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INTERTWINING CONNECTIVITY IN MATROIDS

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ABSTRACT. Let M be a matroid and let Q, R, S and T be subsets of the ground set such that the smallest separation that separates Q from R has order k and the smallest separation that separates S from T has order l. We prove that if $E(M)-(Q\cup R\cup S\cup T)$ is sufficiently large, then there is an element e of M such that, in one of $M\backslash e$ or M/e, both connectivities are preserved.

1. INTRODUCTION

Let M be a matroid with ground set $E(M)$. For any $X \subseteq E(M)$, define $\lambda_M(X) := r_M(X) + r_M(E(M) - X) - r(M)$. For disjoint subsets Q, R of $E(M)$, the *connectivity between Q and R* is

$$\kappa_M(Q, R) := \min\{\lambda_M(X) : Q \subseteq X \subseteq E(M) - R\}.$$

In the paper, we prove

Theorem 1.1. *There is a function $c : \mathbb{N}^2 \to \mathbb{N}$ with the following property. Let M be a matroid, and $Q, R, S, T, F \subseteq E(M)$ sets of elements such that $Q \cap R = S \cap T = \emptyset$ and $F = E(M) - (Q \cup R \cup S \cup T)$. Let $k := \kappa_M(Q, R)$ and $\ell := \kappa(S, T)$. If $|F| \geq c(k, \ell)$, then there is an element $e \in F$ such that one of the following holds:*

(i) $\kappa_{M\backslash e}(Q, R) = k$ *and* $\kappa_{M\backslash e}(S, T) = \ell$*;*
(ii) $\kappa_{M/e}(Q, R) = k$ *and* $\kappa_{M/e}(S, T) = \ell$*.*

This theorem resolves a conjecture of Geelen (private communication). It strengthens a theorem of Huynh and van Zwam [2] who prove the result for a class that includes all representable matroids but does not include all matroids.

The value that we give for $c(k, \ell)$ is unlikely to be tight. The $(k + 1) \times (\ell + 1)$ grid gives an example where the theorem fails with $|F| = 2kl - l - k$. Perhaps this example is extremal?

Conjecture 1.2. *Theorem 1.1 holds with* $|F| = 2kl - l - k + 1$.

2. PROOF OF THEOREM 1.1

For any disjoint subsets Q, R of the ground set of a matroid M, Tutte [3] proved that there is a minor N of M with $E(N) = Q \cup R$ and such that $\kappa(Q, R) = \lambda_N(Q)$, which is a generalization of Menger's theorem to matroids. Equivalently, we have

Lemma 2.1. *Let M be a matroid and Q, R be disjoint subsets of $E(M)$. For any $e \in E(M) - (Q \cup R)$ either $\kappa_{M\backslash e}(Q, R) = \kappa_M(Q, R)$ or $\kappa_{M/e}(Q, R) = \kappa_M(Q, R)$.*

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Let M be a matroid and Q, R be disjoint subsets of $E(M)$. Define $\sqcap_M(Q,R) := r_M(Q) + r_M(R) - r_M(Q \cup R)$. A partition (A, B) of $E(M)$ is *$Q - R$-separating of order $k+1$* if $Q \subseteq A$, $R \subseteq B$ and $\lambda_M(A) \le k$. Let $e \in E(M) - (Q \cup R)$. If $\kappa_{M\backslash e}(Q,R) = \kappa_M(Q,R)$, then e is *deletable with respect to* (Q,R); if $\kappa_{M/e}(Q,R) = \kappa_M(Q,R)$, then e is *contractible with respect to* (Q,R); and if e is both deletable and contractible with respect to (Q,R), then e is *flexible with respect to* (Q,R). Lemma 2.1 implies that for any $e \in E(M) - (Q \cup R)$ either e is deletable with respect to (Q,R) or e is contractible with (Q,R).

Theorem 2.2. *(*[2], *Theorem 3.4.) Let M be a matroid and Q, R be disjoint subsets of $E(M)$, let $k := \kappa(Q,R)$, and let $F \subseteq E(M) - (Q \cup R)$ be a set of non-flexible elements. There are an ordering $(f_1, \cdots, f_n)$ of F and a sequence of $(A_1, \cdots, A_n)$ of subsets of $E(M)$ such that*

(i) *A_i is $Q - R$-separating of order $k+1$ for each $i \in \{1, \cdots, n\}$;*
(ii) *$A_i \subseteq A_{i+1}$ for each $i \in \{1, \cdots, n\}$;*
(iii) *$A_i \cap F = \{f_1, \cdots, f_i\}$ for each $i \in \{1, \cdots, n\}$;*
(iv) *$f_i \in \mathrm{cl}(A_i - \{f_i\}) \cap \mathrm{cl}(E(M) - A_i)$ or $f_i \in \mathrm{cl}^*(A_i - \{f_i\}) \cap \mathrm{cl}^*(E(M) - A_i)$.*

Theorem 2.3. *(*[2], *Lemma 3.6.) Let M be a matroid and Q, R be disjoint subsets of $E(M)$, let $k := \kappa(Q,R)$, and let $(U, E(M) - U)$ be a $Q - R$-separating set of order $k+1$. If $e \in E(M) - (U \cup R)$ is non-contradictable with respect to (Q,R), then e is also non-contradictable with respect to (U,R).*

First we prove that Theorem 1.1 holds for the case $|S| = |T| = \ell$.

Lemma 2.4. *There is a function $c : \mathbb{N}^2 \to \mathbb{N}$ with the following property. Let M be a matroid, and $Q, R, S, T, F \subseteq E(M)$ sets of elements such that $Q \cap R = S \cap T = \emptyset$ and $F = E(M) - (Q \cup R \cup S \cup T)$. Let $k := \kappa_M(Q,R)$ and $\ell := \kappa_M(S,T)$. If $|S| = |T| = \ell$ and $|F| \ge c(k,\ell)$, then there is an element $e \in F$ such that one of the following holds:*

(i) *$\kappa_{M\backslash e}(Q,R) = k$ and $\kappa_{M\backslash e}(S,T) = \ell$;*
(ii) *$\kappa_{M/e}(Q,R) = k$ and $\kappa_{M/e}(S,T) = \ell$.*

Proof. We prove that the result holds for $c(k,\ell) := (2\ell+1)2^{2k+1}$. If F contains some flexible element with respect to (Q,R) or (S,T), then we are done. So we may assume that each element in F is non-flexible with respect to (Q,R) and non-flexible with respect to (S,T). By Lemma 2.1 an element e in F is deletable (or contractible) with respect to (Q,R) if and only if e is contractible (or deletable) with respect to (S,T), for otherwise the lemma holds.

Let $(A_1, \cdots, A_{c(k,\ell)})$ be the nested sequence of $Q - R$ separating sets from Theorem 2.2, let $(B_1, \cdots, B_{c(k,\ell)})$ be their complements, and let $(f_1, \cdots, f_{c(k,\ell)})$ be the corresponding ordering of F. Since $|S| = |T| = \ell$, there is a positive integer i such that $i + 2^{2k+1} \le c(k,\ell)$ and such that $Q \cup R \cup S \cup T \subseteq A_i \cup B_{i+2^{2k+1}}$. Set

$$Q^{'} := A_i,\ R^{'} := B_{i+2^{2k+1}},\ F^{'} := E(M) - (Q^{'} \cup R^{'}),$$
$$A_j^{'} := A_{i+j},\ B_j^{'} := B_{i+j},\ f_j^{'} := f_{i+j},\ \text{for any } 1 \le j \le 2^{2k+1}.$$

That is, $F^{'} = \{f_1^{'}, \cdots, f_{2^{2k+1}}^{'}\}$. By duality and Lemma 2.3, each element in $F^{'}$ is non-flexible with respect to $(Q^{'}, R^{'})$.

Let $(C_1, \cdots, C_{2^{2k+1}})$ be the nested sequence of $S - T$ separating sets from Theorem 2.2 determined by the non-flexible-element set $F^{'}$ with respect to (S,T), let $(D_1, \cdots, D_{2^{2k+1}})$ be their complements, and let $(g_1, \cdots, g_{2^{2k+1}})$ be the corresponding ordering of $F^{'}$. By duality we may assume that g_1 is a deletable element with respect to (S,T). Then (i) $g_1 \in \mathrm{cl}(C_1-\{g_1\})$ and (ii) g_1 is a contractible element with respect to (Q,R). By (i) and the fact that $C_1-\{g_1\} \subseteq Q' \cup R'$ we see that $g_1 \in \mathrm{cl}(Q' \cup R')$. From (ii) we deduce that $g_1 \notin \mathrm{cl}(Q^{'})$ and $g_1 \notin \mathrm{cl}(R^{'})$. Therefore $\sqcap_M(Q^{'} \cup \{g_1\}, R^{'}) = \sqcap_M(Q^{'}, R^{'})+1$. Assume that $g_1 = f_j^{'}$. If $j \leq 2^{2k}$ then set $Q^{''} := A_j^{'}, R^{''} := R^{'}$; else if $j > 2^{2k}$ then set $Q^{''} := Q^{'}, R^{''} := B_{j-1}^{'}$. No matter which case happens, set $F^{''} := E(M) - (Q^{''} \cup R^{''})$. Evidently, $|F^{''}| \geq 2^{2k}$ as $|F^{'}| = 2^{2k+1}$. Replacing $Q^{'}, R^{'}, F^{'}$ with $Q^{''}, R^{''}, F^{''}$ respectively and repeating the above analysis $2k$ times, there are numbers j_1, j_2 with $2k+1 \leq j_1 \leq j_2 \leq 2^{2k+1}$ such that $\sqcap_M(A_{j_1}^{'}, B_{j_2}^{'}) \geq k+1$ or $\sqcap_{M^*}(A_{j_1}^{'}, B_{j_2}^{'}) \geq k+1$, a contradiction to the fact that $\lambda(A_{j_1}^{'}) = k$. So the lemma holds. □

To prove Theorem 1.1 we still need the following lemma.

Lemma 2.5. *(*[1]*, Lemma 4.7.) Let M be a matroid and S,T be disjoint subsets of $E(M)$. There exists sets $S_1 \subseteq S, T_1 \subseteq T$ such that $|S_1| = |T_1| = \kappa(S_1, T_1)$.*

For convenience we restate Theorem 1.1 here.

Theorem 2.6. *There is a function $c : \mathbb{N}^2 \to \mathbb{N}$ with the following property. Let M be a matroid, and $Q,R,S,T,F \subseteq E(M)$ sets of elements such that $Q \cap R = S \cap T = \emptyset$ and $F = E(M) - (Q \cup R \cup S \cup T)$. Let $k := \kappa_M(Q,R)$ and $\ell := \kappa(S,T)$. If $|F| \geq c(k,\ell)$, then there is an element $e \in F$ such that one of the following holds:*

(i) $\kappa_{M\backslash e}(Q,R) = k$ *and* $\kappa_{M\backslash e}(S,T) = \ell$*;*
(ii) $\kappa_{M/e}(Q,R) = k$ *and* $\kappa_{M/e}(S,T) = \ell$*.*

Proof. We prove that the result holds for $c(k,\ell) := (2\ell+1)2^{2k+1}$. By Lemma 2.5 there are sets $S_1 \subseteq S, T_1 \subseteq T$ such that $|S_1| = |T_1| = \kappa_M(S_1,T_1)$. Then Lemma 2.4 implies that there is an element $e_1 \in E(M) - (Q \cup R \cup S_1 \cup T_1)$ such that for some $M_1 \in \{M\backslash e_1, M/e_1\}$ we have $\kappa_{M_1}(Q,R) = k$ and $\kappa_{M_1}(S_1,T_1) = \ell$. Since $\kappa_{M_1}(S_1,T_1) = \ell$ implies $\kappa_{M_1}(S,T) = \ell$, when $e_1 \in F$ the lemma holds. So we may assume that $e_1 \notin F$. That is, $e_1 \in (S \cup T) - (S_1 \cup T_1)$. Since $F \subseteq E(M_1) - (Q \cup R \cup S_1 \cup T_1)$, using Lemma 2.4 again there is an element $e_2 \in E(M_1) - (Q \cup R \cup S_1 \cup T_1)$ such that for some $M_2 \in \{M_1\backslash e_2, M_1/e_2\}$ we have $\kappa_{M_2}(Q,R) = k$ and $\kappa_{M_2}(S_1,T_1) = \ell$. Without loss of generality we may assume that $M_2 = M_1\backslash e_2$. Then $\kappa_{M\backslash e_2}(Q,R) = k$ and $\kappa_{M\backslash e_2}(S_1,T_1) = \ell$ as $\kappa_M(Q,R) = k$ and $\kappa_M(S_1,T_1) = \ell$. Thus, when $e_2 \in F$, the lemma holds. So we may assume that $e_2 \notin F$. Since $(S \cup T) - (S_1 \cup T_1)$ is finite, repeating the above analysis several times we can always find a minor with an element e such that (i) or (ii) holds. The theorem follows from this observation and the fact that the connectivity function is monotone under minors. □

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References

[1] J. Geelen, B. Gerards, G. Whittle, Excluding a planar graph from $GF(q)$-representable matroids, J. Combin. Theory Ser. B, 2007, 97: 971-998.

[2] T. Huynh, S. H. M. van Zwam, Interwining connectivity in representable matroids, SIAM J. Discrete Math., 2014, 28: 188-196.

[3] W. T. Tutte, Menger's Theorem for matroids, J. Res. Nat. Bur. Standards Sect. B, 1965, 69: 49-53.

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