

# A TIGHT ERDŐS-PÓSA FUNCTION FOR WHEEL MINORS

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Abstract. Let  $W_t$  denote the wheel on  $t + 1$  vertices. We prove that for every integer  $t \geq 3$  there is a constant  $c = c(t)$  such that for every integer  $k \geq 1$  and every graph  $G$ , either  $G$  has  $k$  vertex-disjoint subgraphs each containing  $W_t$  as a minor, or there is a subset  $X$  of at most  $ck \log k$  vertices such that  $G - X$  has no  $W_t$  minor. This is best possible, up to the value of  $c$ . We conjecture that the result remains true more generally if we replace  $W_t$  with any fixed planar graph  $H$ .

## 1. Introduction

Let  $H$  be a fixed graph. An  $H$ -model  $\mathcal{M}$  in a graph  $G$  is a collection  $\{S_x \subseteq G : x \in V(H)\}$  of vertex-disjoint connected subgraphs of  $G$  such that  $S_x$  and  $S_y$  are linked by an edge in  $G$  for every edge  $xy \in E(H)$ . The *vertex set*  $V(\mathcal{M})$  of  $\mathcal{M}$  is the union of the vertex sets of the subgraphs in the collection. Two  $H$ -models  $\mathcal{M}$  and  $\mathcal{M}'$  are *disjoint* if  $V(\mathcal{M}) \cap V(\mathcal{M}') = \emptyset$ .

Let  $\nu_H(G)$  be the maximum number of pairwise disjoint  $H$ -models in  $G$ . Let  $\tau_H(G)$  be the minimum size of a subset  $X \subseteq V(G)$  such that  $G - X$  has no  $H$ -model. Clearly,  $\nu_H(G) \leq \tau_H(G)$ . We say that the *Erdős-Pósa property holds for  $H$ -models* if there exists a *bounding function*  $f : \mathbb{N} \rightarrow \mathbb{R}$  such that

$$\tau_H(G) \leq f(\nu_H(G))$$

for every graph  $G$ .

Robertson and Seymour [16] proved that the Erdős-Pósa property holds for  $H$ -models if and only if  $H$  is planar. Their original bounding function was exponential. However, this has been significantly improved by recent breakthrough results of Chekuri and Chuzhoy [3, 4] on the polynomial Grid Theorem.

**Theorem 1.1** (Chekuri and Chuzhoy [3]). *There exist integers  $a, b, c \geq 0$  such that for every planar graph  $H$  on  $h$  vertices, the Erdős-Pósa property holds for  $H$ -models with*

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*bounding function*

$$f(k) = ah^b \cdot k \log^c(k + 1).$$

If we consider  $H$  to be fixed and focus solely on the dependence on  $k$ —which is the point of view we take in this paper—[Theorem 1.1](#) gives a  $O(k \log^c k)$  bounding function. This is remarkably close to being best possible: If  $H$  is planar with at least one cycle, then there is a  $\Omega(k \log k)$  lower bound on bounding functions. This follows easily from the original lower bound of Erdős and Pósa for the case where  $H$  is a triangle [\[6\]](#). Alternatively, one can see this by considering  $n$ -vertex graphs  $G$  with treewidth  $\Omega(n)$  and girth  $\Omega(\log n)$  (which exist [\[13\]](#)), and notice that  $\tau_H(G) = \Omega(n)$  (because removing one vertex decreases treewidth by at most one, and  $G - X$  has treewidth  $O(1)$  when  $G - X$  has no  $H$ -minor, by the Grid Theorem) while  $\nu_H(G) = O(n/\log n)$  (because each  $H$ -model contains a cycle). Thus, a  $O(k \log^c k)$  bound is optimal, up to the value of  $c$ . While no explicit value for  $c$  is given in [\[3\]](#), a quick glance at the proof suggests that it is at least a double-digit integer. In this paper, we put forward the conjecture that a  $O(k \log^c k)$  bound holds with  $c = 1$ .

**Conjecture 1.2.** *For every planar graph  $H$ , the Erdős-Pósa property holds for  $H$ -models with a  $O(k \log k)$  bounding function.*

If true, [Conjecture 1.2](#) would completely settle the growth rate of the Erdős-Pósa functions for  $H$ -models for all planar graphs  $H$  (up to the constant factor depending on  $H$ ). That is, if  $H$  is planar with at least one cycle, then the  $O(k \log k)$  bound would match the  $\Omega(k \log k)$  lower bound mentioned above. And if  $H$  is a forest, it is already known that the right order of magnitude is  $O(k)$ , see [\[9\]](#).

Going back to the  $O(k \log^c k)$  bound of Chekuri and Chuzhoy [\[3\]](#), one could initially hope that a value of  $c = 1$  could be obtained by optimizing the various steps of their proof. However, any constant  $c$  obtained using their general approach necessarily satisfies  $c \geq 2$ . This is because they obtain [Theorem 1.1](#) as a corollary from the following result.

**Theorem 1.3** (Chekuri and Chuzhoy [\[3\]](#)). *There exist integers  $a', b', c' \geq 0$  such that for all integers  $r, k \geq 1$ , every graph  $G$  of treewidth at least*

$$a' r^{b'} \cdot k \log^{c'}(k + 1)$$

*has  $k$  vertex-disjoint subgraphs  $G_1, \dots, G_k$ , each of treewidth at least  $r$ .*

Now, if we fix a planar graph  $H$  and if  $G$  is such that  $\nu_H(G) < k$ , then  $G$  cannot have  $k$  vertex-disjoint subgraphs each of treewidth at least  $r$ , where  $r = r(H)$  is a constant such that every graph with treewidth at least  $r$  contains an  $H$  minor. Note that  $r(H)$  exists by the Grid Theorem of Robertson and Seymour [\[16\]](#). Thus, the above theorem implies that  $G$  has treewidth  $O(k \log^{c'} k)$ . The authors of [\[3\]](#) then apply a standard divide-and-conquer approach on an optimal tree decomposition, and obtain a  $O(k \log^c k)$  bound on  $\tau_H(G)$  (see [\[3, Lemma 5.4\]](#)). This unfortunately results in an extra  $\log k$  factor;

$c = c' + 1$ . On the other hand, we must have  $c' \geq 1$  in [Theorem 1.3](#), as shown again by  $n$ -vertex graphs with treewidth  $\Omega(n)$  and girth  $\Omega(\log n)$ . Hence,  $c \geq 2$ . Therefore, one needs a different approach to prove [Conjecture 1.2](#).

As a side remark, it is natural to conjecture that we could take  $c' = 1$  in [Theorem 1.3](#) (at least, if we forget about the precise dependence on  $r$ ):

**Conjecture 1.4.** *There is a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that for all integers  $r, k \geq 1$ , every graph  $G$  of treewidth at least*

$$f(r) \cdot k \log(k + 1)$$

*has  $k$  vertex-disjoint subgraphs  $G_1, \dots, G_k$ , each of treewidth at least  $r$ .*

As it turns out, this conjecture is implied by our [Conjecture 1.2](#): It suffices to take  $H$  to be the  $r \times r$ -grid, which has treewidth  $r$ . Then either  $\nu_H(G) \geq k$ , in which case we are done, or  $\nu_H(G) < k$ , and then there is a subset  $X$  of  $O(k \log k)$  vertices such that  $G - X$  has no  $H$ -minor, and hence  $G - X$  has treewidth at most  $g(r)$  for some function  $g$  by the Grid Theorem. Adding  $X$  to all bags of an optimal tree decomposition of  $G - X$ , we deduce that  $G$  has treewidth  $O(k \log k)$ . Thus, this is another motivation to study [Conjecture 1.2](#).

While [Conjecture 1.2](#) remains open in general, it is known to hold for some specific graphs  $H$ . For example, the original Erdős-Pósa theorem [\[6\]](#) is simply the assertion that [Conjecture 1.2](#) holds when  $H$  is a triangle. This was recently extended to the case where  $H$  is an arbitrary cycle [\[7\]](#) (see also [\[1, 14\]](#) for related results). The conjecture also holds when  $H$  is a multigraph consisting of two vertices linked by a number of parallel edges [\[2\]](#).

Our main result is that [Conjecture 1.2](#) holds when  $H$  is a wheel. A *wheel* is a graph obtained from a cycle by adding a new vertex adjacent to all vertices of the cycle. We denote by  $W_t$  the wheel on  $t + 1$  vertices.

**Theorem 1.5.** *For each integer  $t \geq 3$ , the Erdős-Pósa property holds for  $W_t$ -models with a  $O(k \log k)$  bounding function.*

We remark that our main theorem implies all the aforementioned special cases. This is because the existence of a  $O(k \log k)$  bounding function for  $H$ -models is preserved under taking minors of  $H$  (see [Lemma 2.7](#)). Our result also have the following consequence.

**Corollary 1.6.** *For every  $t \in \mathbb{N}$  there is a function  $g : \mathbb{N} \rightarrow \mathbb{N}$  with  $g(k) = O(k \log k)$  such that every  $(k \cdot W_t)$ -minor free graph has treewidth at most  $g(k)$ .*

The rest of the paper is organized as follows. In the next section we present some general lemmas about  $H$ -models. Since these lemmas are valid for arbitrary planar graphs  $H$ , they may be useful in attacking [Conjectures 1.2](#) and [1.4](#). In [Section 3](#), we specialize to the case of wheels and prove our main theorem. We conclude with some open problems in [Section 4](#).

## 2. General Tools

In this paper, our graphs are simple (no parallel edges nor loops). Let  $H, G$  be two graphs. We let  $|G|$  denote  $|V(G)|$ . We assume the reader is familiar with the notions of graph minors, tree decompositions, and treewidth (see Diestel [5] for an introduction to the area). We let  $\text{tw}(G)$  denote the treewidth of  $G$ .

An  $H$ -transversal of  $G$  is a set  $X$  of vertices of  $G$  such that  $G - X$  has no  $H$ -model. A graph is *minor-minimal* for a given property if it satisfies the property and none of its proper minors does.

We use the following results. The first is an extension of a classic result of Kostochka [11] and Thomason [18], where in addition the size of the  $K_t$ -model is logarithmic. (For definiteness, all logarithms are in base 2 in this paper.)

**Theorem 2.1** ([12], see also [8, 17]). *There is a function  $\varphi(t) = O(t\sqrt{\log t})$  such that, if an  $n$ -vertex graph has average degree at least  $\varphi(t)$ , then it contains a  $K_t$ -model on  $O(\log n)$  vertices.*

The second is a theorem of Fomin, Lokshtanov, Misra and Saurabh [10], whose original purpose was to show that the algorithmic problem of finding a minimum-size  $H$ -transversal admits a polynomial-size kernel when  $H$  is planar.

**Theorem 2.2** ([10]). *For every planar graph  $H$ , there is a polynomial  $\pi$  such that for every  $k \in \mathbb{N}$ , every graph  $G$  with  $\tau_H(G) = k$  and minor-minimal with this property satisfies  $|G| \leq \pi(k)$ .*

**2.1. Minimal counterexamples to the Erdős-Pósa property.** Let  $H$  be a graph and let  $f: \mathbb{N} \rightarrow \mathbb{R}$  be a function. We say that a graph  $G$  is a *minimal counterexample* to the Erdős-Pósa property for  $H$ -models with bounding function  $f$  if the following properties hold:

- (i)  $\tau_H(G) > f(\nu_H(G))$ ;
- (ii) subject to the above constraint,  $\nu_H(G)$  is minimum;
- (iii) subject to the above constraints,  $|G|$  is minimum;
- (iv) subject to the above constraints,  $|E(G)|$  is minimum.

Notice that the two last requirements of the above definition imply that a minimal counterexample is a minor-minimal graph satisfying requirements (i) and (ii). The following lemma gives a bound on the size of minimal counterexamples.

**Lemma 2.3.** *Let  $H$  be a planar graph and let  $f: \mathbb{N} \rightarrow \mathbb{R}$  be a polynomial non-decreasing function. Then there is a polynomial  $\rho$  such that, for every minimal counterexample  $G$  to the Erdős-Pósa property for  $H$ -models with bounding function  $f$ , we have  $|G| \leq \rho(\nu_H(G))$ .*

*Proof.* Let  $k := \nu_H(G)$ . Let us first show that  $\tau_H(G) = \lfloor f(k) \rfloor + 1$ . Let  $v \in V(G)$ . Observe that  $\tau_H(G) \leq \tau_H(G - v) + 1$ . By minimality of  $G$ ,  $\tau_H(G - v) \leq f(\nu_H(G - v))$ . As  $G - v$  is a minor of  $G$ , we also have  $\nu_H(G - v) \leq \nu_H(G)$ . We deduce  $\tau_H(G) \leq f(k) + 1$ . Since  $G$  is a counterexample, we also have  $\tau_H(G) > f(k)$ . It follows that  $\tau_H(G) = \lfloor f(k) \rfloor + 1$ .

Now,  $\nu_H(G') \leq \nu_H(G)$  holds for every proper minor  $G'$  of  $G$ , and thus  $\tau_H(G') < \tau_H(G)$  (otherwise  $G$  would not be a minimal counterexample). Hence  $G$  is minor-minimal with the property that  $\tau_H(G) = \lfloor f(k) \rfloor + 1$ . By [Theorem 2.2](#) we obtain  $|G| \leq \pi(\lfloor f(k) \rfloor + 1)$  where  $\pi$  is the polynomial given by that theorem. Therefore, it suffices to take  $\rho : t \mapsto \pi(\lfloor f(t) \rfloor + 1)$ .  $\square$

Informally, the following result, originally proved in [\[9\]](#), states that if a graph  $G$  has a large  $H$ -minor-free induced subgraph with a small ‘boundary’, then there is a smaller graph  $G'$  where the values of  $\nu_H$  and  $\tau_H$  are the same.

**Theorem 2.4** ([\[9\]](#)). *For every planar graph  $H$ , there is a computable function  $g' : \mathbb{N} \rightarrow \mathbb{N}$  such that, for every graph  $G$ , if  $J$  is an  $H$ -minor-free induced subgraph of  $G$  such that exactly  $p$  vertices of  $J$  have a neighbor in  $V(G) \setminus V(J)$  and  $|J| \geq g'(p)$ , then there exists a graph  $G'$  such that  $\tau_H(G') = \tau_H(G)$ ,  $\nu_H(G') = \nu_H(G)$ , and  $|G'| < |G|$ .*

We can use [Theorem 2.4](#) to upper bound the size of  $H$ -minor-free induced subgraphs in minimal counterexamples as follows.

**Corollary 2.5.** *For every planar graph  $H$ , there is a computable and non-decreasing function  $g : \mathbb{N} \rightarrow \mathbb{N}$  such that, if  $G$  is a minimal counterexample to the Erdős-Pósa property for  $H$ -models with bounding function  $f$  for some function  $f : \mathbb{N} \rightarrow \mathbb{R}$ , then every  $H$ -minor free induced subgraph  $J$  of  $G$  that has exactly  $p$  vertices with a neighbor in  $V(G) \setminus V(J)$  satisfies  $|J| < g(p)$ .*

*Proof.* Let  $g'$  be the function in [Theorem 2.4](#). We define the function  $g : \mathbb{N} \rightarrow \mathbb{N}$  as follows:  $g(k) = \max_{i \in \{0, \dots, k\}} g'(i)$ . Notice that  $g(k) \geq g'(k)$  holds for every  $k \in \mathbb{N}$  and that  $g$  is non-decreasing. Now, suppose that  $G$  is a graph having an  $H$ -minor free induced subgraph  $J$  with exactly  $p$  vertices having a neighbor in  $V(G) \setminus V(J)$  and such that  $|J| \geq g(p)$ . Then, since  $g(p) \geq g'(p)$ , by [Theorem 2.4](#) there is a graph  $G'$  such that  $\tau_H(G') = \tau_H(G)$ ,  $\nu_H(G') = \nu_H(G)$ , and  $|G'| < |G|$ . In particular,  $G$  cannot be a minimal counterexample to the Erdős-Pósa property for  $H$ -models for any bounding function  $f$ , a contradiction.  $\square$

**2.2. Interplays between treewidth and the Erdős-Pósa property.** Given a planar graph  $H$ , the standard approach to show that  $H$ -models satisfy the Erdős-Pósa property is to first note that  $k \cdot H$  (the disjoint union of  $k$  copies of  $H$ ) is also planar. Thus, if  $\nu_H(G) < k$  for a graph  $G$ , then the treewidth of  $G$  is bounded by a function of  $k$  and  $H$ , by the Grid Theorem [\[16\]](#). Then one can use a tree decomposition of small width to find

a small  $H$ -transversal of  $G$ . This was first used by Robertson and Seymour [16, Theorem 8.8] in their original proof (see also [19, Theorem 3] and the survey [15, Section 3]). It was subsequently used by several authors to obtain improved bounding functions, most notably by Chekuri and Chuzhoy [3] when deriving their [Theorem 1.1](#) from [Theorem 1.3](#).

As it was already mentioned in the introduction when discussing [Conjecture 1.4](#), the reverse direction holds as well: A bounding function for  $H$ -models translates directly to an upper bound on the treewidth of  $(k \cdot H)$ -minor free graphs, up to an additive term depending only on  $H$ :

**Lemma 2.6.** *Let  $H$  be a planar graph, let  $f$  be a bounding function for  $H$ -models, and let  $c = c(H)$  be a constant such that  $\text{tw}(G) \leq c$  for every  $H$ -minor free graph  $G$ . Then, for every  $k \geq 1$ , every  $(k \cdot H)$ -minor free graph  $G$  has treewidth at most  $f(k - 1) + c$ .*

*Proof.* Let  $G$  be a graph not containing  $k \cdot H$  as a minor. Since  $\nu_H(G) \leq k - 1$  and  $f$  is a bounding function for  $H$ -models, we deduce  $\tau_H(G) \leq f(k - 1)$ . That is,  $G$  has a set  $X$  of at most  $f(k - 1)$  vertices such that  $G - X$  is  $H$ -minor free. By definition of  $c$ , we have  $\text{tw}(G - X) \leq c$ . Then  $\text{tw}(G) \leq c + |X| \leq c + f(k - 1)$ , as desired.  $\square$

Thus combining our main result with [Lemma 2.6](#) gives [Corollary 1.6](#) (stated in the introduction).

We also include the following lemma, which states that if  $H'$  is a minor of  $H$ , then a bounding function for  $H'$ -models can be easily obtained from a bounding function for  $H$ -models.

**Lemma 2.7.** *Let  $H$  be a fixed planar graph, let  $f$  be a bounding function for  $H$ -models, and let  $c = c(H)$  be a constant such that  $\text{tw}(G) \leq c$  for all  $H$ -minor free graphs  $G$ . If  $H'$  is a minor of  $H$  with  $q$  connected components, then  $k \mapsto f(k) + (qk - 1)(c + 1)$  is a  $O(f(k))$  bounding function for  $H'$ -models.*

*Proof.* Let  $G$  be a graph with  $\nu_{H'}(G) \leq k$ . As  $H'$  is a minor of  $H$ , we deduce  $\nu_H(G) \leq k$ . By definition of  $f$ , there is a set  $X$  of at most  $f(k)$  vertices such that  $G - X$  is  $H$ -minor free. Hence  $\text{tw}(G - X) \leq c$ . Theorem 8.8 in [16] provides the following upper-bound on  $\tau$  in graphs of bounded treewidth:

$$\tau_{H'}(G - X) \leq (qk - 1)(c + 1).$$

Then,  $\tau_{H'}(G) \leq \tau_{H'}(G - X) + |X| \leq (qk - 1)(c + 1) + f(k)$ . Finally, since  $f(k) \geq k$ , we deduce  $(qk - 1)(c + 1) + f(k) = O(f(k))$ .  $\square$

### 3. The Proof for Wheels

In this section, we prove our main theorem:

**Theorem 1.5.** *For each integer  $t \geq 3$ , the Erdős-Pósa property holds for  $W_t$ -models with a  $O(k \log k)$  bounding function.*

*Proof.* To keep track of the dependencies between the constants that we use, we define them here. Recall that  $t$  denotes the number of spokes of the wheel that we are considering. Let  $\varphi$  and  $\varphi'$  be constants such that every  $n$ -vertex graph of average degree at least  $\varphi$  has a  $K_{t+1}$ -model on at most  $\varphi' \log n$  vertices (both  $\varphi$  and  $\varphi'$  depend on  $t$ , see [Theorem 2.1](#)). Let  $\alpha, \beta \geq 1$  be constants such that  $\rho(n) \leq \alpha n^\beta$ , for every  $n \in \mathbb{N} \setminus \{0\}$ , where  $\rho$  is the polynomial of [Lemma 2.3](#) for  $H = W_t$ . Let  $g$  denote the function from [Corollary 2.5](#) for  $H = W_t$ .

We then set

$$\begin{aligned} c_1 &= g(2t\varphi^2), & p &= g(2c_1\varphi^2), & c_2 &= 4p, \\ \sigma &= \max \{3\varphi'c_2, 2c_2 + tp, (2t^2p + 1)(c_2 + 2c_1\varphi^2)\}, & \text{and} \\ \gamma &= \sigma(\beta + \log \alpha). \end{aligned}$$

Observe that we have  $t < c_1 < p < c_2$ . Let  $f(k) := \gamma \cdot k \log(k + 1)$ , for every  $k \in \mathbb{N}$ . We show that the Erdős-Pósa property holds for  $W_t$ -models with bounding function  $f$ .

Arguing by contradiction, let  $G$  be a minimal counterexample to the Erdős-Pósa property for  $W_t$ -models with bounding function  $f$ . Let  $k := \nu_{W_t}(G)$ . Then  $k \geq 1$  and  $|G|$  is polynomial in  $k$  by [Lemma 2.3](#). That is, letting  $n := |G|$ , we have  $n \leq \alpha k^\beta$ , for the constants  $\alpha$  and  $\beta$  defined above.

We first show that  $G$  cannot contain a  $W_t$ -model of logarithmic size.

**Claim 3.1.**  *$G$  has no  $W_t$ -model of size at most  $\sigma \log n$ .*

*Proof.* Towards a contradiction, we consider a  $W_t$ -model  $\mathcal{M}$  of size at most  $\sigma \log n$ . Notice that  $\log n \leq (\beta + \log \alpha) \log(k + 1)$  can be deduced from the aforementioned upper-bound on  $n$ . Since  $\nu_{W_t}(G - V(\mathcal{M})) \leq k - 1$ , by minimality of  $G$ ,

$$\begin{aligned} \tau_{W_t}(G) &\leq |V(\mathcal{M})| + \tau_{W_t}(G - V(\mathcal{M})) \\ &\leq \sigma \log n + f(k - 1) \\ &\leq \gamma \log(k + 1) + \gamma(k - 1) \log k \\ &\leq \gamma k \log(k + 1) \leq f(k). \end{aligned}$$

However, this contradicts the fact that  $G$  is a minimal counterexample to the Erdős-Pósa property for  $W_t$ -models with bounding function  $f$ .  $\square$

Note that since  $\nu_{W_t}(G) \geq 1$ , we have  $n \geq t + 1 \geq 2$ , and thus  $\log n \geq 1$  (recall that all logarithms are in base 2). Thus, [Claim 3.1](#) implies in particular that  $G$  has no  $W_t$ -model of size at most  $\sigma$ .

Let  $\mathcal{C}$  be a maximum-size collection of vertex-disjoint cycles in  $G$  whose lengths are in the interval  $[c_1, c_2]$ . Let  $\mathcal{P}$  be a maximum-size collection of vertex-disjoint paths of length  $p$  in  $G - V(\mathcal{C})$ , where  $V(\mathcal{C}) := \bigcup_{C \in \mathcal{C}} V(C)$ . (In this paper, the length of a path is defined as its number of edges.) Finally, let  $\mathcal{R}$  be the collection of components of  $G - (V(\mathcal{C}) \cup V(\mathcal{P}))$  and let  $V(\mathcal{R}) := V(G) - (V(\mathcal{C}) \cup V(\mathcal{P}))$ , where  $V(\mathcal{P}) := \bigcup_{P \in \mathcal{P}} V(P)$ . We point out that the cycles in  $\mathcal{C}$  and the paths in  $\mathcal{P}$  are subgraphs of  $G$  but not necessarily induced subgraphs of  $G$ , while the components in  $\mathcal{R}$  are induced subgraphs of  $G$ . We call the elements of  $\mathcal{C} \cup \mathcal{P} \cup \mathcal{R}$  *pieces*.

Observe that, by maximality of  $\mathcal{P}$ , every path in a piece of  $\mathcal{R}$  has length at most  $p - 1$ . This implies that each such piece is  $W_t$ -minor free. Indeed, observe that if such a piece  $R$  has a  $W_t$ -model then  $R$  contains a subgraph consisting of a cycle  $C$  and a rooted tree  $T$  such that  $T$  has at most  $t$  leaves,  $V(T) \cap V(C) = \emptyset$ , and the leaves of  $T$  collectively have at least  $t$  neighbours in  $C$ . The cycle  $C$  has at most  $p$  vertices, and each root-to-leaf path in  $T$  has at most  $p$  vertices. Thus, this gives a  $W_t$ -model with at most  $(t + 1)p$  vertices. However, this contradicts [Claim 3.1](#) since  $(t + 1)p \leq \sigma$ .

Similarly, each piece in  $\mathcal{C}$  (in  $\mathcal{P}$ , respectively) has at most  $c_2$  ( $p$ , respectively) vertices, and these vertices induce a  $W_t$ -minor free subgraph of  $G$ ; otherwise, there would exist a  $W_t$ -model of size at most  $c_2$  (resp.  $p$ ), again a contradiction to [Claim 3.1](#) since  $p \leq c_2 \leq \sigma$ . These facts will be used often in the rest of the proof.

We say that two distinct pieces  $K$  and  $K'$  *touch* if some edge of  $G$  links some vertex of  $K$  to some vertex of  $K'$ . Note that, by construction, two distinct pieces in  $\mathcal{R}$  cannot touch. A piece is said to be *central* if it is a cycle in  $\mathcal{C}$ , a path in  $\mathcal{P}$ , or a piece in  $\mathcal{R}$  that touches at least  $2\varphi$  other pieces. In the next paragraph, we define two auxiliary simple graphs  $H_s$  (for small degrees) and  $H_b$  (for big degrees) that model how the central pieces are connected through the noncentral pieces. To keep track of the correspondence between the edges of  $H_s$  and the noncentral pieces, we put labels on some of these edges.

Initialize both  $H_s$  and  $H_b$  to the graph whose set of vertices is the set of central pieces and whose set of edges is empty. For each pair of central pieces that touch in  $G$ , add an (unlabeled) edge between the corresponding vertices in both  $H_s$  and  $H_b$ .

Next, while there is some noncentral piece  $R \in \mathcal{R}$  that touches two central pieces  $K$  and  $K'$  that are not yet adjacent in  $H_b$ , call  $\mathcal{Z}_R$  the set of central pieces that touch  $R$  and do the following:

- (1) add all (unlabeled) edges to  $H_b$  between pieces of  $\mathcal{Z}_R$  (not already present in  $H_b$ ). This creates a clique on vertex set  $\mathcal{Z}_R$  in  $H_b$ , some of whose edges might have already been there before.
- (2) choose a piece  $K \in \mathcal{Z}_R$  such that the number of newly added edges of  $H_b$  incident to  $K$  is maximum. Add to  $H_s$  every edge that links  $K$  to another piece of  $\mathcal{Z}_R$



(not already present in  $H_s$ ), and label it with  $R$ . This creates a star centered at  $K$  in  $H_s$  with all its edges labeled with the noncentral piece  $R$ .

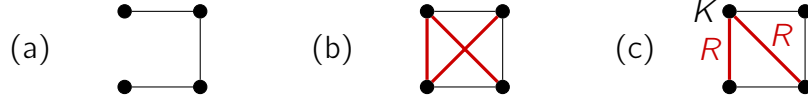


Figure 1. Construction of  $H_s$  and  $H_b$ . (a): the vertices of a set  $\mathcal{Z}_R$  in  $H_s$  and  $H_b$  before step (1). (b):  $H_b[\mathcal{Z}_R]$  after step (1). (c):  $H_s[\mathcal{Z}_R]$  after step (2).

The edges added during these steps are depicted in thick red lines in the example of Figure 1. By construction,  $H_s$  is a subgraph of  $H_b$ . These graphs have the following two crucial properties.

**Claim 3.2.** *If  $H_s$  has a  $W_t$ -model of size  $q$ , then  $G$  has a  $W_t$ -model of size at most  $3c_2q$ .*

*Proof.* Suppose that  $H_s$  has a  $W_t$ -model of size  $q$ . Then there exists a subgraph  $M_s \subseteq H_s$  with  $q$  vertices that can be contracted to  $W_t$ . We may assume that the average degree of  $M_s$  is at most that of  $W_t$ , and hence at most 4. That is,  $|E(M_s)| \leq 2|M_s|$ . From the subgraph  $M_s$ , we construct a subgraph  $M \subseteq G$  that can be contracted to  $M_s$ , and thus also to  $W_t$ .

First, for each central piece  $K \in V(M_s) \cap (\mathcal{C} \cup \mathcal{P})$ , we add all its vertices to  $M$ , as well as  $|K| - 1$  edges from  $K$  in such a way that the subgraph of  $M$  induced by  $V(K)$  is connected. For each central piece  $K \in V(M_s) \cap \mathcal{R}$ , we choose some vertex  $v_K \in K$  and add it to  $M$ . This creates at most  $c_2|M_s| = c_2q$  vertices in  $M$ .

Second, for each unlabeled edge  $KK'$  of  $M_s$  with  $K, K' \in V(M_s) \cap (\mathcal{C} \cup \mathcal{P})$ , we choose some edge of  $G$  linking  $K$  to  $K'$  and add it to  $M$ . This does not create any new vertex in  $M$ .

Third, for each edge  $KK'$  of  $M_s$  that has not been considered so far, we add to  $M$  a path linking some vertex of  $V(M) \cap V(K)$  to some vertex of  $V(M) \cap V(K')$ , as follows. If the edge  $KK'$  is not labeled, then exactly one of its endpoints is a central piece in  $\mathcal{R}$ , say  $K$ . The path we add to  $M$  links  $v_K$  to some vertex of  $K'$  and is a subgraph of  $K$ , except for the last edge and last vertex. Thus, this path has at most  $p - 1$  internal vertices. If the edge  $KK'$  is labeled with the noncentral piece  $R \in \mathcal{R}$ , then this edge is part of a star in  $M_s$  whose edges are all labeled with  $R$ . We may assume without loss of generality that  $K$  is the center of this star. In this case, the path we add to  $M$  links some vertex of  $K$  to some vertex of  $K'$  and has all its internal vertices in  $R$ . Thus, this path has at most  $p$  internal vertices.

In total, the addition of these paths to  $M$  creates at most  $p|E(M_s)| \leq 2c_2|M_s| = 2c_2q$  new vertices in  $M$ . The resulting subgraph  $M$  has at most  $c_2|M_s| + p|E(M_s)| \leq 3c_2q$  vertices. By construction,  $M$  can be contracted to  $M_s$ , as desired.  $\square$

**Claim 3.3.** *The average degree of  $H_b$  is at most  $\varphi$  times the average degree of  $H_s$ . The degree of each central piece of  $\mathcal{R}$  in  $H_s$  is at least  $2\varphi$ .*

*Proof.* First, note that edges that appear in  $H_b$  but not in  $H_s$  must be labeled. Let  $R \in \mathcal{R}$  be a noncentral piece, and let  $r$  be the number of pieces in  $\mathcal{C} \cup \mathcal{P}$  it touches. By definition of noncentral pieces,  $r < 2\varphi$ . When  $R$  is treated in the algorithm used to construct  $H_b$  and  $H_s$ , if  $q$  new edges are added to  $H_b$ , then one of the pieces touched by  $R$  is incident to at least  $2q/r > q/\varphi$  of these new edges and thus at least  $q/\varphi$  new edges are added to  $H_s$ . This proves the first part of the claim.

By definition, a piece  $K$  of  $\mathcal{R}$  is central if it touches at least  $2\varphi$  other pieces. As two pieces of  $\mathcal{R}$  cannot touch,  $K$  touches at least  $2\varphi$  pieces from  $\mathcal{C} \cup \mathcal{P}$ , that is, at least  $2\varphi$  other central pieces. Then in the first step of the construction of  $H_s$ , all edges have been added from  $K$  to these pieces.  $\square$

If the average degree of  $H_s$  is at least  $\varphi$ , then by definition of  $\varphi$  and  $\varphi'$  at the beginning of the proof,  $H_s$  has a  $K_{t+1}$ -model of size at most  $\varphi' \log |H_s|$ , and thus in particular a  $W_t$ -model of size at most  $\varphi' \log |H_s|$ . By [Claim 3.2](#), this gives a  $W_t$ -model of size at most  $3\varphi' c_2 \log |H_s| \leq 3\varphi' c_2 \log n$  in  $G$ , a contradiction to [Claim 3.1](#) since  $3\varphi' c_2 \leq \sigma$ .

Thus, the average degree of  $H_s$  is smaller than  $\varphi$ . Hence, by [Claim 3.3](#), the average degree of  $H_b$  is smaller than  $\varphi^2$ . Then strictly more than half of the central pieces have degree less than  $2\varphi$  in  $H_s$  (otherwise at least half of the vertices of  $H_s$  have degree at least  $2\varphi$ , a contradiction to the fact that  $H_s$  has average degree less than  $\varphi$ ). Similarly, strictly more than half of the central pieces have degree less than  $2\varphi^2$  in  $H_b$ . Thus there is a central piece whose degree in  $H_s$  is less than  $2\varphi$ , and whose degree in  $H_b$  is less than  $2\varphi^2$ . Choose such a piece  $K$ . By [Claim 3.3](#) (second part of the statement),  $K$  is either in  $\mathcal{C}$  or in  $\mathcal{P}$ .

In the rest of the proof we use the fact that  $K$  has degree less than  $2\varphi^2$  in  $H_b$  to find a  $W_t$ -model of size at most  $\sigma \log n$ , contradicting [Claim 3.1](#).

If  $K$  is the unique central piece in  $\mathcal{C} \cup \mathcal{P}$ , then  $V(K)$  is a  $W_t$ -transversal of  $G$  since each piece in  $\mathcal{R}$  is  $W_t$ -minor free. Thus  $\tau_{W_t}(G) \leq |K| \leq c_2 \leq f(k)$ , contradicting the fact that  $G$  is a counterexample.

For each central piece  $K'$  adjacent to  $K$  in  $H_b$ , we consider the collection  $\mathcal{R}_{K,K'}$  of all noncentral pieces  $R \in \mathcal{R}$  that touch both  $K$  and  $K'$  ( $\mathcal{R}_{K,K'}$  might be empty). Then we consider the subgraph  $G_{K'}$  of  $G$  induced by  $V(K) \cup V(K') \cup V(\mathcal{R}_{K,K'})$ .

Let  $q$  be an integer equal to  $t$  if  $K \in \mathcal{C}$ , to  $c_1$  if  $K \in \mathcal{P}$ .

Our next goal is to show that for every central piece  $K'$  adjacent to  $K$  in  $H_b$ , there exists a set of strictly less than  $q$  vertices that separates  $K$  from  $K'$  in  $G_{K'}$ . Thus fix

a piece  $K'$  adjacent to  $K$  in  $H_b$ . By Menger's theorem, it suffices to show that the maximum number of vertex-disjoint  $K$ - $K'$  paths in  $G_{K'}$  is strictly less than  $q$ . Assume for contradiction that  $G_{K'}$  contains  $q$  vertex-disjoint  $K$ - $K'$  paths.

By taking the paths to be as short as possible, we may assume that only their endpoints are in  $K$  and  $K'$ , all their internal vertices are in pieces in  $\mathcal{R}_{K,K'}$ , and each such path intersects at most one piece in  $\mathcal{R}_{K,K'}$  and thus has length at most  $p + 1$ .

Assume first that  $K \in \mathcal{C}$ , and so  $q = t$ . In this case  $G_{K'}$  contains a small  $W_t$ -model as follows. Let  $T$  be a smallest tree in  $K'$  containing all the endpoints of our paths in  $K'$ . The center vertex of the wheel is then modeled by the union of  $T$  and the  $t$   $K$ - $K'$  paths (minus their endpoints in  $K$ ). If  $K' \in \mathcal{C} \cup \mathcal{P}$ , then obviously  $|V(T)| \leq c_2$ , and the model thus has at most  $2c_2 + tp$  vertices. If  $K' \in \mathcal{R}$ , then  $|V(T)| \leq tp$  since each path in  $K'$  has length at most  $p - 1$ ; moreover,  $\mathcal{R}_{K,K'}$  is empty in this case, implying that the model has at most  $c_2 + tp$  vertices. Therefore, in both cases the resulting model has at most  $2c_2 + tp$  vertices, which contradicts [Claim 3.1](#) since  $2c_2 + tp \leq \sigma$ .

Assume now that  $K \in \mathcal{P}$ . Since  $t < c_1$ , by the previous case we may assume that  $K' \in \mathcal{P} \cup \mathcal{R}$ . Since there are  $q = c_1$  vertex-disjoint  $K$ - $K'$  paths in  $G_{K'}$ , two of these paths intersect  $K$  on two vertices that are at distance at least  $c_1 - 1$  on the path  $K$ , which allows us to construct a cycle in  $G_{K'}$  of length at least  $c_1$  and at most  $4p$ : The cycle might use all the vertices of  $K$  and at most  $p$  vertices of  $K'$ , which is at most  $2p$  vertices, and might intersect at most two pieces of  $\mathcal{R}_{K,K'}$ , using at most  $p$  vertices in each of them. This is a contradiction to the maximality of  $\mathcal{C}$ : The length of this cycle is in the interval  $[c_1, c_2]$  and yet the cycle is vertex disjoint from all cycles in  $\mathcal{C}$ .

Therefore, for each  $K'$  adjacent to  $K$  in  $H_b$ , there exists a set  $X(K')$  with less than  $q$  vertices meeting all the  $K$ - $K'$  paths in  $G_{K'}$ .

Let  $X := \bigcup_{K'} X(K')$  where the union is taken over all central pieces  $K'$  adjacent to  $K$  in  $H_b$ . Note that  $|X| \leq 2q\varphi^2$  since there are at most  $2\varphi^2$  such pieces  $K'$  and for every  $K'$  we have  $|X(K')| \leq q$ .

We also note that  $X$  separates  $K$  from all other central pieces in  $G$ . To see this, let  $K''$  be a central piece distinct from  $K$  and let  $Q$  be a  $K''$ - $K$  path in  $G$ . Let  $K'$  be the last central piece that  $Q$  meets before reaching  $K$ . Then  $Q$  contains a  $K'$ - $K$  path that is contained in  $G_{K'}$ , which must contain a vertex from  $X$ .

Let  $J$  be the union of the components of  $G - X$  that intersect  $K$ . Observe that  $V(K)$  is not completely included in  $X$ : If  $K \in \mathcal{C}$ , then  $|K| \geq c_1 > 2t\varphi^2 \geq |X|$ , and if  $K \in \mathcal{P}$ , then  $|K| = p > 2c_1\varphi^2 \geq |X|$ . Thus  $J$  is not empty. Note also that  $X$  separates  $J$  from the rest of the graph.

Suppose that the subgraph  $G'$  of  $G$  induced by  $X \cup V(J)$  is  $W_t$ -minor free. Thus, by [Corollary 2.5](#),  $|G'| < g(|X|)$ . We deduce

$$\begin{aligned} |X| + |J| &< g(|X|) && \text{since } |G'| = |X| + |J| \\ |K| &< g(|X|) && \text{since } |J| \geq |K| - |X| \\ |K| &< g(2q\varphi^2) && \text{since } g \text{ is non-decreasing.} \end{aligned}$$

Hence, if  $K \in \mathcal{C}$ , then  $c_1 \leq |K| < g(2t\varphi^2)$ , and if  $K \in \mathcal{P}$ , then  $p \leq |K| < g(2c_1\varphi^2)$ . Since  $c_1 = g(2t\varphi^2)$  and  $p = g(2c_1\varphi^2)$ , we get a contradiction in both cases.

Thus, we may assume that  $G'$  contains a  $W_t$ -model. Let  $M$  be a subgraph of  $G'$  containing  $W_t$  as a minor with  $|V(M)| + |E(M)|$  minimum. (We remark that here we take  $M$  to be a subgraph instead of just a model as before because we will need to consider the edges of that subgraph in the proof.) To finish the proof, it is now enough to prove that  $M$  has at most  $\sigma \log n$  vertices, since by [Claim 3.1](#) this will give us the desired contradiction.

Let  $R(J) := J[V(\mathcal{R})]$ . Thus  $R(J)$  consists of a number of disjoint pieces or subgraphs of pieces of  $\mathcal{R}$ . Note that  $M$  might use all vertices of  $V(K) \cup X$  (which is fine); what we need to prove is that it cannot use too many vertices of  $R(J)$ .

First, suppose that  $M$  is fully contained in some piece of  $\mathcal{R}$ . Since the vertices of  $M$  can be covered with  $2t$  paths, and each path in the piece has length less than  $p$ , it follows that  $|M| \leq 2tp \leq \sigma$  and we are done.

Thus we may assume that  $M$  is not contained in some piece of  $\mathcal{R}$ , and thus in particular  $M$  is not contained in  $R(J)$  (since  $M$  is connected). By the above remark, we also know that each component of  $M[V(R(J))]$  contains at most  $2tp$  vertices. Since  $M$  has maximum degree at most  $t$  (by minimality of  $|V(M)| + |E(M)|$ ), there are at most  $t|V(K) \cup X|$  edges of  $M$  with one endpoint in  $V(K) \cup X$  and the other in  $R(J)$ . Hence  $M$  intersects  $R(J)$  on at most  $2t^2p|V(K) \cup X|$  vertices. Therefore,  $M$  has at most  $2t^2p|V(K) \cup X| + |V(K) \cup X|$  vertices. Since  $|V(K) \cup X| \leq |K| + |X| \leq c_2 + 2c_1\varphi^2$ , we deduce that  $|M| \leq (2t^2p + 1)(c_2 + 2c_1\varphi^2) \leq \sigma$ , as desired.  $\square$

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#### 4. Conclusion

One obvious extension of our result for wheels would be to prove it for all planar graphs. Note that the first steps of our proof work for any such  $H$ : Starting with  $G$  a minimal counterexample for some bounding function and some value  $k$ , we have that  $G$  has  $n \leq k^{O(1)}$  vertices. Thus, in order to get a contradiction, it is enough to show that

there is a  $O(\log n)$ -size  $H$ -model in  $G$ . Unfortunately, the rest of our proof is specific to wheels and does not generalize.

Let us mention another possible extension of our result. Strengthening the  $O(k \log k)$  bound from [7], Mousset, Noever, Škorić, and Weissenberger [14] recently showed that there is a constant  $c > 0$  such that for every  $\ell \geq 3$ , models of the  $\ell$ -cycle  $C_\ell$  have the Erdős-Pósa property with bounding function  $ck \log k + \ell k$ . In particular, the constant  $c$  in front of the  $k \log k$  term is independent of  $\ell$ . We expect that a similar property holds for wheels:

**Conjecture 4.1.** *There is a constant  $c > 0$  and a function  $g : \mathbb{N} \rightarrow \mathbb{N}$  such that for all integers  $t \geq 3$ ,  $W_t$ -models have the Erdős-Pósa property with bounding function*

$$ck \log k + g(t)k.$$

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