SEMIDEFINITE PROGRAMMING FOR CHANCE CONSTRAINED OPTIMIZATION OVER SEMIALGEBRAIC SETS *

A. M. JASOUR[†], N. S. AYBAT [‡], AND C. M. LAGOA [§]

Abstract. In this paper, "chance optimization" problems are introduced, where one aims at maximizing the probability of a set defined by polynomial inequalities. These problems are, in general, nonconvex and computationally hard. With the objective of developing systematic numerical procedures to solve such problems, a sequence of convex relaxations based on the theory of measures and moments is provided, whose sequence of optimal values is shown to converge to the optimal value of the original problem. Indeed, we provide a sequence of semidefinite programs of increasing dimension which can arbitrarily approximate the solution of the original problem. To be able to efficiently solve the resulting large-scale semidefinite relaxations, a first-order augmented Lagrangian algorithm is implemented. Numerical examples are presented to illustrate the computational performance of the proposed approach.

Key words. Semialgebraic set, Chance constrained, SDP relaxation, Augmented Lagrangian, First-order methods.

1. Introduction. In this paper, we aim at solving *chance optimization problems;* i.e., problems which involve maximization of the probability of a semialgebraic set defined by polynomial inequalities. More precisely, given a probability space $(\mathbb{R}^m, \bar{\Sigma}_q, \bar{\mu}_q)$ with $\bar{\Sigma}_q$ denoting the Borel σ -algebra of \mathbb{R}^m and $\bar{\mu}_q : \bar{\Sigma}_q \to \mathbb{R}_+$ denoting a finite (positive) Borel measure on $\bar{\Sigma}_q$, we focus on the problem given in (1.1) over decision variable $x \in \mathbb{R}^n$.

$$\mathbf{P}^* := \sup_{x \in \mathbb{R}^n} \bar{\mu}_q \left(\bigcup_{k=1,\dots,N} \bigcap_{j=1,\dots,\ell_k} \left\{ q \in \mathbb{R}^m : \mathcal{P}_j^{(k)}(x,q) \ge 0 \right\} \right), \tag{1.1}$$

where $\mathcal{P}_{j}^{(k)}$: $\mathbb{R}^{n} \times \mathbb{R}^{m} \to \mathbb{R}$, $j = 1, 2, ..., \ell_{k}$ and k = 1, ..., N are given polynomials. Let $\mathcal{K}_{k} := \left\{ (x,q) : \mathcal{P}_{j}^{(k)}(x,q) \geq 0, \ j = 1, ..., \ell_{k} \right\}$ and $\mathcal{K} := \bigcup_{k=1}^{N} \mathcal{K}_{k}$. Under the assumption that \mathcal{K} is bounded, we show that by solving a sequence of semidefine programming (SDP) problems of growing dimension, we can construct a sequence $\{\mathbf{y}_{\mathbf{x}}^{d}\}_{d\in\mathbb{Z}_{+}} \subset \mathbb{R}^{\mathbb{N}}$ that has an accumulation point in the weak- \star topology of ℓ_{∞} , and for every accumulation point $\mathbf{y}_{\mathbf{x}}^{*} \in \mathbb{R}^{\mathbb{N}}$, there is a representing finite (positive) Borel measure $\mu_{\mathbf{x}}^{*}$ such that any $x^{*} \in supp(\mu_{\mathbf{x}}^{*})$ is an optimal solution to (1.1), i.e., the supremum \mathbf{P}^{*} is attained at x^{*} , where $\mathbb{R}^{\mathbb{N}}$ denotes the vector space of real sequences. Note that the problem of interest in (1.1), when reformulated in *hypograph* form, can be equivalently written as a chance constrained optimization problem: $\sup_{x \in \mathbb{R}^{n}, \gamma \in \mathbb{R}} \left\{ \gamma : \bar{\mu}_{q} \left(\bigcup_{k=1,...,N} \bigcap_{j=1,...,\ell_{k}} \left\{ q \in \mathbb{R}^{m} : \mathcal{P}_{j}^{(k)}(x,q) \geq 0 \right\} \right\} \geq \gamma \right\}$. First, the emphasis will be placed on the following special case of (1.1), where N = 1,

$$\mathbf{P}^* := \sup_{x \in \mathbb{R}^n} \bar{\mu}_q \bigg(\big\{ q \in \mathbb{R}^m : \mathcal{P}_j(x, q) \ge 0, \quad j = 1, \dots, \ell \big\} \bigg),$$
(1.2)

and then all the results derived for the special case (1.2) will be extended to the case where N > 1.

The potential application area of this problem class is quite large and encompasses many well-known problems in different areas as special cases. For example, designing probabilistic robust controllers [25], model predictive controllers in presence of random disturbances [13, 42, 52], and optimal path planning and obstacle avoidance problems in robotics [14, 15, 19] can be cast as special cases of this framework. Moreover, problems in the area of economics, finance, and trust design [34, 54, 57] can also be formulated as (1.1) and (1.2). Although, in some particular cases, the problem in (1.1) is convex (e.g., see [28, 49]), in general, chance constrained problems are not convex; e.g., see [28] for non-convex chance constrained linear programs. In this paper, we use previous results on moments of measures (e.g., see [31, 32]) to develop a sequence of SDP problems, known as Lasserre's hierarchy [32], whose solutions converge to the solution of (1.1).

^{*}The last two authors have the same contribution.

[†]EE Department, The Pennsylvania State University, PA, USA (jasour@psu.edu)

[±]IE Department, The Pennsylvania State University, PA, USA (nsa10@psu.edu)

EE Department, The Pennsylvania State University, PA, USA (lagoa@psu.edu)

1.1. Previous Work. Several approaches have been proposed to solve chance constrained problems. The main idea behind most of the proposed methods is to find a tractable approximation for chance constraints. One particular method is the so-called *scenario approach*; see [16, 17, 36, 38, 55] and the references therein. In this approach, the probabilistic constraint is replaced by a (large) number of deterministic constraints obtained by drawing independent identically distributed (iid) samples of random parameters. Being a randomized approach, there is always a positive probability of failure (perhaps small). In [7, 8, 9, 10, 11], robust optimization is used to deal with uncertain linear programs (LP). In this method, the uncertain LP is replaced by its robust counterpart, where the worst case realization of uncertain data is considered. The proposed method is not computationally tractable for every type of uncertainty set. A specific case that is tractable is LP with ellipsoidal uncertainty set [7]. In [12, 35, 39], an alternative approach is proposed where one analytically determines an upper bound on the probability of constraint violation. Although this method does provide a convex approximation, it can only be applied to specific uncertainty structures. In [37, 43] the authors propose the so-called Bernstein approximation where a convex conservative approximation of chance constraints is constructed using generating functions. Although approximation is efficiently computable, it is only applicable to problems with convex constraints that are affine in random vector $q \in \mathbb{R}^m$. Moreover, components of q need to be independent and have computable finite generating functions. In [18, 21, 22] convex relaxations of chance constrained problems are presented. The concept of polynomial kinship function is used to estimate an upper bound on the probability of constraint violation. Solutions to a sequence of relaxed problems are shown to converge to a solution of the original problem as the degree of the polynomial kinship function increases along the sequence. In [22, 27], an equivalent convex formulation is provided based on the theory of moments. In this method the probability of a polynomial being negative is approximated by computing polynomial approximations for univariate indicator functions [27].

Distributionally robust chance constrained programming – see [44, 45, 46, 47, 48], is another popular tool for dealing with uncertainty in the problem, where only a finite number of moments m_{α} of the underlying measure $\bar{\mu}_q$ are assumed to be known, i.e., $\{m_\alpha\}_{\alpha\in\mathcal{A}}$ is known for $\mathcal{A}\subset\mathbb{N}^m$ such that $|\mathcal{A}|<\infty$. In this approach robust chance constraints are formulated by considering the worst case measure within a family of measures with moments equal to $\{m_{\alpha}\}_{\alpha \in A}$. However, proposed methods in this literature are mainly limited to linear chance constraints and/or to specific types of uncertainty distributions. For instance, in [44], under the assumption $\bar{m} = E_{\bar{\mu}_q}[q]$ and $\bar{S} = E_{\bar{\mu}_q}[(q - \bar{m})(q - \bar{m})^T]$ are known, the linear chance constraint of the form $\bar{\mu}_q\left(\{q: q^T x \ge 0\}\right) \ge 1 - \epsilon$ is replaced by its robust counterpart: $\inf_{\mu_q \in \mathcal{M}} \mu_q\left(\{q: q^T x \ge 0\}\right) \ge 1 - \epsilon$, where \mathcal{M} is the set of finite (positive) Borel measures on $\bar{\Sigma}_q$ with their means and covariances equal to \bar{m} and \bar{S} , respectively; and it is shown that these robust constraints can be represented as second-order cone constraints for a wide class of probability distributions. In [45], the authors has reviewed and developed different approximation methods for problems with joint chance constraints. In the proposed method, joint chance constraints are decomposed into individual chance constraints, and classical robust optimization approximation is used to deal with the new constraints. In [46] a tractable approximation method for probabilistically dependent linear chance constraints is presented. In [47] linear chance constraints with Gaussian and log-concave uncertainties are addressed, and it is shown that they can be reformulated as semi-infinite optimization problems; moreover, tight probabilistic bounds are provided for the resulting comprehensive robust optimization problems [58, 59]. In [48] an SDP formulation is provided to approximate distributionally robust chance constraints where only the support of $\bar{\mu}_a$, and its first and second order moments are known.

In this paper, we take a different approach to deal with chance constrained problems. The proposed method is based on volume approximation results in [24] and the theory of moments [31, 32]. In [24], a hierarchy of SDP problems are proposed to compute the volume of a given compact semialgebraic set. It is shown that the volume of a semialgebraic can be computed by solving a maximization problem over finite (positive) Borel measures supported on the given set, and restricted by the Lebesgue measure on a simple set containing the semialgebraic set of interest. Building on this result, we propose the *chance optimization* problem over semialgebraic sets –see our preliminary results in [26]. In particular, we address the problem of probability maximization over the union of semialgebraic sets defined by intersections of finite number of polynomial inequalities as in (1.1). Here, one needs to search for the (positive) Borel measure with maximum possible mass on the given semialgebraic set, while simultaneously searching for an upper bound probability measure over a simple set containing the semialgebraic set and restricting the Borel measure.

1.2. The Sequel. The outline of the paper is as follows: in Section 2, the notation adopted in the paper, and preliminary results on measure theory are presented; in Sections 3 and 4, we propose equivalent problems, and sequences of SDP relaxations to (1.2) and (1.1), respectively; and show that the sequence of optimal solutions to SDP relaxations converge to the solutions of the original problems. In Section 5, we implement an efficient first-order algorithm to solve regularized SDP relaxations of the chance constrained problems, and finally, present numerical results, followed by some concluding remarks given in Section 6.

2. Notation and Preliminary Results.

2.1. Notations and Definitions. Throughout the paper, given a sequence $\mathbf{p} = \{p_{\alpha}\}_{\alpha \in \mathcal{A}} \subset \mathbb{R}$ over a countable index set $\mathcal{A} \subset \mathbb{N}^n$, we assume that the elements of \mathcal{A} is sorted according to graded reverse lexicographic order (grevlex): $\mathcal{A} = \{\alpha^{(i)} : i = 1, ..., |\mathcal{A}|\}$ such that $\alpha^{(1)} <_g \ldots <_g \alpha^{(|\mathcal{A}|)}$, where $|\mathcal{A}|$ denotes the cardinality of \mathcal{A} ; and the order on \mathcal{A} also induces an order on the elements of $\mathbf{p} = [p_{\alpha^{(1)}}, \ldots, p_{\alpha^{(|\mathcal{A}|)}}]^T \in$ $\mathbb{R}^{|\mathcal{A}|}$. Throughout the paper the notation $(\mathbf{p})_{\alpha}$ refers to p_{α} . Let $\mathbb{R}[x]$ be the ring of real polynomials in the variables $x \in \mathbb{R}^n$. Given $\mathcal{P} \in \mathbb{R}[x]$, we will represent \mathcal{P} as $\sum_{\alpha \in \mathbb{N}^n} p_{\alpha} x^{\alpha}$ using the standard basis $\{x^{\alpha}\}_{\alpha \in \mathbb{N}^n}$ of $\mathbb{R}[x]$, where $x^{\alpha} := \prod_{j=1}^n x_j^{\alpha_j}$, and $\mathbf{p} = \{p_{\alpha}\}_{\alpha \in \mathbb{N}^n}$ denotes the sequence of polynomial coefficients. Note that \mathbf{p} contains finitely many nonzeros, and we assume that the elements of the coefficient sequence $\mathbf{p} = \{p_{\alpha}\}_{\alpha \in \mathbb{N}^n} \in \mathbb{R},$ are sorted according to grevlex order on the corresponding monomial exponent α . Given $\mathbf{y} = \{y_{\alpha}\}_{\alpha \in \mathbb{N}^n} \subset \mathbb{R}$, let $L_{\mathbf{y}} : \mathbb{R}[x] \to \mathbb{R}$ be a linear map defined as

$$\mathcal{P} \mapsto L_{\mathbf{y}}(\mathcal{P}) = \sum_{\alpha \in \mathbb{N}^n} p_{\alpha} y_{\alpha}, \text{ where } \mathcal{P}(x) = \sum_{\alpha \in \mathbb{N}^n} p_{\alpha} x^{\alpha}.$$
 (2.1)

Given *n* and *d* in \mathbb{N} , we define $S_{n,d} := \binom{d+n}{n}$ and $\mathbb{N}_d^n := \{\alpha \in \mathbb{N}^n : \|\alpha\|_1 \leq d\}$. Let $\mathbb{R}_d[x] \subset \mathbb{R}[x]$ denote the set of polynomials of degree at most $d \in \mathbb{N}$, which is indeed a vector space of dimension $S_{n,d}$. Similar to $\mathcal{P} \in \mathbb{R}[x]$, given $\mathcal{P} \in \mathbb{R}_d[x]$, $\mathbf{p} = \{p_\alpha\}_{\alpha \in \mathbb{N}_d}^n$ is sorted such that $\mathbf{p} = [p_{\alpha^{(1)}}, \ldots, p_{\alpha^{(S_{n,d})}}]^T \in \mathbb{R}^{S_{n,d}}$, where $\mathbb{N}_d^n \ni \mathbf{0} = \alpha^{(1)} <_g \ldots <_g \alpha^{(S_{n,d})}$. Moreover, let $\mathbb{S}^2[x] \subset \mathbb{R}[x]$ be the set of sum of squares (SOS) polynomials. $s : \mathbb{R}^n \to \mathbb{R}$ is an SOS polynomial if it can be written as a sum of *finitely* many squared polynomials, i.e., $s(x) = \sum_{j=1}^{\ell} h_j(x)^2$ for some $\ell < \infty$ and $h_j \in \mathbb{R}[x]$ for $1 \leq j \leq \ell$.

Let $\mathbb{R}^{\mathbb{N}}$ denote the vector space of real sequences, and let $\mathcal{M}(\mathcal{K})$ be the set of finite (positive) Borel measures μ such that $supp(\mu) \subset \mathcal{K}$, where $supp(\mu)$ denotes the support of the measure μ ; i.e., the smallest closed set that contains all measurable sets with strictly positive μ measure. A sequence $\mathbf{y} = \{y_{\alpha}\}_{\alpha \in \mathbb{N}^{n}} \in \mathbb{R}^{\mathbb{N}}$ is said to have a *representing measure*, if there exists a finite Borel measure μ on \mathbb{R}^{n} such that $y_{\alpha} = \int x^{\alpha} d\mu$ for every $\alpha \in \mathbb{N}^{n}$ – see [31, 32]. In this case, \mathbf{y} is called the moment sequence of the measure μ . Given two measures μ_{1} and μ_{2} on a Borel σ -algebra Σ , the notation $\mu_{1} \preccurlyeq \mu_{2}$ means $\mu_{1}(S) \leq \mu_{2}(S)$ for any set $S \in \Sigma$. Moreover, if μ_{1} and μ_{2} are both measures on Borel σ -algebras Σ_{1} and Σ_{2} , respectively, then $\mu = \mu_{1} \times \mu_{2}$ denotes the product measure satisfying $\mu(S_{1} \times S_{2}) = \mu_{1}(S_{1})\mu_{2}(S_{2})$ for any measurable sets $S_{1} \in \Sigma_{1}, S_{2} \in \Sigma_{2}$ [24]. Let $C \subset \mathbb{R}^{n}, \Sigma(C)$ denotes the Borel σ -algebra over C. Given two square symmetric matrices A and B, the notation $A \succeq 0$ denotes that A is positive semidefinite, and $A \succeq B$ stands for A - B being positive semidefinite.

Putinar's property: A closed semialgebraic set $\mathcal{K} = \{x \in \mathbb{R}^n : \mathcal{P}_j(x) \ge 0, j = 1, 2, \dots, \ell\}$ defined by polynomials $\mathcal{P}_j \in \mathbb{R}[x]$ satisfies *Putinar's property* [50] if there exists $\mathcal{U} \in \mathbb{R}[x]$ such that $\{x : \mathcal{U}(x) \ge 0\}$ is compact and $\mathcal{U} = s_0 + \sum_{j=1}^{\ell} s_j \mathcal{P}_j$ for some SOS polynomials $\{s_j\}_{j=0}^{\ell} \subset \mathbb{S}^2[x]$ – see [29, 32, 50]. Putinar's property holds if the level set $\{x : \mathcal{P}_j(x) \ge 0\}$ is compact for some j, or if all \mathcal{P}_j are affine and \mathcal{K} is compact - see [29]. Putinar's property is not a geometric property of the semi-algebraic set \mathcal{K} , but rather an algebraic property related to the representation of the set by its defining polynomials. Hence, if there exits M > 0 such that the polynomial $\mathcal{P}_{\ell+1}(x) := M - ||x||^2 \ge 0$ for all $x \in \mathcal{K}$, then the *new representation* of the set $\mathcal{K} = \{x \in \mathbb{R}^n : \mathcal{P}_j(x) \ge 0, j = 1, 2, \dots, \ell + 1\}$ satisfies Putinar's property.

Moment matrix: Given $d \ge 1$ and a sequence $\{y_{\alpha}\}_{\alpha \in \mathbb{N}^n}$, the moment matrix $M_d(\mathbf{y}) \in \mathbb{R}^{S_{n,d} \times S_{n,d}}$ is a symmetric matrix and its (i, j)-th entry is defined as follows [31, 32]:

$$M_{d}(\mathbf{y})(i,j) := L_{\mathbf{y}}\left(x^{\alpha^{(i)} + \alpha^{(j)}}\right) = y_{\alpha^{(i)} + \alpha^{(j)}}, \quad 1 \le i, j \le S_{n,d},$$
(2.2)

where $\mathbb{N}_d^n = \{\alpha^{(i)}\}_{i=1}^{S_{n,d}}$ such that $\mathbf{0} = \alpha^{(1)} <_g \ldots <_g \alpha^{(S_{n,d})}$ are sorted according to grevlex order.

Let $\mathcal{B}_d^T = \left[x^{\alpha^{(1)}}, \ldots, x^{\alpha^{(S_{n,d})}}\right]^T$ denote the vector comprised of the monomial basis of $\mathbb{R}_d[x]$. Note that the moment matrix can be written as $M_d(\mathbf{y}) = L_{\mathbf{y}} \left(\mathcal{B}_d \mathcal{B}_d^T\right)$; here, the linear map $L_{\mathbf{y}}$ operates componentwise on the matrix of polynomials, $\mathcal{B}_d \mathcal{B}_d^T$. For instance, let d = 2 and n = 2; the moment matrix containing moments up to order 2d is given as

$$M_{2}(\mathbf{y}) = \begin{bmatrix} y_{00} \mid y_{10} \quad y_{01} \mid y_{20} \quad y_{11} \quad y_{02} \\ - & - & - & - & - \\ y_{10} \mid y_{20} \quad y_{11} \mid y_{30} \quad y_{21} \quad y_{12} \\ y_{01} \mid y_{11} \quad y_{02} \mid y_{21} \quad y_{12} \quad y_{03} \\ - & - & - & - & - \\ y_{20} \mid y_{30} \quad y_{21} \mid y_{40} \quad y_{31} \quad y_{22} \\ y_{11} \mid y_{21} \quad y_{12} \mid y_{31} \quad y_{22} \quad y_{13} \\ y_{02} \mid y_{12} \quad y_{03} \mid y_{22} \quad y_{13} \quad y_{04} \end{bmatrix}.$$

$$(2.3)$$

Localizing matrix: Given a polynomial $\mathcal{P} \in \mathbb{R}[x]$, let $\mathbf{p} = \{p_{\gamma}\}_{\gamma \in \mathbb{N}^n}$ be its coefficient sequence in standard monomial basis, i.e., $\mathcal{P}(x) = \sum_{\alpha \in \mathbb{N}^n} p_{\alpha} x^{\alpha}$, the (i, j)-th entry of the *localizing matrix* $M_d(\mathbf{y}; \mathbf{p}) \in \mathbb{R}^{S_{n,d} \times S_{n,d}}$ with respect to \mathbf{y} and \mathbf{p} is defined as follows [31, 32]:

$$M_d(\mathbf{y};\mathbf{p})(i,j) := L_{\mathbf{y}}\left(\mathcal{P}x^{\alpha^{(i)} + \alpha^{(j)}}\right) = \sum_{\gamma \in \mathbb{N}^n} p_\gamma y_{\gamma + \alpha^{(i)} + \alpha^{(j)}}, \quad 1 \le i, j \le S_{n,d}.$$
(2.4)

Equivalently, $M_d(\mathbf{y}; \mathbf{p}) = L_{\mathbf{y}} \left(\mathcal{P} \mathcal{B}_d \mathcal{B}_d^T \right)$, where $L_{\mathbf{y}}$ operates componentwise on $\mathcal{P} \mathcal{B}_d \mathcal{B}_d^T$. For example, given $\mathbf{y} = \{y_\alpha\}_{\alpha \in \mathbb{N}^2}$ and the coefficient sequence $\mathbf{p} = \{p_\alpha\}_{\alpha \in \mathbb{N}^2}$ corresponding to polynomial \mathcal{P} ,

$$\mathcal{P}(x_1, x_2) = a - bx_1 - cx_2^2, \tag{2.5}$$

the localizing matrix for d = 1 is formed as follows

$$M_{1}(\mathbf{y};\mathbf{p}) = \begin{bmatrix} ay_{00} - by_{10} - cy_{02} & ay_{10} - by_{20} - cy_{12} & ay_{01} - by_{11} - cy_{03} \\ ay_{10} - by_{20} - cy_{12} & ay_{20} - by_{30} - cy_{22} & ay_{11} - by_{21} - cy_{13} \\ ay_{01} - by_{11} - cy_{03} & ay_{11} - by_{21} - cy_{13} & ay_{02} - by_{12} - cy_{04} \end{bmatrix}.$$
(2.6)

2.2. Preliminary Results. In this section, we state some standard results found in the literature that will be referred to later in Sections 3 and 4.

LEMMA 2.1. Let μ be a Borel probability measure supported on the hyper-cube $[-1,1]^n$. Its moment sequence $\mathbf{y} \in \mathbb{R}^{\mathbb{N}}$ satisfies $\|\mathbf{y}\|_{\infty} \leq 1$.

Proof. Since $supp(\mu) \subset [-1,1]^n$ and μ is a probability measure, we have $|y_{\alpha}| \leq \int |x^{\alpha}| d\mu \leq \int |x| d\mu \leq 1$ for each $\alpha \in \mathbb{N}^n$. Hence, $\|\mathbf{y}\|_{\infty} \leq 1$. \square

The following lemmas give necessary, and sufficient conditions for \mathbf{y} to have a representing measure μ – for details see [24, 30, 32].

LEMMA 2.2. Let μ be a finite Borel measure on \mathbb{R}^n , and $\mathbf{y} = \{y_\alpha\}_{\alpha \in \mathbb{N}^n}$ such that $y_\alpha = \int x^\alpha d\mu$ for all $\alpha \in \mathbb{N}^n$. Then $M_d(\mathbf{y}) \geq 0$ for all $d \in \mathbb{N}$.

LEMMA 2.3. Let $\mathbf{y} = \{y_{\alpha}\}_{\alpha \in \mathbb{N}^n}$ be a real sequence. If $M_d(\mathbf{y}) \geq 0$ for some $d \geq 1$, then

$$|y_{\alpha}| \leq \max\left\{y_{0}, \max_{i=1,\dots,n} L_{\mathbf{y}}\left(x_{i}^{2d}\right)\right\} \quad \forall \alpha \in \mathbb{N}_{2d}^{n}.$$

LEMMA 2.4. If there exist a constant c > 0 such that $M_d(\mathbf{y}) \succeq 0$ and $|y_\alpha| \leq c$ for all $d \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$, then there exists a representing measure μ with support on $[-1, 1]^n$.

Given polynomials $\mathcal{P}_j \in \mathbb{R}[x]$, let \mathbf{p}_j be its coefficient sequence in standard monomial basis for $j = 1, 2, \ldots, \ell$; consider the semialgebraic set \mathcal{K} defined as

$$\mathcal{K} = \{ x \in \mathbb{R}^n : \mathcal{P}_j(x) \ge 0, \ j = 1, 2, \dots, \ell \}.$$
(2.7)

The following lemma gives a necessary and sufficient condition for \mathbf{y} to have a representing measure μ supported on \mathcal{K} – see [24, 30, 31, 32].

LEMMA 2.5. If \mathcal{K} defined in (2.7) satisfies Putinar's property, then the sequence $\mathbf{y} = \{y_{\alpha}\}_{\alpha \in \mathbb{N}^n}$ has a representing finite Borel measure μ on the set \mathcal{K} , if and only if

$$M_d(\mathbf{y}) \geq 0, \quad M_d(\mathbf{y}; \boldsymbol{p}_j) \geq 0, \quad j = 1, \dots, \ell, \text{ for all } d \in \mathbb{N}.$$

Finally, the following lemma, proven in [24], shows that the Borel measure of a compact set is equal to the optimal value of an infinite dimensional LP problem.

LEMMA 2.6. Let Σ be the Borel σ -algebra on \mathbb{R}^n , and μ_1 be a measure on a compact set $\mathcal{B} \in \Sigma$. Then for any given $\mathcal{K} \in \Sigma$ such that $\mathcal{K} \subseteq \mathcal{B}$, one has

$$\mu_1(\mathcal{K}) = \int_{\mathcal{K}} d\mu_1 = \sup_{\mu_2 \in \mathcal{M}(\mathcal{K})} \left\{ \int d\mu_2 : \mu_2 \preccurlyeq \mu_1 \right\},\,$$

where $\mathcal{M}(\mathcal{K})$ is the set of finite Borel measures on \mathcal{K} .

3. Chance Optimization over a Semialgebraic Set. In this section we focus on the chance optimization problem stated in (1.2). We first provide an equivalent problem over finite (positive) Borel measures as variables, and then we will consider its relaxations in the moment space. Given polynomials $\mathcal{P}_j: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ with degree δ_j for $j = 1, \ldots, \ell$, we define

$$\mathcal{K} = \{ (x,q) \in \mathbb{R}^n \times \mathbb{R}^m : \mathcal{P}_j(x,q) \ge 0, \ j = 1, 2, \dots, \ell \}.$$

$$(3.1)$$

Assumption 1. *K* satisfies Putinar's property.

Remark 3.1. Assumption 1 implies that \mathcal{K} is a compact set; hence the projections of \mathcal{K} onto xcoordinates and onto q-coordinates, i.e., $\Pi_1 =: \{x \in \mathbb{R}^n : \exists q \in \mathbb{R}^m \text{ s.t. } (x,q) \in \mathcal{K}\}$ and $\Pi_2 =: \{q \in \mathbb{R}^n : \exists q \in \mathbb{R}^n : d_q \in \mathcal{K}\}$ \mathbb{R}^m : $\exists x \in \mathbb{R}^n \text{ s.t. } (x,q) \in \mathcal{K} \}$, are also compact. Therefore, after rescaling of polynomials, we assume without loss of generality that $\Pi_1 \subset \chi := [-1,1]^n$ and $\Pi_2 \subset \mathcal{Q} := [-1,1]^m$. Furthermore, instead of working on the original probability space $(\mathbb{R}^m, \overline{\Sigma}_q, \overline{\mu}_q)$, we can adopt a smaller probability space $(\mathcal{Q}, \Sigma_q, \mu_q)$, where $\Sigma_q := \{S \cap \mathcal{Q} : S \in \overline{\Sigma}_q\}$ and $\mu_q(S) := \frac{\overline{\mu}_q(S)}{\overline{\mu}_q(\mathcal{Q})}$ for all $S \in \Sigma_q$. Therefore, we can take for granted that $\mu_q \in \mathcal{M}(\mathcal{Q})$, where $\mathcal{M}(\mathcal{Q})$ is the set of finite Borel measures μ_q such that $supp(\mu_q) \subset \mathcal{Q}$. We also assume that moments of any order of μ_q can be computed.

3.1. An Equivalent Problem. As an intermediate step in the development of convex relaxations of the original problem, a related infinite dimensional problem in the measure space is provided below:

$$\mathbf{P}^*_{\mu_{\mathbf{q}}} := \sup_{\mu,\mu_x} \int d\mu, \tag{3.2}$$

s.t.
$$\mu \preccurlyeq \mu_x \times \mu_q,$$
 (3.2a)

 μ_x is a probability measure, (3.2b)

$$\mu_x \in \mathcal{M}(\chi), \quad \mu \in \mathcal{M}(\mathcal{K}).$$
 (3.2c)

THEOREM 3.1. The optimization problems in (1.2) and (3.2) are equivalent in the following sense:

- i) The optimal values are the same, i.e., P^{*} = P^{*}_{μq}.
 ii) If an optimal solution to (3.2) exists, call it μ^{*}_x, then any x^{*} ∈ supp(μ^{*}_x) is an optimal solution to (1.2).
- iii) If an optimal solution to (1.2) exists, call it x^* , then $\mu_x = \delta_{x^*}$, Dirac measure at x^* , and $\mu = \delta_{x^*} \times \mu_q$ is an optimal solution to (3.2).

Proof. Let $(\mathcal{Q}, \Sigma, \mu_q)$ be the probability space defined in Remark 3.1. Note that since $\mathcal{P}_i(x,q)$ is a polynomial in random vector $q \in \mathbb{R}^m$ for all $x \in \mathbb{R}^n$, it is continuous in q; hence $\mathcal{P}_j(x, .)$ is Borel measurable for all $x \in \mathbb{R}^n$ and $j = 1, \ldots, \ell$. As discussed in Remark 3.1, it can be assumed that $\mathcal{K} \subset \chi \times \mathcal{Q} =$ $[-1,1]^n \times [-1,1]^m$. Define $\mathcal{F} : \mathbb{R}^n \to \Sigma$ as follows

$$\mathcal{F}(x) := \{ q \in \mathbb{R}^m : \mathcal{P}_j(x, q) \ge 0, \ j = 1, 2, \dots, \ell \},$$

$$(3.3)$$

and consider the following problem over the probability measures in $\mathcal{M}(\chi)$:

$$\mathbf{P} := \sup_{\mu_x \in \mathcal{M}(\chi)} \left\{ \int_{\chi} \mu_q(\mathcal{F}(x)) \ d\mu_x : \ \mu_x(\chi) = 1 \right\}.$$
(3.4)



Fig. 3.1: a) Simple chance optimization problem over semialgebraic set \mathcal{K} with random parameter q, and decision variable x, b) Equivalent problem in the measure space over probability measure μ_x as variable for given probability measure μ_q , c) Probability of given semi algebraic set \mathcal{K} for a fixed μ_x is equal to the integral of \mathcal{K} with respect to the measure $\mu_x \times \mu_q$, d) The probability is equal to the volume of the measure μ which is supported on the set \mathcal{K} and has the same distribution as the measure $\mu_x \times \mu_q$ over its support

Note that the optimal value of (1.2) can be written as $\mathbf{P}^* = \sup_{x \in \chi} \mu_q(\mathcal{F}(x))$. Let μ_x be a feasible solution to (3.4). Since $\mu_q(\mathcal{F}(x)) \leq \mathbf{P}^*$ for all $x \in \chi$, we have $\int \mu_q(\mathcal{F}(x)) d\mu_x \leq \mathbf{P}^*$. Thus, $\mathbf{P} \leq \mathbf{P}^*$. Conversely, let $x \in \mathbb{R}^n$ be a feasible solution to the problem in (1.2) and δ_x denote the Dirac measure at x. The objective value of x in (1.2) is equal to $\mu_q(\mathcal{F}(x))$. Moreover, $\mu_x = \delta_x$ is a feasible solution to the problem in (3.2) with objective value equal to $\mu_q(\mathcal{F}(x))$. This implies that $\mathbf{P}^* \leq \mathbf{P}$. Hence, $\mathbf{P}^* = \mathbf{P}$, and (3.4) can be rewritten as

$$\mathbf{P}^* = \sup_{\mu_x \in \mathcal{M}(\chi)} \left\{ \int_{\chi} \int_{\mathcal{F}(x)} d\mu_q d\mu_x : \ \mu_x(\chi) = 1 \right\} = \sup_{\mu_x \in \mathcal{M}(\chi)} \left\{ \int_{\mathcal{K}} d\mu_x \mu_q : \ \mu_x(\chi) = 1 \right\},$$
(3.5)

and using the epigraph formulation shown in Lemma 2.6, we finally obtain

$$\mathbf{P}^* = \sup_{\mu_x \in \mathcal{M}(\chi)} \sup_{\mu \in \mathcal{M}(\mathcal{K})} \int d\mu \quad \text{s.t.} \quad \mu \preccurlyeq \mu_x \times \mu_q, \ \mu_x(\chi) = 1.$$

Therefore, $\mathbf{P}^* = \mathbf{P}^*_{\mu_{\mathbf{q}}}$.

As an example, consider the following chance constrained problem corresponding to the semialgebraic set \mathcal{K} , displayed in Fig.1.a, in the space of $(x,q) \in \mathbb{R} \times \mathbb{R}$. Our objective is to compute an optimal decision x^* that attains $\mathbf{P}^* = \sup_{x \in [-1,1]} \mu_q(\mathcal{F}(x))$, in presence of random variable q with known probability measure μ_q supported on [-1,1]. In other words, x^* should be chosen such that the probability of the random point (x^*,q) belonging to \mathcal{K} becomes maximum. Fig.1.b shows the problem in the measure space, where a probability measure μ_x is assigned to decision variable x. If $x \in [-1,1]$ is chosen randomly according to fixed μ_x , then to calculate the probability of the random event $(x,q) \in \mathcal{K}$, one should compute an integral with respect to measure $\mu_x \times \mu_q$ over the set \mathcal{K} as in (3.5) – see (Fig.1.c). This integral is equal to the volume of a measure which is supported on \mathcal{K} and has the same distribution as $\mu_x \times \mu_q$ on \mathcal{K} – see (Fig.1.d). Hence, for fixed μ_x , one needs to look for the measure μ supported on \mathcal{K} with maximum volume, and bounded above with measure $\mu_x \times \mu_q$. Therefore, searching for μ_x and μ simultaneously leads to the optimization problem (3.2) in the measure space.

3.2. Semidefinite Relaxations. In this section, we provide an infinite dimensional SDP of which feasible region is defined over real sequences in $\mathbb{R}^{\mathbb{N}}$. Unlike the problem (3.2) in which we are looking for measures, in the SDP formulation given in (3.6), we aim at finding moment sequences corresponding to a measure that is optimal to (3.2). After proving the equivalence of (3.2) and (3.6), we next provide a sequence of finite dimensional SDPs and show that the corresponding sequence of optimal solutions can arbitrarily approximate the optimal solution of (3.6), which characterizes the optimal solution of (3.2).

Consider the following infinite dimensional SDP:

$$\mathbf{P}_{\mathbf{y}_{\mathbf{q}}}^{*} := \sup_{\mathbf{y}, \mathbf{y}_{\mathbf{x}} \in \mathbb{R}^{\mathbb{N}}} (\mathbf{y})_{\mathbf{0}}, \tag{3.6}$$

s.t.
$$M_{\infty}(\mathbf{y}) \succeq 0, \ M_{\infty}(\mathbf{y}; \mathbf{p}_j) \succeq 0, \quad j = 1, \dots, \ell,$$
 (3.6a)

$$M_{\infty}(\mathbf{y}_{\mathbf{x}}) \succcurlyeq 0, \ \|\mathbf{y}_{\mathbf{x}}\|_{\infty} \le 1, \ (\mathbf{y}_{\mathbf{x}})_{\mathbf{0}} = 1,$$

$$(3.6b)$$

$$M_{\infty}(\mathbf{A}\mathbf{y}_{\mathbf{x}} - \mathbf{y}) \succeq 0, \tag{3.6c}$$

where $\mathbf{A} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is a linear map depending only on μ_q . Indeed, let $\mathbf{y}_{\mathbf{q}} := \{y_{q_\beta}\}_{\beta \in \mathbb{N}^m}$ be the moment sequence of μ_q . Then for any given $\mathbf{y}_{\mathbf{x}} = \{y_{x_\alpha}\}_{\alpha \in \mathbb{N}^n}$, $\mathbf{A}\mathbf{y}_{\mathbf{x}} = \bar{\mathbf{y}}$ such that $(\bar{\mathbf{y}})_{\theta} = (\mathbf{y}_{\mathbf{q}})_{\beta}(\mathbf{y}_{\mathbf{x}})_{\alpha}$ for all $\theta = (\beta, \alpha) \in \mathbb{N}^m \times \mathbb{N}^n$. Given $\mathbf{y} \in \mathbb{R}^{\mathbb{N}}$, $M_{\infty}(\mathbf{y}) \succeq 0$ means that $M_d(\mathbf{y}) \succeq 0$ for all $d \in \mathbb{Z}_+$.

The following lemma establishes the equivalence of (3.2) and (3.6).

LEMMA 3.2. Suppose that \mathcal{K} satisfies Assumption 1. If an optimal solution to (3.2) exists, call it (μ^*, μ_x^*) , then their moment sequences $(\mathbf{y}^*, \mathbf{y}_x^*)$ is an optimal solution to (3.6). Conversely, if an optimal solution to (3.6) exists, call it $(\mathbf{y}^*, \mathbf{y}_x^*)$, then there exists representing measures μ^* and μ_x^* such that (μ^*, μ_x^*) is optimal to (3.2). Moreover, the optimal values of (3.2) and (3.6) are the same, i.e., $\mathbf{P}_{\mu_{\mathbf{q}}}^* = \mathbf{P}_{\mathbf{y}_{\mathbf{q}}}^*$.

Proof. Suppose that (μ, μ_x) is feasible to (3.2). Let \mathbf{y} and $\mathbf{y}_{\mathbf{x}}$ be the moment sequences corresponding to μ and μ_x , respectively. Lemma 2.5 implies (3.6a); Lemma 2.1 and Lemma 2.2 imply (3.6b). Moreover, let $\bar{\mathbf{y}} = \{\bar{y}_{\alpha}\}_{\alpha \in \mathbb{N}^{n+m}}$ be the moment sequence corresponding to the product measure $\bar{\mu} := \mu_x \times \mu_q$. (3.2a) implies that $\bar{\mu} - \mu$ is a measure; hence, Lemma 2.2 implies $M_{\infty}(\bar{\mathbf{y}} - \mathbf{y}) \succeq 0$. Moreover, the definition of \mathbf{A} implies that $\bar{\mathbf{y}} = \mathbf{A}\mathbf{y}_{\mathbf{x}}$, which gives (3.6c). Since \mathbf{y} is chosen to be the moment sequence of μ , we have $\int d\mu = y_0$. This shows that for each (μ, μ_x) feasible to (3.2), one can construct a feasible solution to (3.6) with the same objective value. Therefore, $\mathbf{P}^*_{\mathbf{y}_{\mathbf{q}}} \geq \mathbf{P}^*_{\mu_{\mathbf{q}}}$. Note that Assumption 1 is not used for this argument. Next, suppose that $(\mathbf{y}, \mathbf{y}_{\mathbf{x}})$ is a feasible solution to (3.6). Since \mathcal{K} satisfies Assumption 1, (3.6a) and

Next, suppose that $(\mathbf{y}, \mathbf{y}_{\mathbf{x}})$ is a feasible solution to (3.6). Since \mathcal{K} satisfies Assumption 1, (3.6a) and Lemma 2.5 together imply that \mathbf{y} has a representing finite Borel measure μ supported on \mathcal{K} , i.e., $\mu \in \mathcal{M}(\mathcal{K})$. Moreover, (3.6b) and Lemma 2.4 together imply that $\mathbf{y}_{\mathbf{x}}$ has a representing probability measure μ_x supported on hyper-cube χ , i.e., $\mu_x \in \mathcal{M}(\chi)$ such that $\mu_x(\chi) = 1$. Hence, the sequence $A\mathbf{y}_{\mathbf{x}}$ has a representing measure $\bar{\mu}$ which is the product measure of μ_x and μ_q , i.e., $\bar{\mu} = \mu_x \times \mu_q$. Furthermore, since $\mathcal{K} \subset \chi \times \mathcal{Q} = [-1, 1]^{n+m}$, (3.6c) implies that $\mu \preceq \bar{\mu}$, which is (3.2a). Finally, the fact that μ is a representing measure of \mathbf{y} implies that $\int d\mu = y_0$. Therefore, $\mathbf{P}^*_{\mathbf{y}_q} \leq \mathbf{P}^*_{\mu_q}$. Combining this with the above result gives us $\mathbf{P}^*_{\mathbf{y}_q} = \mathbf{P}^*_{\mu_q}$. \square

In order to have tractable approximations to the infinite dimensional SDP in (3.6), we consider the following sequence of SDPs, known as Lasserre's hierarchy [32], defined below:

$$\mathbf{P}_d := \sup_{\mathbf{y} \in \mathbb{R}^{S_{n+m,2d}}, \ \mathbf{y}_{\mathbf{x}} \in \mathbb{R}^{S_{n,2d}}} (\mathbf{y})_{\mathbf{0}}, \tag{3.7}$$

s.t.
$$M_d(\mathbf{y}) \geq 0, \ M_{d-r_j}(\mathbf{y}; \mathbf{p}_j) \geq 0, \quad j = 1, \dots, \ell,$$
 (3.7a)

$$M_d(\mathbf{y}_{\mathbf{x}}) \succcurlyeq 0, \|\mathbf{y}_{\mathbf{x}}\|_{\infty} \le 1, (\mathbf{y}_{\mathbf{x}})_{\mathbf{0}} = 1,$$

$$(3.7b)$$

$$M_d(A_d \mathbf{y}_{\mathbf{x}} - \mathbf{y}) \succeq 0, \tag{3.7c}$$

where δ_j is the degree of \mathcal{P}_j , $r_j := \left\lceil \frac{\delta_j}{2} \right\rceil$ for all $1 \leq j \leq \ell$, and $A_d : \mathbb{R}^{S_{n,2d}} \to \mathbb{R}^{S_{n+m,2d}}$ is defined similarly to **A** in (3.6). Indeed, let $\mathbf{y}_{\mathbf{q}} := \{y_{q_\beta}\}_{\beta \in \mathbb{N}_{2d}^m}$ be the truncated moment sequence of μ_q . Then for any given $\mathbf{y}_{\mathbf{x}} = \{y_{x_\alpha}\}_{\alpha \in \mathbb{N}_{2d}^n}$, $A_d \mathbf{y}_{\mathbf{x}} = \mathbf{y}$ such that $(\bar{\mathbf{y}})_{\theta} = (\mathbf{y}_{\mathbf{q}})_{\beta} (\mathbf{y}_{\mathbf{x}})_{\alpha}$ for all $\theta = (\beta, \alpha) \in \mathbb{N}_{2d}^{n+m}$. In the following theorem, it is shown that the sequence of optimal solutions to the SDPs in (3.7) converges to the solution of the infinite dimensional SDP in (3.6). In essence, the following theorem is similar to Theorem 3.2 in [24]; however, for the sake of completeness we give its proof below.

THEOREM 3.3. For all $d \geq 1$, there exists an optimal solution $(\mathbf{y}^d, \mathbf{y}^d_{\mathbf{x}}) \in \mathbb{R}^{S_{n+m,2d}} \times \mathbb{R}^{S_{n,2d}}$ to (3.7) with the optimal value \mathbf{P}_d . Let $\mathcal{S} := \{(\mathbf{y}^d, \mathbf{y}^d_{\mathbf{x}})\}_{d \in \mathbb{Z}_+} \subset \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ be such that each element of \mathcal{S} is obtained by zero-padding, i.e., $(\mathbf{y}^d)_{\alpha} = 0$ for all $\alpha \in \mathbb{N}^{n+m}$ such that $\|\alpha\|_1 > 2d$, and $(\mathbf{y}^d_{\mathbf{x}})_{\alpha} = 0$ for all $\alpha \in \mathbb{N}^n$ such that $\|\alpha\|_1 > 2d$. Then $\{\mathbf{P}_d\}_{d \in \mathbb{Z}_+}$ and \mathcal{S} have the following properties:

- i) $\lim_{d\in\mathbb{Z}_+} \mathbf{P}_d = \mathbf{P}^*$, the optimal value of (1.2),
- ii) There exists an accumulation point of S in the weak-* topology of l_∞ and every accumulation point of S is an optimal solution to (3.6). Hence, there exists corresponding representing measures (μ*, μ_x*) that is optimal to (3.2) and any x* ∈ supp(μ_x*) is optimal to (1.2).

Proof. First, we will show that for all $d \ge 1$, the corresponding feasible region of (3.7) is bounded. Fix $d \ge 1$. Let $(\mathbf{y}, \mathbf{y}_{\mathbf{x}})$ be a feasible solution to (3.7). Then from (3.7b), we have $\|\mathbf{y}_{\mathbf{x}}\|_{\infty} \le 1$. Since μ_q is a probability measure supported on $\mathcal{Q} = [-1, 1]^m$, Lemma 2.1 implies that $\|\mathbf{y}_{\mathbf{q}}\|_{\infty} \le 1$ as well. Moreover, the definition of A_d further implies that $\|A_d\mathbf{y}_{\mathbf{x}}\|_{\infty} \le 1$. Let $\bar{\mathbf{y}} := A_d\mathbf{y}_{\mathbf{x}}$. It follows from (3.7c) that the diagonal elements of $M_d(\bar{\mathbf{y}} - \mathbf{y})$ are nonnegative, i.e., $(\bar{\mathbf{y}})_{2\alpha} - (\mathbf{y})_{2\alpha} \ge 0$ for all $\alpha \in \mathbb{N}_d^{n+m}$. This implies that

$$\max\left\{y_{\mathbf{0}}, \max_{i=1,\dots,n+m} L_{\mathbf{y}}\left(x_{i}^{2d}\right)\right\} \leq \max_{\alpha \in \mathbb{N}_{d}^{n+m}} y_{2\alpha} \leq \max_{\alpha \in \mathbb{N}_{d}^{n+m}} \bar{y}_{2\alpha} \leq \|\bar{\mathbf{y}}\|_{\infty} \leq 1,$$
(3.8)

where the first inequality follows from the fact that

$$\{y_{\mathbf{0}}\} \cup \{L_{\mathbf{y}}\left(x_{i}^{2d}\right): i = 1, \dots, n+m\} \subset \{y_{2\alpha}: \alpha \in \mathbb{N}_{d}^{n+m}\}.$$

From (3.7a), we have $M_d(\mathbf{y}) \succeq 0$. Hence, using Lemma 2.3, (3.8) implies that $|y_{\alpha}| \leq \|\bar{\mathbf{y}}\|_{\infty} \leq 1$ for all $\alpha \in \mathbb{N}_{2d}^{n+m}$. Therefore, the feasible region is bounded. Since the cone of positive semidefinite matrices is a closed set and all the mappings in (3.7) is linear, we also conclude that the feasible region is compact. Hence, there exists an optimal solution $(\mathbf{y}^d, \mathbf{y}^d_{\mathbf{x}})$ to the problem (3.7) for all $d \geq 1$.

Fix $d \ge 1$. Clearly, for any given feasible solution $(\mathbf{y}, \mathbf{y}_{\mathbf{x}})$ to (3.6), by truncating the both sequences to vectors $\mathbf{y} \in \mathbb{R}^{S_{n+m,2d}}$ and $\mathbf{y}_{\mathbf{x}} \in \mathbb{R}^{S_{n,2d}}$, we can construct a feasible solution to (3.7) with the same objective value. Hence, it can be concluded that $\mathbf{P}_d \ge \mathbf{P}^*_{\mathbf{y}_{\mathbf{q}}}$ for all $d \ge 1$. Moreover, the same argument also shows that $\mathbf{P}_d \ge \mathbf{P}_{d'}$ for all $d' \ge d$. Hence, $\{\mathbf{P}_d\}_{d\in\mathbb{Z}_+}$ is a decreasing sequence bounded below by $\mathbf{P}^*_{\mathbf{y}_{\mathbf{q}}}$. Therefore, it is convergent and has a limit such that $\lim_{k\in\mathbb{Z}_+} \mathbf{P}_k \ge \mathbf{P}^*_{\mathbf{y}_{\mathbf{q}}}$.

In order to collect all the optimal solutions corresponding to different d in one space, we extend $(\mathbf{y}^d, \mathbf{y}^d_{\mathbf{x}}) \in \mathbb{R}^{S_{n+m,2d}} \times \mathbb{R}^{S_{n,2d}}$ to vectors in ℓ_{∞} (the Banach space of bounded sequences equipped with the sup-norm) by zero-padding, i.e., we set $(\mathbf{y}^d)_{\alpha} = 0$ for all $\alpha \in \mathbb{N}^{n+m}$ such that $\|\alpha\|_1 > 2d$, and $(\mathbf{y}^d_{\mathbf{x}})_{\alpha} = 0$ for all $\alpha \in \mathbb{N}^n$ such that $\|\alpha\|_1 > 2d$, and $(\mathbf{y}^d_{\mathbf{x}})_{\alpha} = 0$ for all $\alpha \in \mathbb{N}^n$ such that $\|\alpha\|_1 > 2d$, and $(\mathbf{y}^d_{\mathbf{x}})_{\alpha} = 0$ for all $\alpha \in \mathbb{N}^n$ such that $\|\alpha\|_1 > 2d$. Note that ℓ_{∞} is the dual space of ℓ_1 , which is separable; hence, sequential Banach-Alaoglu theorem states that the closed unit ball of ℓ_{∞} , denoted by \mathcal{B}_{∞} , is weak- \star sequentially compact. Since $\{\mathbf{y}^d\}_{d\in\mathbb{Z}_+} \subset \mathcal{B}_{\infty}$ and $\{\mathbf{y}^d_{\mathbf{x}}\}_{d\in\mathbb{Z}_+} \subset \mathcal{B}_{\infty}$, there exists a subsequence $\{d_k\} \subset \mathbb{Z}_+$ such that $\{\mathbf{y}^{d_k}\}_{k\in\mathbb{Z}_+}$ and $\{\mathbf{y}^d_{\mathbf{x}}\}_{k\in\mathbb{Z}_+}$ converge weak- \star to $\mathbf{y}^* \in \mathcal{B}_{\infty}$ and $\mathbf{y}^*_{\mathbf{x}} \in \mathcal{B}_{\infty}$ in the weak- \star topology, respectively. Hence,

$$\lim_{k \in \mathbb{Z}_+} \left(\mathbf{y}^{d_k} \right)_{\alpha} = \left(\mathbf{y}^* \right)_{\alpha}, \quad \forall \; \alpha \in \mathbb{N}^{n+m}, \qquad \lim_{k \in \mathbb{Z}_+} \left(\mathbf{y}^{d_k}_{\mathbf{x}} \right)_{\alpha} = \left(\mathbf{y}^*_{\mathbf{x}} \right)_{\alpha}, \quad \forall \; \alpha \in \mathbb{N}^n.$$
(3.9)

Fix $d \ge 1$, then for all $k \in \mathbb{Z}_+$ such that $d_k \ge d$, we have

$$M_{d}(\mathbf{y}^{d_{k}}) \geq 0, \ M_{d-r_{j}}(\mathbf{y}^{d_{k}}; \mathbf{p}_{j}) \geq 0, \quad j = 1, \dots, \ell,$$

$$M_{d}(\mathbf{y}^{d_{k}}_{\mathbf{x}}) \geq 0, \ \|\mathbf{y}^{d_{k}}_{\mathbf{x}}\|_{\infty} \leq 1, \ (\mathbf{y}^{d_{k}}_{\mathbf{x}})_{\mathbf{0}} = 1,$$

$$M_{d}(\mathbf{A}\mathbf{y}^{d_{k}}_{\mathbf{x}} - \mathbf{y}^{d_{k}}) \geq 0.$$

Since $d \in \mathbb{Z}_+$ is arbitrary, by taking the limit as $k \to \infty$, we see that $(\mathbf{y}^*, \mathbf{y}^*_{\mathbf{x}})$ satisfies all the constraints in (3.6). Therefore, $(\mathbf{y}^*)_{\mathbf{0}} \leq \mathbf{P}^*_{\mathbf{y}_{\mathbf{q}}}$. On the other hand, $(\mathbf{y}^*)_{\mathbf{0}} = \lim_{k \in \mathbb{Z}_+} (\mathbf{y}^{d_k})_{\mathbf{0}} = \lim_{k \in \mathbb{Z}_+} \mathbf{P}_{d_k}$. Moreover, since every subsequence of a convergent sequence converges to the same point, we have $\lim_{k \in \mathbb{Z}_+} \mathbf{P}_k = \lim_{k \in \mathbb{Z}_+} \mathbf{P}_{d_k} = \mathbf{P}^*_{\mathbf{y}_{\mathbf{q}}}$. This shows that the subsequential limit $(\mathbf{y}^*, \mathbf{y}^*_{\mathbf{x}})$ is an optimal solution to (3.6). The rest of the claims follow from our previous results: Theorem 3.1 and Lemma 3.2.

3.3. Discussion on Improving Estimates of Probability. In our numerical experiments, we have observed that the convergence of the upper bound \mathbf{P}_d to the optimum probability \mathbf{P}^* was slow in d when we solved the sequence of SDP relaxations in (3.7). Suppose that the semi-algebraic set $\mathcal{K} := \{(x,q) :$ $\mathcal{P}_j(x,q) \geq 0, \ j = 1, \ldots, \ell\}$ satisfies Putinar's property. The procedure detailed below helped us to get better estimates on the optimum probability \mathbf{P}^* . To make the upcoming discussion easier we make the following assumptions: i) there is a unique $x^* \in \Pi_1$ such that $\mu_q(\mathcal{F}(x^*)) = \mathbf{P}^*$, and there exists some \bar{q} such that $(x^*, \bar{q}) \in \operatorname{relint} \mathcal{K}$, where \mathcal{F} is defined in (3.3), and $\Pi_1 := \{x \in \mathbb{R}^n : \exists q \in \mathbb{R}^m \text{ s.t. } (x,q) \in \mathcal{K}\} \subset \chi := [-1,1]^n$; and ii) $\mu_q \in \mathcal{M}(\mathcal{Q})$ has the following "continuity" property: if $\{S_k\} \subset \Sigma_q$ such that $\lim_{k\to\infty} S_k = S^*$ in the Hausdorff-metric, then $\lim_{k\to\infty} \mu_q(S_k) = \mu_q(S^*)$. Let $(\mathbf{y}^d, \mathbf{y}^d_{\mathbf{x}})$ denote an optimal solution to the SDP relaxation in (3.7), and form $x^d \in \mathbb{R}^n$ using the components of $(\mathbf{y}^d_{\mathbf{x}})_\alpha$ such that $\|\alpha\|_1 = 1$. Clearly, $x^d \in \chi$. Since $\mu_q \in \mathcal{M}(\mathcal{Q})$ is given, we approximate the volume $\int_{\mathcal{F}(x^d)} d\mu_q$ as described in [24] by solving an SDP relaxation for

$$\bar{\mathbf{P}}_d := \sup_{\mu' \in \mathcal{M}(\mathcal{F}(x^d))} \int d\mu' \text{ s.t. } \mu' \preceq \mu_q.$$
(3.10)

Note that this intermediate SDP can be built only after the relaxation in (3.7) is solved. Let \mathbf{P}'_d denote the optimal value of the volume approximation SDP corresponding to (3.10) with relaxation order d. Clearly, $\mathbf{\bar{P}}_d = \mu_q \left(\mathcal{F}(x^d) \right) \geq 0$, and for all d we have $\mathbf{P}_d \geq \mathbf{P}^* \geq \mathbf{\bar{P}}_d$, and $\mathbf{P}_d \geq \mathbf{P}'_d \geq \mathbf{\bar{P}}_d$. Note that since x^* is the unique optimal solution (assumption i), Theorem 3.3 implies that $\lim_{d\to\infty} (\mathbf{y}^d_{\mathbf{x}})_{\alpha} = (\mathbf{y}^*_{\mathbf{x}})_{\alpha}$ for all $\alpha \in \mathbb{N}^n$ such that $\mathbf{y}^*_{\mathbf{x}}$ is the moment sequence corresponding to Dirac measure at x^* . Therefore, from the definition of x^d , it follows that $\lim_{d\to\infty} x^d = x^*$. Also note that since \mathcal{K} is compact (from Putinar's property) and \mathcal{P}_j is a polynomial in (x,q) for all $j = 1, \ldots, \ell$, it follows that the multifunction $\mathcal{F} : \chi \to \Sigma_q$ such that $\mathcal{F}(x) = \{q \in \mathcal{Q} : (x,q) \in \mathcal{K}\}$ with dom $\mathcal{F} = \Pi_1$ is locally bounded, closed-valued, and $\lim_{d\to\infty} \mathcal{F}(x^d) = \mathcal{F}(x^*)$ in Hausdorff metric. Hence, assumption ii implies that $\lim_{d\to\infty} \mathbf{\bar{P}}_d = \lim_{d\to\infty} \mu_q \left(\mathcal{F}(x^d)\right) = \mathbf{P}^*$. Moreover, since $\lim_{d\to\infty} \mathbf{P}_d = \mathbf{P}^*$ (from Theorem 3.3), and $\mathbf{P}_d \geq \mathbf{P}'_d \geq \mathbf{\bar{P}}_d$ for all d, we can conclude that $\lim_{d\to\infty} \mathbf{P}'_d = \mathbf{P}^*$ as well.

We noticed in our numerical experiments that although $\{\mathbf{P}'_d\}_{d\in\mathbb{Z}_+}$ is closer to \mathbf{P}^* when compared to $\{\mathbf{P}_d\}_{d\in\mathbb{Z}_+}$, the convergence of \mathbf{P}'_d to \mathbf{P}^* was still slow in practice as d increases. This phenomena may partly be explained as in [24] by considering the dual problem. Let \mathcal{C} be the Banach space of continuous functions on \mathcal{Q} such that $||f|| := \sup_{q \in \mathcal{Q}} f(q)$ for $f \in \mathcal{C}$, and $\mathcal{C}_+ := \{f \in \mathcal{C} : f \ge 0 \text{ on } \mathcal{Q}\}$. The Lagrangian dual of (3.10) is given below:

$$\bar{\mathbf{P}}_{d}^{\mathbf{Dual}} \coloneqq \inf_{f \in \mathcal{C}_{+}} \int f \, d\mu_{q}, \tag{3.11}$$
s.t. $f \ge 1$ on $\mathcal{F}(x^{d}).$

Moreover, assumption ii ("continuity" of μ_q) and Urysohn's Lemma together imply that $\bar{\mathbf{P}}_d^{\mathbf{Dual}} = \bar{\mathbf{P}}_d$ for all d. Let $\mathcal{I}_{\mathcal{F}(x^d)}$ denote the indicator function of the semi-algebraic set $\mathcal{F}(x^d)$, i.e., $\mathcal{I}_{\mathcal{F}(x^d)}(q) = 1$ if $q \in \mathcal{F}(x^d)$, and 0 otherwise. Indeed, solving the SDP relaxation of (3.10) corresponds in dual space to approximating $\mathcal{I}_{\mathcal{F}(x^d)}$, which is discontinuous on the boundary of the set. Therefore, although there exists a minimizing sequence of functions belonging to \mathcal{C}_+ that approximates $\mathcal{I}_{\mathcal{F}(x^d)}$ from above, the discontinuity on the boundary of $\mathcal{F}(x^d)$ causes the Gibbs phenomenon – see the oscillation observed in Figure 3.3.a. This might be an important factor lurking behind the numerically observed slow convergence of $\{\mathbf{P}'_d\}_{d\in\mathbb{Z}_+}$ to \mathbf{P}^* .

Let $\mathcal{G}^d : \mathcal{Q} \to \mathbb{R}$ such that $\mathcal{G}^d(q) := \prod_{j=1}^{\ell} \mathcal{P}_j(x^d, q)$. To deal with the numerical problems caused by approximating the discontinuous indicator function, we propose to solve

$$\sup_{\tilde{\mu}\in\mathcal{M}(\mathcal{F}(x^d))} \int \mathcal{G}^d \ d\tilde{\mu} \text{ s.t. } \tilde{\mu} \preceq \mu_q.$$
(3.12)

Let μ_d^* denote the optimal solution to (3.12). "Continuity" of μ_q in assumption *ii* implies that \mathcal{G}^d is strictly positive almost everywhere on $\mathcal{F}(x^d)$. Hence, μ_d^* is clearly also optimal to (3.10). Therefore, $\mu_d^*(\mathcal{F}(x^d)) = \mu_q(\mathcal{F}(x^d)) = \bar{\mathbf{P}}_d \to \mathbf{P}^*$ as $d \to \infty$. Let $\mathcal{U}^d(q) = \max\{\mathcal{G}^d(q), 0\}$, and note that \mathcal{U}^d is continuous on the boundary of $\mathcal{F}(x^d)$; hence, it is important to emphasize that solving (3.12) corresponds to approximating the *continuous* function \mathcal{U}^d from above on $\mathcal{F}(x^d)$. These properties of (3.12) motivated us to numerically investigate the behaviour of $\{\tilde{\mathbf{P}}_d\}_{d\in\mathbb{Z}_+}$ sequence, where $\tilde{\mathbf{P}}_d := (\tilde{\mathbf{y}}^d)_0$ and $\tilde{\mathbf{y}}^d$ denotes an optimal solution to the SDP relaxation for (3.12) with order d. In our numerical experiments we observed that $\tilde{\mathbf{P}}_d \to \mathbf{P}^*$; however, this time with a faster convergence rate. To illustrate this behavior numerically, we considered two simple example problems in Section 3.4.

3.4. Simple Examples. In this section, we present two simple example problems that illustrate the effectiveness of the proposed methodology to solve the chance optimization problem in (1.2). The decision variables and the uncertain problem parameters in these examples are low dimensional for illustrative purposes. In the first example, we considered a problem over a semialgebraic set defined by a single polynomial:

$$\sup_{x \in \mathbb{R}} \mu_q \left(\left\{ q \in \mathbb{R} : \mathcal{P}(x, q) \ge 0 \right\} \right), \tag{3.13}$$

where

$$\mathcal{P}(x,q) = \frac{1}{2}q\left(q^2 + \left(x - \frac{1}{2}\right)^2\right) - \left(q^4 + q^2\left(x - \frac{1}{2}\right)^2 + \left(x - \frac{1}{2}\right)^4\right).$$
(3.14)

The uncertain parameter $q \in \mathbb{R}$ has a uniform distribution on [-1,1]. To obtain an approximate solution, we solve the SDP in (3.7) with the minimum relaxation order d = 2 since the degree of the polynomial in (3.14) is 4. The moment vectors $\mathbf{y}_{\mathbf{q}}$, $\mathbf{y}_{\mathbf{x}}$, and \mathbf{y} for the measures μ_q and μ_x , and μ up to order four are

$$\mathbf{y}_{\mathbf{q}}^{T} = \begin{bmatrix} 1, \ 0, \ \frac{1}{3}, \ 0, \ \frac{1}{5} \end{bmatrix}, \quad \mathbf{y}_{\mathbf{x}}^{T} = \begin{bmatrix} 1, \ y_{x_{1}}, \ y_{x_{2}}, \ y_{x_{3}}, \ y_{x_{4}} \end{bmatrix},$$
$$\mathbf{y}^{T} = \begin{bmatrix} y_{00} \mid y_{10}, \ y_{01} \mid y_{20}, \ y_{11}, \ y_{02} \mid y_{30}, \ y_{21}, \ y_{12}, \ y_{03} \mid y_{40}, \ y_{31}, \ y_{22}, \ y_{13}, \ y_{04} \end{bmatrix}.$$

Given moment vectors $\mathbf{y}_{\mathbf{q}}$, the moment vector $\bar{\mathbf{y}}$ for the measure $\overline{\mu} = \mu_x \times \mu_q$ has the form

$$\begin{split} \bar{\mathbf{y}}^{\scriptscriptstyle I} &= [1 \mid y_{x_1}, \; y_{q_1} \mid y_{x_2}, \; y_{x_1}y_{q_1}, \; y_{q_2} \mid y_{x_3}, \; y_{x_2}y_{q_1}, \; y_{x_1}y_{q_2}, \; y_{q_3} \mid y_{x_4}, \; y_{x_3}y_{q_1}, \; y_{x_2}y_{q_2}, \; y_{x_1}y_{q_3}, \; y_{q_4}] \,, \\ &= [1 \mid y_{x_1}, \; 0 \mid y_{x_2}, \; 0, \; \frac{1}{3} \mid y_{x_3}, \; 0, \; \frac{1}{3}y_{x_1}, \; 0 \mid y_{x_4}, \; 0, \; \frac{1}{3}y_{x_2}, \; 0, \; \frac{1}{5}] \,. \end{split}$$

SDP in (3.7) with d = 2 is solved using SeDuMi [53], which is an interior-point solver add-on for Matlab, and the following solution was obtained:

 $\mathbf{y}^{*T} = \begin{bmatrix} 0.66, \ 0.3, \ 0.14, \ 0.16, \ 0.07, \ 0.1, \ 0.08, \ 0.03, \ 0.05, \ 0.04, \ 0.04, \ 0.02, \ 0.02, \ 0.02, \ 0.02 \end{bmatrix}, \\ \mathbf{y}^{*T}_{\mathbf{x}} = \begin{bmatrix} 1, 0.50, 0.25, 0.13, 0.85 \end{bmatrix}.$



Fig. 3.2: \mathbf{P}_d , \mathbf{P}'_d , and $\tilde{\mathbf{P}}_d$ for increasing relaxation order d

We approximate the solution to (1.2) with $y_{x_1}^* = 0.5$ (in Section 3.3 we make a case for this approximation under some simplifying assumptions), and estimate the optimal probability \mathbf{P}^* with $\mathbf{P}_2 = y_{00}^* = 0.66$. To test the accuracy of the results obtained, we used Monte Carlo simulation to estimate \mathbf{P}^* and an optimal solution to (3.13). The details of the Monte Carlo simulation are discussed in Section 5.3.1. This computationally intensive method estimated that $x^* = 0.5$ with optimal probability of 0.25. To obtain better estimates of the optimum probability, one needs to increase the relaxation order d.

against the optimal probability $\mathbf{P}^* = 0.25$ denoted by the green dashed line. For increasing relaxation orders d = 2, ..., 25, we adopted SeDuMi [53] to compute \mathbf{P}_d and \mathbf{P}'_d , the optimal values of the SDP in (3.7), and of the SDP relaxation for the volume problem in (3.10) with relaxation order d, respectively; and also to compute $\tilde{\mathbf{P}}_d = (\tilde{\mathbf{y}}^d)_0$. Similar to the results in [24], Figure 3.2 shows a faster convergence to \mathbf{P}^* for the case when $\int \mathcal{G}^d d\tilde{\mu}$ is maximized as in (3.12). Let $\mathcal{I}_{\mathcal{F}(x^d)}$ denote the indicator function of $\mathcal{F}(x^d)$, i.e., $\mathcal{I}_{\mathcal{F}(x^d)}(q) = 1$ if $q \in \mathcal{F}(x^d)$, and 0 otherwise. As discussed in Section 3.3, $\mathcal{U}^d = \max\{\mathcal{G}^d, 0\}$ is a continuous function while $\mathcal{I}_{\mathcal{F}(x^d)}$ is discontinuous on the boundary of $\mathcal{F}(x^d)$; and this might be a factor affecting the convergence speed. Indeed, Figure 3.3.a displays the degree-100 polynomial approximation f^* to $\mathcal{I}_{\mathcal{F}(x^*)}$, the indicator function of the set $\mathcal{F}(x^*)$, i.e., f^* is a minimizer to $\inf_{f \in \mathbb{R}_d[x]} \{ \int f d\mu_q : f \ge 0 \text{ on } \mathcal{Q}, f \ge 1 \text{ on } \mathcal{F}(x^*) \}$ for d = 100. Note that this problem is a restriction of the Lagrangian dual problem for $\sup\{\int d\mu': \mu \leq \mu_q, \mu' \in \mathcal{M}(\mathcal{F}(x^*))\}$ -indeed, dual variable $f \in \mathcal{C}$ is restricted to be in $\mathbb{R}_d[x]$. On the other hand, Figure 3.3.b displays the degree-100 polynomial approximation h^* to the piecewise-polynomial function $\mathcal{U} = \max\{\mathcal{G}, 0\}$, where $\mathcal{G}(q) =$ $\mathcal{P}(x^*, q)$ and h^* is a minimizer to $\inf_{h \in \mathbb{R}_d[x]} \{ \int h \ d\mu_q : h \ge 0 \text{ on } \mathcal{Q}, h \ge \mathcal{G} \text{ on } \mathcal{F}(x^*) \}$ for d = 100. Similarly, this problem is a restriction of the Lagrangian dual problem for $\sup\{\int \mathcal{G} d\tilde{\mu} : \tilde{\mu} \leq \mu_q, \tilde{\mu} \in \mathcal{M}(\mathcal{F}(x^*))\}$. Note that Figure 3.3 shows that it is easier to approximate the *continuous* function $\mathcal{U} = \max\{\mathcal{G}, 0\}$ than the discontinuous indicator function $\mathcal{I}_{\mathcal{F}(x^*)}$.



(a) f^* : the degree-100 polynomial approximation to $\mathcal{I}_{\mathcal{F}(x^*)}$, (b) h^* : the degree-100 polynomial approximation of the indicator function of $\mathcal{F}(x^*)$ piecewise-polynomial function $\mathcal{U}(q) = \max{\{\mathcal{G}(q), 0\}}$

Fig. 3.3: Comparison of $\sup_{\mu' \in \mathcal{M}(\mathcal{F}(x^*))} \int d\mu'$ s.t. $\mu' \preceq \mu_q$ and $\sup_{\tilde{\mu} \in \mathcal{M}(\mathcal{F}(x^*))} \int \mathcal{G} d\tilde{\mu}$ s.t. $\tilde{\mu} \preceq \mu_q$ from the dual perspective for $\mathcal{G}(q) = \mathcal{P}(x^*, q)$

Next, we considered a problem over a semialgebraic set defined by an intersection of two polynomials:

$$\sup_{x \in \mathbb{R}} \mu_q \left(\left\{ q \in \mathbb{R} : \mathcal{P}_1(x, q) \ge 0, \mathcal{P}_2(x, q) \ge 0 \right\} \right),$$
(3.15)

where

$$\mathcal{P}_1(x,q) = 0.1275 + 0.7x - x^2 - q^2, \quad \mathcal{P}_2(x,q) = -0.1225 + 0.7x + q - x^2 - q^2. \tag{3.16}$$

The uncertain parameter $q \in \mathbb{R}$ has a uniform distribution on [-1,1]. Against the optimal probability $\mathbf{P}^* = 0.25$ denoted by the green dashed line, Figure 3.4 displays two other sequences, $\{\tilde{\mathbf{P}}_d^{(1)}\}_{d\in\mathbb{Z}_+}$ and $\{\tilde{\mathbf{P}}_d^{(2)}\}_{d\in\mathbb{Z}_+}$, in addition to the three sequences defined in Section 3.3: $\{\mathbf{P}_d\}_{d\in\mathbb{Z}_+}, \{\mathbf{P}_d'\}_{d\in\mathbb{Z}_+}, \text{ and } \{\tilde{\mathbf{P}}_d\}_{d\in\mathbb{Z}_+}$. Here, $\tilde{\mathbf{P}}_d^{(1)}$ and $\tilde{\mathbf{P}}_d^{(2)}$ are defined similarly to $\tilde{\mathbf{P}}_d = (\tilde{\mathbf{y}}^d)_0$ by replacing $\mathcal{G}^d(q) = \mathcal{P}_1(x^d, q)\mathcal{P}_2(x^d, q)$ in (3.12) with $\mathcal{P}_1(x^d, q)$, and $\mathcal{P}_2(x^d, q)$, respectively.



Fig. 3.4: \mathbf{P}_d , \mathbf{P}'_d , $\mathbf{\tilde{P}}_d^{(1)}$, $\mathbf{\tilde{P}}_d^{(2)}$, and $\mathbf{\tilde{P}}_d$ for increasing relaxation order d

3.5. Orthogonal Basis. In this paper, all polynomials are expanded in the usual monomial basis, and the SDPs are therefore formulated as optimization problems over ordinary monomial moments. However, one can improve the numerical performance as in [24] by employing an orthogonal basis of polynomials. First, we redefine the moment and localization matrices represented in the given orthogonal basis. Recall that the $S_{n,d} \times S_{n,d}$ -moment matrix represented in monomial basis can be written as $M_d(\mathbf{y}) = L_{\mathbf{y}} \left(\mathcal{B}_d \mathcal{B}_d^T \right)$, where $\mathcal{B}_d^T = \left[x^{\alpha^{(1)}}, \ldots, x^{\alpha^{(S_{n,d})}} \right]^T$ denotes the vector comprised of the elements of the monomial basis of $\mathbb{R}_d[x]$, where $S_{n,d} := \binom{d+n}{n}$ and $\{\alpha^{(i)}\}_{i=1}^{S_{n,d}} = \mathbb{N}_d^n := \{\alpha \in \mathbb{N}^n : \|\alpha\|_1 \leq d\}$ such that $\mathbf{0} = \alpha^{(1)} <_g \ldots <_g \alpha^{(S_{n,d})}$ are sorted in grevlex order. Similarly, given a polynomial $\mathcal{P} \in \mathbb{R}[x]$ with coefficient vector $\mathbf{p} = \{p_\gamma\}_{\gamma \in \mathbb{N}^n}$ with respect to the monomial basis, its $S_{n,d} \times S_{n,d}$ -localizing matrix represented in the monomial basis can be written as $M_d(\mathbf{y}; \mathbf{p}) = L_{\mathbf{y}} \left(\mathcal{P} \mathcal{B}_d \mathcal{B}_d^T \right)$.

Let $\{b_i\}_{i\in\mathbb{N}}$ be an orthogonal basis of univariate polynomials on [-1,1], i.e., $\int_{[-1,1]} b_i(t)b_j(t) dt = 0$ for all $i \neq j$. Without loss of generality, suppose that the degree of b_i is equal to i for all $i \in \mathbb{N}$. Given $n \geq 1$, for all $\alpha \in \mathbb{N}^n$, define $b_\alpha : \mathbb{R}^n \to \mathbb{R}$ such that $b_\alpha(x) := \prod_{i=1}^n b_{\alpha_i}(x_i)$, where α_i and x_i are the *i*-th components of $\alpha \in \mathbb{N}^n$ and $x \in \mathbb{R}^n$, respectively. Clearly $\{b_\alpha : \alpha \in \mathbb{N}^n_d\}$ is an orthogonal basis of multivariate polynomials on $[-1, 1]^n$ with degree at most d, i.e., $\int_{[-1,1]^n} b_{\alpha^{(i)}}(x) \ b_{\alpha^{(j)}}(x) \ dx = 0$ for all $1 \leq i \neq j \leq S_{n,d}$. Let \mathcal{B}^o_d denote the vector of polynomials in $\mathbb{R}_d[x]$ defined as $\mathcal{B}^{oT}_d = [b_{\alpha^{(1)}}(x), \ b_{\alpha^{(2)}}(x), \ \dots, b_{\alpha^{(S_{n,d})}}(x)]$; and $T_d \in \mathbb{R}^{S_{n,d} \times S_{n,d}}$ denote the one-to-one correspondence such that $\mathcal{B}^o_d = T_d \mathcal{B}_d$. Moreover, for a given sequence $\mathbf{y} = \{y_\alpha\}_{\alpha \in \mathbb{N}^n}$, let $L^o_{\mathbf{y}} : \mathbb{R}[x] \to \mathbb{R}$ be a linear map defined as

$$\mathcal{P} \mapsto L^{o}_{\mathbf{y}}(\mathcal{P}) = \sum_{\alpha \in \mathbb{N}^{n}} p^{o}_{\alpha} y_{\alpha}, \quad \text{where} \quad \mathcal{P}(x) = \sum_{\alpha \in \mathbb{N}^{n}} p^{o}_{\alpha} b_{\alpha}(x).$$
 (3.17)

Given $y \in \mathbb{R}^{S_{n,2d}}$ such that $y^T = [y_{\alpha^{(1)}}, \ldots, y_{\alpha^{(S_{n,2d})}}]^T$, define its extension $\mathbf{y} = \{y_{\alpha}\}_{\alpha \in \mathbb{N}^n}$ such that $y_{\alpha} = 0$ for all $\alpha \in \mathbb{N}^n$ with $\|\alpha\|_1 > 2d$. For $\bar{y} := T_{2d}^{-1}y$, define its extension $\bar{\mathbf{y}}$ similarly. Then for all $\mathcal{P} \in \mathbb{R}_d[x]$, we have $L_{\mathbf{y}}^{\sigma}(\mathcal{P}) = L_{\bar{\mathbf{y}}}(\mathcal{P})$. In the rest of the paper, we abuse the notation and write $\bar{\mathbf{y}} = T_{2d}^{-1}\mathbf{y}$. Then the moment matrix operator, $M_d^{\sigma}(\mathbf{y})$, for the given orthogonal basis is defined as

$$M_d^o(\mathbf{y}) := L_{\mathbf{y}}^o\left(\mathcal{B}_d^o \ \mathcal{B}_d^{oT}\right) = L_{T_{2d}^{-1}\mathbf{y}}\left(T_d \mathcal{B}_d \ \mathcal{B}_d^T T_d^T\right) = T_d M_d\left(T_{2d}^{-1}\mathbf{y}\right) T_d^T.$$
(3.18)

For example for d = 2 and n = 2, the moment matrix under the orthogonal basis formed by Chebyshev polynomials of the first kind can be written as follows

$$M_{2}^{o}(\mathbf{y}) = \begin{bmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & \frac{y_{00} + y_{20}}{2} & y_{11} & \frac{y_{10} + y_{30}}{2} & \frac{y_{01} + y_{21}}{2} & y_{12} \\ y_{01} & y_{11} & \frac{y_{00} + y_{02}}{2} & y_{21} & \frac{y_{10} + y_{12}}{2} & \frac{y_{01} + y_{03}}{2} \\ y_{20} & \frac{y_{10} + y_{30}}{2} & y_{21} & \frac{y_{00} + y_{40}}{2} & \frac{y_{11} + y_{31}}{2} & y_{22} \\ y_{11} & \frac{y_{01} + y_{21}}{2} & \frac{y_{10} + y_{12}}{2} & \frac{y_{11} + y_{31}}{2} & \frac{y_{00} + y_{00} + y_{02} + y_{02} + y_{22}}{4} & \frac{y_{11} + y_{13}}{2} \\ y_{02} & y_{12} & \frac{y_{01} + y_{03}}{2} & y_{22} & \frac{y_{11} + y_{13}}{2} & \frac{y_{00} + y_{00} + y_{04}}{2} \end{bmatrix} .$$
(3.19)

Let $\mathcal{P} \in \mathbb{R}[x]$ be a given polynomial with degree δ , and $\mathbf{p} = \{p_{\alpha}\}_{\alpha \in \mathbb{N}^n}$ denote its coefficient sequence with respect to the standard monomial basis, i.e., $\mathcal{P}(x) = \sum_{\alpha \in \mathbb{N}^n} p_{\alpha} x^{\alpha}$. For a given orthogonal basis, the localization matrix operator is defined as

$$M_d^o(\mathbf{y}; \mathbf{p}) := L_{\mathbf{y}}^o \left(\mathcal{P} \mathcal{B}_d^o \ \mathcal{B}_d^{oT} \right) = L_{T_{2d+\delta}^{-1} \mathbf{y}} \left(T_d \mathcal{P} \mathcal{B}_d \ \mathcal{B}_d^T T_d^T \right) = T_d M_d \left(T_{2d+\delta}^{-1} \mathbf{y}; \mathbf{p} \right) T_d^T.$$
(3.20)

Let $r := \lceil \frac{\delta}{2} \rceil$. It is important to note that since T_{2d} is invertible, $\{\mathbf{y} : M_d^o(\mathbf{y}) \succeq 0, M_{d-r}^o(\mathbf{y}; \mathbf{p}) \succeq 0\}$ and $\{\mathbf{y} : M_d(\mathbf{y}) \succeq 0, M_{d-r}(\mathbf{y}; \mathbf{p}) \succeq 0\}$ are isomorphic. Hence, one can reformulate the SDP relaxation in (3.7) using the new moment and localization matrix operators defined in (3.18) and (3.20), respectively; and the resulting problem stated in the given orthogonal basis is equivalent to (3.7). In order to illustrate the effect of orthogonal polynomial basis on the numerical behavior of the proposed method, we compared the two formulations of the simple example in (3.13): the first formulation is given in (3.7) using monomial basis, and the second formulation is obtained by replacing $M_d(.)$ and $M_{d-r_j}(.; \mathbf{p}_j)$ in (3.7) with $M_d^o(.)$ and $M_{d-r_j}^o(.; \mathbf{p}_j)$, i.e., moment and localizing matrices in Chebyshev polynomial basis representations. In order to avoid matrix inversions as in (3.18) and in (3.20), we used Chebfun package [60], which can efficiently manipulate univariate Chebyshev polynomials, to form $M_d^o(.)$ and $M_{d-r_j}^o(.; \mathbf{p}_j)$ that use multivariate Chebyshev polynomials in a numerically stable way; and solved the resulting SDP problems represented in the Chebyshev polynomial basis is used as opposed to the standard monomial basis as relaxation order d increases. For the problems in Chebyshev basis, the approximation $(x^o)^d$ to the optimal decision x^* is formed similarly as x^d – see Section 3.3. For this example x^d and $(x^o)^d$ sequences were close.



Fig. 3.5: \mathbf{P}_d for monomial and Chebyshev polynomial bases

4. Chance Optimization over a Union of Sets. We now focus on the more general setting of the chance optimization problem in (1.1). Given polynomials $\mathcal{P}_j^k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ with degree $\delta_j^{(k)}$ for $j = 1, \ldots, \ell_k$ and k = 1, ..., N, the semi-algebraic set of interest is $\mathcal{K} = \bigcup_{k=1}^{N} \mathcal{K}_k$, where

$$\mathcal{K}_k = \left\{ (x,q) \in \mathbb{R}^n \times \mathbb{R}^m : \mathcal{P}_j^{(k)}(x,q) \ge 0, \ j = 1, \dots, \ell_k \right\}, \quad k = 1, \dots, N.$$

$$(4.1)$$

Similar to the previous section, we need Putinar's property to hold for \mathcal{K}_k for all $k = 1, \ldots, N$. With the following assumption, we can ensure this.

Assumption 2. $\mathcal{K} = \bigcup_{k=1}^{N} \mathcal{K}_k$ is bounded, where \mathcal{K}_k is defined in (4.1). Hence, as discussed in Remark 3.1, we can assume without loss of generality that $\mathcal{K} \subseteq \chi \times \mathcal{Q}$ and the probability measure $\mu_q \in \mathcal{M}(\mathcal{Q})$, where $\chi = [-1,1]^n$ and $\mathcal{Q} = [-1,1]^m$. Therefore, for all $(x,q) \in \mathcal{K}$, we have $||x||_2^2 + ||q||_2^2 \le m + n$. Define $\mathcal{P}_0^{(k)}(x,q) := m + n - \sum_{i=1}^n x_i^2 - \sum_{i=1}^m q_i^2$ for all k = 1, ..., N. \mathcal{K}_k can be represented as $\mathcal{K}_k = \left\{ (x,q) : \mathcal{P}_j^{(k)}(x,q) \ge 0, \ j = 0, ..., \ell_k \right\}$ -note that index j starts from 0. Since polynomials are continuous in (x, q), the new representation of \mathcal{K}_k satisfies Putinar's property for each k and we still have $\mathcal{K} = \bigcup_{k=1}^{N} \mathcal{K}_k$.

The objective of this section is to provide a sequence of SDP relaxations to the chance optimization problem in (1.1) with N > 1, and show that the results presented in the previous sections can be easily extended for this case. More precisely, we start by providing an equivalent problem in the measure space and then develop relaxations based on moments of measures.

4.1. An Equivalent Problem. As an intermediate step in the development of convex relaxations of (1.1), an equivalent problem in the measure space is provided below.

$$\mathbf{P}_{\mu_{\mathbf{q}}}^{*} := \sup_{\mu_{k}, \ \mu_{x}} \sum_{k=1}^{N} \int d\mu_{k}, \tag{4.2}$$

s.t.
$$\sum_{k=1}^{N} \mu_k \preccurlyeq \mu_x \times \mu_q, \tag{4.2a}$$

 μ_x is a probability measure, (4.2b)

$$\mu_x \in \mathcal{M}(\chi), \quad \mu_k \in \mathcal{M}(\mathcal{K}_k) \quad k = 1, \dots, N.$$
 (4.2c)

This problem is equivalent to the problem addressed in this paper in the following sense.

THEOREM 4.1. The optimization problems in (1.1) and (4.2) are equivalent in the following sense:

- i) The optimal values are the same, i.e. P^{*} = P^{*}_{µq}.
 ii) If an optimal solution to (4.2) exists, call it µ^{*}_x, then any x^{*} ∈ supp(µ^{*}_x) is an optimal solution to (1.1).
- iii) If an optimal solution to (1.1) exists, call it x^* , then Dirac measure at x^* , $\mu_x = \delta_{x^*}$ and $\mu = \delta_{x^*} \times \mu_q$ is an optimal solution to (4.2).

Proof. Let \mathbf{P}^* denote the optimal value of (1.1), and $\mathcal{K} = \bigcup_{k=1}^N \mathcal{K}_k$, where \mathcal{K}_k is defined in (4.1). It can be proven as in Theorem 3.1 that

$$\mathbf{P}^* = \sup_{\mu_x \in \mathcal{M}(\chi)} \sup_{\mu \in \mathcal{M}(\mathcal{K})} \int d\mu \quad \text{s.t.} \quad \mu \preccurlyeq \mu_x \times \mu_q, \ \mu_x(\chi) = 1.$$
(4.3)

Let $\{\mu_k\}_{k=1}^N$ and μ_x be a feasible solution to (4.2) with objective value P. Since $\mu_k \in \mathcal{M}(\mathcal{K}_k) \subset \mathcal{M}(\mathcal{K})$ for all k = 1, ..., N, we have $\sum_{k=1}^{N} \mu_k \in \mathcal{M}(\mathcal{K})$. Hence, $\left(\sum_{k=1}^{N} \mu_k, \mu_x\right)$ is a feasible solution to (4.3) with objective value P, as well. Clearly, this shows that $\mathbf{P}_{\mu_{\mathbf{q}}}^* \leq \mathbf{P}^*$, where $\mathbf{P}_{\mu_{\mathbf{q}}}^*$ denotes the optimal value of (4.2).

Suppose that (μ, μ_x) is a feasible solution to (4.3) with objective value P. Define $\{\mu_k\}_{k=1}^N$ as follows

$$\mu_k(S) := \mu\left(S \cap \left(\mathcal{K}_k \setminus \bigcup_{j=0}^{k-1} \mathcal{K}_j\right)\right), \quad \forall S \in \Sigma(\mathcal{K}),$$
(4.4)

for all k = 1, ..., N, where $\mathcal{K}_0 := \emptyset$ and $\Sigma(\mathcal{K})$ denotes the Borel σ -algebra over \mathcal{K} . Definition in (4.4) implies that $\mu_k \in \mathcal{M}(\mathcal{K}_k)$ for all k = 1, ..., N, and $\sum_{k=1}^N \mu_k(S) = \mu(S)$ for all $S \in \Sigma(\mathcal{K})$. Hence, $\{\mu_k\}_{k=1}^N$ and μ_x form a feasible solution to (3.2) with objective value equal to P. Therefore, $\mathbf{P}_{\mu_{\mathbf{q}}}^* = \mathbf{P}^*$. \square

4.2. Semidefinite Relaxations. In this section, a sequence of semidefinite programs is provided which can arbitrarily approximate the optimal solution of (4.2). As before, this is done by considering moments of measures instead of the measures themselves. Define the following optimization problem indexed by the relaxation order d.

$$\mathbf{P}_{\mathbf{d}} := \sup_{\mathbf{y}_k \in \mathbb{R}^{S_{n+m,2d}}, \ \mathbf{y}_{\mathbf{x}} \in \mathbb{R}^{S_{n,2d}}} \sum_{k=1}^{N} \left(\mathbf{y}_k \right)_{\mathbf{0}},\tag{4.5}$$

s.t.
$$M_d(\mathbf{y}_k) \succeq 0, \ M_{d-r_j^{(k)}}\left(\mathbf{y}_k; \mathbf{p}_j^{(k)}\right) \succeq 0, \quad j = 1, \dots, l_k, \quad k = 1, \dots, N$$
 (4.5a)

$$M_d(\mathbf{y}_{\mathbf{x}}) \succeq 0, \ \|\mathbf{y}_{\mathbf{x}}\|_{\infty} \le 1, \ (\mathbf{y}_{\mathbf{x}})_{\mathbf{0}} = 1,$$

$$(4.5b)$$

$$M_d\left(A_d\mathbf{y}_{\mathbf{x}} - \sum_{k=1}^{N} \mathbf{y}_k\right) \succeq 0,\tag{4.5c}$$

where $\delta_j^{(k)}$ is the degree of $\mathcal{P}_j^{(k)}, r_j^{(k)} := \left\lceil \frac{\delta_j^{(k)}}{2} \right\rceil$ for all $1 \le j \le \ell_k$ and $1 \le k \le N$; and $A_d : \mathbb{R}^{S_{n,2d}} \to \mathbb{R}^{S_{n+m,2d}}$ is defined similarly to A in (3.6). Indeed, let $\mathbf{y}_{\mathbf{q}} := \{y_{q_{\beta}}\}_{\beta \in \mathbb{N}_{2d}^{m}}$ be the truncated moment sequence of μ_{q} .

Then for any given $\mathbf{y}_{\mathbf{x}} = \{y_{x_{\alpha}}\}_{\alpha \in \mathbb{N}_{2d}^{n}}, \mathbf{y} = A_{d}\mathbf{y}_{\mathbf{x}}$ such that $y_{\theta} = y_{q_{\beta}}y_{x_{\alpha}}$ for all $\theta = (\beta, \alpha) \in \mathbb{N}_{2d}^{n+m}$. Next, we show that the sequence of optimal solutions to the SDPs in (4.5) converges to the solution of the infinite dimensional SDP in (4.2). More precisely, we have the following result.

THEOREM 4.2. For all $d \ge 1$, there exists an optimal solution $(\{\mathbf{y}_k^d\}_{k=1}^N, \mathbf{y}_{\mathbf{x}}^d)$ to (4.5) with the optimal value \mathbf{P}_d . Moreover,

i) $\lim_{d \in \mathbb{Z}_+} \mathbf{P}_d = \mathbf{P}^*$, the optimal value of (1.1). ii) Let $S := \left\{ \left(\{\mathbf{y}_k^d\}_{k=1}^N, \mathbf{y}_{\mathbf{x}}^d \} \right\}_{d \in \mathbb{Z}_+}$ such that each element is obtained by zero-padding \mathbf{y}^d and \mathbf{y}_k^d for $1 \le k \le 1$. N. There exists an accumulation point of S in the weak- \star topology of ℓ_{∞} , and for every accumulation point of S, there exists corresponding representing measures $(\{\mu_k^*\}_{k=1}^N, \mu_x^*)$ that is optimal to (4.2) and any $x^* \in supp(\mu_x^*)$ is optimal to (1.1). Proof. Let $\{\mathbf{y}_k\}_{k=1}^N \subset \mathbb{R}^{S_{n+m,2d}}$ and $\mathbf{y}_{\mathbf{x}} \in \mathbb{R}^{S_{n,2d}}$ be a feasible solution to (4.5). As in Theorem 3.3, it

can be shown that

$$\max\left\{\left(\mathbf{y}\right)_{\mathbf{0}}, \max_{i=1,\dots,n+m} L_{\mathbf{y}}\left(x_{i}^{2d}\right)\right\} \leq 1,$$

$$(4.6)$$

where $\mathbf{y} := \sum_{k=1}^{N} \mathbf{y}_k$. Note that $L_{\mathbf{y}}\left(x_i^{2d}\right) = \sum_{k=1}^{N} L_{\mathbf{y}_k}\left(x_i^{2d}\right)$, and $\{L_{\mathbf{y}_k}\left(x_i^{2d}\right)\}_{i=1}^{n+m}$ is a subset of diagonal elements of $M_d(\mathbf{y}_k) \succeq \mathbf{0}$ for each $k \in \{1, \ldots, N\}$. Hence, $L_{\mathbf{y}_k}\left(x_i^{2d}\right) \ge 0$ for all $i \in \{1, \ldots, n+m\}$ and $k \in \{1, \ldots, N\}$. $\{1,\ldots,N\}.$ Therefore, (4.6) implies that $\max\left\{(\mathbf{y}_k)_{\mathbf{0}}, \max_{i=1,\ldots,n+m} L_{\mathbf{y}_k}\left(x_i^{2d}\right)\right\} \leq 1 \text{ for all } k \in \{1,\ldots,N\}.$ Lemma 2.3 implies that $|(y_k)_{\alpha}| \leq 1$ for all $\alpha \in \mathbb{N}_{2d}^{n+m}$. Therefore, the feasible region is bounded. The rest of the proof is exactly the same as in Theorem 3.3. \square

5. Implementation and Numerical Results. In previous sections, we showed that chance optimization problem in (1.1) can be relaxed to a sequence of SDPs. In this section, we go one step further to improve approximation quality of the relaxed problems in practice and implement an efficient first-order algorithm to solve the resulting SDP relaxations.

5.1. Regularized Chance Optimization Using Trace Norm. As shown in Theorem 3.1 and Theorem 4.1, if the chance optimization problems in (1.2) and (1.1) have unique optimal solution x^* , then the optimal distribution μ_x^* is a Dirac measure whose mass is concentrated on the single point x^* , i.e., its support is the singleton $\{x^*\}$. Such distributions, have moment matrices with rank one. To improve the solution quality of the algorithm, one can incorporate this observation in the formulation of the relaxed problem. For the sake of notational simplicity, in this section we will consider the regularized version of chance optimization problem (3.7) for presenting the algorithm:

$$\min_{\mathbf{y}\in\mathbb{R}^{S_{n+m,2d}}, \mathbf{y}_{\mathbf{x}}\in\mathbb{R}^{S_{n,2d}}} \omega_r \operatorname{Tr}(M_d(\mathbf{y}_{\mathbf{x}})) - (\mathbf{y})_{\mathbf{0}} \text{ subject to } (3.7a), (3.7b), (3.7c)$$
(5.1)

for some $\omega_r > 0$, where $\mathbf{Tr}(.)$ denotes the trace function. Our objective is to achieve the maximum probability with a low-rank moment matrix $M_d(\mathbf{y}^*_{\mathbf{x}})$, hopefully with rank 1. To this end, we regularize the objective with trace norm. Since $M_d(\mathbf{y}^*_{\mathbf{x}}) \geq 0$, $\mathbf{Tr}(M_d(\mathbf{y}^*_{\mathbf{x}}))$ is equal to sum of singular values of $M_d(\mathbf{y}^*_{\mathbf{x}})$, which is called the nuclear norm of $M_d(\mathbf{y}^*_{\mathbf{x}})$. This is a well known approach for obtaining low-rank solutions. Indeed, the nuclear norm is the convex envelope of the rank function and, in practice, produces good results; see [20] and [51] for details.

To be able to solve the SDP in (5.1) involving large scale matrices in practice, one need to implement an efficient convex optimization algorithm. Recently, a first-order augmented Lagrangian algorithm ALCC has been proposed in [2] to deal with regularized conic convex problems. We will adapt this algorithm to solve SDPs of the form in (5.1). In the following section, we briefly discuss the algorithm ALCC.

5.2. First-Order Augmented Lagrangian Algorithm. Consider the optimization problem:

$$(P): p^* = \min\{\rho(x) + \gamma(x): A(x) - b \in \mathcal{C}\},$$
(5.2)

where $\gamma : \mathbb{R}^n \to \mathbb{R}$ is a convex function such that $\nabla \gamma$ is Lipschitz continuous with constant $L_{\gamma}, \rho : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a closed convex function such that $\Delta := \operatorname{dom}(\rho)$ is convex compact set, $A : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear map*, and $\mathcal{C} \subset \mathbb{R}^m$ is a closed convex cone. Let $\mathcal{C}^* := \{\theta \in \mathbb{R}^n : \langle z, \theta \rangle \ge 0, \forall z \in C\}$ denote the dual cone of \mathcal{C} , and B > 0 denote the diameter of Δ , i.e., $B = \max\{\|x-y\|_2 : x, y \in \Delta\}$; and we assume that B is given. Given a penalty parameter $\nu > 0$ and Lagrangian dual multiplier $\theta \in \mathcal{C}^*$, the augmented Lagrangian for (P) in (5.2) is given by

$$\mathcal{L}(x;\nu,\theta) := \frac{1}{\nu} \left(\rho(x) + \gamma(x)\right) + \frac{1}{2} d_{\mathcal{C}}(A(x) - b - \theta)^2, \tag{5.3}$$

where $d_{\mathcal{C}}: \mathbb{R}^m \to \mathbb{R}$ denotes the distance function to cone \mathcal{C} , i.e., $d_{\mathcal{C}}(\bar{z}) := \|\bar{z} - \Pi_{\mathcal{C}}(\bar{z})\|_2$, and $\Pi_{\mathcal{C}}(\bar{z}) := argmin\{\|z - \bar{z}\|_2 : z \in \mathcal{C}\}$ denotes the Euclidean projection of \bar{z} onto \mathcal{C} . Given $\nu_k > 0$ and $\theta_k \in \mathcal{C}^*$, we define $\mathcal{L}_k(x) := \mathcal{L}(x;\nu_k,\theta_k)$ and $\mathcal{L}_k^* := \min_x \mathcal{L}_k(x)$. Let $f_k : \mathbb{R}^n \to \mathbb{R}$ such that $f_k(x) := \frac{1}{\nu_k}\gamma(x) + \frac{1}{2}d_{\mathcal{C}}(A(x) - b - \theta)^2$; hence, $\mathcal{L}_k^* = \min_x \frac{1}{\nu_k}\rho(x) + f_k(x)$. It is important to note that f_k is a convex function with Lipschitz continuous gradient $\nabla f_k(x) = \frac{1}{\nu_k}\gamma(x) - A^*(\Pi_{\mathcal{C}^*}(\theta_k + b - A(x)))$; and the Lipschitz constant of ∇f_k is equal to $L_k := \frac{1}{\nu_k}L_{\gamma} + \sigma_{\max}^2(A)$, where $A^* : \mathbb{R}^m \to \mathbb{R}^n$ denotes the adjoint operator of $A : \mathbb{R}^n \to \mathbb{R}^m$, and $\sigma_{\max}(A)$ denotes the maximum singular value of the linear map A. Therefore, given $\epsilon_k > 0$, an ϵ_k -optimal solution, \tilde{x}_k , to $\mathcal{L}_k^* := \min_x \mathcal{L}_k(x)$ can be efficiently computed such that $\mathcal{L}_k(\tilde{x}_k) - \mathcal{L}_k^* \leq \epsilon_k$ using an Accelerated Proximal Gradient (APG) algorithm [6, 40, 41, 56] within $\ell_k^{\max}(\epsilon_k) := B \sqrt{\frac{2L_k}{\epsilon_k}}$ APG iterations. In each APG iteration, ∇f_k , $\Pi_{\mathcal{C}^*}$ and proximal map of ρ are all evaluated *once*.

ALCC algorithm proposed in [2] can generate a minimizing sequence $\{x_k\}$ to (P) in (5.2) by *inexactly* solving a sequence of subproblems $\min_x \mathcal{L}_k(x)$. In particular, given inexact computation parameters $\alpha_k > 0$ and $\eta_k > 0$, x_k is computed such that either one of the following conditions holds:

$$\mathcal{L}_k(x_k) - \mathcal{L}_k^* \le \frac{\alpha_k}{\nu_k},\tag{5.4}$$

$$\exists s_k \in \partial \mathcal{L}_k(x_k) \quad \text{such that} \quad \|s_k\|_2 \le \frac{\eta_k}{\nu_k},$$
(5.5)

where $\partial \mathcal{L}_k(x_k)$ denotes the subdifferential of \mathcal{L}_k at x_k – the inexact optimality criteria in (5.4) and (5.5) have been successfully implemented in other first-order augmented Lagrangian algorithms in [3, 4, 5] as well. Then dual Lagrangian multiplier is updated: $\theta_{k+1} = \frac{\nu_k}{\nu_{k+1}} \prod_{C^*} (\theta_k + b - A(x_k))$. For given $c, \beta > 1$, fix the parameter sequence as follows: $\nu_k = \beta^k \nu_0$, $\alpha_k = \frac{1}{k^{2(1+\epsilon)}\beta^k} \alpha_0$, and $\eta_k = \frac{1}{k^{2(1+\epsilon)}\beta^k} \eta_0$ for all $k \ge 1$; and let $\{x_k, \theta_k\} \subset \Delta \times C^*$ be the primal-dual ALCC iterate sequence. **Theorem 3.10** in [2] shows that $\lim_k \theta_k \nu_k$ exists and it is an optimal solution to the dual problem. Moreover, **Theorem 3.8** shows that for all $\epsilon > 0$, x_k is ϵ -feasible, i.e., $d_{\mathcal{C}}(Ax_k - b) \le \epsilon$, and ϵ -optimal, i.e., $|\rho(x_k) + \gamma(x_k) - p^*| \le \epsilon$ within $\log(1/\epsilon)$ ALCC iterations, i.e., $k = \mathcal{O}(\log(1/\epsilon))$, which requires $\mathcal{O}(\epsilon^{-1}\log(\epsilon^{-1}))$ APG iterations in total. Moreover, every limit point of $\{x_k\}$ is optimal (when $A \in \mathbb{R}^{m \times n}$ is surjective, the techniques used for proving **Theorem 4** in [3] can be used to improve the rate result to $\mathcal{O}(1/\epsilon)$).

Now consider the following problem $p^* = \min_{x \in \Delta} \{\gamma(x) : A(x) - b \in \mathcal{C}\}$, where $\Delta \subset \mathbb{R}^n$ is a compact convex set. Note that this problem can be written as a special case of (5.2) by setting $\rho(x) = \mathbf{1}_{\Delta}(x)$, the

indicator function of the set Δ , i.e., $\mathbf{1}_{\Delta}(x) = 0$, if $x \in \Delta$, and equal to $+\infty$, if $x \notin \Delta$. In Figure 5.1, we present the ALCC algorithm customized to solve $p^* = \min_{x \in \Delta} \{\gamma(x) : A(x) - b \in \mathcal{C}\}$. Note that Step 11 and Step 12 in Figure 5.1 are the bottleneck steps (one $\nabla \gamma$ evaluation and two projections: one onto \mathcal{C}^* , and one onto Δ) – in Step 11 ∇f_k is evaluated at $x_{\ell}^{(2)}$, and then in Step 12 $x_{\ell}^{(1)}$ is computed via a projected gradient step of length $1/L_k$. In this customized version, ALCC iterate x_k is set to $x_{\ell}^{(1)}$ whenever either $\ell > \ell_k^{\max}$ or $\|x_{\ell}^{(1)} - x_{\ell}^{(2)}\|_2 \leq \frac{\eta_k}{\nu_k}$. Note that $\ell_k^{\max} := k^{1+c}\beta^k B \sqrt{\frac{2\nu_0 L_k}{\alpha_0}}$, which is equal to $\ell_k^{\max}(\epsilon_k)$ when $\epsilon_k = \frac{\alpha_k}{\nu_k}$. Therefore, if $\ell > \ell_k^{\max}$, then $\mathcal{L}_k(x_k) - \mathcal{L}_k^* \leq \frac{\alpha_k}{\nu_k}$ – this follows from the complexity of Accelerated Proximal Gradient algorithm (lines 9-19 in Figure 5.1) running on $\min \mathcal{L}_k(x)$; next we'll show that if $\|x_{\ell}^{(1)} - x_{\ell}^{(2)}\|_2 \leq \frac{1}{2L_k} \frac{\eta_k}{\nu_k}$, then (5.5) holds. For $\rho(x) = \mathbf{1}_{\Delta}(x)$, we have $\mathcal{L}_k(x) = \rho(x) + f_k(x)$. Suppose that for some ℓ , $\|x_{\ell}^{(1)} - x_{\ell}^{(2)}\|_2 \leq \frac{1}{2L_k} \frac{\eta_k}{\nu_k}$, then $1 \Delta(x_{\ell}^{(2)} - \nabla f_k(x_{\ell}^{(2)})/L_k)$, where $L_k := \frac{1}{\nu_k} L_\gamma + \sigma_{\max}^2(A)$ is the Lipschitz constant of ∇f_k . One can easily show that $x_{\ell}^{(2)} - \nabla f_k(x_{\ell}^{(2)})/L_k - x_{\ell}^{(1)} \in \partial \rho(x_{\ell}^{(1)})$; and since ρ is the indicator function, we also have $L_k\left(x_{\ell}^{(2)} - x_{\ell}^{(1)}\right) - \nabla f_k(x_{\ell}^{(2)}) \in \partial \rho(x_{\ell}^{(1)})$. Hence, $s_k := L_k\left(x_{\ell}^{(2)} - x_{\ell}^{(1)}\right) + \nabla f_k(x_{\ell}^{(1)}) - \nabla f_k(x_{\ell}^{(2)}) \in \partial P_k(x_{\ell}^{(1)})$. Since ∇f_k is Lipschitz continuous, we have $\|\nabla f_k(x_{\ell}^{(1)}) - \nabla f_k(x_{\ell}^{(2)})\|_2 \leq L_k \|x_{\ell}^{(2)} - x_{\ell}^{(1)}\|_2$. Therefore, we have $\|s_k\|_2 \leq 2L_k \|x_{\ell}^{(2)} - x_{\ell}^{(1)}\|_2 \leq \frac{\eta_k}{\nu_k}$.

Algorithm ALCC $(x_0, \nu_0, \alpha_0, L_{\gamma}, B)$

1: $k \leftarrow 1, \theta_1 \leftarrow \mathbf{0}$ 2: $\eta_0 \leftarrow 0.5 \|\nabla \gamma(x_0) - \nu_0 A^* (\Pi_{C^*} (b - A(x_0)))\|_2$ 3: while $k \ge 1$ do
$$\begin{split} & \stackrel{-}{\ell} \leftarrow 0, t_1 \leftarrow 1, \\ & x_0^{(1)} \leftarrow x_{k-1}, x_1^{(2)} \leftarrow x_{k-1} \\ & L_k \leftarrow \frac{1}{\nu_k} L_\gamma + \sigma_{\max}^2(A), \ \ell_k^{\max} \leftarrow k^{1+c} \beta^k B \sqrt{\frac{2\nu_0 L_k}{\alpha_0}} \end{split}$$
4: 5:6: $\nu_k \leftarrow \beta^k \nu_0, \, \alpha_k \leftarrow \frac{1}{k^{2(1+c)} \beta^k} \alpha_0, \, \eta_k \leftarrow \frac{1}{k^{2(1+c)} \beta^k} \eta_0$ 7: $\mathrm{STOP} \gets \mathbf{false}$ 8: 9: while STOP = false do10: $\ell \leftarrow \ell + 1$ $g_{\ell} \leftarrow \frac{1}{\nu_{k}} \nabla \gamma \left(x_{\ell}^{(2)} \right) - A^{*} \left(\Pi_{C^{*}} \left(\theta_{k} + b - A \left(x_{\ell}^{(2)} \right) \right) \right)$ 11: $x_{\ell}^{(1)} \leftarrow \Pi_{\Delta} \left(x_{\ell}^{(2)} - g_{\ell} / L_k \right)$ 12: $\begin{array}{l} \text{if } \|x_{\ell}^{(1)} - x_{\ell}^{(2)}\|_{2} \leq \frac{1}{2L_{k}} \frac{\eta'_{k}}{\nu_{k}} \text{ or } \ell > \ell_{k}^{\max} \text{ then} \\ \text{STOP} \leftarrow \textbf{true} \\ x_{k} \leftarrow x_{\ell}^{(1)} \\ \text{end if} \end{array}$ 13:14: 15:16:end If $t_{\ell+1} \leftarrow \left(1 + \sqrt{1 + 4 t_{\ell}^2}\right)/2$ 17: $x_{\ell+1}^{(2)} \leftarrow x_{\ell}^{(1)} + \left(\frac{t_{\ell}-1}{t_{\ell+1}}\right) \left(x_{\ell}^{(1)} - x_{\ell-1}^{(1)}\right)$ 18:end while 19: $\theta_{k+1} \leftarrow \frac{\nu_k}{\nu_{k+1}} \prod_{C^*} (\theta_k + b - A(x_k))$ 20:21: end while

Semidefinite program of (5.1) is a special case of the conic convex problem in (5.2), where $\gamma(\mathbf{y}_{\mathbf{x}}, \mathbf{y}) = c_r^T \mathbf{y}_{\mathbf{x}} + c_p^T \mathbf{y}$ for some $c_r \in \mathbb{R}^{S_{n,2d}}$ and $c_p \in \mathbb{R}^{S_{n+m,2d}}$ since the objective of (5.1) is linear in $(\mathbf{y}, \mathbf{y}_{\mathbf{x}})$; hence, $L_{\gamma} = 0$, the conic constraint $A(.) - b \in C$ in (5.2) is a linear matrix inequality (LMI), with $\mathcal{C} = \mathcal{C}^*$ being the cone of positive semidefinite matrices \mathbb{S}_+ , and the compact set $\Delta = \{(\mathbf{y}, \mathbf{y}_{\mathbf{x}}) : \|\mathbf{y}\|_{\infty} \leq 1, \|\mathbf{y}_{\mathbf{x}}\|_{\infty} \leq 1, \|\mathbf{y}_{\mathbf{x}}\|_{\infty} \leq 1, \|\mathbf{y}_{\mathbf{x}}\|_{\infty} = 1\}$. Hence, $\Pi_{\mathcal{C}}(.) = \Pi_{\mathcal{C}^*}(.)$ can be computed using one eigenvalue decomposition, and $\Pi_{\Delta}(.)$ is very efficient and can be computed in linear time. In our numerical experiments in Section 5.3, we used $\|x_k - x_{k-1}\|_2/(1 + \|x_{k-1}\|_2) \leq \text{tol as the stopping condition for ALCC.}$

Fig. 5.1: first-order Augmented Lagrangian algorithm for Conic Convex (ALCC) problems

5.3. Numerical Examples. In this section, four numerical examples are presented that illustrate the performance of the proposed methodology, discussed in Sections 3 and 4. We compared the augmented Lagrangian algorithm, ALCC, presented in Section 5.2 with GloptiPoly, which is a Matlab-based toolbox aimed at optimizing moments of measures [23], to compute approximate solutions to the chance constrained problems in (1.1) and (1.2). In all the tables, for problems of the form (1.2), i.e., N = 1, \mathbf{P}_d , \mathbf{P}_d , $\mathbf{\bar{P}}_d$, and $\mathbf{\bar{P}}_d$ denote the optimal probability estimates defined similarly as in Section 3.3 for x^d obtained by solving the regularized problem in (5.1); for problems of the form (1.1), i.e., N > 1, these estimates can be defined naturally using $(\mathbf{y}^d, \mathbf{y}^d_{\mathbf{x}})$ with $\mathbf{y}^d := \sum_{k=1}^N \mathbf{y}^d_k$; and $d \in \mathbb{Z}_+$ denotes the relaxation order. In order to compute \mathbf{P}^* and $\mathbf{\bar{P}}_d$, we used Monte Carlo simulation discussed in Section 5.3.1. In all the tables, iter denotes the total number of algorithm iterations, and \mathbf{cpu} denotes the computing time in *seconds* required for computing \mathbf{P}_d ; \mathbf{n}_{var} denotes the number of variables, i.e., total number of moments used. For ALCC iter is the total number of APG iterations, and for GloptiPoly it denotes the total number of SeDuMi [53] iterations.

5.3.1. Monte Carlo Simulation. To test the accuracy of the results obtained using ALCC and GloptiPoly, we used Monte Carlo integration to estimate an optimal solution and the corresponding optimal probability. Let $\mathcal{K} \subset \mathbb{R}^n \times \mathbb{R}^m$ be the given semialgebraic set such that $\Pi_1 := \{x \in \mathbb{R}^n : \exists q \in \mathbb{R}^m \text{ s.t. } (x,q) \in \mathcal{K}\} \subset \chi := [-1,1]^n$, and $\Pi_2 := \{q \in \mathbb{R}^m : \exists q \in \mathbb{R}^m \text{ s.t. } (x,q) \in \mathcal{K}\} \subset \mathcal{Q} := [-1,1]^m$. Define $\mathcal{F} : \chi \to \Sigma_q$,

$$\mathcal{F}(x) := \{ q \in \mathcal{Q} : (x, q) \in \mathcal{K} \}.$$
(5.6)

First, we uniformly grid χ into \overline{N} grid-points (\overline{N} depending on the desired precision). Let $\{x^{(i)}\}_{i=1}^{\overline{N}} \subset \chi$ denote the points in the uniform grid. Next, for each grid point $x^{(i)}$, we sample from the distribution induced by the given finite Borel measure μ_q supported on \mathcal{Q} . Let $\{q^{(i,k)}\}_{k=1}^{N_i}$ be N_i i.i.d. sample of random parameter q. Then we approximate $\mu_q(\mathcal{F}(x^{(i)}))$ by

$$P_{N_i}^{(i)} := \frac{1}{N_i} \sum_{k=1}^{N_i} \mathbf{1}_{\mathcal{K}} \left(x^{(i)}, q^{(i,k)} \right), \quad \text{where} \quad \mathbf{1}_{\mathcal{K}} \left(x, q \right) = \begin{cases} 1, & \text{if } (x,q) \in \mathcal{K}; \\ 0, & \text{otherwise.} \end{cases}$$

Because of law of large numbers, $\lim_{N_i \nearrow \infty} P_{N_i}^{(i)} = \mu_q(\mathcal{F}(x^{(i)}))$. For each $x^{(i)}$, we chose sample size N_i such that $P_{N_i}^{(i)}$ becomes stagnant to further increase in N_i . Finally, we approximate x^* by $x^{(i^*)}$, where $i^* \in \operatorname{argmax}\{P_{N_i}^{(i)}: 1 \le i \le \overline{N}\}$. It is clear that what we used is a *naive* method, and it can be made much more efficient by using an adaptive gridding scheme on χ . On the other hand, as the dimensions n and m are very small for the problems discussed in the numerical section, this naive method served its purpose.

5.3.2. Example 1: A Simple Semialgebraic Set. Consider the chance optimization problem

$$\sup_{x \in \mathbb{R}^5} \mu_q \left(\left\{ q \in \mathbb{R}^5 : \mathcal{P}(x, q) \ge 0 \right\} \right), \tag{5.7}$$

where

$$\mathcal{P}(x,q) = 0.185 + 0.5x_1 - 0.5x_2 + x_3 - x_4 + 0.5q_1 - 0.5q_2 + q_3 - q_4 - x_1^2 - 2x_1q_1 - x_2^2 - 2x_2q_2 - x_3^2 - 2x_3q_3 - x_4^2 - 2x_4q_4 - x_5^2 + 2x_5q_5 - q_1^2 - q_2^2 - q_3^2 - q_4^2 - q_5^2,$$

and the uncertain parameters q_1, q_2, q_3, q_4, q_5 have a uniform distribution: $q_1 \sim U[-1, 0], q_2 \sim U[0, 1], q_3 \sim U[-0.5, 1], q_4 \sim U[-1, 0.5], q_5 \sim U[0, 1] - U[a, b]$ denotes the uniform distribution between a and b. The k-th moment of uniform distribution U[a,b] is $(\mathbf{y_q})_k = \frac{b^{k+1} - a^{k+1}}{(b-a)(k+1)}$. The optimum solution and corresponding optimal probability are obtained by Monte Carlo method: $x_1^* = 0.75, x_2^* = -0.75, x_3^* = 0.25, x_4^* = -0.25, x_5^* = 0.5$, and $P^* = 0.75$. To obtain an approximate solution, we solve the SDP in (3.7) using GloptiPoly and ALCC. For ALCC, we set ν_0 to $1, 5 \times 10^{-2}$ and 5×10^{-3} when d is equal to 1, 2, and 3, respectively, and to $l = 1 \times 10^{-2}$. The results for relaxation order d = 1, 2, 3 are shown in Table 5.1. As in Figure 3.2, when compared to \mathbf{P}_d , $\tilde{\mathbf{P}}_d$ approximates \mathbf{P}^* better, i.e., when max{ $\int \mathcal{P}(x^d, q) d\tilde{\mu} : \tilde{\mu} \leq \mu_q, \tilde{\mu} \in \mathcal{M}(\mathcal{F}(x^d))$ } is solved instead of max{ $\int d\mu' : \mu' \leq \mu_q, \mu' \in \mathcal{M}(\mathcal{F}(x^d))$ }. We reported results up to order d = 3, because for larger d, GloptiPoly did not terminate in 24 hours.

ALCC				GloptiPoly				
d	1	2	3	d	1	2	3	
$\mathbf{n}_{\mathrm{var}}$	87	1127	8463	$\mathbf{n}_{\mathrm{var}}$	87	1127	8463	
iter	169	624	1207	iter	18	25	41	
cpu	0.9	28.1	785.9	cpu	0.5	12.3	15324.3	
$\mathbf{x_1}$	0.742	0.745	0.757	$\mathbf{x_1}$	0.467	0.710	0.742	
$\mathbf{x_2}$	-0.777	-0.701	-0.721	$\mathbf{x_2}$	-0.467	-0.710	-0.742	
\mathbf{x}_3	0.213	0.226	0.216	x ₃	0.163	0.245	0.249	
x 4	-0.239	-0.250	0.236	X 4	-0.163	-0.245	-0.249	
\mathbf{x}_{5}	0.500	0.551	0.557	\mathbf{x}_{5}	0.319	0.475	0.495	
\mathbf{P}_d	0.991	0.971	0.961	\mathbf{P}_d	1	1	1	
\mathbf{P}_d'	1	1	1	\mathbf{P}_d'	1	1	1	
$ ilde{\mathbf{P}}_d$	0.996	0.7739	0.6919	$ ilde{\mathbf{P}}_d$	0.9652	0.7768	0.7031	
$ar{\mathbf{P}}_d$	0.7504	0.7459	0.7459	$ar{\mathbf{P}}_d$	0.5067	0.7484	0.7535	

Table 5.1: ALCC and GloptiPoly results for Example 1

5.3.3. Example 2: Union of Simple Sets. Given the following polynomials

$$\mathcal{P}^{(1)}(x,q) = -0.263 + 0.4x_1 - 0.4x_2 + 0.8x_3 - 0.8x_4 + 1.2x_5 + 0.1q_1 + 0.08q_2 + 0.04q_3 + 0.4q_4 + 0.6q_5 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 - 0.5q_1^2 - 0.4q_2^2 - 0.1q_3^2 - q_4^2 - q_5^2,$$
$$\mathcal{P}^{(2)}(x,q) = -2.06 + 0.4x_1 - 0.8x_2 + 3.2x_3 - 1.6x_4 + 3.6x_5 - 0.4q_1 - 0.4q_2 - 0.2q_3 - 0.2q_4 - 0.8q_5 - x_1^2 - 2x_2^2 - 4x_3^2 - 2x_4^2 - 3x_5^2 - q_1^2 - q_2^2 - q_3^2 - q_4^2 - q_5^2,$$

consider the chance optimization problem

$$\sup_{\mathbf{x}\in\mathbb{R}^5}\mu_q\left(\bigcup_{j=1,2}\left\{q\in\mathbb{R}^5:\ \mathcal{P}^{(j)}(x,q)\ge 0\right\}\right),\tag{5.8}$$

where $q_i \sim U[-0.5, 0.5]$ for all i = 1, ..., 5, i.e., the uncertain parameters q_i are uniformly distributed on [-0.5, 0.5]. The optimum solution and corresponding optimal probability are obtained by Monte Carlo method: $x_1^* = 0.2$, $x_2^* = -0.2$, $x_3^* = 0.4$, $x_4^* = -0.4$, $x_5^* = 0.6$, and $\mathbf{P}^* = 0.80$. To obtain an approximate solution, we solve the SDP in (4.5) using ALCC, where we set ν_0 to 1, 1×10^{-1} and 1×10^{-3} when d is equal to 1, 2, and 3, respectively, and tol $= 1 \times 10^{-2}$. The results for relaxation order d = 1, 2, 3 are shown in Table 5.2. Let $\mathcal{F}^{(k)}(x) =: \{q \in \mathcal{Q} : \mathcal{P}^{(k)}(x,q) \ge 0\}$ for k = 1, 2. The probability estimates $\tilde{\mathbf{P}}_d$ reported in Table 5.2 are computed by solving the SDP relaxation for

$$\max\left\{\int \mathcal{P}^{(1)}(x^d, q) \ d\tilde{\mu}_1 + \int \mathcal{P}^{(2)}(x^d, q) \ d\tilde{\mu}_2 : \ \tilde{\mu}_1 + \tilde{\mu}_2 \preceq \mu_q, \ \tilde{\mu}_1 \in \mathcal{M}(\mathcal{F}^{(1)}(x^d)), \ \tilde{\mu}_2 \in \mathcal{M}(\mathcal{F}^{(2)}(x^d))\right\}.$$

For this example, GloptiPoly fails to extract the optimum solution.

5.3.4. Example 3: Portfolio Selection Problem. We aim at selecting a portfolio of financial assets to maximize the probability of achieving a return higher than a specified amount r^* . Suppose that for each asset i = 1, ..., N, its uncertain rate of return is a random variable $\xi_i(q)$; and let $(\mathcal{Q}, \Sigma_q, \mu_q)$ denote the underlying probability space. In this context x_i denotes the percentage of money invested in asset i. More precisely, we solve the following problem:

$$\sup_{x \in \mathbb{R}^N} \mu_q \left(\left\{ q \in \mathbb{R}^N : \sum_{i=1}^N \xi_i(q) x_i \ge r^* \right\} \right) \quad \text{s.t.} \quad \sum_{i=1}^N x_i \le 1, \quad x_i \ge 0 \quad \forall \ i \in \{1, \dots, N\}.$$
(5.9)

ALCC								
d	1	2	3					
$\mathbf{n}_{\mathrm{var}}$	153	2128	16478					
iter	979	1467	1875					
cpu	6.5	102.2	434.7					
\mathbf{x}_1	0.209	0.328	0.201					
$\mathbf{x_2}$	-0.202	-0.174	-0.201					
x ₃	0.397	0.466	0.430					
X 4	-0.400	-0.405	-0.401					
\mathbf{x}_{5}	0.667	0.638	0.591					
\mathbf{P}_d	1	0.997	0.981					
\mathbf{P}_d'	1	1	1					
$ ilde{\mathbf{P}}_d$	0.9973	0.8610	0.8926					
$ar{\mathbf{P}}_d$	0.8937	0.8745	0.8984					

Table 5.2: ALCC results for Example 2

In our example problem, $r^* = 1.5$, N = 4, $\xi_1(q) = 1 + q_1$, $\xi_2(q) = 1 + q_2$, $\xi_3(q) = 0.9 + q_3$, $\xi_4(q) = 0.9 + q_4$, where $\{q_i\}_{i=1}^4$ are independent, and $q_1 \sim \text{Beta}(3 - \sqrt{2}, 3 + \sqrt{2})$, $q_2 \sim \text{Beta}(4, 4)$, $q_3 \sim \text{Beta}(3 + \sqrt{2}, 3 - \sqrt{2})$, $q_4 \sim U[0.5, 1]$. The k-th moment of Beta distribution $\text{Beta}(\alpha, \beta)$ over [0,1] is $y_k = \frac{\alpha+k-1}{(\alpha+\beta+k-1)}y_{k-1}$ and $y_0 = 1$. We will solve an equivalent problem in the form of (1.2) with $\ell = 7$, where $\mathcal{P}_j(x, q) = x_j$ for $j = 1, \ldots, 4$, $\mathcal{P}_5(x, q) = 1 - \sum_{i=1}^4 x_i$, $\mathcal{P}_6(x, q) = 8 - \sum_{i=1}^4 x_i^2 - \sum_{i=1}^4 q_i^2$, and $\mathcal{P}_7(x, q) = \sum_{i=1}^4 \xi_i(q)x_i - r^*$. Since any $(x, q) \in \mathcal{K}$ satisfies $x \in \chi$ and $q \in \mathcal{Q}$, we added polynomial $\mathcal{P}_6(x, q)$ to assure that the resulting representation of the semialgebraic set \mathcal{K} satisfies Putinar's property. The optimum solution and the corresponding optimal probability are computed approximately by Monte Carlo method: $x_1^* = 0$, $x_2^* = 0$, $x_3^* = 0.3$, $x_4^* = 0.7$, and $P^* = 0.89$. To obtain an approximate solution, we solve the SDP relaxation in (3.7) using GloptiPoly and ALCC. For ALCC, we set ν_0 to 1×10^{-2} , 1×10^{-2} and 1×10^{-3} when d is equal to 1, 2, and 3, respectively, and tol = 1×10^{-3} . The results for relaxation order d = 1, 2, 3 are shown in Table 5.3. We reported results up to order d = 3, because for larger d, GloptiPoly did not terminate in 24 hours.

ALCC				GloptiPoly				
d	1	2	3	d	1	2	3	
$\mathbf{n}_{\mathrm{var}}$	60	565	3213	$\mathbf{n}_{\mathrm{var}}$	60	565	3213	
iter	573	388	2227	iter	15	20	48	
cpu	3.625	16.426	756.798	cpu	0.509	2.617	1025.045	
$\mathbf{x_1}$	0.004	0.009	0.002	$\mathbf{x_1}$	0.133	0.0462	0.003	
$\mathbf{x_2}$	0.012	0.009	0.006	x ₂	0.192	0.154	0.075	
\mathbf{x}_3	0.438	0.449	0.299	x ₃	0.295	0.297	0.210	
\mathbf{x}_4	0.5007	0.522	0.677	\mathbf{x}_4	0.325	0.493	0.710	
\mathbf{P}_d	0.996	0.994	0.980	\mathbf{P}_d	1	1	0.999	
\mathbf{P}_d'	1	1	0.9716	\mathbf{P}_d'	0.9071	0.9997	0.9896	
$ ilde{\mathbf{P}}_d$	0.7928	0.8177	0.8220	$ ilde{\mathbf{P}}_d$	0.3808	0.7753	0.8395	
$ar{\mathbf{P}}_d$	0.7405	0.8655	0.8422	$ar{\mathbf{P}}_d$	0.3865	0.8267	0.8675	

Table 5.3: ALCC and GloptiPoly results for Example 3

5.3.5. Example 4: Nonlinear Control Problem. In this example, we consider the controller design problem for the following uncertain nonlinear dynamical system. For a given control parameter vector

ALCC				GloptiPoly				
d	2	3	4	d	2	3	4	
$\mathbf{n}_{\mathrm{var}}$	365	1800	6600	$\mathbf{n}_{\mathrm{var}}$	365	1800	6600	
iter	416	4300	5325	iter	19	26	36	
cpu	14.934	897.708	5318.387	cpu	1.3	99.2	10389.8	
$\mathbf{K_1}$	0	-0.244	-0.683	$\mathbf{K_1}$	0	-0.492	-0.796	
$\mathbf{K_2}$	0	0.468	0.476	$\mathbf{K_2}$	0	0.439	0.487	
$\mathbf{K_3}$	0	-0.868	-0.868	\mathbf{K}_{3}	0	-0.823	-0.891	
\mathbf{P}_d	0.238	0.996	0.983	\mathbf{P}_d	1	1	1	
\mathbf{P}_d'	0.65	0.9	0.982	\mathbf{P}_d'	0.65	0.959	0.999	
$ar{\mathbf{P}}_d$	0.061	0.445	0.685	$\bar{\mathbf{P}}_d$	0.061	0.508	0.766	

Table 5.4: ALCC and GloptiPoly results for Example 4

 $K \in \mathbb{R}^3$, let the system $x(k)^T = [x_1(k), x_2(k), x_3(k)] \in \mathbb{R}^3$ satisfy

$$u(k) = K_1 x_1(k) + K_2 x_2(k) + K_3 x_3(k),$$

$$x_1(k+1) = \Delta x_2(k),$$

$$x_2(k+1) = x_1(k) x_3(k),$$

$$x_3(k+1) = 1.2 x_1(k) - 0.5 x_2(k) + x_3(k) + u(k),$$

(5.10)

for k = 0, 1, where $x_1(0) \sim U[-1, 1]$, $x_2(0) \sim U[-1, 1]$, $x_3(0) \sim U[-1, 1]$, $\Delta \sim U[-0.4, 0.4]$, i.e., initial state vector x(0), and model parameter Δ are uncertain and uniformly distributed. The objective is to lead the system using state feedback control u(k) to the cube centered at the origin with the edge length of 0.2 in at most 2 steps by properly choosing the control decision variables $\{K_i\}_{i=1}^3$ such that $-1 \leq K_i \leq 1$. The equivalent chance problem is stated in (5.11), where $\mathbf{e}^T = [1, 1, 1]$.

$$\sup_{K \in \mathbb{R}^3} \mu_q \left(\left\{ \left(x(0), \Delta \right) : -0.1 \mathbf{e} \le x(2) \le 0.1 \mathbf{e} \right\} \right),$$
s.t. $\{ x(k), u(k) \}_{k=0}^2$ satisfy (5.10),
 $-\mathbf{e} \le K \le \mathbf{e}.$
(5.11)

The following optimal solution and the corresponding optimal probability are computed by Monte Carlo method: $K_1^* = -1$, $K_2^* = 0.5$, $K_3^* = -0.9$, and $\mathbf{P}^* = 0.84$. To obtain an equivalent SDP formulation for the chance constrained problem in (5.11), x(2) is explicitly written in terms of control vector $K \in \mathbb{R}^3$ and uncertain parameters, x(0) and Δ , using the dynamic system given in (5.10):

$$\begin{aligned} x_1(2) &= \Delta x_1(0)x_3(0), \\ x_2(2) &= (1.2+K_1)\Delta x_1(0)x_2(0) + (K_2-0.5)\Delta x_2(0)^2 + (1+K_3)\Delta x_2(0)x_3(0), \\ x_3(2) &= (1+2K_3+K_3^2) x_3(0) + (K_2-0.5K_3-0.5+1.2\Delta+K_1\Delta+K_2K_3) x_2(0) \\ &+ (1.2+K_1+1.2K_3+K_1K_3) x_1(0) + (K_2-0.5) x_1(0)x_3(0). \end{aligned}$$

Based on the obtained polynomials, the minimum relaxation order for this problem is 2. To obtain an approximate solution, we solve the SDP in (3.7) using GloptiPoly and ALCC. For ALCC, we set ν_0 to 5×10^{-3} , 5×10^{-3} and 1×10^{-3} when d is equal to 2, 3 and 4, respectively, and tol = 1×10^{-3} . The results for relaxation order d = 2, 3, 4 are shown in Table 5.4.

5.3.6. Example 5: Run time. In this example, for fixed degree of the relaxation order d, we examined how the run times of ALCC algorithm scale as the problem size increases. For this purpose, we consider the following problem: Given $n \ge 1$, we set $\mathcal{P} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\mathcal{P}(x,q) = 0.81 - \sum_{i=1}^n (x_i - q_i)^2$; and solve

$$\sup_{x \in R^n} \mu_q \left(\{ q \in R^n : \mathcal{P}(x, q) \ge 0 \} \right).$$
(5.12)

The numerical results for increasing n and fixed relaxation order d = 1 are displayed in Table 5.5. For each n, ALCC recovered the optimal decision value: $x^* = 0$.

ALCC									
n	5	10	20	30	40	50	60	70	80
d	1	1	1	1	1	1	1	1	1
$\mathbf{n}_{\mathrm{var}}$	10	20	40	60	80	100	120	140	160
iter	82	140	97	182	201	175	191	186	208
cpu	0.3969	1.5349	3.5542	14.2899	27.7978	37.2624	60.4454	83.3669	122.7844

Table 5.5: ALCC for increasing problem in Example 5

6. Conclusion. In this paper, "chance optimization" problems are introduced, where one aims at maximizing the probability of a set defined by polynomial inequalities. These problems are, in general, nonconvex and computationally hard. A sequence of semidefinite relaxations is provided whose sequence of optimal values is shown to converge to the optimal value of the original problem. To solve the semidefinite programs of increasing size obtained by relaxing the original chance optimization problem, a first-order augmented Lagrangian algorithm is implemented which enables us to solve much larger size semidefinite programs that interior point methods can deal with. Numerical examples are provided that show that one can obtains reasonable approximations to the optimal solution and the corresponding optimal probability even for lower order relaxations.

REFERENCES

- [1] N. I. Akhiezer, I. M. Glazman, "Theory of Linear Operators in Hilbert Space", Courier Dover Publications, 1993.
- [2] N. S. Aybat, G. Iyengar, "An Augmented Lagrangian Method for Conic Convex Programming," preprint (2013), arXiv:1302.6322v1.
- [3] N. S. Aybat and G. Iyengar, "A Unified Approach for Minimizing Composite Norms," Mathematical Programming Journal, Series A, Vol. 44, pp. 181-226, 2014.
- [4] N. S. Aybat and G. Iyengar, "A First-Order Augmented Lagrangian Method for Compressed Sensing," SIAM Journal on Optimization, Vol. 22, pp. 429-459, 2012.
- [5] N. S. Aybat, G. Iyengar and Z. Wang, "An Asynchronous Distributed Proximal Method for Composite Convex Optimization," to appear in the Proceedings of the 32th International Conference on Machine Learning, Lille, Franse, 2015, preprint available at arXiv:1409.8547v1.
- [6] A. Beck, M. Teboulle, "A fast iterative shrinkage-thresholding algorithm for linear inverse problems," SIAM Journal on Imaging Sciences, V 2, pp. 183-202, 2009.
- [7] A. Ben-Tal, A. Nemirovski, "Robust solutions of uncertain linear programs", Operations Research Letters, Vol. 25, pp. 1-13, 1999.
- [8] A. Ben-Tal, E. L. Ghaoui, A. Nemirovski, "Robust Semidefinite Programming", Handbook on Semidefinite Programming, KluwerAcademis Publishers, 2000.
- [9] A. Ben-Tal, A. Nemirovski, C. Roos, "Robust solutions of uncertain quadratic and conic-quadratic problems", SIAM Journal on Optimization, Vol. 13, pp. 535-560, 2002.
- [10] A. Ben-Tal, A. Goryashko, E. Guslitzer, A. Nemirovski, "Adjustable robust solutions of uncertain linear programs", *Mathematical Programming*, Vol. 99, pp. 351-376, 2004.
- [11] A. Ben-Tal and A. Nemirovski, "Robust solutions of Linear Programming Problems Contaminated with Uncertain Data", Mathematical Programming, Vol. 88, pp. 411-424, 2000.
- [12] D. Bertsimas and M. Sim, "The price of robustness", *European Journal of Operations Research*, Vol. 52, pp. 35-53, 2004.
 [13] L. Blackmore, M. Ono, A. Bektassov, B. Williams, "A Probabilistic Particle-Control Approximation of Chance-Constrained
- Stochastic Predictive Control", IEEE Transactions on Robotics, Vol. 26, Iss. 3, 2010.
 [14] L. Blackmore, M. Ono, B. Williams, "Chance-Constrained Optimal Path Planning With Obstacles", IEEE Transactions on Robotics, Vol. 27, No. 6, pp. 1080-1094, 2011.
- [15] L. Blackmore, H. Li, B. Williams, "A probabilistic approach to optimal robust path planning with obstacles", American Control Conference, Minneapolis, MN,June 2006.
- [16] G. Calafiore and M. C. Campi, "Uncertain convex programs: Randomized solutions and confidence levels", Mathematical Programing, Vol. 102, Springer, pp. 25-46, 2004.
- [17] G. Calafiore and M. C. Campi, "The scenario approach to robust control design", IEEE Transactions on Automatic Control, Vol. 51, No. 5, pp. 742-753, 2006.
- [18] F. Dabbene, C. Feng, and C. M. Lagoa, "Robust and Chance-Constrained Optimization under Polynomial Uncertainty," Proceedings of the 2009 American Control Conference, St. Louis, Missouri, 2009.
- [19] N. E. Du Toit, J. W. Burdick, "Probabilistic Collision Checking With Chance Constraints", IEEE Transactions on Robotics, Vol. 27, Iss. 4, 2011.
- [20] M. Fazel, H. Hindi, and S. Boyd, "Log-det Heuristic for Matrix Rank Minimization with Applications to Hankel and Euclidean Distance Matrices", Proceedings of American Control Conference, Denver, Colorado, June 2003.
- [21] C. Feng, C. M. Lagoa, "Distributional robustness Analysis for polynomial uncertainty", Proceedings of IEEE Conference

on Decision and Control, Shanghai, P.R. China, December 16-18, 2009.

- [22] C. Feng, F. Dabbene, and C. M. Lagoa, "A Kinship Function Approach to Robust and Probabilistic Optimization under Polynomial Uncertainty", *IEEE Transactions on Automatic Control*, Vol. 56, No. 7, 2011.
- [23] D. Henrion, J. B. Lasserre, J. Loefberg. "GloptiPoly 3: moments, optimization and semidefinite programming" Optimization Methods and Software, Vol. 24, Nos. 4-5, pp. 761-779, 2009.
- [24] D. Henrion, J. B. Lasserre, C. Savorgnan, "Approximate volume and integration for basic semialgebraic sets", SIAM Review, Vol. 51, No. 4, pp. 722-743, 2009.
- [25] A. M. Jasour, C. Lagoa, "Convex Relaxations of a Probabilistically Robust Control Design Problem", 52st IEEE Conference on Decision and Control, Florence, Italy, 2013.
- [26] A. M. Jasour, C. Lagoa, "Semidefinite relaxation of chance constrained algebraic problems", 51st IEEE Conference on Decision and Control, Maui, Hawaii, 2012.
- [27] C. M. Lagoa, F. Dabbene, and R. Tempo, "Hard Bound on Probability of Performance with Applications to Circuit Analysis," *IEEE Transactions on Circuits and Systems*, Vol. 55, No. 10, pp. 3178-3187, 2008.
- [28] C. M. Lagoa, X. Li, and M. Sznaier, "Probabilistically constrained linear programs and risk-adjusted controller design", SIAM J. Optim., Vol. 15, No. 3, pp. 938–951, 2005.
- [29] R. Laraki, J. B. Lasserre, "Semidefinite Programming for Min-Max Problems and Games", Mathematical Programming, Volume 131, Issue 1-2, pp 305-332, 2012.
- [30] J. B. Lasserre, "A semidefinite programming approach to the generalized problem of moments", Mathematical Programming, Vol 112, Issue 1, pp 65-92, 2008.
- [31] J. B. Lasserre, "Global optimization with polynomials and the problem of moments", SIAM J. Optim., Vol. 11, pp. 796–817, 2001.
- [32] J. B. Lasserre, "Moments Positive Polynomials and Their Applications", Imperial College Press, 2010.
- [33] M. Laurent, "Sums of Squares, Moment Matrices and optimization Over Polynomials", The IMA Volumes in Mathematics and its Applications, Vol 149, pp 157-270, 2009.
- [34] X. Li, Z. Qin, L. Yang, "A chance-constrained portfolio selection model with risk constraints", Applied Mathematics and Computation, vol 217, pp. 949-951, 2010.
- [35] L. B. Miller and H. Wagner, "Chance-constrained programming with joint constraints", European Journal of Operations Research, Vol. 13, pp. 930-945, 1965.
- [36] A. Nemirovski, "On safe tractable approximations of chance constraints", European Journal of Operations Research, Vol. 219, No. 3, pp. 707-718, 2012.
- [37] A. Nemirovski, A. Shapiro, "Convex Approximations of Chance Constrained Programs", SIAM Journal on Optimization, Vol. 17, No. 4, pp. 969-996, 2006.
- [38] A. Nemirovski, A. Shapiro, "Scenario Approximations of Chance Constraints", Probabilistic and Randomized Methods for Design under Uncertainty, Springer, pp. 3-48, 2004.
- [39] A. Nemirovski, "On tractable approximations of randomly perturbed convex constraints", Proceedings of the 42nd IEEE Conference on Decision and Control, Maui, HI, pp.2419–2422, 2003.
- [40] Y. Nesterov, "Smooth minimization of nonsmooth functions," Mathematical Programming, Series A, pp. 127-152, 2005.
- [41] Y. Nesterov, "Introductory Lectures on Convex Optimization: A Basic Course," Kluwer Academic Publishers, 2004.
- [42] M. Ono, L. Blackmore, and B. C. Williams, "Chance Constrained Finite Horizon Optimal Control with Nonconvex Constraints", American Control Conference, Canada, 2012.
- [43] J. Pinter, "Deterministic approximations of probability inequalities", European Journal of Operations Research, Vol. 33, pp. 219-239, 1989.
- [44] G. C. Calafiore and L. El Ghaoui. On distributionally robust chance-constrained linear programs. Journal of Optimization Theory and Applications, 130(1):1-22, 2006.
- [45] W. Chen, M. Sim, J. Sun, and C.-P. Teo. From CVaR to uncertainty set: Implications in joint chance-constrained optimization. Operations Research, 58(2):470-485, 2010.
- [46] S.-S. Cheung, A. Man-Cho So, and K. Wang. Linear matrix inequalities with stochastically dependent perturbations and applications to chance-constrained semidefinite optimization. SIAM Journal on Optimization, 22(4):1394-1430, 2012.
- [47] H. Xu, C. Caramanis, S. Mannor. Optimization under probabilistic envelope constraints. Operations Research, 60(3), 682-699, 2012.
- [48] S. Zymler, D. Kuhn, and B. Rustem. Distributionally robust joint chance constraints with second-order moment information. Mathematical Programming, 137(1-2):167-198, 2013.
- [49] A. Prekopa, "Stochastic Programming", Kluwer Academic Publishers, 1995.
- [50] M. Putinar, "Positive polynomials on compact semi-algebraic sets", Indiana Univ. Math. J. 42, 969-984, 1993.
- [51] B. Recht, M. Fazel, P. Parrilo, "Guaranteed Minimum-Rank Solutions of Linear Matrix Equations via Nuclear Norm Minimization", SIAM Review, Vol. 52, No. 3, pp. 471-501, 2010.
- [52] S. K. Shah, C.D. Pahlajani, N. A. Lacock, H. G. Tanner, "Stochastic receding horizon control for robots with probabilistic state constraints", IEEE International Conference on Robotics and Automation (ICRA), Saint Paul, MN, May 2012.
- [53] J.F. Sturm, "Using SeDuMi 1.02, A Matlab toolbox for optimization over symmetric cones", Optimization Methods and Software, Vol. 11, No. 1-4, pp. 625-653, 1999.
- [54] W. Tang, Q. Han, G. Li, "The portfolio selection problems with chance-constrained", IEEE International Conference on Systems, Man, and Cybernetics, Tucson, AZ, 2001.
- [55] R. Tempo, G. Calafiore, and F. Dabbene, "Randomized Algorithms for Analysis and Control of Uncertain Systems", Communications and Control Engineering Series, Springer-Verlag, London, 2005.
- [56] P. Tseng, "On Accelerated Proximal Gradient Methods for Convex-Concave Optimization," submitted to SIAM Journal on Optimization, 2008.

- [57] B. Xu, J. Yu, Y. Meng, "The bi-objective stochastic chance-constrained optimization model of multi-project and multi-item investment combination based on the view of real options", IEEE International Conference on Industrial Engineering and Engineering Management, Hong Kong, 2009.
- [58] Ben-Tal A., Bertsimas D., Brown D., "A soft robust model for optimization under ambiguity," Oper. Res. 58(4):12201234, 2010.
- [59] Ben-Tal A., Boyd S., Nemirovski A., "Extending scope of robust optimization: Comprehensive robust counterparts of uncertain problems," Math. Programming Ser. B 107(12):6389, 2006.
- [60] Battles, Z. and Trefethen, L. N., "An extension of Matlab to continuous functions and operators," SIAM J. Sci. Comp. 25, pp. 17431770, 2004.

Appendix A. Sample GloptiPoly Code for Chance Optimization. In this section, we provide the Gloptipoly code for solving the simple problem given in (3.13) and (3.14) of Section 3.4.

```
>> d=2; %relaxation order
>> % mu_s: slack measure, mu_s = mux muq - mu, y_s: moments of mu_s
>> mpol x_s q_s; mu_s = meas([x_s;q_s]); y_s=mom(mmon([x_s;q_s],2*d));
>> % mu: measure supported on p>=0, y: moments of mu
>> mpol x q; mu = meas([x;q]); y=mom(mmon([x;q],2*d));
>> p=0.5*q*(q<sup>2</sup>+(x-0.5)<sup>2</sup>)-(q<sup>4</sup>+q<sup>2</sup>*(x-0.5)<sup>2</sup>+(x-0.5)<sup>4</sup>);
>> % mux: measure, yx: moments of mux
>> mpol xm; mux= meas([xm]); yx=mom(mmon([xm],2*d));
>> % yq: moments of uniform distribution muq on [-1,1]
>> yq=[1;0;1/3;0;0.2];
>> % yxq : moments of upper bound measure mux muq
>> yxq = [yx(1)*yq(1);yx(2)*yq(1);yx(1)*yq(2);yx(3)*yq(1);yx(2)*yq(2);
>> yx(1)*yq(3);yx(4)*yq(1);yx(3)*yq(2);yx(2)*yq(3);yx(1)*yq(4);
>> yx(5)*yq(1);yx(4)*yq(2);yx(3)*yq(3);yx(2)*yq(4);yx(1)*yq(5)];
>> Pd=msdp(max(mass(mu)),mass(mux)==1,p>=0,y_s==yxq - y,-1<=yx,yx<=1,d);msol(Pd);
>> y=double(mvec(mu)); yx=double(mvec(mux)); % results
>> Decision= yx(2)
>> Probability = y(1)
                                Fig. A.1: GloptiPoly Code in Matlab for Example 1
```