

# Lazy Local Search Meets Machine Scheduling

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## Abstract

We study the *restricted case* of SCHEDULING ON UNRELATED PARALLEL MACHINES. In this problem, we are given a set of jobs  $J$  with processing times  $p_j$  and each job may be scheduled only on some subset of machines  $S_j \subseteq M$ . The goal is to find an assignment of jobs to machines to minimize the time by which all jobs can be processed. In a seminal paper, Lenstra, Shmoys, and Tardos [LST87] designed an elegant 2-approximation for the problem in 1987. The question of whether approximation algorithms with *better* guarantees exist for this classic scheduling problem has since remained a source of mystery.

In recent years, with the improvement of our understanding of Configuration LPs, it now appears an attainable goal to design such an algorithm. Our main contribution is to make progress towards this goal. When the processing times of jobs are either 1 or  $\epsilon \in (0, 1)$ , we design an approximation algorithm whose guarantee tends to  $1 + \sqrt{3}/2 \approx 1.8660254$ , for the interesting cases when  $\epsilon \rightarrow 0$ . This improves on the  $2 - \epsilon_0$  guarantee recently obtained by Chakrabarty, Khanna, and Li [CKL15] for some constant  $\epsilon_0 > 0$ .

**Keywords:** scheduling, unrelated parallel machines, restricted assignment, configuration linear programs.

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# 1 Introduction

We study a special case of the problem of SCHEDULING ON UNRELATED PARALLEL MACHINES. An instance  $\mathcal{I}$  of this problem consists of machines  $M$ , jobs  $J$ , and a collection of positive processing times  $\{p_{ij}\}_{i \in M, j \in J}$  for every machine–job pair. The goal is to assign all the jobs to the available machines such that the *makespan* of the resulting schedule is as small as possible. The makespan of a schedule  $\sigma : J \mapsto M$  is defined as

$$\max_{i \in M} \sum_{j \in \sigma^{-1}(i)} p_{ij}.$$

It is one of the major open questions [WS11] in the field of approximation algorithms to better understand the approximability of this problem. Curiously, the best known hardness result *and* approximation algorithm for the problem today were simultaneously established in the 1987 paper of Lenstra, Shmoys and Tardos [LST87].

Given the challenging nature of the problem, the road to a better understanding of its approximability has focused on two special cases that each isolate two difficult aspects associated with it—on the one hand a given job  $j \in J$  may be assigned with a finite processing time to an unbounded number of machines, and on the other hand, its processing time may vary considerably across those machines. In the *graph balancing* case, only instances where every job may be assigned to at most two machines with finite processing time are considered. In the *restricted* case, often referred to as the RESTRICTED ASSIGNMENT MAKESPAN MINIMIZATION problem, the input processing times obey the condition  $p_{ij} \in \{\infty, p_j\}$  for each  $i \in M$  and  $j \in J$ , where  $p_j$  is a *machine independent* processing time for job  $j$ . This has the natural interpretation that each job has a fixed processing time but may only be scheduled on some subset of the machines. The latter special case is the focus of this work.

The elegant 2-approximation of Lenstra et al. [LST90] works by rounding extreme point solutions to a linear program called the Assignment LP. As this linear program has a matching integrality gap, one of the natural directions was to develop a stronger convex relaxation for the *restricted* case. An important step forward was made by Bansal and Sviridenko [BS06] who, among other things, introduced the Configuration LP for the problem, which has exponentially many decision variables. At the time, however, it was not clear if this new linear program was indeed stronger than the Assignment LP in the sense of a worst case integrality gap. A breakthrough in this direction was achieved by Svensson [Sve12] who proved that the integrality of the Configuration LP is no worse than  $33/17 \approx 1.94$ , and, therefore, strictly better than the Assignment LP. Tantalizingly, however, the proof of his result did not lead to an approximation algorithm with the same (or even similar) guarantee. This is a fairly strange situation for a problem, as we usually expect integrality gap upper bounds to accompany an approximation algorithm; indeed, it is often established as a *consequence* of the latter.

The difficulties in turning the non-constructive aspects of Svensson’s result into an efficient algorithm mirror the situation faced in the RESTRICTED MAX-MIN FAIR ALLOCATION problem. In the latter problem, following a long line of work [BS06, Fei08, HSS11, AFS12, PS12, AKS15], a non-constructive integrality gap upper bound on the Configuration LP by Asadpour, Feige and Saberi [AFS12] was turned into an approximation algorithm [AKS15]. Although one might reasonably hope for a similar resolution in the *restricted* case, it has proved to be elusive thus far. Indeed, there have been works aimed at obtaining better-than-2 approximation algorithms for special cases,

and the gap between the  $33/17 \approx 1.94$  (recently improved to  $11/6 \approx 1.83$  [JR16]) integrality gap upper bound and the 2 approximation algorithm of Lenstra et al. [LST90] persists. Ebenlendr, Křéal and Sgall [EKS08] studied the case when jobs may be assigned to at most two machines and gave a 1.75 approximation in this case. Recently, Chakrabarty, Khanna, and Li [CKL15] designed a  $2 - \epsilon_0$  approximation algorithm, for some constant  $\epsilon_0 > 0$ , for the so-called  $(1, \epsilon)$ -case where processing times of jobs are drawn from a set of size two.

The special significance of  $(1, \epsilon)$ -case is that it already proves to be hard from the perspective of the Configuration LP—the best known 1.5 factor integrality gap instances are of this type—and seems to adequately capture the difficulty of the *restricted* case in general. It is also interesting in its own right, and in a sense it is the simplest case of the problem that is not yet fully understood. Indeed, the case when processing times of all the jobs are equal can be solved optimally in polynomial time: this amounts to finding a maximum flow in an appropriate flow network with jobs as sources, machines as sinks, and setting the sink capacities to be uniformly equal to some guess on the optimum makespan.

**Our Results.** After normalizing the job sizes, we assume without loss of generality that the jobs are of size 1 or  $\epsilon$  for some  $0 < \epsilon < 1$ . Our main result is a new purely flow based local search algorithm for the  $(1, \epsilon)$ -case.

**Theorem 1.1.** *Let  $0 < \epsilon < 1$ . For an arbitrary but fixed  $\zeta > 0$ , the  $(1, \epsilon)$ -case of RESTRICTED ASSIGNMENT MAKESPAN MINIMIZATION admits a polynomial time  $1 + R(\epsilon, \zeta)$  approximation algorithm where*

$$R(\epsilon, \zeta) \triangleq \frac{1}{2} (\sqrt{3 - 2\epsilon} + \epsilon) + \zeta.$$

From the point of view of better-than-2 approximation algorithms, the hard case is when  $\epsilon \rightarrow 0$ . For this range of values, the approximation ratio guaranteed by Theorem 1.1 tends to  $1 + \sqrt{3}/2 \approx 1.87$ . By balancing against a simple algorithm based on bipartite matching we also derive an approximation guarantee independent of the size of small jobs.

**Theorem 1.2.** *Let  $0 < \epsilon < 1$ . For an arbitrary but fixed  $\zeta > 0$ , the  $(1, \epsilon)$ -case of RESTRICTED ASSIGNMENT MAKESPAN MINIMIZATION admits a polynomial time  $17/9 + \zeta$  approximation algorithm. Note that  $17/9 \approx 1.89$ .*

**Our Techniques.** We now give a very high level overview of the ideas behind the proof of Theorem 1.1 assuming that the optimum makespan is 1 for simplicity.

Our local search algorithm continually increases the number of jobs scheduled by an assignment  $\sigma : M \mapsto J \cup \{\text{TBD}\}$ <sup>1</sup> that satisfies the  $1 + R$  makespan bound i.e.,  $\sum_{j \in \sigma^{-1}(i)} p_{ij} \leq 1 + R$  for each  $i \in M$ . The algorithm takes a job  $j_0$  such that  $\sigma(j_0) = \text{TBD}$  and attempts to assign it to one of the machines  $M$  while respecting the makespan bound of  $1 + R$ . In general, it may be required to modify the assignment  $\sigma$ , which we call a partial schedule, before  $j_0$  can be successfully assigned along with the rest of the jobs in  $\sigma^{-1}(M)$ . The algorithm identifies a set of machines  $M_0$  such that sufficiently reducing the load on any of the machines  $i \in M_0$  suffices to assign  $j_0$  successfully. Once again, to reduce the load on one of the machines in  $M_0$  a new set of machines  $M_1$  that is disjoint from  $M_0$  is identified. In this way, in general there will be a sequence of disjoint machine sets  $M_0, M_1, \dots, M_\ell$  such that reducing the load on some  $M_i$  allows the load to be reduced on some

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<sup>1</sup>TBD for “to be decided”

machine in  $M_0 \cup \dots \cup M_{i-1}$ . At some point, it may turn out that a lot of machines in  $M_\ell$  have very little load, call them *free* machines, and therefore, it is possible for many of the jobs currently scheduled on some machine in  $M_0 \cup \dots \cup M_{\ell-1}$  to be relocated to free machines in  $M_\ell$ , which represents progress towards our goal of eventually scheduling  $j_0$ .

The first property we require is *large progress* which ensures that many free machines in  $M_\ell$  implies that proportionally many jobs from  $\sigma^{-1}(M_0 \cup \dots \cup M_{\ell-1})$  can be relocated. Further, such relocations should be performed in a way as to maintain the same property for the machine sets of smaller indices.

Large progress by itself would not guarantee that the algorithm terminates quickly if it does not happen often enough. To ensure that we find many free machines *frequently*, a second property ensures  $|M_i| \geq \mu |M_0 \cup \dots \cup M_{i-1}|$  so that large progress happens at least once every  $O_\mu(\log |M|)$  sets that are encountered.

These two properties—and maintaining them as the partial schedule is continually modified by the algorithm—roughly correspond to the core technical difficulties of our approach. The proof of the first property (see Section 3.3.4) makes fairly extensive use of properties of maximum flows, while the second property (see Section 3.3.5) is based on new ideas for constructing dual unboundedness certificates for the Configuration LP. It is interesting to note that the construction of such certificates in the analysis, to prove the required property about the local search algorithm, effectively amounts to a second algorithm that is merely used to determine an assignment of values to the dual variables in the proof.

Our final algorithm is a highly structured local search that performs job relocations *only* through flow computations in two kinds of flow networks. In essence, our approach can be seen as effectively reducing the scheduling problem we started with to polynomially many maximum flow computations in a structured fashion. The sets alluded to earlier correspond to certain “reachability graphs” associated with  $\epsilon$  jobs whereas the number of such sets at any time is bounded by a function of the number of 1 jobs in the instance. We also stress that identifying such sets, which allow our analyses to prove these properties, requires a careful design of the algorithm in the first place, and is not clear at the outset that this can be achieved.

**Organization.** The rest of the paper is organized as follows. In Section 2, we briefly go over some notation and state the Configuration LP for our problem. In Section 3.1 we explain some of the basic concepts such as flow networks and subroutines for job relocations used in the final local search algorithm. The algorithm and its running time analysis are presented in Sections 3.2 and 3.3 respectively, followed by the proof of the main theorem in the paper in Section 3.4. Proofs of some statements missing in the body of the paper appear in Appendix A.

## 2 Preliminaries

### 2.1 Notation and Conventions

Let  $\mathcal{I}$  be the given instance of RESTRICTED ASSIGNMENT MAKESPAN MINIMIZATION. By scaling all the processing times in the instance, we assume without loss of generality that  $p_{\max} \stackrel{\Delta}{=} \max\{p_j \mid j \in J\} = 1$ . We use OPT to denote the optimal makespan for  $\mathcal{I}$ .

For a subset of jobs  $S \subseteq J$ , we use the notation  $p(S)$  and  $p_i(S)$  to refer to  $\sum_{j \in S} p_j$  and  $\sum_{j \in S} p_{ij}$  respectively. We define  $\Gamma : J \mapsto 2^M$  to be a function that maps each job  $j \in J$  to the set of

all machines that it can be assigned to with a finite processing time. The input processing times  $\{p_{ij}\}_{i \in M, j \in J}$  and  $\{p_j\}_{j \in J}$  satisfy

$$p_{ij} = \begin{cases} p_j, & \text{if } i \in \Gamma(j), \\ \infty, & \text{else.} \end{cases} \quad \forall i \in M \forall j \in J.$$

For a collection of indexed sets  $\{S_0, \dots, S_\ell\}$  and  $0 \leq i \leq \ell$  we use the notation  $S_{\leq i}$  to refer to the set  $\bigcup_{j=0}^i S_j$ .

## 2.2 The Configuration Linear Program

The Configuration LP is a feasibility linear program parametrized by a guess  $\tau$  on the value of the optimal makespan for  $\mathcal{I}$ , and is simply denoted by  $CLP(\tau)$ . A *configuration* for machine  $i$  is a set of jobs with a total processing time at most  $\tau$  on machine  $i$ . The collection of all such configurations for  $i$  is denoted as  $\mathcal{C}(i, \tau)$ .  $CLP(\tau)$  ensures that each machine receives at most one configuration fractionally, while ensuring that every job is assigned, also in the fractional sense. The constraints of  $CLP(\tau)$  are described in (2.1).

$$\begin{aligned} \sum_{C \in \mathcal{C}(i, \tau)} x_{iC} &\leq 1, \quad \forall i \in M, \\ \sum_{i \in M} \sum_{C \in \mathcal{C}(i, \tau) : j \in C} x_{iC} &\geq 1, \quad \forall j \in J, \\ x &\geq 0. \end{aligned} \tag{2.1}$$

We can also write the dual of  $CLP(\tau)$  as follows.

$$\begin{aligned} \max \quad & \sum_{j \in J} z_j - \sum_{i \in M} y_i \\ & y_i \geq \sum_{j \in C} z_j, \quad \forall i \in M \text{ and } C \in \mathcal{C}(i, \tau), \\ & y, z \geq 0. \end{aligned} \tag{2.2}$$

Let  $\tau^*$  be the smallest value of  $\tau$  for which  $CLP(\tau)$  is feasible. We refer to  $\tau^*$  as the value of the Configuration LP. Observe that  $\tau^*$  is a lower bound on  $\text{OPT}$ . As  $p_{\max} = 1$ ,  $\tau^*$  must be at least 1.

## 3 The $(1, \epsilon)$ Case

Let  $0 < \epsilon < 1$ . In the  $(1, \epsilon)$ -case of RESTRICTED ASSIGNMENT MAKESPAN MINIMIZATION, jobs  $j \in J$  have one of only two possible sizes: 1 or  $\epsilon$ . We partition the jobs accordingly into the sets  $J_b \triangleq \{j \in J \mid p_j = 1\}$  and  $J_s \triangleq J \setminus J_b$ , which we will refer to as the sets of *big* and *small* jobs respectively.

For the rest of the section fix some  $0 < \epsilon < 1$  and  $\zeta > 0$ . As  $\epsilon$  and  $\zeta$  are fixed constants, we refer to  $R(\epsilon, \zeta)$  as simply  $R$ . To prove Theorem 1.1 we describe an algorithm that terminates in polynomial time with a schedule of makespan at most  $\tau^* + R$  for the given instance  $\mathcal{I}$ . This algorithm, described in Section 3.2, is a local search algorithm which continually increases the size of a *partial schedule*.

**Definition 3.1** (Partial schedule). A partial schedule is a map  $\sigma : J \mapsto M \cup \{\text{TBD}\}$  such that

- (a)  $\forall i \in M, p_i(\sigma^{-1}(i)) \leq \tau^* + R,$
- (b)  $J_b \subseteq \sigma^{-1}(M),$  and
- (c)  $\forall i \in M, |\sigma^{-1}(i) \cap J_b| \leq 1.$

The *size* of a partial schedule  $\sigma$  is  $|\sigma^{-1}(M)|.$

*Remark 3.2.* Following the description and analysis of our algorithms, it will become clear that solving the Configuration LP on the input instance  $\mathcal{I}$ , in order to determine  $\tau^*$ , is not necessary. For the moment, however, it may be assumed that it is somehow known. We remark that  $\tau^*$  can be computed in polynomial time upto any desired accuracy  $\nu > 0$  by using the ellipsoid algorithm with an appropriate separation oracle [BS06].

Of course, a partial schedule  $\sigma$  of size  $|J|$  is a schedule of makespan at most  $\tau^* + R$  for our instance  $\mathcal{I}$ . The following statement ensures that partial schedules exist in the first place.

**Lemma 3.3.** *Suppose  $1 \leq \tau^* < 2$ . Then, there is a map from  $J_b$  to  $M$  such that i) no two big jobs are mapped to the same machine, and ii) every big job  $j \in J_b$  is mapped to a machine  $i_j \in M$  such that  $i_j \in \Gamma(j)$ . Furthermore, such a map can be computed in polynomial time.*

### 3.1 Flow Networks for Job Relocations

Let  $\sigma$  be some partial schedule. We now define several quantities whose description depends on  $\sigma$ .

**Definition 3.4** (Job Assignment Graphs).  $G_\sigma = (M \cup J, E)$  is a directed bipartite graph with machines and jobs in  $\mathcal{I}$  as vertices. The edge set of  $G_\sigma$  is defined as

$$E \triangleq \{(i, j) \mid \exists i \in M, j \in J : \sigma(j) = i\} \cup \{(j, i) \mid \exists i \in M, j \in J : \sigma(j) \neq i, i \in \Gamma(j)\}.$$

We define the graph of small job assignments  $G_\sigma^s \triangleq G_\sigma \setminus J_b$  and the graph of big job assignments  $G_\sigma^b \triangleq G_\sigma \setminus J_s.$

**Definition 3.5** (Big and Small Machines). Let  $M_\sigma^b \triangleq \{i \in M \mid \sigma^{-1}(i) \cap J_b \neq \emptyset\}$  and  $M_\sigma^s \triangleq M \setminus M_\sigma^b,$  which we refer to as *big machines* and *small machines* respectively.

We need to define two flow networks which will facilitate the movement of the two kinds of jobs we have in our instance  $\mathcal{I}$ . We will speak of *the* maximum flow in these flow networks, even though it may not necessarily be unique. In such cases it is implicitly assumed that there is fixed rule to obtain a particular maximum flow given a flow network. We also assume that flow decompositions of flows in such networks contain only source to sink paths (no cycles). First, we define the flow network for big jobs.

**Definition 3.6** (Flow Network for Big Jobs). For collections of machines  $S \subseteq M_\sigma^b,$  and  $T \subseteq M_\sigma^s,$  the flow network  $H_\sigma^b(S, T)$  is defined on the directed graph  $G_\sigma^b$  as follows. Each machine to job arc has a capacity of 1 whereas all other arcs have infinite capacity.  $S$  and  $T$  are the sources and sinks respectively in this flow network. Sinks have vertex capacities of 1. The value of maximum flow in this flow network is denoted as  $|H_\sigma^b(S, T)|.$

The interpretation of the flow network  $H_\sigma^b(S, T)$  is that it allows us to compute the maximum number of vertex disjoint paths in the graph  $G_\sigma^b$  between the sets of vertices  $S$  and  $T$ .

**Proposition 3.7.** *For any  $S \subseteq M_\sigma^b$  and  $T \subseteq M_\sigma^s$ , there are  $|H_\sigma^b(S, T)|$  vertex disjoint paths in  $G_\sigma^b$  with sources in  $S$  and sinks in  $T$ .*

These vertex disjoint paths suggest an update of the partial schedule  $\sigma$  in the natural way. Algorithm 3.1 formalizes the description of this task and Proposition 3.8 follows easily.

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**Algorithm 3.1** BigUpdate( $\sigma, X$ ): Update  $\sigma$  using flow paths in  $X$ .

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**Require:**  $\sigma$  is a partial schedule and  $X$  is a flow in  $H_\sigma^b(M_\sigma^b, M_\sigma^s)$  where

$$\forall f \in X, \quad p(\sigma^{-1}(f \cap M_\sigma^s)) \leq \tau^* + R - 1.$$

$P \leftarrow$  Vertex disjoint paths corresponding to  $X$  as ensured by Proposition 3.7.

**for all**  $p = i_0, j_0, \dots, j_{k_p-1}, i_{k_p} \in P$  **do**

**for**  $\ell = 0, \dots, k_p - 1$  **do**

$\sigma(j_\ell) \leftarrow i_{\ell+1}$ .

**end for**

**end for**

**return**  $\sigma$ .

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**Proposition 3.8.** *For any partial schedule  $\sigma$ , and flow  $X$  in  $H_\sigma^b(M_\sigma^b, M_\sigma^s)$  such that  $\forall f \in X$ ,  $p(\sigma^{-1}(f \cap M_\sigma^s)) \leq \tau^* + R - 1$ , BIGUPDATE( $\sigma, X$ ) returns a partial schedule  $\sigma'$  such that*

(a)  $\sigma'^{-1}(M) = \sigma^{-1}(M)$ , and

(b)  $\forall f = i_0, j_0, \dots, j_{k-1}, i_k \in X$ ,  $\sigma'^{-1}(i_0) \cap J_b = \emptyset$ .

Now we define the flow network for small jobs.

**Definition 3.9** (Flow Network for Small Jobs). For two *disjoint* collections of machines  $S \subseteq M_\sigma^s$  and  $T \subseteq M$ , the flow network  $H_\sigma^s(S, T)$  is defined on the directed graph  $G_\sigma^s$  as follows. The arcs going from machines to jobs have capacity  $\epsilon$  while arcs going from jobs to machines have infinite capacity.  $S$  and  $T$  are the sources and sinks respectively in this flow network. The sinks have *vertex capacities* are set as follows:

$$\forall i \in T, \quad c(i) = \begin{cases} 1 + \tau^* + R - p(\sigma^{-1}(i)) - \epsilon, & \text{if } i \in M_\sigma^b, \\ \tau^* + R - p(\sigma^{-1}(i)), & \text{else.} \end{cases}$$

The value of the maximum flow in this network is denoted as  $|H_\sigma^s(S, T)|$ .

By construction it is clear that the maximum flow in both flow networks (3.6) and (3.9) is finite. By the max-flow min-cut theorem, infinite capacity arcs going from the source-side to the sink-side will therefore not cross any minimum capacity. We will use this fact later in our proof.

Algorithm 3.2 interprets flows in  $H_\sigma^s(S, T)$  as a collection of updates for  $\sigma$ . Proposition 3.10 is a statement about the partial schedule  $\sigma'$  output by SMALLUPDATE(...), and the flow  $X'$  computed at the end of the **while** loop in the procedure. For convenience we let  $f^{\text{source}}$  and  $f^{\text{sink}}$  denote the source and sink vertices, respectively, of a flow path  $f$ .

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**Algorithm 3.2**  $\text{SmallUpdate}(\sigma, S, T)$ : Update  $\sigma$  using  $H_\sigma^s(S, T)$ .

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**Require:**  $\sigma$  is a partial schedule,  $S \subseteq M_\sigma^s$ ,  $T \subseteq M \setminus S$ .

$X_0 \leftarrow$  Maximum flow in the network  $H_\sigma^s(S, \{i \in T \mid p(\sigma^{-1}(i)) \leq \tau^* + R - \epsilon\})$ .

$X \leftarrow$  Augment  $X_0$  to a maximum flow in the network  $H_\sigma^s(S, T)$ .

**while**  $\exists f = i_0, j_0, \dots, j_{k_f-1}, i_{k_f} \in X : p(\sigma^{-1}(f^{\text{sink}})) \leq \tau^* + R - \epsilon$  **do**

**for**  $\ell = 0, \dots, k_f - 1$  **do**

$\sigma(j_\ell) = i_{\ell+1}$ .

**end for**

$X \leftarrow X \setminus \{f\}$ .

**end while**

**return**  $\sigma$ .

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**Proposition 3.10.** *Let  $S \subseteq M_\sigma^s$ ,  $T \subseteq M \setminus S$ , and  $X$  be the maximum flow in  $H_\sigma^s(S, T)$ . Then,  $\text{SMALLUPDATE}(\sigma, S, T)$  returns a partial schedule  $\sigma'$  and computes a maximum flow  $X'$  in  $H_{\sigma'}^s(S, T)$  at the end of the **while** loop in Algorithm 3.2 such that*

(a)  $\sigma'^{-1}(M) = \sigma^{-1}(M)$ ,

(b)  $\forall i \in S, p(\sigma'^{-1}(i)) - \epsilon \cdot |\{f \in X' \mid f^{\text{source}} = i\}| = p(\sigma^{-1}(i)) - \epsilon \cdot |\{f \in X \mid f^{\text{source}} = i\}|$ , and

(c)  $\forall f \in X, p(\sigma'^{-1}(f \cap T)) > \tau^* + R - \epsilon$ .

(d) *There is no path in the graph  $G_{\sigma'}^s$ , from  $S$  to some machine  $i \in T$  such that*

$$p(\sigma'^{-1}(i)) \leq \tau^* + R - \epsilon.$$

*Proof.* The first three properties follow directly from the updates performed in the **while** loop of Algorithm 3.2. To see that  $X'$  is a maximum flow in  $H_{\sigma'}^s(S, T)$ , first observe that for each update of the partial schedule  $\sigma$  maintained by the algorithm, along some flow path  $f \in X$  from  $\sigma^{(b)}$  to  $\sigma^{(a)}$  in a single iteration of the **while** loop (superscripts for before and after), the graph  $G_{\sigma^{(a)}}^s$  can be obtained from  $G_{\sigma^{(b)}}^s$  by simply reversing the directions of arcs of the flow path in question. Suppose that  $X$  is a maximum flow in  $H_{\sigma^{(b)}}^s(S, T)$ . By the max-flow min-cut theorem, the maximum flow is associated with a minimum capacity cut of equal value, and the latter observation implies that both the flow value in  $X \setminus \{f\}$  and the corresponding cut capacity in the new network  $H_{\sigma^{(a)}}^s(S, T)$  is less, than the flow value in  $X$  and its corresponding cut capacity in  $H_{\sigma^{(b)}}^s(S, T)$ , by  $\epsilon$ . This implies that  $X \setminus \{f\}$  is a maximum flow in  $H_{\sigma^{(a)}}^s(S, T)$ .

The final property follows from the particular way in which the maximum flow  $X$  in  $H_\sigma^s(S, T)$  is constructed. Note that Algorithm 3.2 first computes a flow  $X_0$  that maximizes the flow to machines in  $\{i \in T \mid p(\sigma^{-1}(i)) \leq \tau^* + R - \epsilon\}$ , and then augments  $X_0$  to a maximum flow in  $H_\sigma^s(S, T)$ . Suppose to the contrary that there is a path  $P$  in  $G_{\sigma'}^s$ , from  $S$  to a machine  $i$  such that  $p(\sigma'^{-1}(i)) \leq \tau^* + R - \epsilon$ . We know that one can obtain  $G_{\sigma'}^s$  from  $G_\sigma^s$  by reversing the arc directions of all the flow paths in  $X \setminus X'$  i.e., the paths that were used to update the partial schedule in the **while** loop. So each arc in  $P$  is either i) present in some path of  $X \setminus X'$  in the opposite direction, or ii) not present in the paths of  $X \setminus X'$  in the opposite direction and, therefore, also present in  $G_\sigma^s$ . Now consider the *residual flow network* of the flow  $X \setminus X'$  in  $H_\sigma^s(S, T)$ . It is now easy to see that  $P$  is an augmenting path in this residual flow network because  $p(\sigma^{-1}(i)) + \epsilon \cdot |\{f \in X \setminus X' \mid f^{\text{sink}} = i\}| = p(\sigma'^{-1}(i)) \leq \tau^* + R - \epsilon$ , and, therefore,  $c(i) \geq \tau^* + R - p(\sigma^{-1}(i)) \geq \epsilon \cdot |\{f \in X \setminus X' \mid f^{\text{sink}} = i\}| + \epsilon$ . This, however, contradicts the maximality of the flow  $X_0$  computed in the first step of the algorithm.  $\square$

We now have the tools necessary to state our local search algorithm.

### 3.2 Flow Based Local Search

In this section we describe the local search algorithm that takes as input  $\tau^*$ , a partial schedule  $\sigma$ , and a small job  $j_0 \in J_s \setminus \sigma^{-1}(M)$ . It outputs a partial schedule  $\sigma'$  such that  $\sigma'^{-1}(M) = \sigma^{-1}(M) \cup \{j_0\}$ . The algorithm is parameterized by three constants  $0 < \mu_1, \mu_2, \delta \leq 1$ . They are defined as:

$$\begin{aligned}\mu_1 &\triangleq \min\{1, \zeta\}/4, \\ \mu_2 &\triangleq \min\{\delta, \zeta\}/4, \\ \delta &\triangleq (\sqrt{3 - 2\epsilon} - 1) / 2.\end{aligned}\tag{3.1}$$

The algorithm maintains certain sets of machines in *layers*  $L_0, \dots, L_\ell$  throughout its execution, where  $\ell$  is some dynamically updated index variable that always points to the last layer. A layer  $L_i$  is a tuple  $(A_i, B_i)$  where  $A_i \subseteq M_\sigma^s$ , and  $B_i \subseteq M$ , except  $L_0$  which is defined to be  $(\{j_0\}, B_0)$  for some  $B_0 \subseteq M$ . In addition to layers, the algorithm also maintains a collection of machines  $\{I_i\}_{i=0}^\ell$  that will be disjoint from the machines in  $L_{\leq \ell}$ .

We will describe the algorithm in a procedural style, and in the course of the execution the algorithm, sets and other variables will be modified. Function calls are specified in the pseudocode assuming call-by-value semantics. Concretely, this means that function calls have no side effects besides the assignment of the returned values at the call site. We abuse notation slightly and use  $L_i$  to also refer to  $A_i \cup B_i$  (for  $i = 0$ , as  $A_0$  is not a set of machines, we use  $L_0$  to just mean  $B_0$ ), so that  $L_{\leq \ell}$  refers to a set of machines. For a subset of machines  $N \subseteq M$  we use  $R_\sigma^s(N)$  denote the set of all machines reachable from vertices  $N$  in the graph  $G_\sigma^s$ . Note that  $N \subseteq R_\sigma^s(N)$  always holds.

The description is now found in Algorithm 3.3. We refer to the **while** loop in Step 3 of Algorithm 3.3 as the *main loop* of the algorithm. Observe that, in Step 4, Algorithm 3.4 is used a subroutine, which constructs and returns a new layer while potentially modifying the partial schedule  $\sigma$  maintained by Algorithm 3.3.

### 3.3 Running Time Analysis

The *state of the algorithm* is defined to be the dynamic tuple  $\mathcal{S} \triangleq (\sigma, \ell, \{L_i\}_{i=0}^\ell, \{I_i\}_{i=0}^\ell)$  which contains the variables and sets that are maintained by Algorithm 3.3. In the analysis it will be useful to compare quantities before and after certain operations performed by the algorithm, and we will consistently use  $\mathcal{S}$  and  $\mathcal{S}'$  to refer to the state of the algorithm before and after such an operation. For example, if  $\mathcal{S}$  and  $\mathcal{S}'$  denote the states of the algorithm before Step 4 and after Step 5 in Algorithm 3.3 respectively, then  $\ell' = \ell + 1$ ,  $I_{\ell'} = \emptyset$ , etc.

#### 3.3.1 Basic Invariants of the Algorithm

By observing the description of Algorithms 3.3 and 3.4 we can conclude certain basic properties which will come in handy when reasoning about its running time.

**Proposition 3.11.** *Consider some state  $\mathcal{S}$  of the algorithm.*

- (a) *The sets in the collection  $\{A_i\}_{i=1}^\ell \cup \{B_i\}_{i=0}^\ell \cup \{I_i\}_{i=0}^\ell$  are pairwise disjoint subsets of  $M$ .*

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**Algorithm 3.3** LocalSearch( $\tau^*, \sigma, j_0$ ): Extend the partial schedule  $\sigma$  to include small job  $j_0$ .

---

**Require:**  $\sigma$  is a partial schedule,  $j_0 \in J_s \setminus \sigma^{-1}(M)$ .

```

1: Set  $A_0 \leftarrow \{j_0\}$ ,  $B_0 \leftarrow R_\sigma^s(\Gamma(j_0))$ . ▷ Construction of layer  $L_0$ 
2: Set  $\ell \leftarrow 0$  and  $I_0 \leftarrow \emptyset$ .
3: while  $\nexists i \in B_0$  such that  $p(\sigma^{-1}(i)) \leq \tau^* + R - \epsilon$  do ▷ Main loop
4:    $(\sigma, A_{\ell+1}, B_{\ell+1}) \leftarrow \text{BUILD LAYER}(\sigma, \{L_i\}_{i=0}^\ell, \{I_i\}_{i=0}^\ell)$ . ▷ Construction of layer  $L_{\ell+1}$ 
5:   Set  $\ell \leftarrow \ell + 1$  and  $I_{\ell+1} \leftarrow \emptyset$ .
6:   while  $\ell \geq 1$  and  $|\{i \in A_\ell \mid p(\sigma^{-1}(i)) \leq \tau^* + R - 1\}| \geq \mu_2 |A_\ell|$  do
7:     Set  $I \leftarrow \{i \in A_\ell \mid p(\sigma^{-1}(i)) \leq \tau^* + R - 1\}$ .
8:      $(I'_0, \dots, I'_\ell, X) \leftarrow \text{CANONICAL DECOMPOSITION}(\sigma, \{L_i\}_{i=0}^\ell, \{I_i\}_{i=0}^\ell, I)$ .
9:     Set  $I_i \leftarrow I'_i$  for all  $1 \leq i \leq \ell$ .
10:    if  $\exists 1 \leq r \leq \ell : |I_r| \geq \mu_1 \mu_2 |B_{r-1} \cap M_\sigma^b|$  then
11:      Choose the smallest such  $r$ .
12:       $\sigma \leftarrow \text{BIG UPDATE}(\sigma, \{f \in X \mid f \cap I_r \neq \emptyset\})$ .
13:       $\sigma \leftarrow \text{SMALL UPDATE}(\sigma, A_{r-1}, B_{r-1})$  unless  $r = 1$ .
14:       $B_{r-1} \leftarrow R_\sigma^s(A_{r-1}) \setminus (A_{r-1} \cup L_{\leq r-2} \cup I_{\leq r-2})$  unless  $r = 1$ .
15:      Discard all layers with indices greater than  $r - 1$ .
16:      Set  $\ell \leftarrow r - 1$ .
17:    end if
18:  end while
19: end while
20: Update  $\sigma$  using a path from  $j_0$  to  $i$  in  $G_\sigma^s$  where  $p(\sigma^{-1}(i)) \leq \tau^* + R - \epsilon$ .
21: return  $\sigma$ .
22:
23: function CANONICALDECOMPOSITION( $\sigma, \{L_i\}_{i=0}^\ell, \{I_i\}_{i=0}^\ell, I$ )
24:   Let  $X$  be the maximum flow in  $H_\sigma^b(B_0 \cap M_\sigma^b, I_{\leq \ell} \cup I)$ .
25:   for  $1 \leq i \leq \ell - 1$  do
26:     Augment  $X$  to a maximum flow in  $H_\sigma^b(B_{\leq i} \cap M_\sigma^b, I_{\leq \ell} \cup I)$ .
27:   end for
28:   for  $1 \leq i \leq \ell$  do
29:     Set  $I'_i$  to be the collection of sinks used by flow paths from  $X$  with sources in  $B_{i-1}$ 
30:   end for
31:   return  $(\emptyset, I'_1, \dots, I'_\ell, X)$ .
32: end function

```

---

---

**Algorithm 3.4** BuildLayer( $\sigma, \{L_i\}_{i=0}^\ell, \{I_i\}_{i=0}^\ell$ ): Construct and return a new layer.

---

```

1: Let  $S \leftarrow \emptyset$ .
2: while  $\exists i \in M : \text{ISADDABLEQ}(i, \sigma, \{L_i\}_{i=0}^\ell, \{I_i\}_{i=0}^\ell, S)$  do
3:    $\sigma \leftarrow \text{SMALLUPDATE}(\sigma, S \cup \{i\}, T \setminus \{i\})$  where  $T \leftarrow M \setminus (L_{\leq \ell} \cup I_{\leq \ell} \cup S)$ .
4:    $S \leftarrow S \cup \{i\}$ .
5: end while
6:  $A_{\ell+1} \leftarrow S$ .
7:  $B_{\ell+1} \leftarrow R_\sigma^s(A_{\ell+1}) \setminus (A_{\ell+1} \cup L_{\leq \ell} \cup I_{\leq \ell})$ .
8: return  $(\sigma, A_{\ell+1}, B_{\ell+1})$ .
9:
10: function ISADDABLEQ( $i, \sigma, \{L_i\}_{i=0}^\ell, \{I_i\}_{i=0}^\ell, S$ ) ▷ Decide if  $i$  can be added to  $S$ 
11:   if  $i \notin M_\sigma^s \setminus (L_{\leq \ell} \cup I_{\leq \ell} \cup S)$  then
12:     return False.
13:   end if
14:   Set  $T \leftarrow M \setminus (L_{\leq \ell} \cup I_{\leq \ell} \cup S)$ .
15:   if  $|H_\sigma^b(B_{\leq \ell} \cap M_\sigma^b, S \cup \{i\})| = |H_\sigma^b(B_{\leq \ell} \cap M_\sigma^b, S)| + 1$  then
16:     if  $|H_\sigma^s(S \cup \{i\}, T \setminus \{i\})| \geq |H_\sigma^s(S, T)| + (p(\sigma^{-1}(i)) - (\tau^* - 1 + R - \delta))$  then
17:       return True.
18:     end if
19:   end if
20:   return False.
21: end function

```

---

(b) For each  $i = 1, \dots, \ell$ , the sets  $A_i$  have not been modified since the last time  $L_i$  was initialized in some execution of Step 4 of Algorithm 3.3. Similarly, for  $i = 0$ , the sets  $A_0$  and  $B_0$  have not been modified since the execution of Step 1.

(c) For each  $i \in B_{\leq \ell}$ ,  $p(\sigma^{-1}(i)) > \tau^* + R - \epsilon$ .

(d) For every  $j \in \sigma^{-1}(L_{\leq \ell}) \cap J_s$ ,  $\Gamma(j) \subseteq L_{\leq \ell} \cup I_{\leq \ell}$ .

*Proof.* (a) For a newly constructed layer  $L_{\ell+1}$  in Step 4 of Algorithm 3.3, the sets  $A_{\ell+1}$  and  $B_{\ell+1}$  satisfy the properties by the construction of the set  $S$  and the setting in Step 7 Algorithm 3.4. Further,  $I_{\ell+1}$  is initialized to the empty set in Step 5 of Algorithm 3.3. In future updates of  $B_{\ell+1}$  (if any) in Step 14 of Algorithm 3.3, let  $\mathcal{S}$  and  $\mathcal{S}'$  be the states before Step 13 and after Step 14 of Algorithm 3.3 respectively. From the description of Algorithm 3.2, we see that  $\sigma^{-1}(A_{r-1}) \supseteq \sigma'^{-1}(A_{r-1}) = \sigma'^{-1}(A'_{r-1})$ , which then implies that  $B_{r-1} \supseteq B'_{r-1}$  from the assignment in Step 14.

(b) This follows directly from the description of the algorithm.

(c) When a new layer  $L_{\ell+1}$  is constructed during a call to BUILD LAYER(...), at the end of the **while** loop in Step 2 of Algorithm 3.4, we show in Claim 3.19, that a maximum flow  $X$  in  $H_\sigma^s(S, T)$  is computed where  $T = M \setminus (L_{\leq \ell} \cup I_{\leq \ell} \cup S)$ . So we can apply Proposition 3.10(d) and conclude that after the assignment in Step 7, there is no machine  $i \in B_{\ell+1}$  such that  $p(\sigma^{-1}(i)) \leq \tau^* + R - \epsilon$ . We can argue in exactly the same way in Steps 13 and 14 of Algorithm 3.3.

(d) This follows from Step 7 of Algorithm 3.4 and Step 14 of Algorithm 3.3.  $\square$

**Definition 3.12** (Collapsibility of a layer). Layer  $L_0$  is *collapsible* if there is an  $i \in B_0$  such that  $p(\sigma^{-1}(i)) \leq \tau^* + R - \epsilon$ . For  $\ell \geq 1$ ,  $L_\ell$  is *collapsible* if  $A_\ell$  contains at least  $\mu_2|A_\ell|$  machines  $i$  such that  $p(\sigma^{-1}(i)) \leq \tau^* + R - 1$ .

Note the correspondence between Definition 3.12 and the conditions in Steps 3 and 6 of Algorithm 3.3.

**Lemma 3.13.** *At the beginning of each iteration of the main loop of Algorithm 3.3, none of the layers  $L_0, \dots, L_\ell$  are collapsible. In particular, for all  $1 \leq i \leq \ell$ ,*

$$|\{i' \in A_i \mid p(\sigma^{-1}(i')) \leq \tau^* + R - 1\}| < \mu_2|A_i|.$$

*Proof.* In the first iteration of the main loop of the algorithm,  $\ell = 0$ , and the statement follows from the condition of the main loop of the algorithm. Assume the statement to hold at the beginning of some iteration of the main loop of the algorithm with the state  $(\sigma, \ell, \{L_i\}_{i=0}^\ell, \{I_i\}_{i=0}^\ell)$ . Observe that a new layer  $L_{\ell+1}$  is created in Step 4 and it is tested for collapsibility in Step 6. Suppose that the **while** loop in the latter step executes at least once (if it happens infinitely many times, we are done). Let  $r$  denote the choice made in Step 11 in the last execution of the step. The layers  $L_0, \dots, L_{r-2}$  continue to be non-collapsible by induction and Proposition 3.11(b), and layer  $L_{r-1}$  is not collapsible because execution exits the **while** loop.  $\square$

**Lemma 3.14.** *At the beginning of each iteration of the main loop of the algorithm, for every  $0 \leq i \leq \ell - 1$ ,*

$$|I_{i+1}| < \mu_1\mu_2|B_i \cap M_\sigma^b|.$$

*Proof.* The sets  $I_0, \dots, I_\ell$  maintained by the algorithm start out initialized to  $\emptyset$  in Step 5 of Algorithm 3.3 when the corresponding layer is created. They are modified only within the **while** loop of Step 6 of Algorithm 3.3 through the computation of the canonical decomposition and assignment in Step 9. Within this loop, in Step 11, the smallest  $1 \leq r \leq \ell$  such that  $|I_r| \geq \mu_1\mu_2|B_{r-1} \cap M_\sigma^b|$  is chosen; layers with indices greater than  $r - 1$  are discarded in Step 15; and  $\ell$  is set to  $r - 1$  at the end of the loop in Step 16. Therefore, the claim follows.  $\square$

### 3.3.2 Maximum Flows and Canonical Decompositions

We now recall some properties about network flows that will be of use later on in the proof our main theorem. For basic concepts related to flows, such as residual flow networks and augmenting paths, we refer the reader to the textbook by Cormen, Leiserson, Rivest and Stein [CLRS09].

**Proposition 3.15.** *Let  $S \subseteq M_\sigma^b$  and  $T \subseteq M_\sigma^s$ . Let  $S' \subseteq M_\sigma^s$  and  $T' \subseteq M$  such that  $S' \cap T' = \emptyset$ .*

- (a) *Let  $X$  be the maximum flow in  $H_\sigma^b(S, T)$  and let  $C_X$  denote the minimum capacity cut corresponding to  $X$  i.e., the set of vertices reachable from  $S$  in the residual flow network of  $X$ . Then,  $|H_\sigma^b(S, T \cup \{i\})| > |H_\sigma^b(S, T)|$  for all  $i \in C_X \setminus S$ .*
- (b) *Let  $Y$  be the maximum flow in  $H_\sigma^s(S', T')$  and let  $C_Y$  denote the minimum capacity cut corresponding to  $Y$ . For any  $i \in M \setminus C_Y$  such that  $i$  is not used as a sink by a flow path in  $Y$ , let the corresponding maximum flow in  $H_\sigma^s(S' \cup \{i\}, T' \setminus \{i\})$  be  $Y'$  and minimum capacity cut be  $C_{Y'}$ . Then,  $C_Y \subset C_{Y'}$  and this inclusion is strict.*

We state some consequences of the description of the procedure `CANONICALDECOMPOSITION(...)` in Algorithm 3.3.

**Proposition 3.16.** *For a given state  $\mathcal{S}$  and  $I \subseteq M_\sigma^s$ , let  $(I'_0, \dots, I'_\ell, X)$  be the tuple returned by `CANONICALDECOMPOSITION` $(\sigma, \{L_i\}_{i=0}^\ell, \{I_i\}_{i=0}^\ell, I)$ . Then,  $X$  is a maximum flow in the network*

$$H_\sigma^b(B_{\leq \ell-1} \cap M_\sigma^b, I_{\leq \ell} \cup I),$$

such that

- (a)  $I'_i$  is the collection of sinks used by flow paths from  $X$  with sources in  $B_{i-1}$  for all  $i = 1, \dots, \ell$ , and  $I'_0 = \emptyset$ ,
- (b)  $|H_\sigma^b(B_{\leq i} \cap M_\sigma^b, I'_{\leq i+1})| = |H_\sigma^b(B_{\leq i} \cap M_\sigma^b, I_{\leq \ell} \cup I)|$ , for all  $i = 0, \dots, \ell - 1$ , and
- (c)  $|H_\sigma^b(B_{\leq i} \cap M_\sigma^b, I'_{\leq i+1})| = |H_\sigma^b(B_{\leq i} \cap M_\sigma^b, I'_{\leq \ell})|$ , for all  $i = 0, \dots, \ell - 1$ .

### 3.3.3 Relating Set Sizes within Layers

**Lemma 3.17.** *Suppose that  $1 \leq \tau^* < 2$ . At the beginning of each iteration of the main loop of the algorithm,  $|B_0 \cap M_\sigma^b| \geq 1$ .*

*Proof.* After the execution of Step 1 of Algorithm 3.3,  $|A_0| = 1$  and  $|B_0 \cap M_\sigma^b| \geq 1$ . The latter inequality follows from the feasibility of  $CLP(\tau^*)$  and the fact that  $\tau^* + R - \epsilon \geq \tau^*$ . We omit its proof here since it is similar to Lemma 3.3. In the main loop of the algorithm, consider the first time (if at all) the schedule of big jobs on machines in  $B_0$  is altered in Step 12. Then it must be that  $|I_1| \geq \mu_1 \mu_2 |B_0 \cap M_\sigma^b| > 0$  from Step 10. Using Proposition 3.16(b) in Step 8,  $X$  contains a set of flow paths connecting sources in  $B_0 \cap M_\sigma^b$  to  $I_1$ . Then, Proposition 3.8(b) implies that  $|B'_0 \cap M_\sigma^b| < |B_0 \cap M_\sigma^b|$  after Step 12. Then, the condition of the main loop of the algorithm is no longer satisfied since  $p(\sigma'^{-1}(i)) \leq \tau^* + R - 1 \leq \tau^* + R - \epsilon$  for some  $i \in B'_0$ . The condition of the **while** loop in Step 6 is also not satisfied because  $\ell = 0$  after Step 16. Therefore, the main loop is exited in this case.  $\square$

**Lemma 3.18.** *At the beginning of each iteration of the main loop of the algorithm, for every  $1 \leq i \leq \ell$ ,*

$$|B_i \cap M_\sigma^b| > (\delta(1 - \mu_2) - 2\mu_2) \cdot |A_i|.$$

*Proof.* We first prove a general claim which will then ease the proof the lemma.

*Claim 3.19.* The **while** loop in Step 2 of Algorithm 3.4 that iteratively builds  $S$  and modifies  $\sigma$  satisfies the following invariant, where,  $T \triangleq M \setminus (L_{\leq \ell} \cup I_{\leq \ell} \cup S)$ , as defined in Step 3, and  $X$  denotes the maximum flow in the network  $H_\sigma^s(S, T)$ .

$$\begin{aligned} \epsilon |X| &\geq \sum_{i' \in S} (p(\sigma^{-1}(i')) - (\tau^* - 1 + R - \delta)), \\ \forall f \in X, \quad f^{\text{sink}} &\in M_\sigma^b. \end{aligned}$$

*Proof.* Before the first iteration of the **while** loop,  $S = \emptyset$  and the statement is vacuously true. Suppose it is true before some iteration for a set  $S$ . Let  $T$  and  $X$  be as in the claim. If  $i \in M$  is chosen in Step 2 then, from the description of the procedure ISADDABLEQ(...) in Algorithm 3.4, we can conclude that

1.  $i \in M_\sigma^s \setminus (L_{\leq \ell} \cup I_{\leq \ell} \cup S)$ ,
2.  $|H_\sigma^b(B_{\leq \ell} \cap M_\sigma^b, S \cup \{i\})| = |H_\sigma^b(B_{\leq \ell} \cap M_\sigma^b, S)| + 1$ , and
3.  $|H_\sigma^s(S \cup \{i\}, T \setminus \{i\})| \geq |H_\sigma^s(S, T)| + (p(\sigma^{-1}(i)) - (\tau^* - 1 + R - \delta))$ .

By the induction hypothesis and the first property,  $i$  cannot be a sink of some flow path in  $X$ . So,  $X$  is a valid flow in  $H_\sigma^s(S \cup \{i\}, T \setminus \{i\})$ . Using the third property we therefore conclude that  $X$  can be augmented to a maximum flow  $X'$  in  $H_\sigma^s(S \cup \{i\}, T \setminus \{i\})$  such that

$$\epsilon|X'| \geq \epsilon|X| + (p(\sigma^{-1}(i)) - (\tau^* - 1 + R - \delta)) \geq \sum_{i' \in S \cup \{i\}} (p(\sigma^{-1}(i')) - (\tau^* - 1 + R - \delta)),$$

where the second inequality uses the induction hypothesis. In Step 3, a call to SMALLUPDATE(...) is made. In this call, a maximum flow in  $H_\sigma^s(S \cup \{i\}, T \setminus \{i\})$ , say  $\bar{X}$ , is computed at the beginning of the **while** loop in Algorithm 3.2. By using Proposition 3.10(b), we can conclude that a partial schedule  $\sigma'$  and a maximum flow  $\bar{X}'$  in  $H_{\sigma'}^s(S \cup \{i\}, T \setminus \{i\})$  are computed at the of the **while** loop which satisfy the property

$$\epsilon|\bar{X}'| \geq \sum_{i' \in S \cup \{i\}} (p(\sigma'^{-1}(i')) - (\tau^* - 1 + R - \delta)).$$

Furthermore, Proposition 3.10(c) implies that  $f^{\text{sink}} \in M_\sigma^b$  for all  $f \in \bar{X}'$ . This is because the vertex capacities of small machine sinks  $i'$  is defined to be  $\tau^* + R - p(\sigma'^{-1}(i'))$  in Definition 3.9,  $p(\sigma'^{-1}(i')) > \tau^* + R - \epsilon$ , and flow paths carry flows of value  $\epsilon$ .  $\square$

Let  $L_0, \dots, L_\ell$  denote the set of layers at the beginning of the current iteration. Fix some  $1 \leq i \leq \ell$ . By Lemma 3.13,

$$|\{i' \in A_i \mid p(\sigma^{-1}(i')) \leq \tau^* + R - 1\}| < \mu_2|A_i|.$$

Now, consider the iteration (some previous one) in which  $L_i$  was constructed and let  $\sigma^{(b)}$  be the partial schedule at the end of Step 6 in Algorithm 3.4 during the corresponding call to BUILDLAYER(...). Using Claim 3.19, after the assignments in Steps 6 and 7,  $X$  is a maximum flow in  $H_{\sigma^{(b)}}^s(A_i, B_i)$  such that

$$\epsilon|X| \geq \sum_{i' \in A_i} (p(\sigma^{(b)-1}(i')) - (\tau^* - 1 + R - \delta)) > \delta(1 - \mu_2)|A_i| - 2\mu_2|A_i|,$$

where we use Lemma 3.13 in the final step along with the bound  $(\tau^* - 1 + R - \delta) \leq 2$ . Now consider a sink  $f^{\text{sink}} \in M_{\sigma^{(b)}}^b$  used by some flow path  $f \in X$ . By Proposition 3.10(c),  $p(\sigma^{(b)-1}(f \cap T)) > \tau^* + R - \epsilon$ . Definition 3.9 states that the vertex capacity  $c(f^{\text{sink}}) = 1 + \tau^* + R - p(\sigma'^{-1}(f \cap T)) - \epsilon$  since  $f^{\text{sink}} \in M_{\sigma^{(b)}}^b$  from Claim 3.19. Thus,  $c(f^{\text{sink}}) < 1$ . This proves that at the iteration in which  $L_i$  was constructed, by flow conservation,  $|B_i \cap M_{\sigma^{(b)}}^b| > (\delta(1 - \mu_2) - 2\mu_2)|A_i|$ .

In the intervening iterations, the variable  $r$  in Step 11 of Algorithm 3.3 might have been chosen to be  $i+1$ , and, therefore,  $|B_i \cap M_\sigma^b|$  may have reduced in Step 12 and  $|\{i' \in A_i \mid p(\sigma^{-1}(i')) > \tau^* + R - 1\}|$  may have reduced in Step 13. In such an event, all the layers following layer  $L_i$  would have been discarded in Step 15. But in our current iteration, the set of layers is  $L_0, \dots, L_\ell$ . So, it only remains to prove that  $|B_\ell \cap M_\sigma^b| > (\delta(1 - \mu_2) - 2\mu_2) |A_\ell|$  in the current iteration after one or more events where  $r$  was chosen to be  $\ell + 1$  in the intervening iterations. The claim is true in this case too by arguing as follows. In each intervening iteration, where  $r$  was chosen to be  $\ell + 1$ , after Step 12, the partial schedule changes from  $\sigma^{(b)}$  to  $\sigma^{(a)}$ . Due to the way sink capacities were set in Definition 3.9, the new flow network  $H_{\sigma^{(a)}}^s(A_\ell, B_\ell)$  can be obtained from the old flow network  $H_{\sigma^{(b)}}^s(A_\ell, B_\ell)$  by increasing the capacities of the machines  $M_{\sigma^{(b)}}^b \setminus M_{\sigma^{(a)}}^b$  (those machines in  $B_\ell$  from which big jobs were moved in Step 12) by  $\epsilon$ . Using the same arguments as before after applying Lemma 3.13 proves the lemma.  $\square$

### 3.3.4 Maintaining Multiple Sets of Disjoint Paths

We now prove an invariant of Algorithm 3.3 concerning the updates performed in Step 12 through the statement  $\sigma \leftarrow \text{BIGUPDATE}(\sigma, \{f \in X \mid f \cap I_r \neq \emptyset\})$ .

**Theorem 3.20.** *At the beginning of each iteration of the main loop of the algorithm, for every  $0 \leq i \leq \ell - 1$ ,*

$$|H_\sigma^b(B_{\leq i} \cap M_\sigma^b, A_{i+1} \cup I_{\leq i+1})| \geq |A_{i+1}|.$$

*Furthermore, at the beginning of each execution of the **while** loop in Step 6 of Algorithm 3.3, for all  $0 \leq i \leq \ell - 1$ ,  $|H_\sigma^b(B_{\leq i} \cap M_\sigma^b, A_{i+1} \cup I_{\leq i+1})| \geq |A_{i+1}|$ .*

*Proof.* Consider the first statement. Before the first iteration of the main loop, there is nothing to prove. Assume the statement to be true at the beginning of iteration  $u$  of the main loop for some  $u \geq 1$ . We will now show that the statement holds at the end of iteration  $u$  as well.

Following Step 4 of iteration  $u$ , the newly created layer  $L_{\ell+1}$  has the property  $I_{\ell+1} = \emptyset$  and  $|H_\sigma^b(B_{\leq \ell} \cap M_\sigma^b, A_{\ell+1})| = |A_{\ell+1}|$ , by the construction of set  $S$  in Step 2 of Algorithm 3.4. For  $0 \leq i \leq \ell - 1$ , the statement holds by the induction hypothesis and the fact that  $\sigma$  was not changed in Step 4 of Algorithm 3.3 in a way that affects the graph  $G_\sigma^b$  (indeed, only small jobs are moved). As in Step 5 of the algorithm, we also update  $\ell$  to  $\ell + 1$  in this proof, and so we have at the end of Step 5, for all  $0 \leq i \leq \ell - 1$ ,

$$|H_\sigma^b(B_{\leq i} \cap M_\sigma^b, A_{i+1} \cup I_{\leq i+1})| \geq |A_{i+1}|. \quad (3.2)$$

Now iteration  $u$  of the main loop could potentially involve one or more iterations of the **while** loop in Step 6. If there are none, we are done using (3.2). The rest of the proof follows from Claim 3.21 which completes the induction on  $u$ , and also proves the second statement in Theorem 3.20.  $\square$

*Claim 3.21.* At the end of the execution of some iteration of the **while** loop in Step 6 of Algorithm 3.3, for all  $0 \leq i \leq \ell - 1$ ,  $|H_\sigma^b(B_{\leq i} \cap M_\sigma^b, A_{i+1} \cup I_{\leq i+1})| \geq |A_{i+1}|$ , assuming it holds at the beginning of the same iteration of the **while** loop.

*Proof.* Assume the statement to be true at the beginning of some iteration of the **while** loop in question as stated in the hypothesis. We will now show that the statement holds at the end of that iteration as well.

As in Step 7, let  $I \triangleq \{i \in A_\ell \mid p(\sigma^{-1}(i)) \leq \tau^* + R - 1\}$ . In Step 8, we have the statement

$$(I'_0, \dots, I'_\ell, X) \leftarrow \text{CANONICALDECOMPOSITION}(\sigma, \{L_i\}_{i=0}^\ell, \{I_i\}_{i=0}^\ell, I),$$

which computes a *specific* maximum flow  $X$  in the network  $H_\sigma^b(B_{\leq \ell-1}, I'_{\leq \ell})$  which has the properties guaranteed by Proposition 3.16. Recall that the sets  $I'_i$  are precisely the sinks used by flow paths from  $X$  with sources in  $B_{i-1}$ . Additionally, for the purposes of this proof, define  $X_i$  to be the set of flow paths from  $X$  that end in sinks  $I'_i$ , for each  $1 \leq i \leq \ell$ . Observe that  $X = X_1 \cup \dots \cup X_\ell$ . Let  $r$  be the choice made in Step 11.

In the algorithm, in Step 12, observe that only the flow paths  $X_r$  are used to modify the partial schedule from  $\sigma$  to  $\sigma'$  and therefore also change the graph  $G_\sigma^b$  to  $G_{\sigma'}^b$ . Step 13 does not alter the set  $A_{r-1}$ , even though it alters the partial schedule  $\sigma'$  by moving small jobs. Since it will not be important to consider these updates for the proof of Claim 3.21, which is a statement about the flow network for big jobs, we simply denote by  $\sigma'$  the partial schedule at the end of this step. After this update, all layers following layer  $L_{r-1}$  are discarded in Step 15 and  $\ell$  is updated to  $r - 1$  in Step 16. So, to prove Claim 3.21, we only need to verify that for all  $0 \leq i \leq r - 2$ ,

$$|H_{\sigma'}^b(B_{\leq i} \cap M_{\sigma'}^b, A_{i+1} \cup I'_{\leq i+1})| \geq |A_{i+1}|. \quad (3.3)$$

Fix  $i$  for the rest of the proof. Since  $0 \leq i \leq r - 2 \leq \ell - 2$ , we have, by hypothesis,  $|H_\sigma^b(B_{\leq i} \cap M_\sigma^b, A_{i+1} \cup I_{\leq i+1})| \geq |A_{i+1}|$ . So let  $Y$  be the maximum flow in  $H_\sigma^b(B_{\leq i} \cap M_\sigma^b, A_{i+1} \cup I_{\leq i+1})$ , with at least  $|A_{i+1}|$  flow paths. We now interpret the flow paths  $X$  and  $Y$  as vertex disjoint paths in  $G_\sigma^b$  using Proposition 3.7.

For a collection of vertex disjoint paths  $P$  in  $G_\sigma^b$  let  $S_P \subseteq M_\sigma^b$  and  $T_P \subseteq M_\sigma^s$  denote the set of source and sink vertices respectively used by paths in  $P$ . Now, if it happens that  $S_Y \subseteq S_{X_{\leq i+1}}$  then we are done. This is because there must be some  $X' \subseteq X_{\leq i+1}$  such that  $X'$  has cardinality at least  $|S_Y| = |Y| \geq |A_{i+1}|$ . Also,  $i + 1 \leq r - 1$  and therefore the paths  $X_r$  used to update the partial schedule  $\sigma$  are disjoint from the paths  $X'$  and hence,  $X'$  continues to be present in the new graph  $G_{\sigma'}^b$  following the update.

Our goal is to show this generally in Claim 3.22. Note that it immediately implies that even after updating the partial schedule to  $\sigma'$  we will have a collection of at least  $|Y| \geq |A_{i+1}|$  many vertex disjoint paths connecting the sources of  $Y$  to sinks in  $A_{i+1} \cup I'_{\leq i+1}$  in  $G_{\sigma'}^b$ . This proves (3.3) and completes the proof of the claim.  $\square$

*Claim 3.22 (Source Alignment Lemma).* There is a collection of vertex disjoint paths  $D$  in  $G_\sigma^b$  such that  $X_r \subseteq D$ ,  $S_Y \subseteq S_{D \setminus X_r}$ , and each source in  $S_Y$  is connected by  $D$  to a sink in  $A_{i+1} \cup I'_{\leq i+1}$ .

*Proof.* Recall the two sets of disjoint paths  $X = X_1 \cup \dots \cup X_\ell$  and  $Y$  we have defined in the graph  $G_\sigma^b$ . Paths in  $X_i$  connect sources in  $B_{i-1} \cap M_\sigma^b$  to sinks in  $I'_i$ , whereas paths in  $Y$  connect sources from  $B_{\leq i} \cap M_\sigma^b$  to sinks in  $A_{i+1} \cup I'_{\leq i+1}$ . We now describe a finite procedure which demonstrates the existence of paths  $D$  as in the claim.

```

D ← X.
while ∃s ∈ SY \ SD do
  v ← s.
  while ¬(v ∈ SD ∨ v ∈ TY) do
    if there is an incoming arc of some path p ∈ D at v then
      Set v to be the previous vertex along p.
    else if there is an outgoing arc of some path q ∈ Y at v then
      Set v to be the next vertex along q.
    end if
  end while
  Augment D using the set of edges traversed in the above loop.
end while
return D.

```

Before we analyze this procedure, we wish to point out that the augmenting step is well-defined. Suppose  $D$  and  $Y$  are sets of disjoint paths before some iteration of the outer **while** loop; this is clearly true before the first iteration. Let  $s \in S_Y \setminus S_D$  be chosen as the starting vertex for  $v$ . If at no point during the execution of the inner **while** loop, the **if** condition was satisfied, then it is trivial to see that the augmentation step is well-defined since the set of edges that are traversed in this case simply corresponds to some path  $y \in Y$ , whose source is  $s$ , and whose edges do not intersect with the edges of paths from  $D$ . On the other hand, consider the first time the **if** condition is satisfied. From that point onwards it is easy to see that the inner loop simply traverses the edges of some particular path  $x \in D$  in reverse order until it reaches the source of  $x$ , and the inner loop terminates. Here, we made use of the induction hypothesis that  $D$  is a collection of vertex disjoint paths. Therefore, we can conclude that the total set of edges traversed in this case are composed of two disjoint parts: i) edges from the prefix of the path  $y \in Y$ , whose source is  $s$ , followed by ii) edges from the prefix of the path  $x \in D$  in reverse order. Furthermore, the unique vertex, say  $v^*$ , at which the two parts have an incoming arc must be a machine vertex (since job vertices in  $G_\sigma^b$  have at most one incoming arc and the two parts are disjoint). Also,  $v^* \in M_\sigma^b$  since  $v^* \notin T_Y$ , and the paths  $y$  and  $x$  must intersect at the unique edge  $e^* = (v^*, j^*)$  where  $j^* \in J_b : \sigma(j^*) = v^*$ . Thus, deleting the set of edges traversed in the second part, and adding the set of edges traversed in the first part corresponds to a valid augmentation of  $D$ .

We now prove that this procedure terminates after finite iterations of the outer **while** loop. We claim that, in each iteration, either  $|S_Y \setminus S_D|$  decreases, or  $|S_Y \setminus S_D|$  stays the same and the quantity

$$\mathcal{Q} \triangleq \sum_{y \in Y} \sum_{x \in D} \mathcal{N}(x, y)$$

decreases, where  $\mathcal{N}(x, y)$  is defined as the total number of non-contiguous intersections between a pair of paths  $x$  and  $y$  in  $G_\sigma^b$  (see Figure 1). Consider a particular iteration of the outer loop. If the **if** condition is never satisfied during the execution of the inner loop, then, by the arguments above, the number of disjoint paths in  $D$  increases after the augmentation, and further the vertex  $s$  chosen in the outer loop becomes a new source of  $D$  after the augmentation. On the other hand, suppose that the path chosen for augmentation is composed of two parts arising from two paths  $y \in Y$  and  $x \in D$  as argued before. Further, let  $s_x$  be the source of  $x$ , and suppose that  $s_x \in S_Y$ , as otherwise, once again  $|S_Y \setminus S_D|$  decreases after augmenting  $D$ . Let  $y' \in Y$  be the path with source

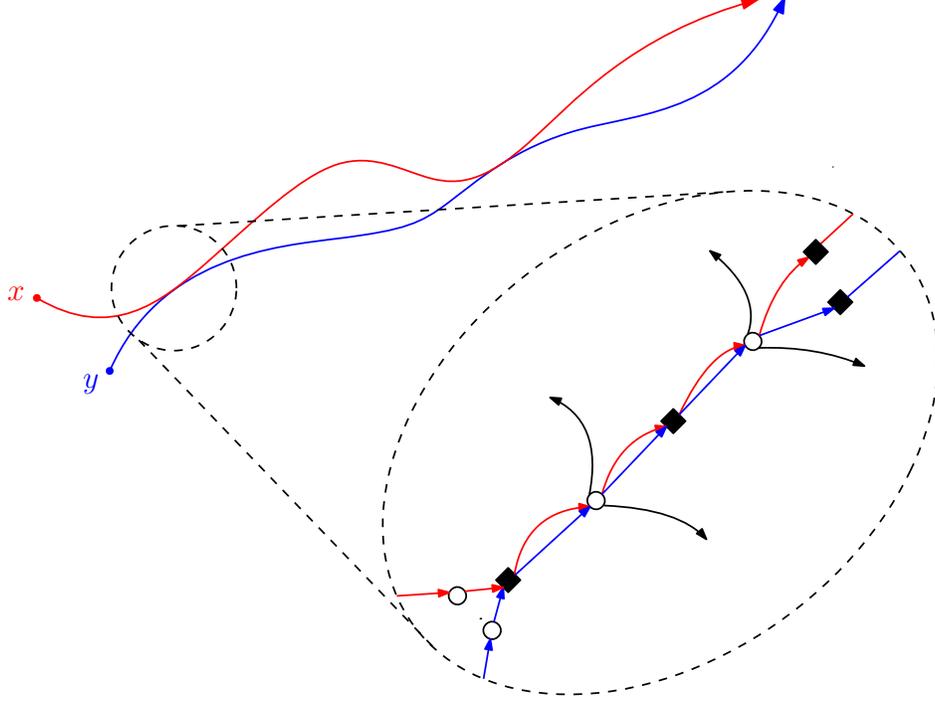


Figure 1: The number  $\mathcal{N}(x, y)$  of non-contiguous intersections between the pair of paths  $x \in D$  and  $y \in Y$  depicted here is 2. Arcs from  $G_\sigma^b$  that are neither present in  $D$  nor in  $Y$  are shown in black.

$s_x$ . After augmenting  $D$  it is seen that  $\sum_{x \in D'} \mathcal{N}(x, y') < \sum_{x \in D} \mathcal{N}(x, y')$ , and for all other paths  $y'' \in Y \setminus \{y'\}$ ,  $\sum_{x \in D'} \mathcal{N}(x, y'') \leq \sum_{x \in D} \mathcal{N}(x, y'')$ , thereby proving that the procedure eventually terminates. For an example execution of a single iteration of the outer **while** loop of the procedure, see Figure 2.

At the end of the procedure, we have  $S_Y \subseteq S_D$ . As  $S_Y \subseteq B_{\leq i}$ , by the invariant of the procedure that we prove below in Claim 3.23, the paths in  $D$  with sources  $S_Y$  end in sinks from  $A_{i+1} \cup I'_{\leq i+1}$ . Also, by Claim 3.23,  $X_r \subseteq D$  (because  $D_3 = X_{\geq i+2} \supseteq X_r$ ) and  $S_Y \subseteq S_{D \setminus X_r}$  (because  $B_{\geq i+1} \supseteq S_{D_3} \supseteq S_{X_r}$ ), which proves the claim.

*Claim 3.23 (The  $D$  Invariant).*  $D$  is a collection of vertex disjoint paths in  $G_\sigma^b$  which can be partitioned into  $D_1 \cup D_2 \cup D_3$  such that:

- (a)  $S_{D_1} \subseteq B_{\leq i}$ ,  $T_{D_1} = I'_{\leq i+1}$ ,
- (b)  $S_{D_2} \subseteq B_{\leq i}$ ,  $T_{D_2} \subseteq A_{i+1}$ , and
- (c)  $D_3 = X_{\geq i+2}$ .

*Proof.* Before the first iteration of the algorithm,  $D$  is initialized to  $X_{\leq \ell}$  and therefore admits the decomposition into  $D_1 = X_{\leq i+1}$ ,  $D_2 = \emptyset$ ,  $D_3 = X_{\geq i+2}$  which satisfy all of the above invariants. Notice here that  $i \leq r - 2 \leq \ell - 2$ , so that this decomposition is well-defined.

Assume the  $D$  invariant to hold for a collection of disjoint paths  $D$  at the beginning of some iteration of outer **while** loop. Following the augmentation let  $D'$  be the resulting collection of



that would lead to a contradiction to the property of the canonical decomposition  $I'_1 \cup \dots \cup I'_\ell$  that  $|H_\sigma^b(B_{\leq i}, I'_{\leq i+1})| = |H_\sigma^b(B_{\leq i}, I'_{\leq \ell})|$  (note here again that  $i+1 \leq r-1 \leq \ell-1$ ) by Proposition 3.16(c). Therefore, we also have that  $S_{D'_1}, S_{D'_2} \subseteq B_{\leq i}$ . Finally, since we did not encounter any edges of  $D_3$  during the augmentation process, we not only have that  $S_{D'_3} \subseteq B_{\geq i+1}$  but that  $D_3 = D'_3$ .  $\square$

$\square$

### 3.3.5 Proportionally Many Options for Big Jobs

**Theorem 3.24.** *Suppose that  $1 \leq \tau^* < 2$ . At the beginning of each iteration of the main loop of the algorithm, for every  $0 \leq i \leq \ell - 1$ ,*

$$|A_{i+1}| \geq \mu_1 |B_{\leq i}|.$$

*The statement also holds at the beginning of each iteration of the **while** loop of Step 6 of Algorithm 3.3.*

*Proof.* Let  $L_0, \dots, L_\ell$  be the set of layers maintained by the algorithm at the beginning of iteration of the main loop. It is sufficient to prove that following the construction of layer  $L_{\ell+1}$  in Step 4 of Algorithm 3.3,

$$|A_{\ell+1}| \geq \mu_1 |B_{\leq \ell}|.$$

The rest follows by applying Proposition 3.11(b). Suppose that the set  $S$  at the end of the **while** loop in Step 2 of Algorithm 3.4 is smaller than  $\mu_1 |B_{\leq \ell}|$ . We now describe an assignment of values to the variables  $(y, z)$  from the dual of  $CLP(\tau^*)$  (defined in Section 2.2) in four parts. Then, we will show that the final setting of dual variables  $(\bar{y}, \bar{z})$  obtained in this way satisfies  $\sum_{j \in J} \bar{z}_j - \sum_{i \in M} \bar{y}_i > 0$ , while also respecting the constraints of (2.2). It then follows that the dual of  $CLP(\tau^*)$  is unbounded because for any  $\lambda > 0$ ,  $(\lambda \bar{y}, \lambda \bar{z})$  is a feasible dual solution as well. Therefore, by weak duality,  $CLP(\tau^*)$  must be infeasible, a contradiction. We now proceed to execute this strategy.

**Part I: Layers** We set positive dual values to all machines that appear in the sets  $L_{\leq \ell} \cup I_{\leq \ell}$ , and the corresponding jobs, as follows:

$$y_i^{(1)} = \begin{cases} \tau^*, & i \in L_{\leq \ell} \cup I_{\leq \ell}, \\ 0, & \text{else.} \end{cases} \quad z_j^{(1)} = \begin{cases} R - \delta, & \exists i \in L_{\leq \ell} : j \in \sigma^{-1}(i) \cap J_b, \\ \epsilon, & \exists i \in L_{\leq \ell} : j \in \sigma^{-1}(i) \cap J_s, \\ 0, & \text{else.} \end{cases}$$

The objective function of the assignment  $(y^{(1)}, z^{(1)})$  can be lower bounded as:

$$\sum_{j \in J} z_j^{(1)} - \sum_{i \in M} y_i^{(1)} \geq (2R - \delta - \epsilon - 1)|B_{\leq \ell}| - \tau^* |I_{\leq \ell}| - (1 - \mu_2)(1 - R)|A_{\leq \ell}| - \mu_2 \tau^* |A_{\leq \ell}|.$$

Let us explain the lower bound. For each machine  $i \in B_{\leq \ell}$ ,  $p(\sigma^{-1}(i)) > \tau^* + R - \epsilon$  from Proposition 3.11(c). This allows us to derive  $\sum_{j \in \sigma^{-1}(i)} z_j^{(1)} - y_i^{(1)} > \tau^* + R - \epsilon - (1 - (R - \delta)) - \tau^* = 2R - \delta - \epsilon - 1$ . For the machines  $i \in I_{\leq \ell}$ , we have the trivial lower bound  $\sum_{j \in \sigma^{-1}(i)} z_j^{(1)} - y_i^{(1)} \geq -\tau^*$ . Next, for each machine  $i \in A_{\leq \ell}$  such that  $p(\sigma^{-1}(i)) > \tau^* - 1 + R$ , we have  $\sum_{j \in \sigma^{-1}(i)} z_j^{(1)} - y_i^{(1)} \geq$

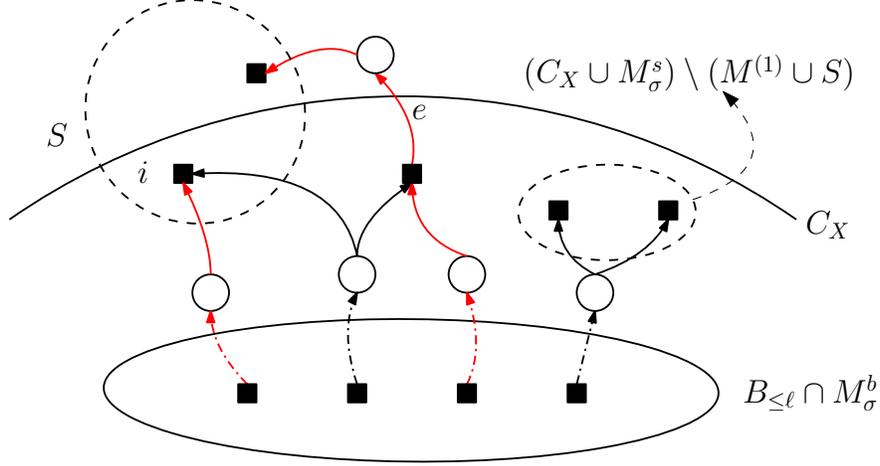


Figure 3: The cut  $C_X$  arising from the flow  $X$ , whose paths are colored red, in the flow network  $H_\sigma^b(B_{\leq \ell} \cap M_\sigma^b, S)$  is shown here. All machines except those inside dotted circles are big machines. Every big machine is matched to a distinct big job except for the machine with the outgoing machine-job arc  $e$ , which crosses  $C_X$ . The matching edges are dot-dashed and  $i$  has unit capacity.

$\tau^* - 1 + R - \tau^* = -1 + R$ , whereas for the rest of the machines in  $A_{\leq \ell}$ , we use the trivial lower bound  $-\tau^*$ . Thus, using Lemmas 3.13 and 3.14, we have

$$\sum_{j \in J} z_j^{(1)} - \sum_{i \in M} y_i^{(1)} \geq (2R - \delta - \epsilon - 1 - \tau^* \mu_1 \mu_2) |B_{\leq \ell}| - (1 - \mu_2)(1 - R) |A_{\leq \ell}| - \mu_2 \tau^* |A_{\leq \ell}|. \quad (3.4)$$

At this point we have assigned a positive  $z_j^{(1)}$  value to big jobs  $j$  assigned by  $\sigma$  to machines in  $L_{\leq \ell}$ . However there could potentially be machines  $i \in M \setminus (L_{\leq \ell} \cup I_{\leq \ell})$  such that  $i \in \Gamma(j)$  as well. Therefore, the current assignment  $(y^{(1)}, z^{(1)})$  does not necessarily constitute a dual feasible solution since it might violate the inequality  $y_i \geq \sum_{j \in C} z_j$ , for a configuration  $C = \{j\} \in \mathcal{C}(i, \tau^*)$  consisting of a single big job. We now fix this in the next part. For convenience, let  $M^{(1)} \triangleq L_{\leq \ell} \cup I_{\leq \ell}$ .

**Part II: Approximate Matchings** Consider the flow network of big jobs  $H_\sigma^b(B_{\leq \ell} \cap M_\sigma^b, S)$  that was used to construct the set  $S$ . By the construction of the set  $S$  in the algorithm, there is a flow  $X$  in this network of value  $|S|$ . This flow naturally defines a minimum capacity cut: the cut  $C_X$  is defined as the set of reachable jobs and machines from  $B_{\leq \ell} \cap M_\sigma^b$  in the *residual flow network* corresponding to  $X$  in  $H_\sigma^b(B_{\leq \ell} \cap M_\sigma^b, S)$ . Let  $M^{(2)} \triangleq (C_X \cap M_\sigma^b) \setminus M^{(1)}$ . We extend the assignment  $(y^{(1)}, z^{(1)})$  described in the first part in the following way.

$$y_i^{(2)} = \begin{cases} R - \delta, & i \in M^{(2)}, \\ 0, & \text{else.} \end{cases} \quad z_j^{(2)} = \begin{cases} R - \delta, & \exists i \in M^{(2)} : j \in \sigma^{-1}(i) \cap J_b, \\ 0, & \text{else.} \end{cases}$$

The capacity of the cut  $C_X$  is  $|S|$  by the max-flow min-cut theorem. This in particular implies that no job-machine arcs can cross this cut as such arcs have infinite capacity. In other words, the only arcs crossing  $C_X$  are machine-job arcs and machine-supersink arcs where the machine arises from  $S$  (recall that sink vertices have vertex capacity 1 in  $H_\sigma^b(B_{\leq \ell} \cap M_\sigma^b, S)$  according to Definition 3.6).

```

 $\rho \leftarrow \sigma.$ 
 $U \leftarrow S.$ 
 $V \leftarrow M \setminus (M^{(1)} \cup U).$ 
 $Y \leftarrow$  Maximum flow in  $H_\sigma^s(U, V).$ 
 $C_Y \leftarrow$  Mincut corresponding to  $Y$  in  $H_\sigma^s(U, V).$ 
while  $\exists i \in (C_X \cap M_\rho^s) \setminus (M^{(1)} \cup C_Y)$  do
    Augment  $Y$  to a maximum flow in  $H_\rho^s(U \cup \{i\}, V \setminus \{i\}).$   $\triangleright$  This is well-defined
     $C_Y \leftarrow$  Mincut corresponding to  $Y$  in  $H_\rho^s(U \cup \{i\}, V \setminus \{i\}).$ 
    for  $f \in Y$  : the sink of  $f$  belongs to  $((C_X \cap M_\rho^s) \setminus (M^{(1)} \cup C_Y))$  do
        Update  $\rho$  by using the flow path  $f.$ 
         $Y \leftarrow Y \setminus \{f\}.$ 
    end for  $\triangleright C_Y$  is still the mincut corresponding to  $Y$ 
     $U \leftarrow U \cup \{i\}.$ 
     $V \leftarrow V \setminus \{i\}.$ 
end while
return  $C_Y$ 

```

Figure 4: Mincut Growing Procedure

For every big machine  $i$  that is present in  $C_X$ , the corresponding big job assigned to  $i$  by  $\sigma$  is also present in  $C_X$  with the exception of at most  $|S|$  big jobs as shown in Figure 3 (corresponding to the machine-job arcs that cross the cut  $C_X$ ). Therefore, the total loss incurred in this step is at most  $|S|$ . In other words,

$$\sum_{j \in J} z_j^{(2)} - \sum_{i \in M} y_i^{(2)} \geq -(R - \delta)|S|. \quad (3.5)$$

**Part III: Growing Mincuts** In this part and the next, we assign positive dual values to machines in  $S$ , machines in  $(C_X \cap M_\sigma^s) \setminus M^{(1)}$ , and some other machines, to complete the description of our dual assignment. To make such an assignment, we will use the algorithm described in Figure 4 in the analysis.

The properties of the above procedure that we require in the proof are encapsulated in the following claim which we will prove inductively.

*Claim 3.25.* The **while** loop of the above procedure maintains the following invariants.

- (a)  $\rho$  is a partial schedule.
- (b)  $Y$  is a maximum flow in  $H_\rho^s(U, V)$  and  $C_Y$  is the corresponding mincut.
- (c) The value of maximum flow  $Y$  can be upper bounded as

$$|H_\rho^s(U, V)| \leq (\tau^* + R)|S| + \sum_{i \in U \setminus S} (p(\rho^{-1}(i)) - (\tau^* - 1 + R - \delta)).$$

- (d) There is no flow path  $f \in Y$  that ends in a sink belonging to  $(C_X \cap M_\rho^s) \setminus (M^{(1)} \cup C_Y)$ .
- (e) For each  $i \in (C_X \cap M_\rho^s) \setminus (M^{(1)} \cup C_Y)$ ,

$$|H_\rho^s(U \cup \{i\}, V \setminus \{i\})| < |H_\rho^s(U, V)| + (p(\rho^{-1}(i)) - (\tau^* - 1 + R - \delta)).$$

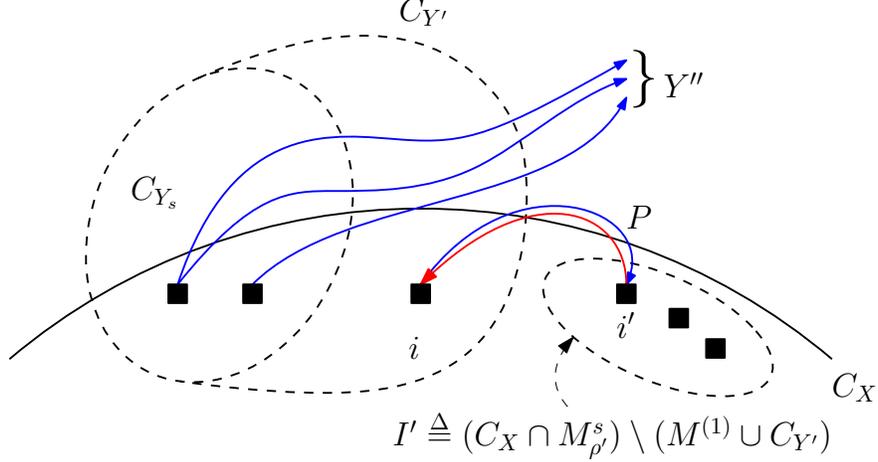


Figure 5: The induction step in the proof of Claim 3.25. For an  $i \in (C_X \cap M_{\rho_s}^s) \setminus (M^{(1)} \cup C_{Y_s})$ ,  $Y'$  is the maximum flow in  $H_{\rho_s}^s(U_s \cup \{i\}, V_s \setminus \{i\})$  shown in blue. It is partitioned as  $Y'' \cup P$ . The direction of the arcs in  $P$  is reversed, as shown in red, during the **for** loop from the procedure defined in Part III. The mincuts corresponding to the flows  $Y'$  and  $Y_s$  obey the inclusion  $C_{Y'} \supset C_{Y_s}$ .

*Proof.* Before the first iteration of the **while** loop, Claim 3.25(c) is satisfied because  $U = S$  and  $\rho = \sigma$  is a partial schedule; Claim 3.25(d) is satisfied because  $Y$  is a flow that has the properties guaranteed by Claim 3.19; Claim 3.25(e) follows from the fact that the **while** loop in Step 2 of Algorithm 3.4 was exited. The last claim needs some more explanation. Note that every  $i \in (C_X \cap M_{\sigma}^s) \setminus (M^{(1)} \cup C_Y)$  satisfies the following two properties:

- $i \in M_{\sigma}^s \setminus (L_{\leq \ell} \cup I_{\leq \ell} \cup S)$ , and
- $|H_{\sigma}^b(B_{\leq \ell} \cap M_{\sigma}^b, S \cup \{i\})| = |H_{\sigma}^b(B_{\leq \ell} \cap M_{\sigma}^b, S)| + 1$ .

The second property follows from Proposition 3.15(a) because  $i \in C_X$ . Therefore, for all such  $i$ , it must be the case that  $H_{\sigma}^s(S \cup \{i\}, T \setminus \{i\}) < H_{\sigma}^s(S, T) + (p(\sigma^{-1}(i)) - (\tau^* - 1 + R - \delta))$ .

Suppose that the statement is true until the beginning of some iteration of the **while** loop. Let  $Y_s$  be the maximum flow in  $H_{\rho_s}^s(U_s, V_s)$  and  $C_{Y_s}$  be the corresponding minimum cut maintained by this procedure. We now show it holds at the end of that iteration as well. Let the machine chosen in this iteration be  $i \in (C_X \cap M_{\rho_s}^s) \setminus (M^{(1)} \cup C_{Y_s})$ . The augmentation step is well defined because  $Y_s$  is a flow in  $H_{\rho_s}^s(U_s, V_s)$  that does not use any flow path with a sink belonging to  $(C_X \cap M_{\rho_s}^s) \setminus (M^{(1)} \cup C_{Y_s})$  as guaranteed by Claim 3.25(d). Therefore,  $Y_s$  is also a feasible flow in  $H_{\rho_s}^s(U_s \cup \{i\}, V_s \setminus \{i\})$ , which can be augmented to a maximum flow in that network, say  $Y'$ . Let  $C_{Y'}$  be the corresponding mincut that is computed in the procedure. We remark that  $C_{Y_s} \subset C_{Y'}$  where the inclusion is strict, because  $i \in C_{Y'} \setminus C_{Y_s}$ , which follows from Proposition 3.15(b). We use this fact later.

The execution now enters a **for** loop that modifies the flow  $Y$  and the partial schedule  $\rho$  maintained by the procedure, which currently assume the values  $Y'$  and  $\rho_s$  respectively. Let  $Y'' \subseteq Y'$  and  $\rho'$  be the state of  $Y$  and  $\rho$  at the *end* of the **for** loop respectively, so that  $P \triangleq Y' \setminus Y''$  is precisely the set of flow paths used to update the partial schedule  $\rho$  maintained by the procedure. It is seen that the  $G_{\rho'}^s$  is obtained from the graph  $G_{\rho_s}^s$  by reversing the directions of the arcs contained in the paths of  $P$ . For an illustration, see Figure 5.

As only flow paths  $P \subseteq Y'$  ending in small machine sinks were used to update  $\rho$ ,  $\rho'$  is still a partial schedule (recall that  $Y'$  is a flow in the network  $H_{\rho_s}^s(U_s \cup \{i\}, V \setminus \{i\})$ ), which proves Claim 3.25(a). We also claim that Claim 3.25(b) holds.

*Claim 3.26.*  $Y''$  is a maximum flow in  $H_{\rho'}^s(U_s \cup \{i\}, V_s \setminus \{i\})$  and  $C_{Y'}$  is the corresponding minimum cut.

*Proof.* The first part is true, because, after every single iteration of the **for** loop, the value of the flow  $Y$  decreases by  $\epsilon$  and so does the capacity of the arcs in  $H_{\rho_s}^s(U \cup \{i\}, V \setminus \{i\})$  crossing the cut  $C_Y$ ; since the value of the flow  $Y'$  was equal to the capacity of the minimum cut  $C_{Y'}$  before the **for** loop, by the max-flow min-cut theorem, the claim follows (we apply the converse of the max-flow min-cut theorem at the end).

The second part is true as well, because the set of vertices reachable from  $U_s \cup \{i\}$  in the reduced flow network corresponding to the flow  $Y'$  in  $H_{\rho_s}^s(U_s \cup \{i\}, V_s \setminus \{i\})$  (defined to be  $C_{Y'}$ ) is the *same* as the set of vertices reachable from  $U_s \cup \{i\}$  in the reduced flow network corresponding to the flow  $Y''$  in  $H_{\rho'}^s(U_s \cup \{i\}, V_s \setminus \{i\})$ .  $\square$

Next, using Claims 3.25(e) and 3.25(c), we see that

$$\begin{aligned} |H_{\rho_s}^s(U_s \cup \{i\}, V_s \setminus \{i\})| &< |H_{\rho_s}^s(U_s, V_s)| + \underbrace{(p(\rho_s^{-1}(i)) - (\tau^* - 1 + R - \delta))}_{(*)} \\ &\leq (\tau^* + R)|S| + \sum_{i' \in U_s \setminus S} (p(\rho_s^{-1}(i')) - (\tau^* - 1 + R - \delta)) + (*) \\ &= (\tau^* + R)|S| + \sum_{i' \in (U_s \cup \{i\}) \setminus S} (p(\rho_s^{-1}(i')) - (\tau^* - 1 + R - \delta)). \end{aligned}$$

In the final equality, note that  $i \notin S$  because  $i \notin C_{Y'}$  and  $C_{Y'} \supseteq U_s \supseteq S$  (using the fact mentioned in our earlier remark). In each iteration of the **for** loop, we saw that the value of the quantities  $|H_{\rho_s}^s(U \cup \{i\}, V \setminus \{i\})|$  and  $p(\rho^{-1}(i))$  reduces exactly by  $\epsilon$  so that at the end of the **for** loop, we have

$$|H_{\rho'}^s(U_s \cup \{i\}, V_s \setminus \{i\})| \leq (\tau^* + R)|S| + \sum_{i' \in (U_s \cup \{i\}) \setminus S} (p(\rho'^{-1}(i')) - (\tau^* - 1 + R - \delta)),$$

which proves Claim 3.25(c).

At the end of the **for** loop, Claim 3.25(d) is true as well because for every  $f \in Y''$  the sink of  $f$  does not belong to  $I' \triangleq (C_X \cap M_{\rho'}^s) \setminus (M^{(1)} \cup C_{Y'})$  by the postcondition of the loop, and using Claim 3.26. It only remains to prove Claim 3.25(e). Suppose towards contradiction that there is an  $i' \in I'$  such that

$$|H_{\rho'}^s(U_s \cup \{i, i'\}, V_s \setminus \{i, i'\})| \geq |H_{\rho'}^s(U_s \cup \{i\}, V_s \setminus \{i\})| + (p(\rho'^{-1}(i')) - (\tau^* - 1 + R - \delta)).$$

Let  $F$  be some maximum flow in  $H_{\rho'}^s(U_s \cup \{i, i'\}, V_s \setminus \{i, i'\})$  obtained by augmenting  $Y''$  (which is well-defined because  $Y''$  does not have flow paths that use vertices in  $I'$  as sinks). Construct a new flow network  $H'$  from  $H_{\rho_s}^s(U_s \cup \{i, i'\}, V_s \setminus \{i, i'\})$  by adding a copy of the vertex  $i'$  and call it  $i'_{\text{dummy}}$  (with identical neighborhood structure and vertex capacity). Interpret the flow  $P$  in  $H'$  so that none of the flow paths use  $i'$  as as sink (they may use  $i'_{\text{dummy}}$  however). In the *residual flow network* corresponding to this flow in  $H'$ , use the flow paths of  $F$  to augment the flow. This is well-defined

because of the way the graphs  $G_{\rho'}^s$  and  $G_{\rho_s}^s$  are related (recall that, to obtain the former from the latter, we just need to reverse the directions of the arcs of paths in  $P$ ). It is important to note here that the resulting flow only contains paths and no cycles. Through this process we obtain a flow of value  $|F| + |P|$  in the network  $H'$ . By assumption,

$$\begin{aligned} |F| + |P| &\geq |Y''| + \underbrace{(p(\rho'^{-1}(i')) - (\tau^* - 1 + R - \delta))}_{(*)} + |P| \\ &= |Y'| + (*) \\ &= |Y_s| + f_i + (*). \end{aligned}$$

The first equality follows from the definition of  $P$ ; the second equality follows from the fact that  $Y_s$  was augmented in the flow network  $|H_{\rho_s}^s(U_s \cup \{i\}, V_s \setminus \{i\})|$  to  $Y'$  so that  $Y'$  has exactly  $f_i$  value flow paths with sources at  $i$ , and  $|Y_s|$  value flow paths with sources in  $U_s$  (here we make use of the fact that  $Y_s$  is a maximum flow). Therefore, we have a flow of value at least  $|Y_s| + f_i + (*)$  in the network  $H'$ . Since the latter flow was constructed by augmenting maximum flows, we can deduce that it is composed of  $|Y_s|$  value flow paths originating at  $U_s$ ,  $f_i$  value flow paths originating at  $i$  and the rest originating at  $i'$ . Deleting all flow paths leading to  $i'_{\text{dummy}}$ , we have a resulting flow of value at least  $|Y_s| + f_i + (*) - l_{i'}$ , where  $l_{i'}$  is the value of flow paths in  $P$  that end in  $i'$ . Owing to the way in which we updated  $\rho$  in the **for** loop, we can see that  $l_{i'} = p(\rho'^{-1}(i')) - p(\rho_s^{-1}(i'))$ . Therefore there is a flow of value at least  $|Y_s| + (p(\rho_s^{-1}(i')) - (\tau^* - 1 + R - \delta))$  in the network  $H_{\rho_s}^s(U_s \cup \{i'\}, V_s \setminus \{i'\})$ , which then implies

$$|H_{\rho_s}^s(U_s \cup \{i'\}, V_s \setminus \{i'\})| \geq |H_{\rho_s}^s(U_s, V_s)| + (p(\rho_s^{-1}(i')) - (\tau^* - 1 + R - \delta)),$$

contradicting Claim 3.25(e). □

Returning to our proof, we now run this procedure with one modification: we add  $i$  to the set  $U$  maintained by the procedure only if, in addition to the condition in the **while** loop, the new set  $C'_Y$  would have a size at least  $|C_Y| + 2$ . Let  $\rho_f$  and  $C_{Y,f}$  be the reallocation policy and the cut at the end of the execution of this modified procedure. We have as a postcondition that executing the body of the **while** loop once with an  $i \in (C_X \cap M_{\rho_f}^s) \setminus (M^{(1)} \cup C_{Y,f})$  would result only in a set of size  $|C_{Y,f}| + 1$  (note that  $i$  would be the new element in that case). Extend the dual assignment as follows.

$$y_i^{(3)} = \begin{cases} \tau^*, & i \in C_{Y,f}, \\ 0, & \text{else.} \end{cases} \quad z_j^{(3)} = \begin{cases} \epsilon, & j \in \rho_f^{-1}(i) \cap J_s : i \in C_{Y,f}, \\ 0, & \text{else.} \end{cases}$$

We now need to bound the total loss incurred in this part of the proof. Suppose the **while** loop in the procedure executes  $t \geq 0$  times. Let  $U_f$ ,  $V_f$ , and  $Y_f$  be the state of the (remaining) variables at the end of the procedure. For convenience assume that  $U_f = S \cup \{i_1, \dots, i_t\}$ , where the numbering follows the order in which the machines are added to the variable  $U$  in the **while** loop. By Claim 3.25(b),  $Y_f$  is a maximum flow in  $H_{\rho_f}^s(U_f, V_f)$  and  $C_{Y,f}$  is the corresponding mincut. Let  $P \triangleq (C_{Y,f} \setminus U_f) \cap M_{\rho_f}^s$  and  $Q \triangleq C_{Y,f} \cap M_{\rho_f}^b$ . Since the size of the mincut in the variable  $C_Y$  increased by at least 2 in each iteration, we have at the end that  $|P| + |Q| \geq t$ .

By the max-flow min-cut theorem, the value of the maximum flow equals the capacity of the

minimum cut, and therefore, by Claim 3.25(c),

$$\begin{aligned}
& (\tau^* + R)|S| + \sum_{j=1}^t (p(\rho_f^{-1}(i_j)) - (\tau^* - 1 + R - \delta)) \\
& > \sum_{i \in S \cup \{i_1, \dots, i_t\} \cup P \cup Q} c_i + \sum_{i \in P} (\tau^* + R - p(\rho_f^{-1}(i))) + \sum_{i \in Q} (\tau^* + 1 + R - p(\rho_f^{-1}(i)) - \epsilon),
\end{aligned}$$

where  $c_i$  is the total capacity of machine-job arcs with  $i$  as one endpoint crossing the minimum cut  $C_{Y_f}$ . The terms on the left together upper bound the value of maximum flow in the final network  $H_{\rho_f}^s(U_f, V_f)$ , whereas the terms on the right count the contributions to the minimum cut arising from machine-job arcs and machine-sink arcs. Splitting the first sum on the right,

$$\begin{aligned}
& (\tau^* + R)|S| + \sum_{j=1}^t (p(\rho_f^{-1}(i_j)) - (\tau^* - 1 + R - \delta)) \\
& > \sum_{i \in S} c_i + \sum_{i \in \{i_1, \dots, i_t\} \cup P} c_i + \sum_{i \in Q} c_i + \sum_{i \in P} (\tau^* + R - p(\rho_f^{-1}(i))) + \sum_{i \in Q} (\tau^* + 1 + R - p(\rho_f^{-1}(i)) - \epsilon).
\end{aligned}$$

After rearranging the terms,

$$\begin{aligned}
& (\tau^* + R)|S| - t(\tau^* - 1 + R - \delta) > \sum_{i \in S} c_i - \sum_{i \in \{i_1, \dots, i_t\} \cup P} (p(\rho_f^{-1}(i)) - c_i) \\
& \quad - \sum_{i \in Q} (p(\rho_f^{-1}(i)) - 1 - c_i) + (|P| + |Q|)(\tau^* + R) - |Q|\epsilon,
\end{aligned}$$

we derive

$$\begin{aligned}
& - \sum_{i \in S} c_i + \sum_{i \in \{i_1, \dots, i_t\} \cup P} (p(\rho_f^{-1}(i)) - c_i) + \sum_{i \in Q} (p(\rho_f^{-1}(i)) - 1 - c_i) \\
& > -(\tau^* + R)|S| + t(\tau^* - 1 + R - \delta) + (|P| + |Q|)(\tau^* + R) - |Q|\epsilon \\
& \geq -(\tau^* + R)|S| + t(\tau^* - 1 + R - \delta) + (|P| + |Q|)(\tau^* + R - \epsilon).
\end{aligned} \tag{3.6}$$

We demonstrate that the assignment  $(y^{(3)}, z^{(3)})$  amortizes itself locally using (3.6).

$$\begin{aligned}
& \sum_{j \in J} z_j^{(3)} - \sum_{i \in M} y_i^{(3)} \\
&= \sum_{i \in S \cup \{i_1, \dots, i_t\} \cup P} (p(\rho_f^{-1}(i)) - c_i) + \sum_{i \in Q} (p(\rho_f^{-1}(i)) - 1 - c_i) - \tau^*(|S| + t + |P| + |Q|) \\
&\geq \underbrace{- \sum_{i \in S} c_i + \sum_{i \in \{i_1, \dots, i_t\} \cup P} (p(\rho_f^{-1}(i)) - c_i) + \sum_{i \in Q} (p(\rho_f^{-1}(i)) - 1 - c_i) - \tau^*(|S| + t + |P| + |Q|)}_{(3.6)} \\
&> -(\tau^* + R)|S| + t(\tau^* - 1 + R - \delta) + (|P| + |Q|)(\tau^* + R - \epsilon) - \tau^*(|S| + t + |P| + |Q|) \\
&= -(\tau^* + R)|S| + t(\tau^* - 1 + R - \delta) + (|P| + |Q|)(R - \epsilon) - \tau^*(|S| + t) \\
&= -(2\tau^* + R)|S| + t(R - \delta - 1) + (|P| + |Q|)(R - \epsilon) \\
&\geq -(2\tau^* + R)|S| + t(R - \delta - 1) + t(R - \epsilon) \\
&= -(2\tau^* + R)|S| + t \underbrace{(2R - \delta - \epsilon - 1)}_{\geq 0 \text{ follows from Claim A.1}} \\
&\geq -(2\tau^* + R)|S|. \tag{3.7}
\end{aligned}$$

**Part IV: The Rest** As noted in Part III, we may now have machines  $i \in (C_X \cap M_{\rho_f}^s) \setminus (M^{(1)} \cup C_{Y,f})$  that increase the size of the set  $C_{Y,f}$  described in the previous part by one. Let  $M^{(4)}$  denote the set of such machines; note that they must necessarily be a subset of  $M_\sigma^s$  (which is the same as  $M_{\rho_f}^s$ ). By the postcondition of the modified procedure, we deduce that each machine in  $M^{(4)}$  has at least  $\tau^* - 1 + R - \delta$  processing time small jobs assigned to it by  $\rho_f$  such that each of those jobs can be assigned to only machines in  $C_{Y,f} \cup M^{(1)}$  besides itself. Let

$$S_i \triangleq \{j \in \rho_f^{-1}(i) \cap J_s \mid \Gamma(j) \subseteq \{i\} \cup C_{Y,f} \cup M^{(1)}\}.$$

We set the dual values of these machines as follows.

$$y_i^{(4)} = \begin{cases} \sum_{j \in \rho_f^{-1}(i)} z_j^{(4)}, & i \in M^{(4)}, \\ 0, & \text{else.} \end{cases} \quad z_j^{(4)} = \begin{cases} \epsilon, & j \in S_i : i \in M^{(4)}, \\ 0, & \text{else.} \end{cases}$$

**The Dual Assignment** Before we describe our final dual assignment  $(\bar{y}, \bar{z})$ , let us note that the supports of  $(y^{(1)}, z^{(1)})$ ,  $(y^{(3)}, z^{(3)})$  and  $(y^{(4)}, z^{(4)})$  are disjoint by construction. Further, observe that the support of  $(y^{(2)}, z^{(2)})$  may only intersect with the support of  $(y^{(3)}, z^{(3)})$ , and is disjoint from the other two. However, we can assume without loss of generality that they too are disjoint, as machines that receive both positive  $y^{(2)}$  and  $y^{(3)}$  values will only help us in the later arguments. The reasoning is that, for a machine  $i$  such that  $y_i^{(3)} = \tau^*$  and  $y_i^{(2)} = R - \delta$ , we will only consider the contribution of  $y_i^{(3)}$  to the final assignment  $\bar{y}$  in the feasibility whereas we will count both contributions towards the objective function i.e., we prove that the dual objective function of the final assignment is positive even after counting an extra contribution of  $R - \delta$  for such machines. Note that there can be no jobs in the intersection of the supports of the dual assignments from the second and third parts. So we assume that the supports of the dual assignments from the four parts are disjoint. Set  $(\bar{y}, \bar{z})$  to be the union of the four assignments in the natural way.

**Feasibility** Our assignment  $(\bar{y}, \bar{z})$  to the dual variables is such that  $\sum_{j \in C} \bar{z}_j \leq \tau^*$  for every  $i \in M, C \in \mathcal{C}(i, \tau^*)$  because  $\bar{z}_j \leq p_j$  for every  $j \in J$ . Therefore, the constraints of (2.2) involving machines  $i$  for which  $\bar{y}_i = \tau^*$  are satisfied.

This leaves us to only consider the machines whose dual values were set in Parts II and IV. Let  $i \in M^{(2)}$  and  $C \in \mathcal{C}(i, \tau^*)$ . By Proposition 3.11(d), the construction of the cut  $C_{Y,f}$  (note that infinite capacity job-machine arcs cannot cross this cut), and the dual setting of  $z^{(4)}$  (where we assigned positive  $z_j^{(4)}$  values only to jobs in  $S_i$  for some  $i \in M^{(4)}$ ), there can be no  $j \in C \cap J_s$  such that  $\bar{z}_j > 0$ . As  $\tau^* < 2$ , there is at most one big job in a configuration. Since it is assigned a dual value of  $R - \delta$ , all constraints involving such machines are satisfied. Now let  $i \in M^{(4)}$  and  $C \in \mathcal{C}(i, \tau^*)$ . Recall that  $\bar{y}_i > \tau^* - 1 + R - \delta$ . If  $C$  contains a big job then  $\sum_{j \in C} \bar{z}_j \leq R - \delta + \tau^* - 1$ . If  $C$  does not contain big jobs, then,

$$\sum_{j \in C} \bar{z}_j = \sum_{j \in C: \bar{z}_j > 0} \bar{z}_j \leq \sum_{j \in \sigma^{-1}(i)} \bar{z}_j = \bar{y}_i.$$

The inequality in the middle deserves explanation. This follows from the assertion that any job  $j \in C \cap J_s$  such that  $\bar{z}_j > 0$  must be part of  $\sigma^{-1}(i)$  by Proposition 3.11(d), the construction of the cut  $C_{Y,f}$ , and the dual setting of  $z^{(4)}$ .

**Positivity** Now that we have described our dual assignment, we show  $\sum_{j \in J} \bar{z}_j - \sum_{i \in M} \bar{y}_i > 0$  by counting the contributions to the objective function from the dual variable settings in each of the four previous parts.

From (3.4) and (3.5), the total gain in the first and second part is at least

$$(2R - \delta - \epsilon - 1 - \tau^* \mu_1 \mu_2) |B_{\leq \ell}| - (1 - \mu_2)(1 - R) |A_{\leq \ell}| - \mu_2 \tau^* |A_{\leq \ell}| - |S|.$$

In the third part, using (3.7), the total loss is at most  $(2\tau^* + R)|S|$ . In the fourth part there is no net loss or gain. So, we can lower bound the objective function value of our dual assignment  $(\bar{y}, \bar{z})$  as follows, making use of Lemma 3.17 and Lemma 3.18 in the second inequality.

$$\begin{aligned} & \sum_{j \in J} \bar{z}_j - \sum_{i \in M} \bar{y}_i \\ & \geq (2R - \delta - \epsilon - 1 - \tau^* \mu_1 \mu_2) |B_{\leq \ell}| - (1 - \mu_2)(1 - R) |A_{\leq \ell}| - \mu_2 \tau^* |A_{\leq \ell}| - |S| - (2\tau^* + R) |S| \\ & \geq (2R - \delta - \epsilon - 1 - \tau^* \mu_1 \mu_2 - (1 + 2\tau^* + R) \mu_1) |B_{\leq \ell}| - (1 - \mu_2)(1 - R) |A_{\leq \ell}| - \mu_2 \tau^* |A_{\leq \ell}| \\ & \geq (2R - \delta - \epsilon - 1 - \tau^* \mu_1 \mu_2 - (1 + 2\tau^* + R) \mu_1) (\delta(1 - \mu_2) - 2\mu_2) \cdot |A_{\leq \ell}| - ((1 - \mu_2)(1 - R) - \mu_2 \tau^*) |A_{\leq \ell}| \\ & \geq \underbrace{\left( (2R - \delta - \epsilon - 1 - \tau^* \mu_1 \mu_2 - (1 + 2\tau^* + R) \mu_1) (\delta(1 - \mu_2) - 2\mu_2) - (1 - \mu_2)(1 - R) - \mu_2 \tau^* \right)}_{(*)} |A_{\leq \ell}|. \end{aligned}$$

Substituting the values of  $R, \mu_1, \mu_2, \delta$  as defined in the statement of Theorem 1.1 and (3.1), one can verify (see Claim A.1) that the bracketed expression  $(*)$  is strictly positive for every  $1 \leq \tau^* < 2$ ,  $0 < \epsilon < 1$  and  $\zeta > 0$ . □

### 3.3.6 Polynomial Bound on Loop Iterations

**Corollary 3.27.** *Suppose  $1 \leq \tau^* < 2$ . In each execution of the **while** loop in Step 6 of Algorithm 3.3, the **if** condition in Step 10 is satisfied.*

*Proof.* Consider the beginning of some iteration of the **while** loop in Step 6 of Algorithm 3.3 with a state  $\mathcal{S}$ . By the condition in Step 6,  $\ell \geq 1$  and  $|I| \geq \mu_2 |A_\ell|$ , where  $I \triangleq \{i \in A_\ell \mid p(\sigma^{-1}(i)) \leq \tau^* + R - 1\}$ . Applying Theorem 3.20,  $|H_\sigma^b(B_{\leq \ell-1} \cap M_\sigma^b, A_\ell \cup I_{\leq \ell})| \geq |A_\ell|$ . Since  $|I| \geq \mu_2 |A_\ell|$  and  $I \subseteq A_\ell$  this means that

$$|H_\sigma^b(B_{\leq \ell-1} \cap M_\sigma^b, I \cup I_{\leq \ell})| \geq \mu_2 |A_\ell| \geq \mu_1 \mu_2 |B_{\leq \ell-1} \cap M_\sigma^b|,$$

where the second inequality follows from Theorem 3.24. By Proposition 3.16(a), at least one of the sets  $I'_i$  computed in Step 8 of Algorithm 3.3 must be of size at least  $\mu_1 \mu_2 |B_{i-1} \cap M_\sigma^b|$  for some  $1 \leq i \leq \ell$ .  $\square$

Given the state  $\mathcal{S}$  of the algorithm at some point during its execution, the *signature of a layer*  $L_i$  is defined as

$$s_i \triangleq \left\lceil \log_{\frac{1}{1-\mu_1\mu_2}} \left( \left( \frac{1}{\eta} \right)^i |B_i \cap M_\sigma^b| \right) \right\rceil + i,$$

where  $\eta \triangleq (\delta(1 - \mu_2) - 2\mu_2) \mu_1 > 0$  by Claim A.2. The *signature vector* corresponding to the given state is then defined as a vector in the following way:

$$s \triangleq (s_0, \dots, s_\ell, \infty).$$

**Lemma 3.28.** *Suppose  $1 \leq \tau^* < 2$ . The signature vector satisfies the following properties.*

- (a) *At the beginning of each iteration of the main loop of the algorithm,  $\ell = O(\log |J_b|)$ .*
- (b) *The coordinates of the signature vector are well-defined and increasing at the beginning of each iteration of the main loop of the algorithm.*

*Proof.* Consider the beginning of some iteration of the main loop of the local search algorithm. Let  $L_0, \dots, L_\ell$  be the set of layers maintained by the algorithm. Let  $0 \leq i \leq \ell$ . Observe that from the moment layer  $L_i$  was constructed until now,  $A_i$  remains unmodified (even though the assignment of jobs by  $\sigma$  to machines in  $A_\ell$  may have changed). This is because  $A_i$  can be modified only if, in some intervening iteration of the main loop, the variable  $r$  from Step 11 of Algorithm 3.3 is chosen to be  $i'$  for some  $i' \leq i$ . But in that case we discard all the layers  $L_{i'}, \dots, L_\ell$  in Step 15 and this includes layer  $L_i$  as well. Therefore, for  $0 \leq i \leq \ell - 1$ ,

$$|B_{i+1} \cap M_\sigma^b| \stackrel{\text{Lem 3.18}}{>} (\delta(1 - \mu_2) - 2\mu_2) |A_{i+1}| \stackrel{\text{Thm 3.24}}{\geq} (\delta(1 - \mu_2) - 2\mu_2) \cdot \mu_1 |B_{\leq i}| \stackrel{(*)}{\geq} \eta |B_{\leq i} \cap M_\sigma^b|.$$

The second inequality above uses the fact that the layers  $L_0, \dots, L_\ell$  were not modified since construction as argued previously. As the final term in the chain of inequalities above is at least  $\eta |B_i \cap M_\sigma^b|$ , this proves (b). As the sets  $B_0, \dots, B_\ell$  are disjoint by construction,

$$|B_{\leq i+1} \cap M_\sigma^b| = |B_{i+1} \cap M_\sigma^b| + |B_{\leq i} \cap M_\sigma^b| \stackrel{\text{Using } (*)}{\geq} (1 + \eta) |B_{\leq i} \cap M_\sigma^b|.$$

As  $|B_0 \cap M_\sigma^b| \geq 1$  by Lemma 3.17,  $\ell$  is  $O(\log_{1+\eta} |M_\sigma^b|) = O(\log |J_b|)$ , which proves (a).  $\square$

**Lemma 3.29.** *Suppose  $1 \leq \tau^* < 2$ . Only  $\text{poly}(|J_b|)$  many signature vectors are encountered during the execution of Algorithm 3.3.*

*Proof.* By Lemma 3.28(a), and the definition of  $s_i$ , each coordinate is at most  $O(\log |J_b|)$ . Lemma 3.28(b) also implies that the coordinates of the signature vector are increasing at the beginning of the main loop. So every signature vector encountered at the beginning of the main loop can be unambiguously described as a subset of a set of size  $O(\log |J_b|)$ .  $\square$

**Lemma 3.30.** *Suppose  $1 \leq \tau^* < 2$ . The signature vector decreases in lexicographic value across each iteration of the main loop of the local search algorithm.*

*Proof.* Consider the beginning of some iteration of the main loop of the local search algorithm with the state  $\mathcal{S}$ . So,  $L_0, \dots, L_\ell$  are the set of layers at the beginning of the iteration. During the iteration, a single new layer  $L_{\ell+1}$  is created in Step 4, and zero or more layers from the set  $\{L_1, \dots, L_\ell, L_{\ell+1}\}$  are discarded in Step 15. We consider two cases accordingly.

- **No layer is discarded.** Therefore, at the end of this iteration, we will have layers  $L_0, \dots, L_{\ell+1}$  and we can apply Lemma 3.28(b) at the beginning of the next iteration to prove this claim. Note here that we used the converse of Corollary 3.27 to deduce that the **while** loop in Step 6 of Algorithm 3.3 did not execute since no layer was discarded.
- **At least one layer is discarded.** During each iteration of the **while** loop in Step 6, for some  $1 \leq r \leq \ell + 1$  as chosen in Step 11, the **if** condition in Step 10 is satisfied by Corollary 3.27. Therefore, the size of  $|B_{r-1} \cap M_\sigma^b|$  reduces to at most  $(1 - \mu_1 \mu_2)|B_{r-1} \cap M_\sigma^b|$ , and the  $(r - 1)$ -th coordinate of the signature vector reduces by at least one, whereas the coordinates of the signature vector of the layers preceding  $r - 1$  are unaffected. In other words, the signature vector at the beginning of the next iteration of the main loop (if any) would be

$$s' = (s_0, \dots, s_{r-2}, s'_{r-1}, \infty),$$

where  $r \leq \ell + 1$  and  $s'_{r-1} \leq s_{r-1} - 1$ .

$\square$

An immediate corollary of Lemma 3.29 and Lemma 3.30 is that the local search algorithm described in Section 3.2 terminates after  $\text{poly}(|J_b|)$  iterations of the main loop under the assumption  $1 \leq \tau^* < 2$ . Notice, however, that all statements proved in Section 3.3.6 also hold merely given the *conclusions* of Lemma 3.17 and Theorem 3.24 without necessarily assuming that  $\tau^* \in [1, 2)$ .

### 3.4 Proof of the Main Theorem

*Proof of Theorem 1.1.* Let  $\mathcal{I}$  be the given instance of the  $(1, \epsilon)$  case of RESTRICTED ASSIGNMENT MAKESPAN MINIMIZATION and  $\text{OPT}$  denote the optimum makespan. Assume for the moment that  $\tau^*$  is known by solving the Configuration LP. If  $\tau^* \geq 2$ , then the algorithm of Lenstra, Shmoys and Tardos [LST90] for this problem gives an  $\text{OPT} + p_{\max}$  approximation guarantee, which is of course at most  $1.5\text{OPT}$  in this case.

Suppose instead that  $1 \leq \tau^* < 2$ . Start with a partial schedule  $\sigma$  guaranteed by Lemma 3.3. Let  $\sigma$  and  $j_0 \in J_s \setminus \sigma^{-1}(M)$  denote the input partial schedule and small job to Algorithm 3.3. From its description it is clear that the partial schedule maintained by it is modified either in Step 20 or within the main loop. In the main loop, this occurs in exactly three places: Step 3 of Algorithm 3.4; Steps 12 and 13 of Algorithm 3.3. From Lemma 3.29 and Lemma 3.30, we deduce that the main

loop is exited after  $\text{poly}(|J_b|)$  iterations. Using Proposition 3.10(a), Proposition 3.8(a), and Step 20 of the local search algorithm, the output partial schedule  $\sigma'$  therefore satisfies the property

$$\sigma'^{-1}(M) = \sigma^{-1}(M) \cup \{j_0\}.$$

Repeating this algorithm a total of  $|J_s|$  times yields a schedule of makespan at most  $\tau^* + R$  for  $\mathcal{I}$  in polynomial time.

However, it is not necessary to know  $\tau^*$  in advance by solving the Configuration LP. Suppose that  $\tau \in [1, 2)$  is a guess on the value of  $\tau^*$ . Let  $\mathcal{A}(\tau)$  denote the following algorithm. Run the procedure outlined above after substituting  $\tau$  in place of  $\tau^*$  with two modifications to Algorithm 3.3: if  $|B_0 \cap M_\sigma^b| = 0$  after Step 1, or if  $|A_{\ell+1}| < \mu_1 |B_{\leq \ell}|$  in any execution of Step 4, then terminate the procedure with an error claiming that the guessed value  $\tau < \tau^*$ .

Suppose  $\mathcal{A}(\tau)$  returns a schedule during a binary search over the range  $\tau \in [1, 2)$ , then it is guaranteed to have makespan at most  $\tau + R$ . Note that the range of possible values for  $\tau^*$  is discrete ( $1 + k\epsilon$  or  $k\epsilon$  for  $k \in \mathbb{Z}$ ). As the running time analysis in Section 3.3.6 of Algorithm 3.3 depends only on conclusions of Lemma 3.17 and Theorem 3.24,  $\mathcal{A}(\tau)$  is always guaranteed to terminate in polynomial time irrespective of whether a schedule is returned or not. If  $\mathcal{A}(\tau)$  does not return a meaningful result during the binary search then  $\tau^* \geq 2$ , and it suffices to return the schedule computed by the algorithm of Lenstra et al. [LST90].  $\square$

### 3.5 Balancing Against Bipartite Matching

The approximation guarantee of the local search algorithm from Section 3.2 deteriorates with increasing  $\epsilon$ . There is however a simple algorithm that performs better for the case of large  $\epsilon$ .

**Theorem 3.31.** *Let  $0 < \epsilon < 1$ . The  $(1, \epsilon)$  case of RESTRICTED ASSIGNMENT MAKESPAN MINIMIZATION admits a  $2 - \epsilon$  approximation algorithm.*

*Proof.* Let  $\text{OPT}$  denote the makespan of an optimum solution to the input instance. Guess  $\text{OPT}$  through binary search. Construct a bipartite graph with  $\lfloor \text{OPT}/\epsilon - 1 \rfloor$  small nodes and 1 big node for each machine in the input instance. Each small job is connected by an edge to all the nodes of all the machines it can be assigned to with a finite processing time. Each big job is connected by an edge to all the big nodes of all the machines it can be assigned to with a finite processing time. It is easy to see that there is a perfect matching in this bipartite graph which corresponds to a schedule of makespan at most

$$\left( \left\lfloor \frac{\text{OPT}}{\epsilon} - 1 \right\rfloor \right) \epsilon + 1 \leq \text{OPT} - \epsilon + 1 \leq \text{OPT} + (1 - \epsilon)\text{OPT} = (2 - \epsilon)\text{OPT}.$$

$\square$

*Proof of Theorem 1.2.* Run the algorithm in Theorem 1.1 with a parameter  $\zeta'$  on the input instance to obtain a schedule with makespan at most  $(1 + R(\epsilon, \zeta'))\text{OPT}$ . Run the algorithm in Theorem 3.31 to get a  $(2 - \epsilon)\text{OPT}$  makespan schedule. The better of the two schedules has an approximation guarantee that is no worse than

$$\min \left\{ 1 + \frac{1}{2} (\epsilon + \sqrt{3 - 2\epsilon}) + \zeta', 2 - \epsilon \right\}.$$

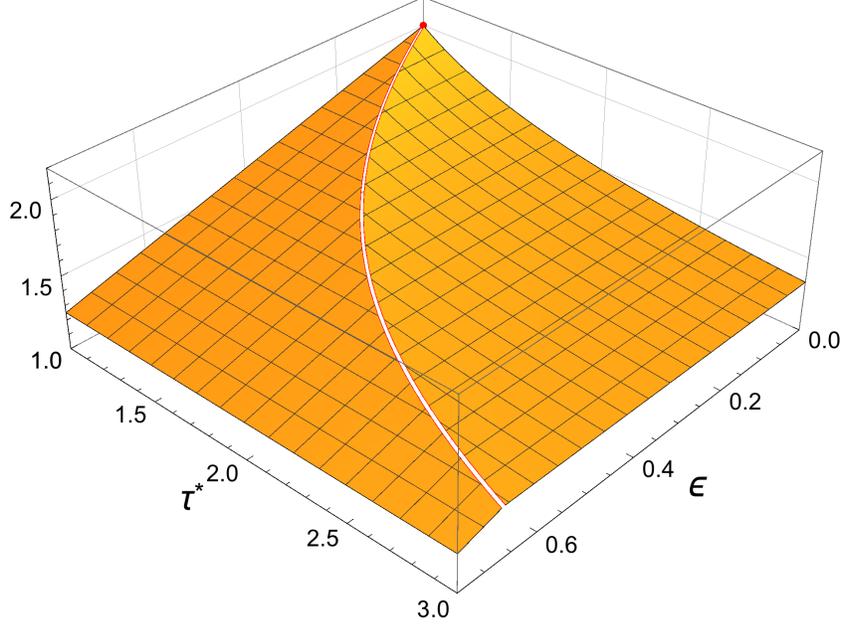


Figure 6: The previous profile of the approximation guarantee as a function of  $1 \leq \tau^* \leq 3$  and  $0 < \epsilon \leq 3/4$ . The two surfaces making up the profile correspond to the guarantees  $2 - \epsilon$  from Theorem 3.31 and  $1 + 1/\tau^*$  from the algorithm of Lenstra, Shmoys and Tardos [LST90]. The work of Chakrabarty, Khanna and Li [CKL15] provided a  $2 - \epsilon_0$  guarantee for some positive  $\epsilon_0 > 0$ , which is indicated as a red dot at the apex of the profile.

Suppose that  $2 - \epsilon \geq 17/9 + \zeta$ . Then,  $\epsilon \leq 1/9 - \zeta$ . So,

$$1 + \frac{1}{2} (\epsilon + \sqrt{3 - 2\epsilon}) + \zeta' \leq 1 + \frac{1}{2} (\epsilon + \sqrt{3 - 2\epsilon}) \Big|_{\epsilon=1/9} + \zeta' = 1 + \frac{1}{2} \cdot \left( \frac{1}{9} + \frac{5}{3} \right) + \zeta' = \frac{17}{9} + \zeta,$$

for  $\zeta' = \zeta$ . □

## 4 Conclusion

In this paper we presented a purely flow based local search algorithm for the  $(1, \epsilon)$ -case of RESTRICTED ASSIGNMENT MAKESPAN MINIMIZATION. The guarantees achieved by our algorithm improve significantly over the previous best one due to Chakrabarty et al. [CKL15]. For an illustration of the approximation profile for the problem, see Figures 6 and 7.

We remark that the ideas presented in this paper do *not* crucially depend on the fact that the instances contain exactly two different job sizes. Nevertheless, we have chosen to present our results in the  $(1, \epsilon)$ -case as there are still certain obstructions which prevent us from achieving  $2 - \epsilon_0$  guarantees for the *restricted* case in general. A second reason is that the algorithm for the case with two different job sizes admits a clean description in terms of maximum flows, as we have seen earlier in Section 3.2.

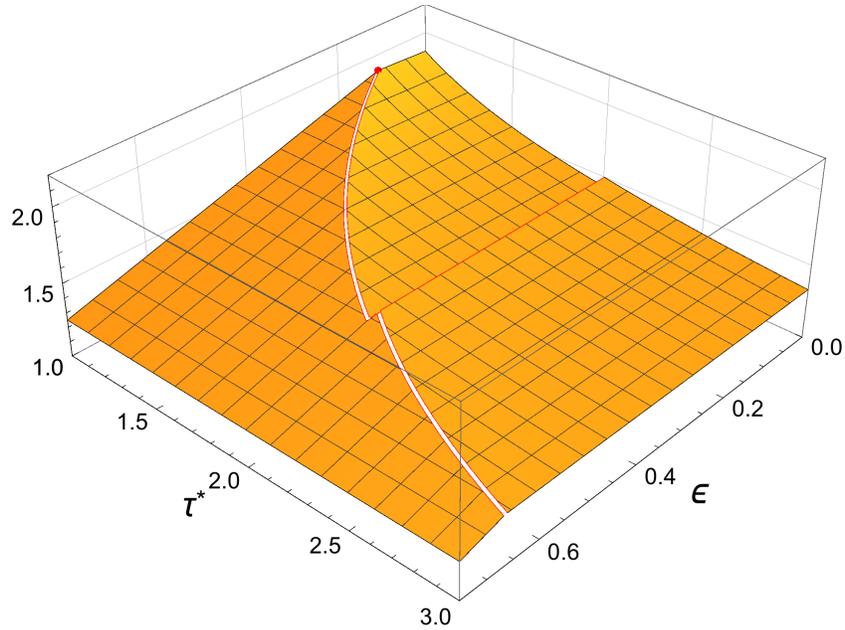


Figure 7: Now, following Theorem 1.2, the worst case guarantee is greatest (roughly 1.89) for instances with  $\tau^* = 1$  and  $\epsilon \approx 1/9$  as shown in the figure. The third surface arises from the guarantee of  $1 + R(\epsilon, \zeta)/\tau^*$  for  $1 \leq \tau^* < 2$  from the proof of Theorem 1.1.

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## A Appendix

*Proof of Lemma 3.3.* Consider the bipartite graph  $G = (M \cup J_b, E)$  where there is an edge  $\{i, j\} \in E$  if and only if  $i \in \Gamma(j)$ . A perfect matching in  $G$  of size  $|J_b|$  corresponds to such a map. If there is no such perfect matching, by Hall’s condition, there is a set  $S \subseteq J_b$  such that  $|N_G(S)| < |S|$ . Consider the following setting  $(y^*, z^*)$  of variables in the dual of  $CLP(\tau^*)$ .

$$y_i^* = \begin{cases} 1, & \text{if } i \in N_G(S), \\ 0, & \text{else.} \end{cases} \quad \text{and} \quad z_j^* = \begin{cases} 1, & \text{if } i \in S, \\ 0, & \text{else.} \end{cases}$$

It is now easily verified that  $(y^*, z^*)$  is a feasible solution to the dual of  $CLP(\tau^*)$  defined in (2.2). We use here the fact that configurations  $C \in \mathcal{C}(\tau^*, i)$  for any machine  $i \in M$  contain at most one big job since  $\tau^* < 2$ . As the objective function value  $\sum_{j \in J} z_j^* - \sum_{i \in M} y_i^*$  attained by this feasible solution  $(y^*, z^*)$  is strictly positive, it follows that the dual of  $CLP(\tau^*)$  is unbounded—for any  $\lambda > 0$ ,  $(\lambda y^*, \lambda z^*)$  is a feasible dual solution as well. Therefore, by *weak duality*,  $CLP(\tau^*)$  must be infeasible, a contradiction.  $\square$

*Proof of Proposition 3.7.* By flow decomposition, the maximum flow in  $H_\sigma^b(S, T)$  has flow paths  $p_1, \dots, p_{|H_\sigma^b(S, T)|}$ , each of which sends one unit of flow from some vertex in  $S$  to some vertex in  $T$ . The flow paths may not share a vertex in  $T$  as sinks have unit vertex capacities in  $H_\sigma^b(S, T)$  as

defined in Definition 3.6. Each machine  $i \in M$  has at most one outgoing edge with unit capacity in  $G_\sigma^b$  due to Definition 3.1(c) and Definition 3.4. So the flow paths may also not intersect in some vertex in  $M \setminus T$  since there is at most one outgoing arc with unit capacity. Similarly, they may not share a vertex in  $J_b$  as there is only one incoming arc of unit capacity to a vertex in  $J_b$  in  $G_\sigma^b$  using Definition 3.1 and Definition 3.4.  $\square$

*Claim A.1.*  $\forall 1 \leq \tau^* < 2, 0 < \epsilon < 1, \zeta > 0,$

$$\left( (2R - \delta - \epsilon - 1 - \tau^* \mu_1 \mu_2 - (1 + 2\tau^* + R)\mu_1) \cdot (\delta(1 - \mu_2) - 2\mu_2) - (1 - \mu_2)(1 - R) - \mu_2 \tau^* \right) > 0,$$

where  $\mu_1 = \min\{1, \zeta\}/4, \mu_2 = \min\{\delta, \zeta\}/4, \delta = (\sqrt{3 - 2\epsilon} - 1)/2,$  and  $R = (\epsilon + \sqrt{3 - 2\epsilon})/2 + \zeta.$

*Proof.* The statement is true if it is true for  $\tau^* = 2.$  To prove that the bracketed expression is positive we substitute the values of  $\mu_1, \mu_2, \delta$  and  $R$  from (3.1) and the statement of Theorem 1.1, and additionally set  $\tau^* = 2$  to get the statement

$$\begin{aligned} & -\frac{1}{2} \min \left\{ \frac{1}{2} (\sqrt{3 - 2\epsilon} - 1), \zeta \right\} - \frac{1}{2} (2\zeta + \epsilon + \sqrt{3 - 2\epsilon} - 2) \left( \frac{1}{4} \min \left\{ \frac{1}{2} (\sqrt{3 - 2\epsilon} - 1), \zeta \right\} - 1 \right) \\ & \quad + \frac{1}{16} \left( \frac{1}{4} (\sqrt{3 - 2\epsilon} + 3) \min \left\{ \frac{1}{2} (\sqrt{3 - 2\epsilon} - 1), \zeta \right\} - \sqrt{3 - 2\epsilon} + 1 \right) \times \\ & \left( \min\{1, \zeta\} \left( \min \left\{ \frac{1}{2} (\sqrt{3 - 2\epsilon} - 1), \zeta \right\} + 2\zeta + \epsilon + \sqrt{3 - 2\epsilon} + 10 \right) - 4(4\zeta + \sqrt{3 - 2\epsilon} - 1) \right) > 0. \end{aligned}$$

Using a standard computer algebra system for eliminating quantifiers over reals, we can verify the truth of the above statement for all  $0 < \epsilon < 1$  and  $\zeta > 0.$   $\square$

*Claim A.2.*  $\forall 0 < \epsilon < 1, \zeta > 0,$

$$\delta(1 - \mu_2) - 2\mu_2 > 0,$$

where  $\mu_2 = \min\{\delta, \zeta\}/4,$  and  $\delta = (\sqrt{3 - 2\epsilon} - 1)/2.$

*Proof.* Substituting the values of  $\mu_2$  and  $\delta,$  the statement reads

$$4(\sqrt{3 - 2\epsilon} - 1) > (\sqrt{3 - 2\epsilon} + 3) \min \left\{ \frac{1}{2} (\sqrt{3 - 2\epsilon} - 1), \zeta \right\}.$$

It suffices to verify that statement assuming that the min term always evaluates to the first argument, which then reduces to  $\epsilon + 3\sqrt{3 - 2\epsilon} > 4.$  Let  $f(\epsilon)$  denote the expression on the left. Then,  $f'(\epsilon) = 1 - 3/(\sqrt{3 - 2\epsilon})$  is negative over the range  $[0, 1],$   $f(0) - 4 > 0$  and  $f(1) - 4 = 0.$   $\square$