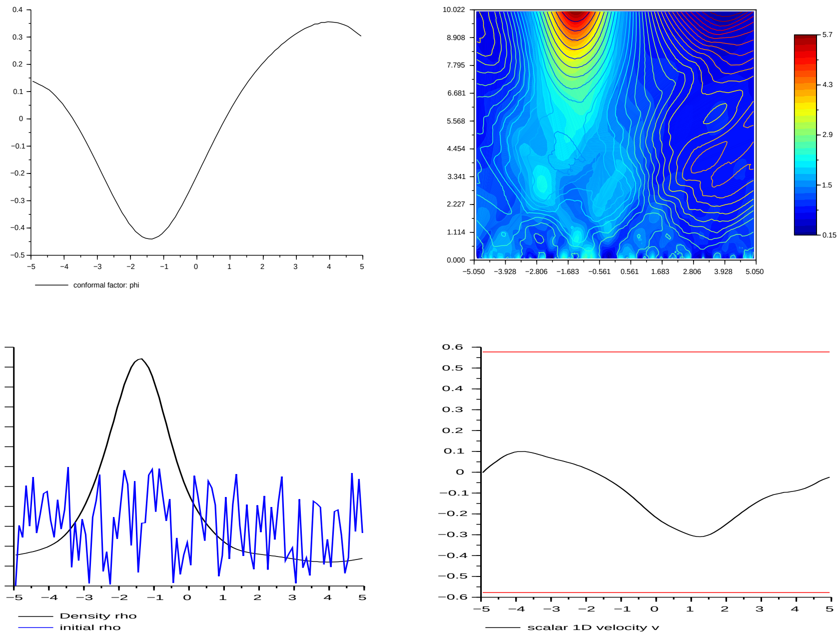

FIG. 4.1. *Numerical $\phi(t=10,\cdot)$ (left), ρ (right, in (x,t)-plane, $t \le 10$); initial and final states $\rho(t=10,\cdot), v(t=10,\cdot)$ (bottom) for a gravitational collapse of random initial data with $\sigma^2 = \frac{1}{3}$.*

Initial data for the conformal factor was chosen as a stationary distribution satisfying (4.2) with $\partial_t \phi(t=0,\cdot) = 0$. Numerical results (with identical grid parameters) at time $t = 10$ are displayed on Fig. 4.2. The initial mass repartition is shaken, an oscillatory movement takes place for a short time, but dynamics stabilize later onto a new mass distribution, less massive, with residual mass flowing outside, toward each boundary of the computational domain. The scalar velocity around $x = 0$ at $t = 10$ is close to zero, thanks to the well-balanced property of our discretization. As in the former simulation, a (less piked) polytrope emerged, with a density distribution strongly condensed at its center, in accordance with the infinite polytropic index.

5. Conclusion and outlook. In the realm of $1+1$-dimensional general relativity, several numerical computations were achieved, first within the "external field approximation", keeping the space-time metric static despite the dynamics of matter, and then by assuming the time-dependent metric given in a conformal gauge (2.5), resulting in a nonlinear coupled Euler-Liouville system (2.10)–(2.8). The result of the gravitational collapse displayed on Fig. 4.1 isn't a black hole as such a metric g cannot have an "event horizon" [31]. Instead, the Schwarzschild gauge defined in (2.3) would be well-suited for such a delicate computation, at the price of handling stronger nonlinearities and coordinate shocks, see Remark 2. Hereafter we summarize quickly the coupled system which expresses the dynamics of a perfect fluid in such a time-dependent metric, starting with Christoffel symbols (compare with (2.6)),

$$\Gamma^t = \frac{1}{2\alpha}\begin{pmatrix} \partial_t \alpha & \partial_x \alpha \\ \partial_x \alpha & \partial_t (1/\alpha) \end{pmatrix}, \qquad \Gamma^x = \frac{\alpha}{2}\begin{pmatrix} \partial_x \alpha & \partial_t (1/\alpha) \\ \partial_t (1/\alpha) & \partial_x (1/\alpha) \end{pmatrix}.$$

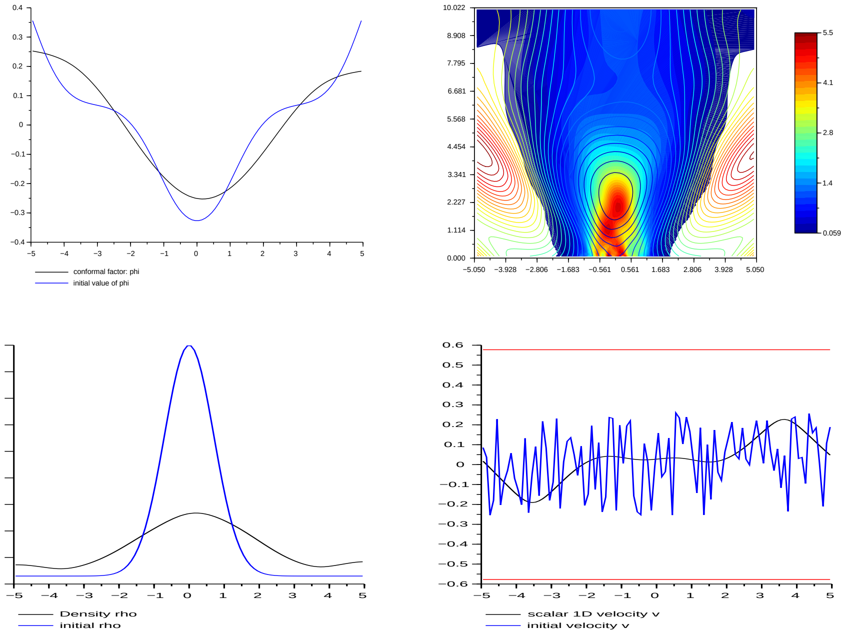

FIG. 4.2. *Numerical $\phi(t = 10, \cdot)$ (left), ρ (right, in (x, t)-plane, $t \leq 10$); initial and final states $\rho(t = 10, \cdot), v(t = 10, \cdot)$ (bottom) for a random perturbation of Gaussian mass with $\sigma^2 = \frac{1}{3}$.*

The Lorentz factor allows to define a slight variant of the scalar velocity v, so

$$\gamma = \frac{1}{\sqrt{1 - v^2}}, \qquad u := \gamma \left(\frac{1}{\sqrt{\alpha}}, \sqrt{\alpha} v \right).$$

According to this normalization, the mass-energy tensor rewrites, [50, 63],

$$\begin{cases} T^{tt} &= (\rho + p) u^t u^t + p\, g^{tt} &= (\rho + p) \frac{\gamma^2}{\alpha} - \frac{p}{\alpha}, \\ T^{tx} &= (\rho + p) u^t u^x + p\, g^{tx} &= (\rho + p)\ \gamma^2 v, \\ T^{xx} &= (\rho + p) u^x u^x + p\, g^{xx} &= \alpha(\rho + p)(\gamma v)^2 + p\alpha, \end{cases}$$

so that $\mathcal{T} = p - \rho$, and relying again on (2.9), Euler equations read now,

$$\begin{cases} \partial_t(T^{tt}) + \partial_x(T^{tx}) + \frac{1}{2\alpha} \left[2T^{tx} \partial_x \alpha + (T^{tt} - \frac{T^{xx}}{\alpha^2}) \partial_t \alpha \right] &= 0, \\ \partial_t(T^{tx}) + \partial_x(T^{xx}) + \frac{\alpha}{2}(T^{tt} - \frac{T^{xx}}{\alpha^2}) \partial_x \alpha + T^{tx} \partial_t(1/\alpha) &= 0. \end{cases} \tag{5.1}$$

By multiplying (5.1) by $(\alpha, 1/\alpha)$, a quasi-conservative form is derived:

$$\boxed{\begin{cases} \partial_t(\alpha T^{tt}) + \partial_x(\alpha T^{tx}) = \mathcal{V} \cdot \partial_t (\log \sqrt{\alpha}), \\ \partial_t(T^{tx}/\alpha) + \partial_x(T^{xx}/\alpha) = \mathcal{V} \cdot \partial_x \log(1/\sqrt{\alpha}), \\ \mathcal{V} = (\rho + p)\gamma^2(1 + v^2),\ \mathcal{T} = p - \rho, \end{cases} \quad \partial_{tt} \left(\frac{1}{\alpha} \right) - \partial_{xx} \alpha = \mathcal{T}} \tag{5.2}$$

supplemented by a quasilinear wave equation for the metric components, see (2.3).

Appendix A. Intrinsic derivative and WB scheme on a curved surface. Here, we recall both the geometric objects of constant use in relativistic applications and some numerical tools which are well-adapted to their treatment.

A.1. Scalar law on a static Riemannian surface of $\mathbb{R}^3$. In order to understand a simple model of 1D flow on a curved surface $\mathcal{S}$ of (time-independent, smooth) elevation $x \mapsto a(x)$, we introduce the following parametrization:

$$\mathcal{S} : \vec{r}(t,x) = \big(t, x, a(x)\big), \qquad \vec{r}_t = (1,0,0), \quad \vec{r}_x = (0,1,a').$$

The standard scalar product of $\mathbb{R}^3$ is denoted by "$\cdot$": the first fundamental form (the Riemannian metric) is the bilinear form which relates tangent vectors of $\mathcal{S}$ and their corresponding scalar product. Its 2×2 matrix reads in standard notation:

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \qquad E = \|\vec{r}_t\|^2 = 1, F = \vec{r}_t \cdot \vec{r}_x = 0, G = \|\vec{r}_x\|^2 = 1 + |a'|^2.$$

At each point $p \in \mathcal{S}$ corresponding to Cartesian coordinates $t, x \in \mathbb{R}^2$, we define a vector field $Y(t,x)$ in the tangent plane $T_p\mathcal{S}$ spanned by $\vec{r}_t(t,x), \vec{r}_x(t,x)$:

$$Y(t,x) = (u(t,x), \mathcal{F}(t,x)), \qquad \mathcal{F} \text{ a smooth function of } u.$$

Denoting "div_g" the Riemannian divergence operator induced by the metric g, the scalar conservation law we consider is simply,

$$\forall (t,x) \in \mathbb{R}^2, \qquad \mathsf{div}_g(Y)(t,x) = 0. \tag{A.1}$$

In order to compute this divergence, we need the covariant derivative operators on $\mathcal{S}$, thus we first recall the expression of Christoffel symbols (matrices) Γ_t, Γ_x:

$$\begin{cases} \Gamma_t = \begin{pmatrix} \Gamma^t_{tt} & \Gamma^t_{tx} \\ \Gamma^x_{tt} & \Gamma^x_{tx} \end{pmatrix} = \frac{1}{2D} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \times \begin{pmatrix} \partial_t E & \partial_x E \\ 2\partial_t F - \partial_x E & \partial_t G \end{pmatrix} \\ \Gamma_x = \begin{pmatrix} \Gamma^t_{xt} & \Gamma^t_{xx} \\ \Gamma^x_{xt} & \Gamma^x_{xx} \end{pmatrix} = \frac{1}{2D} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \times \begin{pmatrix} \partial_x E & 2\partial_x F - \partial_t G \\ \partial_t G & \partial_x G \end{pmatrix} \end{cases} \tag{A.2}$$

where $D = EG - F^2 = G = 1 + |a'|^2$ stands for the determinant of the metric g.

$$\Gamma^t = 0, \qquad \Gamma^x = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial_x G}{2G} \end{pmatrix}, \quad \Gamma^x_{xx} = \frac{\partial_x G}{2G} = \frac{1}{2}\partial_x \left(\log |G|\right) = \partial_x \left(\log \sqrt{G}\right).$$

Geodesic second-order differential equations easily follow, since $G = G(x)$:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\sqrt{G}\frac{\mathrm{d}x}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{G}\frac{\mathrm{d}x}{\mathrm{d}t}\right) = 0.$$

Now we can compute both the covariant derivatives, denoted by ∇_t and ∇_x:

$$\begin{cases} \nabla_t Y = (\vec{r}_t, \vec{r}_x)\left[\begin{pmatrix} \partial_t u \\ \partial_t \mathcal{F} \end{pmatrix} + \Gamma^t \begin{pmatrix} u \\ \mathcal{F} \end{pmatrix}\right] = \partial_t u \; \vec{r}_t + \partial_t \mathcal{F} \; \vec{r}_x \\ \nabla_x Y = (\vec{r}_t, \vec{r}_x)\left[\begin{pmatrix} \partial_x u \\ \partial_x \mathcal{F} \end{pmatrix} + \Gamma^x \begin{pmatrix} u \\ \mathcal{F} \end{pmatrix}\right] = \partial_x u \; \vec{r}_t + (\partial_x \mathcal{F} + \Gamma^x_{xx}\mathcal{F}) \; \vec{r}_x. \end{cases}$$

Finally, the scalar law on $\mathcal{S}$ rewrites as follows:

$$\mathsf{div}_g(Y) = \partial_t u + \underbrace{\partial_x \mathcal{F} + \Gamma^x_{xx}\mathcal{F}}_{\textit{intrinsic derivative}} = 0.$$

Now we can insert all the available values at hand:

$$\partial_t u + \partial_x \mathcal{F} = -\mathcal{F}\ \partial_x \left(\log \sqrt{1+|a'|^2} \right). \tag{A.3}$$

This is a balance law for which one can deduce a conservative expression:

$$\partial_t u + \partial_x \mathcal{F} + \mathcal{F} \frac{\partial_x \sqrt{G}}{\sqrt{G}} = 0 = \partial_t \left(u\sqrt{1+|a'|^2} \right) + \partial_x \left(\mathcal{F}\sqrt{1+|a'|^2} \right), \tag{A.4}$$

because G is a function of x only. The conservation law (A.4) is a particular case of a well-known formula[3]. Yet we can look at the behavior of numerical schemes on these 2 writings (A.3) and (A.4) of the same geometric conservation law (A.1).

Appendix B. Numerical approximation in non-resonant regime.

Let the flux term $\mathcal{F}$ create ony convection from left to right only (for instance, pick $\mathcal{F} = \alpha u$, $\alpha \geq c > 0$). As usual, call $u_j^n \simeq u(t^n, x_j)$, $t^n = n\Delta t$, $x_j = j\Delta x$.

B.1. Conservative equation. The most natural discretization for (A.4) reads

$$(u\sqrt{G})_j^{n+1} = (u\sqrt{G})_j^n - \frac{\Delta t}{\Delta x}\left[(\mathcal{F}\sqrt{G})_j^n - (\mathcal{F}\sqrt{G})_{j-1}^n \right],$$

where one can divide by $\sqrt{G}_j = \sqrt{G(x_j)} > 1$ in order to get:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}_j^n - \mathcal{F}_{j-1}^n \frac{\sqrt{G}_{j-1}}{\sqrt{G}_j} \right]. \tag{B.1}$$

B.2. WB scheme for the balance law. Based on *e.g.* [23], the way a WB scheme approximates the balance law (A.3) usually involves a modified state in the upwind direction, which results from the integration of the stationary equation:

$$u_j^{n+1} = u_j^n - \Delta t \underbrace{\left[\frac{\mathcal{F}_j^n - \mathcal{F}_{j-\frac{1}{2}}^n}{\Delta x} \right]}_{approx.\ intrinsic\ space-derivative}. \tag{B.2}$$

The stationary part of (A.3) rewrites $\partial_x \log \mathcal{F} = -\partial_x \log \sqrt{G}$, hence we have the expression of the numerical flux at each interface $x_{j-\frac{1}{2}}$ of the computational grid:

$$\mathcal{F}_{j-\frac{1}{2}}^n = \mathcal{F}(\Delta x) = \mathcal{F}(0)\frac{\sqrt{G(0)}}{\sqrt{G(\Delta x)}} = \mathcal{F}_{j-1}^n \frac{\sqrt{G}_{j-1}}{\sqrt{G}_j}.$$

Hence **the WB discretization (B.2) is identical to (B.1)**. On the particular example of the scalar law (A.1) posed on the static, smoothly curved, surface $\mathcal{S}$ for which the metric g depends only on x, the WB scheme for (A.3) delivers a **finite-difference approximation** of the "intrinsic derivative" appearing in $\nabla_x Y$ which is consistent with the conservative form (A.4). This isn't true for a time-split approximation.

Appendix C. Absorbing boundaries for $1+1$ Liouville equation.

Besides the general framework [18], one can derive numerical transparent boundary for the 1D Liouville equation (2.8) by means of rather easy considerations.

[3] See `http://en.wikipedia.org/wiki/List_of_formulas_in_Riemannian_geometry`

C.1. Formal continuous computation. Let's consider the issue of solving a 1D, possibly semi-linear, wave equation in a bounded portion $x \in (-\ell, \ell)$ of an infinite domain, typically the whole real line, $x \in \mathbb{R}$,

$$\partial_{tt}\phi - \partial_{xx}\phi + f(t,x,\phi) = 0, \qquad \phi(t=0,\cdot),\ \partial_t\phi(t=0,\cdot) \text{ given.}$$

To ensure overall consistency of the computational approximation, its outgoing signals in $x = \pm\ell$ must not be reflected inside the bounded interval $(-\ell, \ell)$. Since the characteristics' slope is ± 1 independently of $f(t,x,\cdot)$, incoming and outgoing waves can be separated by defining left- and right-going components,

$$\phi_<(t,x) = \partial_t\phi + \partial_x\phi, \qquad \phi_>(t,x) = \partial_t\phi - \partial_x\phi, \tag{C.1}$$

for which it comes easily that, always neglecting the forcing term,

$$\partial_t\phi_< - \partial_x\phi_< = \Box\phi = 0, \qquad \partial_t\phi_> + \partial_x\phi_> = \Box\phi = 0.$$

Transparent boundary conditions mean that, in practice, one requires:

$$\forall t > 0, \qquad \phi_<(t, x=\ell) = 0, \qquad \phi_>(t, x=-\ell) = 0. \tag{C.2}$$

C.2. Algorithmic implications. Implementing these local expressions (C.2) is quite straightforward as it boils down to approximate (C.1) by finite-differences while maintaining consistency with (4.1). The wave equation is reversible, so centered-differences are well-suited, and a convenient discretization of $\phi_<$ reads,

$$\forall n \in \mathbb{N}, \qquad (\phi_<)_j^n \simeq \frac{\phi_j^{n+1} - \phi_j^{n-1}}{2\Delta t} + \frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta x}.$$

Imposing that this quantity vanishes on the right of the computational domain, say $\ell = J\Delta x$, yields a value for ϕ_{J+1}^n, which can be inserted further inside (4.1),

$$\phi_J^{n+1} = \frac{2\phi_J^n + (a-1)\phi_J^{n-1} - 2a^2(\phi_J^n - \phi_{J-1}^n) - \Delta t^2 f(t^n, \ell, \phi_J^n)}{1+a}, \quad a = \frac{\Delta t}{\Delta x},$$

together with a similar expression for the condition on the left side, $x = -\ell$.

Appendix D. Geodesics of an empty spacetime in conformal gauge.

As the space-time is empty, $\mathcal{T} = 0$ and the metric $g(t,x)$ is in the conformal gauge, with $\phi(t,x)$ its conformal factor, its Ricci scalar $\mathcal{R}$ yieds a 1D wave equation,

$$\mathcal{R} = \frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0, \qquad \phi(t,x) = \phi(t-x). \tag{D.1}$$

Geodesics, parametrized by arc-length (proper time) τ satisfy the second-order ODE's, (where $<\cdot,\cdot>$ stand for the $\mathbb{R}^2$ scalar product)

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + \left\langle \Gamma^t\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}x}{\mathrm{d}\tau}\right), \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}x}{\mathrm{d}\tau}\right) \right\rangle = 0, \quad \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} + \left\langle \Gamma^x\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}x}{\mathrm{d}\tau}\right), \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}x}{\mathrm{d}\tau}\right) \right\rangle = 0, \tag{D.2}$$

Thanks to the simple structure of Christoffel symbols (2.6),

$$\Gamma^t \begin{pmatrix} \mathrm{d}t/d\tau \\ \mathrm{d}x/\mathrm{d}\tau \end{pmatrix} = \begin{pmatrix} \partial_t\phi \cdot \mathrm{d}t/\mathrm{d}\tau + \partial_x\phi \cdot \mathrm{d}x/\mathrm{d}\tau \\ \partial_x\phi \cdot \mathrm{d}t/\mathrm{d}\tau + \partial_t\phi \cdot \mathrm{d}x/\mathrm{d}\tau \end{pmatrix} = \begin{pmatrix} \mathrm{d}\phi(t,x)/\mathrm{d}\tau \\ ... \end{pmatrix},$$

and similarly,

$$\Gamma^x \begin{pmatrix} \mathrm{d}t/\mathrm{d}\tau \\ \mathrm{d}x/\mathrm{d}\tau \end{pmatrix} = \begin{pmatrix} \partial_x\phi \cdot \mathrm{d}t/\mathrm{d}\tau + \partial_t\phi \cdot \mathrm{d}x/\mathrm{d}\tau \\ \partial_t\phi \cdot \mathrm{d}t/\mathrm{d}\tau + \partial_x\phi \cdot \mathrm{d}x/\mathrm{d}\tau \end{pmatrix} = \begin{pmatrix} \ldots \\ \mathrm{d}\phi(t,x)/\mathrm{d}\tau \end{pmatrix}.$$

Geodesic equations (D.2) rewrite accordingly,

$$0 = \frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + \frac{\mathrm{d}t}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \frac{\mathrm{d}x}{\mathrm{d}\tau}(\ldots), \qquad 0 = \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} + \frac{\mathrm{d}x}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \frac{\mathrm{d}t}{\mathrm{d}\tau}(\ldots). \tag{D.3}$$

Following [60], we now switch to "null coordinates", $\zeta = t + x$, $\hat{\zeta} = t - x$. At this level, ϕ being solution of (D.1) satisfy $\partial_x\phi = -\partial_t\phi$, and

$$(\ldots) = \phi'(t-x)\frac{\mathrm{d}(t-x)}{\mathrm{d}\tau} = \phi'(\hat{\zeta})\frac{\mathrm{d}\hat{\zeta}}{\mathrm{d}\tau}.$$

Adding and subtracting both equations in (D.3) yield:

$$\frac{\mathrm{d}^2\hat{\zeta}}{\mathrm{d}\tau^2} + \frac{\mathrm{d}\hat{\zeta}}{\mathrm{d}\tau} \cdot \underbrace{\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} - \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)}_{=0} = 0, \qquad \frac{\mathrm{d}^2\zeta}{\mathrm{d}\tau^2} + 2\frac{\mathrm{d}\phi(\hat{\zeta})}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}\hat{\zeta}}{\mathrm{d}\tau} = 0.$$

The first equation gives that $\hat{\zeta}(\tau) = A\tau + B$, with $A, B \in \mathbb{R}^2$ some integration constants. Since $\mathrm{d}\hat{\zeta}/\mathrm{d}\tau \equiv A$, the second one can be integrated once:

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} + 2A\phi(\hat{\zeta}) + C = \frac{\mathrm{d}\zeta}{\mathrm{d}\tau} + 2A\phi(A\tau + B) + C = 0.$$

Let Φ be an antiderivative of ϕ,

$$\hat{\zeta}(\tau) = A\tau + B, \qquad \zeta(\tau) = -2\Phi(A\tau + B) + C\tau + D.$$

Since $\tau = \frac{\hat{\zeta} - B}{A}$, an unparameterized expression follows,

$$\zeta = -2\Phi(\hat{\zeta}) + \alpha\hat{\zeta} + \beta, \qquad \alpha, \beta \in \mathbb{R}^2.$$

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