

ROBINSON-SCHENSTED-KNUTH ALGORITHM, JEU DE TAQUIN AND KEROV-VERSHIK MEASURES ON INFINITE TABLEAUX

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ABSTRACT. We investigate *Robinson-Schensted-Knuth algorithm* (RSK) and Schützenberger’s *jeu de taquin* in the infinite setup. We show that the recording tableau in RSK defines an *isomorphism* of the following two dynamical systems: (i) a sequence of i.i.d. random letters equipped with Bernoulli shift, and (ii) a random infinite Young tableau (with the distribution given by *Vershik-Kerov measure*, corresponding to some Thoma character of the infinite symmetric group) equipped with jeu de taquin transformation. As a special case we recover the results on non-colliding random walks and multidimensional Pitman transform.

1. INTRODUCTION

We start with a rather informal introduction; the formal definitions and some missing notation are postponed until Section 2.

1.1. Characters of the infinite symmetric groups. The notion of *irreducible representations* turns out to be not very suitable in the case of infinite groups and it is more convenient to replace it by the notion of *indecomposable characters* (the name *extremal characters* is also frequently used). The *indecomposable characters* of the *infinite symmetric group* \mathfrak{S}_∞ were classified by Thoma [Tho64]; he showed that there is a bijective correspondence between such characters and triples (α, β, γ) such that

$$\begin{aligned}\alpha &= (\alpha_1, \alpha_2, \dots) \text{ with } \alpha_1 \geq \alpha_2 \geq \dots \geq 0, \\ \beta &= (\beta_1, \beta_2, \dots) \text{ with } \beta_1 \geq \beta_2 \geq \dots \geq 0,\end{aligned}$$

are weakly decreasing sequences of non-negative numbers and $\gamma \geq 0$ is a non-negative number such that

$$\alpha_1 + \alpha_2 + \dots + \beta_1 + \beta_2 + \dots + \gamma = 1.$$

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10	22				
6	21				
5	12				
4	9	20	62	78	99
3	8	13	33	52	71
1	2	7	24	29	40

(a)

9	26	33	87	89	133
8	21	25	79	85	93
6	14	20	54	56	62
4	12	15	37	47	52
3	7	13	22	38	39
1	2	5	10	32	34

(b)

Figure 1. Simulated infinite Young tableaux, sampled according to Vershik-Kerov measure $\mathfrak{M}_{\alpha,\beta,\gamma}$ for two choices of the parameters: (a) $\alpha = (0.1, 0.1, 0.1, 0, 0, \dots)$, $\beta = (0.5, 0.2, 0, 0, \dots)$, $\gamma = 0$; (b) $\alpha = (0, 0, \dots)$, $\beta = (0.5, 0, 0, \dots)$, $\gamma = 0.5$.

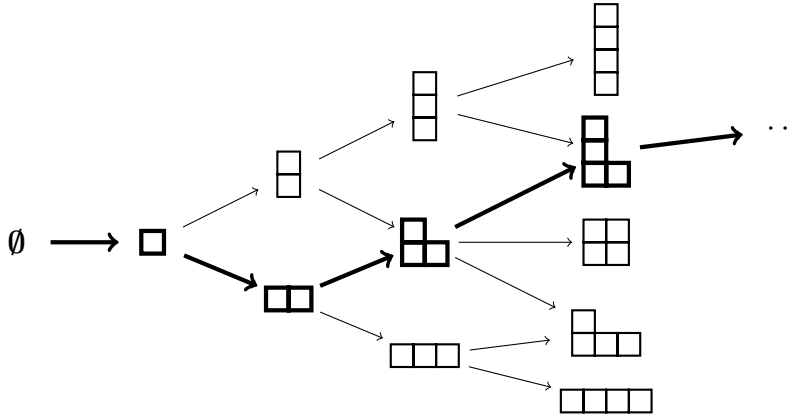


Figure 2. The Young graph. The highlighted diagrams form a path corresponding to the infinite Young tableau from Figure 1a.

The corresponding character will be denoted by $\chi_{\alpha,\beta,\gamma}$. The set of such triples (α, β, γ) is called *Thoma simplex*.

The meaning of the parameters in Thoma's characterization remained rather mysterious until Vershik and Kerov [VK81] related them to asymptotics of some random *infinite Young tableaux*. We shall review this relationship in the following.

1.2. Infinite Young tableaux and Vershik-Kerov measures. Vershik and Kerov [VK81] noticed that there is a natural bijective correspondence between the indecomposable characters of the infinite symmetric group \mathfrak{S}_∞ and *indecomposable central measures* on the set \mathbb{T} of *infinite Young tableaux*; thus Thoma's classification is equivalent to studying properties of some *random infinite Young tableaux* (see Figure 1). These indecomposable central measures are in the focus of the current paper. The measure on \mathbb{T} corresponding to the character $\chi_{\alpha,\beta,\gamma}$ will be denoted by $\mathfrak{M}_{\alpha,\beta,\gamma}$; we will call it *Vershik-Kerov measure*.

Any infinite Young tableau $t \in \mathbb{T}$ can be alternatively viewed as an infinite path $(\emptyset = \lambda^0 \nearrow \lambda^1 \nearrow \dots)$ in Young graph, see Figure 2. In the same paper [VK81], Vershik and Kerov found a beautiful interpretation of the parameters α and β in Thoma simplex as *asymptotic frequencies* of boxes appearing in appropriate rows and columns in such a sequence $(\lambda^0 \nearrow \lambda^1 \nearrow \dots)$ of Young diagrams. We postpone the details of this result and we provide it as Fact 5.1.

1.3. Generalized RSK algorithm. Usually, a *semistandard tableau* — or, shortly, *tableau* — is defined as a filling of the boxes of a Young diagram (with the letters from some alphabet) in such a way that the rows and columns are, *roughly speaking*, increasing. This definition creates no difficulties as long as we consider tableaux in which the entries do not repeat. If the entries repeat, the traditional approach is to require that each row should be *weakly increasing* and each column *strongly increasing*; in other words a letter can appear several times in one row and can appear at most once in one column.

Kerov and Vershik [KV86] took a different approach: they declared that each letter of the alphabet can be either a *row letter* (such a letter can appear several times in a row but can appear at most once in a column) or a *column letter* (such a letter can appear several times in a column but can appear at most once in a row). They also described how *Robinson-Schensted-Knuth algorithm* (RSK) can be adapted to this more general setup; we recall this construction in Section 3.4. As we shall see below, this generalization was essential in order to give a new interpretation of the parameters of Thoma simplex.

This generalization of RSK appeared also in the work of Berele and Remmel [BR85] as well as Berele and Regev [BR87], but only in a special case of finite alphabets in which any row letter is smaller than any column letter which is not sufficient for our purposes.

1.4. RSK and Vershik-Kerov measures.

1.4.1. *RSK as a homomorphism.* In the usual setup, Robinson-Schensted-Knuth algorithm applied to a *finite* sequence gives as an output a *pair* of tableaux, namely the *insertion tableau* and the *recording tableau*. Kerov and Vershik [KV86] applied Robinson-Schensted-Knuth algorithm to an *infinite* sequence (w_1, w_2, \dots) of letters from an arbitrary alphabet \mathbb{A} which consists of row letters and column letters. In this infinite setup the notion of the insertion tableau does not make sense and the outcome of Robinson-Schensted-Knuth algorithm $\text{RSK}(w_1, w_2, \dots) \in \mathbb{T}$ is defined as just the *recording tableau* (which is an *infinite Young tableau*, see Figure 1).

Kerov and Vershik [KV86] proved that if (W_1, W_2, \dots) is a sequence of random, independent, identically distributed letters with the distribution \mathcal{M} , then the distribution of the random infinite Young tableau $\text{RSK}(W_1, W_2, \dots)$ coincides with the *indecomposable central measure* $\mathfrak{M}_{\alpha, \beta, \gamma}$ (*Vershik-Kerov measure*) corresponding to some element (α, β, γ) of Thoma simplex given as follows: $\alpha_1 \geq \alpha_2 \geq \dots$ are the probabilities of the atoms of the measure \mathcal{M} on the row letters; $\beta_1 \geq \beta_2 \geq \dots$ are the probabilities of the atoms of the measure \mathcal{M} on the column letters; γ is the total probability of the continuous part of \mathcal{M} . We present this result in full detail in Fact 3.1. This result gives another interpretation of the parameters in Thoma simplex as probabilities of atoms of the measure \mathcal{M} on the alphabet \mathbb{A} .

Notice that the extension of RSK algorithm to *column letters* was essential in order to recover all elements of Thoma's simplex. It is worth pointing out that the above result of Kerov and Vershik [KV86] — contrary to the results presented in the current paper — holds in general and does not require any additional assumptions on the alphabet \mathbb{A} and the probability distribution \mathcal{M} of the letters.

The above result of Kerov and Vershik provides a very concrete realization (or, viewed alternatively, an equivalent definition) of all indecomposable central measures on the set \mathbb{T} of infinite Young tableaux. In other words: RSK provides a convenient *homomorphism* between the following two probability spaces: from (i) the very simple *product space* $(\mathbb{A}^{\mathbb{N}}, \mathcal{B}, \mathcal{M}^{\mathbb{N}})$ (i.e., i.i.d. letters), to (ii) the probability space $(\mathbb{T}, \mathcal{F}, \mathfrak{M}_{\alpha, \beta, \gamma})$ of infinite Young tableaux equipped with some Vershik-Kerov measure. The original paper Kerov and Vershik [KV86] presents some applications of this homomorphism.

1.4.2. *RSK as an isomorphism.* It is a natural to ask if this homomorphism is, in fact, an *isomorphism*. In the current paper we give a positive answer to this question under additional assumptions about the structure of the alphabet and the probability measure on it. Namely, for a special choice of the alphabet \mathbb{J} (*the jeu de taquin alphabet*) which can be informally visualized

as

$$(1.1) \quad \underbrace{1 < 2 < 3 < \dots < \dots < 0.1 < \dots < 0.9 < \dots < \dots}_{\text{row letters}} < \underbrace{\dots < -3 < -2 < -1}_{\text{column letters}},$$

and with a special choice of the probability distribution $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{J}}$ on \mathbb{J} (for details see Section 3.2) the following result holds true.

Theorem 1.1 (RSK is an isomorphism of probability spaces). *Let (α, β, γ) be an element of Thoma simplex. Then RSK is an isomorphism between the following two probability spaces:*

- $(\mathbb{J}^{\mathbb{N}}, \mathcal{B}, (\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{J}})^{\mathbb{N}})$, i.e., a sequence of i.i.d. random letters of the jeu de taquin alphabet with the distribution $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{J}}$;
- $(\mathbb{T}, \mathcal{F}, \mathfrak{M}_{\alpha, \beta, \gamma})$, i.e., infinite Young tableaux with Vershik-Kerov measure $\mathfrak{M}_{\alpha, \beta, \gamma}$.

Here and through the whole paper the symbol \mathcal{B} will refer to the product σ -algebra on appropriate product space. The σ -algebra \mathcal{F} on \mathbb{T} will be defined in Section 2.2.

So, it is natural to ask *what is the inverse to this isomorphism?* In order to answer this question we will have to study *jeu de taquin* for infinite Young tableaux.

1.5. Jeu de taquin. *Jeu de taquin* (literally, *teasing game*) was introduced by Schützenberger [Sch77] for finite (semistandard) tableaux. It turned out to be a powerful tool of algebraic combinatorics, in particular for problems related to the representation theory of symmetric groups and Robinson-Schensted-Knuth algorithm. In our previous paper [RS15] we investigated a generalization of jeu de taquin to the setup of *infinite Young tableaux*. We will recall it briefly.

Consider an *infinite Young tableau* $t \in \mathbb{T}$, see Figure 3a. We remove the bottom-left corner box (the box which contains the number 1); in this way an empty space is created. We start sliding the boxes according to the rules presented in Figure 4, i.e., we always slide one of the following two boxes: the one on the right or the one on the top of the empty space, always choosing the box which has smaller contents. As we continue sliding, the empty space keeps moving to the top or to the right, see Figure 3b.

The outcome of jeu de taquin is twofold. Firstly, it is the path of the empty space $\mathbf{p}(t) = (\mathbf{p}_1(t), \mathbf{p}_2(t), \dots)$, which will be called *jeu de taquin path*. (A careful reader might object that for some tableaux the jeu de taquin path is a *finite* sequence, see Figure 5. We will show in Theorem 1.4 that in the cases of our interest this is not the case.)

18	29	43	66	85		
13	24	35	56	83		
12	19	20	47	57		
10	15	17	25	42	61	119
5	9	14	22	30	48	58
4	6	7	11	26	27	41
1	2	3	8	16	21	28

(a)

18	29	43	66	110		
13	24	35	56	85		
12	19	20	47	83		
10	15	17	42	57	61	119
5	9	14	25	30	48	58
4	6	11	22	26	27	41
2	3	7	8	16	21	28

(b)

Figure 3. (a) A part of an infinite Young tableau t . The highlighted boxes form the beginning of the jeu de taquin path $\mathbf{p}(t)$. (b) The outcome of sliding of the boxes along the highlighted jeu de taquin path. The outcome of the jeu de taquin transformation $J(t)$ is obtained by subtracting 1 from every entry.

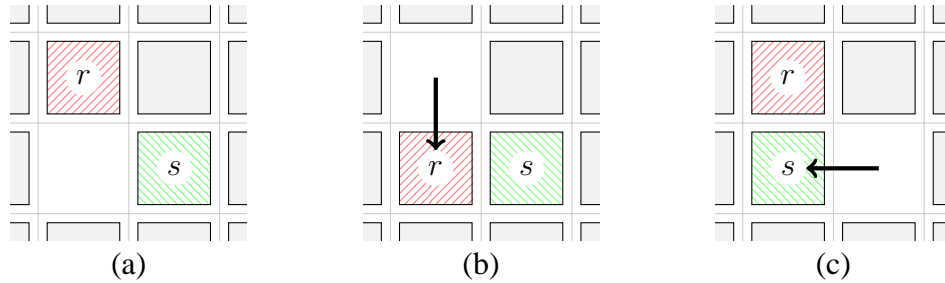


Figure 4. Elementary step of the jeu de taquin transformation: (a) the initial configuration of boxes, (b) the outcome of the slide in the case when $r < s$, (c) the outcome of the slide in the case when $s < r$.

Secondly, after performing all slides of jeu de taquin, we obtain an object which looks almost like an infinite Young tableau (see Figure 3b) except that the numbering of boxes starts with 2 instead of 1. Let us subtract 1 from every entry of this “tableau”; the outcome is a true infinite Young tableau which we denote by $J(t)$. The map $t \mapsto J(t)$ will be called *jeu de taquin transformation*.

These two outcomes of jeu de taquin are in the focus of the current paper. In the following we will discuss them in more detail.

12	53					
11	47					
10	42					
6	32	41	50			
5	9	13	21	60	61	89
4	7	8	17	20	24	30
1	2	3	16	19	23	27

Figure 5. Example of an infinite Young tableau t for which the corresponding jeu de taquin path $\mathbf{p}(t) = (\mathbf{p}_1(t), \dots, \mathbf{p}_\ell(t))$ is finite. Theorem 1.4 shows that this cannot happen for the random Young tableaux considered in the current paper.

1.6. The dynamical system of jeu de taquin. As we just mentioned, one of the outcomes of jeu de taquin applied to an infinite tableau $t \in \mathbb{T}$ is another infinite tableau $J(t) \in \mathbb{T}$. This setup naturally raises questions about the iterations of the jeu de taquin map

$$t, J(t), J(J(t)), \dots$$

or, in other words, about the *dynamical system of jeu de taquin*. More precisely, we consider the set \mathbb{T} of infinite Young tableaux equipped with some Vershik-Kerov measure $\mathfrak{M}_{\alpha,\beta,\gamma}$, thus we consider the *measure-preserving dynamical system* $(\mathbb{T}, \mathcal{F}, \mathfrak{M}_{\alpha,\beta,\gamma}, J)$. Some basic properties of this dynamical system are summarized by the following theorem.

Theorem 1.2. *Jeu de taquin transformation $J : \mathbb{T} \rightarrow \mathbb{T}$ on the probability space $(\mathbb{T}, \mathcal{F}, \mathfrak{M}_{\alpha,\beta,\gamma})$ of infinite Young tableaux equipped with an arbitrary Vershik-Kerov measure is*

- *measure preserving,*
- *ergodic (i.e., every measurable set $E \in \mathcal{F}$ which is J -invariant fulfills $\mathfrak{M}_{\alpha,\beta,\gamma}(E) \in \{0, 1\}$; for an introduction to the ergodic theory see [Sil08]).*

The following extension of Theorem 1.1 holds true.

Theorem 1.3 (RSK is an isomorphism of dynamical systems). *Let (α, β, γ) be an element of Thoma simplex. Then RSK is an isomorphism of the following dynamical systems:*

- $(\mathbb{J}^{\mathbb{N}}, \mathcal{B}, (\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{J}})^{\mathbb{N}}, S)$, i.e., a sequence of i.i.d. random letters from the jeu de taquin alphabet, equipped with Bernoulli shift $S : \mathbb{J}^{\mathbb{N}} \rightarrow \mathbb{J}^{\mathbb{N}}$, defined by

$$S(w_1, w_2, \dots) := (w_2, w_3, \dots);$$

- $(\mathbb{T}, \mathcal{F}, \mathfrak{M}_{\alpha, \beta, \gamma}, J)$, i.e., infinite Young tableaux with Vershik-Kerov measure $\mathfrak{M}_{\alpha, \beta, \gamma}$ equipped with jeu de taquin transformation.

In the following we will show explicitly the inverse map to this isomorphism. In order to do this we will have to investigate jeu de taquin paths.

1.7. Asymptotes of jeu de taquin paths. For a box $\square = (x, y)$ of a tableau we denote by $x(\square) := x$ the index of column of the box and by $y(\square) := y$ the index of row of the box (the numbering of rows and columns starts with 1).

As we already mentioned, one of the outcomes of jeu de taquin applied to an infinite tableau $t \in \mathbb{T}$ is the jeu de taquin path $\mathbf{p}(t)$. The following theorem describes the asymptotic behavior of jeu de taquin paths on random tableaux.

Theorem 1.4 (Asymptotics of a jeu de taquin path). *Let T be a random infinite Young tableau distributed according to some Vershik-Kerov measure $\mathfrak{M}_{\alpha, \beta, \gamma}$.*

Then, almost surely, jeu de taquin path $\mathbf{p}(T) = (\mathbf{p}_1(T), \mathbf{p}_2(T), \dots)$ is an infinite sequence (i.e., the situation from Figure 5 is not possible).

Furthermore, almost surely, exactly one of the following three events holds true.

- (A) The path stabilizes in some row k ; in other words $y(\mathbf{p}_i(T)) = k$ holds true for almost all i . This event happens with probability α_k .
- (B) The path stabilizes in some column k ; in other words $x(\mathbf{p}_i(T)) = k$ holds true for almost all i . This event happens with probability β_k .
- (C) The path has some asymptotic slope; in other words the limit

$$\lim_{i \rightarrow \infty} \frac{\mathbf{p}_i(T)}{\|\mathbf{p}_i(T)\|}$$

exists and thus is equal to $(\cos \Theta(T), \sin \Theta(T))$ for some $0 < \Theta(T) < \frac{\pi}{2}$. This event happens with probability γ .

This theorem is illustrated in Figure 6 where some sample jeu de taquin paths are shown together with their asymptotes (dashed lines). The set of all possible asymptotes is visualized in Figure 7.

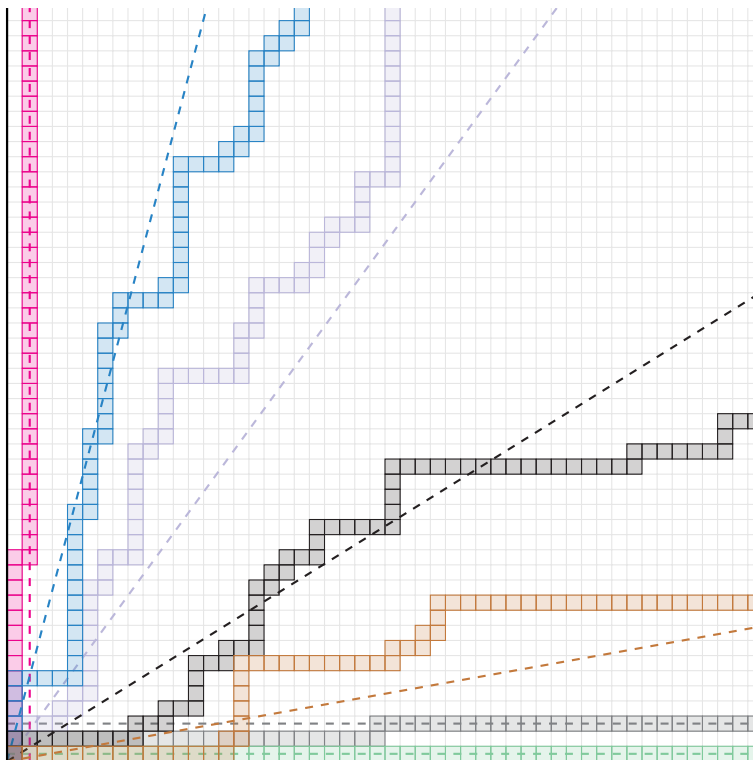


Figure 6. Simulated jeu de taquin paths and their asymptotes (dashed lines). Horizontal asymptotes correspond to case (A), vertical asymptotes correspond to case (B), sloped asymptotes correspond to case (C) of Theorem 1.4.

The probability distribution of slopes Θ of jeu de taquin paths in case (C) is universal (in the sense that it does not depend on α, β, γ) and is known explicitly; we postpone presentation of its details until Proposition 7.2.

1.8. The inverse of RSK. Recall that \mathbb{J} is the jeu de taquin alphabet shown in Eq. (1.1); the details of its definition are postponed to Section 3.2. We shall define now a function $\Psi : \mathbb{T} \rightarrow \mathbb{J}$. Let t be an infinite Young tableau. We will use notations of Theorem 1.4.

$$\Psi(t) := \begin{cases} k & \text{if case (A) holds,} \\ -k & \text{if case (B) holds,} \\ F_{\Theta}(\Theta(t)) & \text{if case (C) holds,} \end{cases}$$

where F_{Θ} is the cumulative distribution function of the distribution of Θ (see Proposition 7.2).

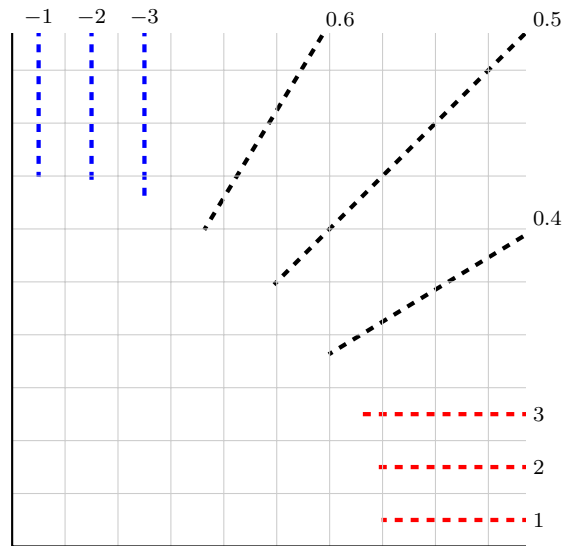


Figure 7. Possible asymptotes for a *jeu de taquin* path and the corresponding values (elements of the alphabet \mathbb{J}) of the function Ψ . For details see Theorem 1.4.

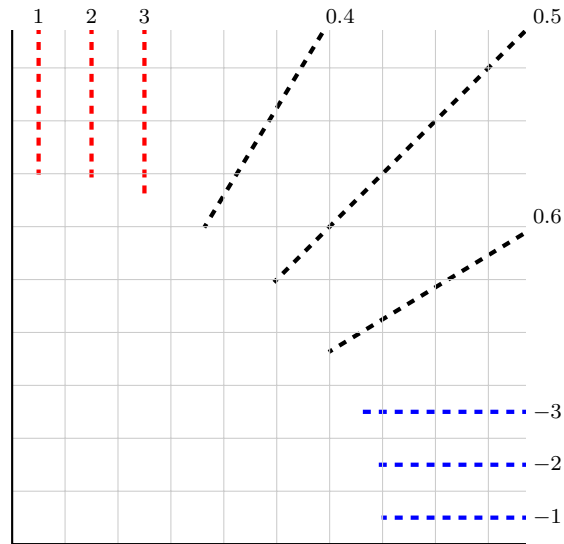


Figure 8. Possible asymptotes for *Schensted insertion* and the corresponding values (elements of the alphabet \mathbb{I}) of the letter w . For details see Theorem 6.4.

This function Ψ is visualized in Figure 7. Theorem 1.4 shows that (with respect to the probability measure $\mathfrak{M}_{\alpha,\beta,\gamma}$) this function is well-defined almost everywhere.

Theorem 1.5. *The inverse of RSK map from Theorem 1.1 and Theorem 1.3 is given (almost surely) by asymptotic slopes of jeu de taquin in consecutive iterations of jeu de taquin transformation J . More explicitly,*

$$(1.2) \quad \text{RSK}^{-1}(t) := \left(\Psi(t), \Psi(J(t)), \Psi(J(J(t))), \dots \right) \in \mathbb{J}^{\mathbb{N}}.$$

It should be stressed that the above theorem holds *only in the almost sure sense*, i.e., it states that the equalities

$$\begin{aligned} \text{RSK} \circ \text{RSK}^{-1} &= \text{Id}, \\ \text{RSK}^{-1} \circ \text{RSK} &= \text{Id}, \end{aligned}$$

hold true except for measure-zero sets.

1.9. Special case: Plancherel measure. One of Thoma characters, the one corresponding to $\alpha = \beta = (0, 0, \dots)$, $\gamma = 1$ plays a special role. The corresponding indecomposable central measure is the celebrated *Plancherel measure* on the set \mathbb{T} of infinite Young tableaux. The jeu de taquin alphabet \mathbb{J} in this case can be identified simply with the unit interval $(0, 1)$ equipped with the Lebesgue measure and one does not have to consider the subtleties related to row letters and column letters. This case was considered in our previous paper [RS15]; in particular Theorems 1.1 to 1.5 were all proved there in this special case. The proofs for the general case presented in the current paper will heavily use the results from that paper (see Fact 5.5).

1.10. Special case: non-colliding random walks and Pitman transform. We consider the special case when $\alpha = (\alpha_1, \dots, \alpha_\ell, 0, 0, \dots)$ has only finitely many non-zero entries, and $\beta = (0, 0, \dots)$, $\gamma = 0$ are zero. In particular, this means that as the jeu de taquin alphabet we can take $\mathbb{J} = [\ell] = \{1, \dots, \ell\}$, thus we recover the usual version of RSK without column letters.

In this case, a random infinite word (W_1, W_2, \dots) of i.i.d. letters with distribution $\mathcal{M}_{\alpha,\beta,\gamma}^{\mathbb{J}}$ can be identified with a random walk X in \mathbb{Z}_+^ℓ . The recording tableau $\text{RSK}(W_1, W_2, \dots)$ has boxes only in the first ℓ rows, thus the corresponding path $(\lambda^0 \nearrow \lambda^1 \nearrow \dots) := \text{RSK}(W_1, W_2, \dots)$ in the Young graph can be also viewed as a random walk Λ in \mathbb{Z}_+^ℓ .

This setup has been studied by O'Connell and Yor [OY02] who introduced a certain path-transformation $G^{(\ell)}$, called *generalized Pitman transform*, with the property that the transformed walk $G^{(\ell)}(X)$ has the same law as the original walk $X = (X_1, \dots, X_\ell)$ conditioned never to exit the Weyl chamber $\{x : x_1 \geq \dots \geq x_\ell\}$; such a walk can be alternatively viewed

as a collection of ℓ random walks X_1, \dots, X_ℓ which are conditioned to be *non-colliding*, i.e. $X_1 \geq \dots \geq X_\ell$. This path-transformation has been further studied by O’Connell [O’C03] who has shown that Pitman transform is nothing else but RSK transform in disguise, i.e., $\Lambda = G^{(\ell)}(X)$. He also proved that the inverse of the map $G^{(\ell)} = \text{RSK}$ exists and he found it explicitly. Clearly, his result is a special case of Theorem 1.1, however it is not immediate that his formula [O’C03, Corollary 3.2] for RSK^{-1} is equivalent to the one given in the current paper (Theorem 1.5).

1.11. Outline of the paper. The main results of the paper (which were presented in this Introduction) will be proved in Section 7. All proofs will base on key Theorem 7.1 which gives a detailed information about the jeu de taquin path for some special random infinite Young tableau. It will be convenient to prove this result in an equivalent form as Theorem 6.4; essentially most of the current paper is just a preparation for the proof of this Theorem 6.4. We review it briefly.

Section 2 contains some missing notation from this Introduction and presents some wider context.

In Section 3 we present how some classical combinatorial notions can be adapted to the more general setup of alphabets containing row letters and column letters.

Section 4 concerns some basic properties of jeu de taquin.

Section 5 concerns typical shape of some random Young diagrams and the asymptotic determinism of Schensted insertion in the special case related to the Plancherel measure.

In Section 6 we show the key technical result, Theorem 6.4 which concerns asymptotic determinism of Schensted insertion in the general case.

Finally, Section 7 contains the proofs of the main results.

2. PRELIMINARIES:

YOUNG DIAGRAMS, YOUNG TABLEAUX

2.1. Young diagrams, Young graph. The set of Young diagrams with n boxes will be denoted by \mathbb{Y}_n ; the set of all Young diagrams will be denoted by \mathbb{Y} .

The set \mathbb{Y} of Young diagrams carries in a natural way the structure of a directed graph, which will be called *Young graph*, see Figure 2. Namely, for a pair of Young diagrams we write $\lambda \nearrow \mu$ if the diagram μ is obtained from λ by adding exactly one box. The *empty Young diagram* with no boxes will be denoted by \emptyset .

From the perspective of the asymptotic representation theory, it is very interesting to investigate the boundary of this graph. This motivates investigation of *infinite paths in this graph* which, as we shall see, correspond to *infinite Young tableaux*.

2.2. Infinite Young tableaux. We use the notation $\mathbb{N} = \{1, 2, 3, \dots\}$ for the set of the natural numbers. We use so defined natural numbers to index rows and columns of Young diagrams and tableaux; in particular the first row (column) corresponds to the number 1, etc.

An *infinite Young tableau* t is a function $\mathbb{N}^2 \ni (x, y) \mapsto t_{x,y} \in \{1, 2, 3, \dots, \infty\}$. We interpret it as a filling of the boxes of the first quadrant of the plane; the boxes filled with the symbol ∞ can be interpreted as empty boxes (see Figure 1a). We require that each finite entry (an element of the set $\{1, 2, \dots\}$) appears in *exactly* one box and that each row and each column is weakly increasing (from left to right and from bottom to top). This definition differs slightly from the one from our previous paper [RS15], where no empty boxes were allowed.

An infinite Young tableau can be viewed alternatively, as follows. There is a bijective correspondence between infinite Young tableaux and *infinite paths in the Young graph*

$$(2.1) \quad \emptyset = \lambda^0 \nearrow \lambda^1 \nearrow \dots$$

This correspondence is defined as follows: for an infinite tableau t we define the Young diagram λ^i as the collection of boxes with entries $\leq i$.

The *set of infinite Young tableaux* will be denoted by \mathbb{T} . It is equipped with its natural measurable structure, namely, the minimal σ -algebra \mathcal{F} of subsets of \mathbb{T} such that all the coordinate functions $t \mapsto t_{x,y}$ are measurable.

3. ALPHABETS WITH ROW LETTERS AND COLUMN LETTERS

3.1. Alphabets with row letters and column letters. Let $\mathbb{A} = \mathbb{A}_c \sqcup \mathbb{A}_r$ be an *alphabet* (i.e., a linearly ordered set). The elements of \mathbb{A}_r will be called *row letters* while the elements of \mathbb{A}_c will be called *column letters* (in the original paper [KV86, Section 1] these were called, respectively, *positive* and *negative*, which is not very convenient for our purposes). We define the relationships $<_r$ and $<_c$ by

$$\begin{aligned} a <_r b &\iff (a < b) \vee [(a = b) \wedge a \in \mathbb{A}_r], \\ a <_c b &\iff (a < b) \vee [(a = b) \wedge a \in \mathbb{A}_c] \end{aligned}$$

for any $a, b \in \mathbb{A}$. Notice that for any $a, b \in \mathbb{A}$ *exactly one* of the following statements is true: $a <_r b$ or $b <_c a$.

We may also consider $\mathbb{A} = \mathbb{A}_c \sqcup \mathbb{A}_0 \sqcup \mathbb{A}_r$; the elements of \mathbb{A}_0 will be called *neutral letters*. In this case the relationships $a <_r a$ and $a <_c a$ are

not well-defined for $a \in \mathbb{A}_0$. This will not create any problems as long as any element of \mathbb{A}_0 appears in the words and tableaux which we consider at most once. Alternatively, any element of \mathbb{A}_0 can be regarded either as an element of \mathbb{A}_r or \mathbb{A}_c .

3.2. The jeu de taquin alphabet. For our purposes, the most important example of an alphabet is $\mathbb{J} = \mathbb{J}_r \sqcup \mathbb{J}_0 \sqcup \mathbb{J}_c$ with $\mathbb{J}_r = \{1, 2, 3, \dots\}$, $\mathbb{J}_0 = (0, 1) \subset \mathbb{R}$ and $\mathbb{J}_c = \{\dots, -3, -2, -1\}$ with the linear order defined as follows: on each of the sets \mathbb{J}_r , \mathbb{J}_0 , \mathbb{J}_c we consider the natural order; we declare any element of \mathbb{J}_r smaller than any element of \mathbb{J}_0 , which is smaller than any element of \mathbb{J}_c . This alphabet will be called *the jeu de taquin alphabet*; it can be visualized informally as Eq. (1.1).

If (α, β, γ) belongs to Thoma simplex, we define the following probability measure $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{J}}$ on \mathbb{J} :

- for $i \in \{1, 2, 3, \dots\} = \mathbb{J}_r$ we set $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{J}}(i) = \alpha_i$;
- for $-i \in \{-1, -2, -3, \dots\} = \mathbb{J}_c$ we set $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{J}}(-i) = \beta_i$;
- on $\mathbb{J}_0 = (0, 1)$ we take as $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{J}}$ the absolutely continuous measure on the unit interval $(0, 1)$ with constant density γ .

This alphabet and probability measure were used in Theorem 1.1 and Theorem 1.3.

3.3. Tableaux. A (*semistandard*) *tableau* in our new set up is defined as a filling of the entries of a Young diagram with the property that each row is $<_r$ -*increasing* (from left to right) and each column is $<_c$ -*increasing* (from bottom to top), see Figure 9. This definition is equivalent to the one of Kerov and Vershik [KV86, Section 2].

3.4. Robinson-Schensted-Knuth algorithm. We assume that the reader is familiar with the details of Robinson-Schensted-Knuth algorithm, which are described in several well-known sources such as [Ful97, Knu73, Sta99, Sag01]. We provide only a brief overview below.

The (*row*) *insertion procedure* applied to a tableau t and a letter $w \in \mathbb{A}$ produces a new tableau denoted $t \leftarrow w$. The new tableau is computed by performing a succession of bumping steps whereby w is inserted (by a procedure which we call *elementary insertion*) into the first row of the diagram, bumping an existing entry from the first row into the second row, which results in an entry of the second row being bumped to the third row, and so on, until finally the entry being bumped settles down in an unoccupied position outside the diagram.

The *elementary insertion* has to be adjusted to our new setup: we insert the new letter into the row as much to the right as possible, so that the row remains $<_r$ -*increasing* and no gaps are created, see Figures 9 and 10.

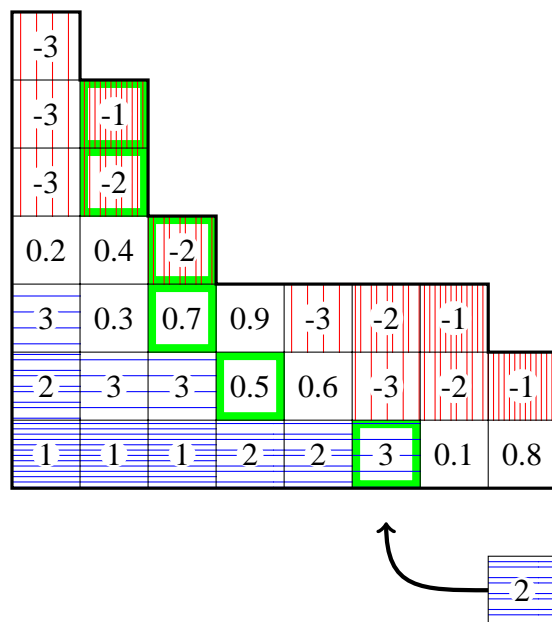


Figure 9. Example of a tableau in the jeu de taquin alphabet \mathbb{J} , see Section 3.2. Row letters are marked by horizontal lines, column letters are marked by vertical lines. Highlighted boxes form the bumping route when letter 2 is inserted into tableau.

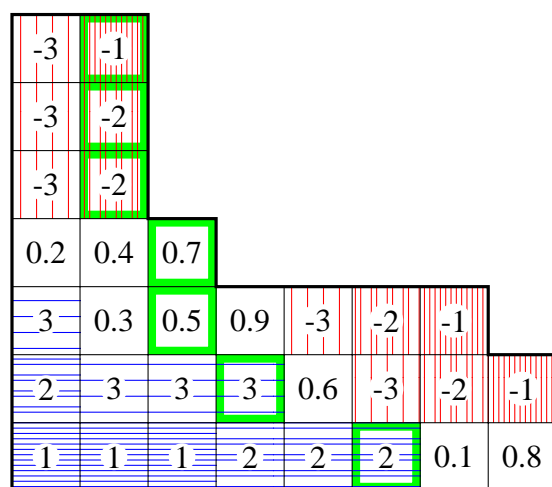


Figure 10. The outcome of insertion of letter 2 into the tableau from Figure 9. Highlighted boxes form the bumping route.

This definition is equivalent to the one from the work of Kerov and Vershik [KV86, Section 2].

The *insertion tableau* $P(w_1, \dots, w_n)$ associated to a finite word is defined as the outcome of iterative insertion of the letters into the empty tableau:

$$(3.1) \quad P(w_1, \dots, w_n) := \left(\left((\emptyset \leftarrow w_1) \leftarrow w_2 \right) \leftarrow \dots \right) \leftarrow w_n.$$

The RSK *shape* of a finite word (w_1, \dots, w_n) is defined as the Young diagram, equal to the shape of $P(w_1, \dots, w_n)$.

The *recording tableau* $Q(w_1, w_2, \dots)$ associated to the (finite, respectively, infinite) word (w_1, w_2, \dots) is defined as the (finite, respectively, infinite) Young tableau which corresponds to the (finite, respectively, infinite) path $\lambda^0 \nearrow \lambda^1 \nearrow \dots$ in Young graph defined as follows: λ^k is the RSK shape of the prefix (w_1, \dots, w_k) .

If w_1, w_2, \dots is an infinite word, we define the outcome of *Robinson-Schensted-Knuth algorithm* as the corresponding recording tableau.

$$\text{RSK}(w_1, w_2, \dots) := Q(w_1, w_2, \dots) \in \mathbb{T}.$$

3.5. Robinson-Schensted-Knuth algorithm as a homomorphism of probability spaces. We present now the precise form of the result of Kerov and Vershik which we discussed in Section 1.4. We will use this result several times: roughly speaking, whenever a random infinite Young tableau distributed according to some indecomposable central measure (Vershik-Kerov measure) has to be used, we will use a concrete realization of such a random tableau on the probability space of a sequence of i.i.d. random letters.

Note that the result below applies, in particular, to the special cases when (a) the alphabet $\mathbb{A} = \mathbb{J}$ is the jeu de taquin alphabet equipped with the probability measure $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{J}}$ or, (b) when the alphabet $\mathbb{A} = \mathbb{I}$ is the insertion alphabet equipped with the probability measure $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{I}}$ (the definition of this alphabet is postponed until Section 6.1). In fact, these are the only two cases which will be used in the current paper, so the reader can focus her attention on them.

Fact 3.1 (RSK is a *homomorphism* of probability spaces, Kerov and Vershik [KV86, Theorem 2]). *Let alphabet $\mathbb{A} = \mathbb{A}_r \sqcup \mathbb{A}_0 \sqcup \mathbb{A}_c$ with a probability measure \mathcal{M} be given. Let $\alpha_1 \geq \alpha_2 \geq \dots$ be the probabilities (listed in the weakly decreasing order) of the atoms of the measure \mathcal{M} restricted to \mathbb{A}_r and let $\beta_1 \geq \beta_2 \geq \dots$ be the probabilities (listed in the weakly decreasing order) of the atoms of the measure \mathcal{M} restricted to \mathbb{A}_c . Let γ be the total probability of the continuous part of \mathcal{M} . We assume that the probability measure \mathcal{M} restricted to \mathbb{A}_0 has no atoms.*

Let W_1, W_2, \dots be a sequence of random, independent, identically distributed letters from \mathbb{A} with distribution \mathcal{M} . Then the distribution of the recording tableau $Q(W_1, W_2, \dots) \in \mathbb{T}$ coincides with Vershik-Kerov measure $\mathfrak{M}_{\alpha, \beta, \gamma}$.

In other words, RSK is a homomorphism between the following two probability spaces:

- $(\mathbb{A}^\infty, \mathcal{B}, \mathcal{M}^\infty)$, i.e., sequences of i.i.d. random letters;
- $(\mathbb{T}, \mathcal{F}, \mathfrak{M}_{\alpha, \beta, \gamma})$, i.e., random infinite Young tableaux with Vershik-Kerov measure $\mathfrak{M}_{\alpha, \beta, \gamma}$.

In order to recover this formulation from the original work of Kerov and Vershik, one should simply declare that any element of \mathbb{A}_0 is either a row or a column letter. Since, almost surely, any neutral letter appears in the sequence W_1, W_2, \dots at most once, this does not create any difficulties.

3.6. Greene's theorem.

Fact 3.2 (Greene's theorem). *Let w be a finite word in some alphabet $\mathbb{A} = \mathbb{A}_r \sqcup \mathbb{A}_c$. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be the RSK shape associated to w .*

Then for each $k \geq 1$, the sum of the lengths of the first k rows, $\lambda_1 + \dots + \lambda_k$, is equal to the length of the longest subsequence of w which can be decomposed into k disjoint $<_r$ -increasing subsequences.

Also, the sum of the lengths of the first k columns, $\lambda'_1 + \dots + \lambda'_k$, is equal to the length of the longest subsequence of w which can be decomposed into k disjoint $<_c$ -decreasing subsequences.

For the proof of this result for alphabets containing row letters and column letters we refer to the work of Kerov and Vershik [KV86, Proposition 1].

3.7. Standardization of a sequence. In the current paper we will use generalizations of several classical results concerning RSK in the setup of alphabets involving row letters and column letters. In the following we present a simple technical tool which will be used in order to show that a given result in the generalized setup is, in fact, equivalent to its classical version.

Let $w = (w_1, \dots, w_n)$ with $w_1, \dots, w_n \in \mathbb{A}$. We assume that each neutral letter appears at most once in w . Let $\pi = (\pi_1, \dots, \pi_n)$ be a tuple of some abstract elements which are all different. We define a linear order on $\{\pi_1, \dots, \pi_n\}$ by setting for all $1 \leq i < j \leq n$:

$$(3.2) \quad \pi_i < \pi_j \iff w_i <_r w_j;$$

in other words it is a lexicographic order in which we first compare w_i with w_j with respect to the usual order $<$; if they are equal then we compare the

indices i and j in the usual order (for $w_i \in \mathbb{A}_r$) or in the opposite order (for $w_i \in \mathbb{A}_c$).

The tuple $\pi(w_1, \dots, w_n) := (\pi_1, \dots, \pi_n)$, called *standardization* of w , is uniquely determined up to an order-preserving isomorphism; it can be identified with a permutation. The following Lemma 3.3 shows that with respect to RSK, the original tuple and its standardization have similar properties; the advantage of the tuple π is that its entries are not repeated, thus we avoid the difficulties related to column letters and row letters and we can apply some classical results directly.

Lemma 3.3. *The recording tableaux corresponding to the words w and its standardization $\pi(w)$ are equal.*

Proof. In order to show that the recording tableaux are equal, it is enough to show that for each $1 \leq k \leq n$, RSK shapes associated to the prefixes (w_1, \dots, w_k) and (π_1, \dots, π_k) are equal.

Since there is a bijective correspondence between $<_r$ -increasing subsequences of w and $<$ -increasing subsequences of π , i.e., for any $i_1 < \dots < i_\ell$

$$w_{i_1} <_r \dots <_r w_{i_\ell} \iff \pi_{i_1} < \dots < \pi_{i_\ell},$$

Greene's theorem (Fact 3.2) finishes the proof. \square

4. ELEMENTARY PROPERTIES OF JEU DE TAQUIN

4.1. Lazy version of jeu de taquin. It will be convenient to work with a modified version of the jeu de taquin path in which time is reparametrized. We call this the *natural parametrization* of the jeu de taquin path. To define it, for a given tableau $t \in \mathbb{T}$ let $\mathbf{q}_n(t) = \mathbf{p}_{K(n)}$ where $K(n)$ is the maximal number k such that $t_{\mathbf{p}_k} \leq n$, i.e., the tableau entry in position \mathbf{p}_k is smaller or equal than n . The reparametrized sequence $(\mathbf{q}_n)_{n \geq 1}$ is simply a slowed-down or “lazy” version of the jeu de taquin path: as n increases it either jumps to its right or up if in the growth process (2.1) a box was added in one of those two positions, and stays put at other times.

4.2. Finite version of jeu de taquin. For a finite Young tableau t with $n \geq 1$ boxes, just like for the infinite case considered in Section 1.5, we remove the corner box, we perform the sequence of slidings (which is now a *finite* sequence), and we subtract 1 from every entry of the resulting “tableau”. The resulting Young tableau with $n - 1$ boxes will be denoted by $j(t)$.

4.3. Schützenberger's jeu de taquin. We will use the special name *Schützenberger's jeu de taquin* (which maps the set of *skew tableaux* to the set of tableaux; this map associates to a skew tableau its rectification, see [Ful97, Section 1.2] and [Sag01, Section 3.7]) in order to distinguish it from *jeu de taquin transformation* considered in the current paper (which

is a map J , respectively j , on the set of infinite, respectively finite, Young tableaux). In particular, the finite *jeu de taquin transformation* j can be described equivalently as the composition of (i) removal of the corner box, (ii) Schützenberger’s *jeu de taquin*, (iii) subtracting 1 from each entry.

4.4. Duality between jeu de taquin and one-directional shift. In the setup when the alphabet \mathbb{A} consists only of row letters, this result has been proved by Schützenberger [Sch63]; we will use its generalized version for alphabets consisting of row and column letters.

Lemma 4.1 (Duality between jeu de taquin and one-directional shift). *Let the alphabet $\mathbb{A} = \mathbb{A}_r \sqcup \mathbb{A}_0 \sqcup \mathbb{A}_c$ be given and let $w_1, \dots, w_n \in \mathbb{A}$. We assume that each neutral letter appears at most once in this tuple.*

Then

$$Q(w_2, w_3, \dots, w_n) = j(Q(w_1, w_2, \dots, w_n)),$$

where j is the finite version of the jeu de taquin map.

Proof. In Section 3.7 we defined the standardization $(\pi_1, \dots, \pi_n) = \pi(w_1, \dots, w_n)$. One can easily show that $(\pi_2, \dots, \pi_n) = \pi(w_2, \dots, w_n)$ (more precisely, we can define $\pi(w_2, \dots, w_n) := (\pi_2, \dots, \pi_n)$ and check that it fulfills the requirement (3.2) from the definition; notice that $\pi(w_2, \dots, w_n)$ is defined only up to an order-preserving isomorphism). Lemma 3.3 shows that the corresponding recording tableaux are equal:

$$\begin{aligned} Q(w_1, \dots, w_n) &= Q(\pi_1, \dots, \pi_n), \\ Q(w_2, \dots, w_n) &= Q(\pi_2, \dots, \pi_n). \end{aligned}$$

Thus it is enough to show the lemma for the tuple $(w'_1, \dots, w'_n) := (\pi_1, \dots, \pi_n)$. Since π_1, \dots, π_n are distinct, this is the setup considered by Schützenberger, see [Sag01, Proposition 3.9.3]. \square

5. GROWTH OF RANDOM YOUNG DIAGRAMS

5.1. Lengths of rows and columns of random Young diagrams. The following is the classical result of Vershik and Kerov (which we discussed already in Section 2.2) about the asymptotic growth of a random Young diagram distributed according to some indecomposable central measure.

Fact 5.1 (Vershik and Kerov [VK81, Corollary 5]). *Let (α, β, γ) be an element of Thoma simplex and let $(\Lambda^0 \nearrow \Lambda^1 \nearrow \dots) \in \mathbb{T}$ be a random infinite tableau with the distribution given by Vershik-Kerov measure $\mathfrak{M}_{\alpha, \beta, \gamma}$.*

Then, almost surely, for each $i \in \{1, 2, \dots\}$

$$\lim_{n \rightarrow \infty} \frac{\Lambda_i^n}{n} = \alpha_i,$$

$$\lim_{n \rightarrow \infty} \frac{(\Lambda^n)'_i}{n} = \beta_i,$$

where Λ_i^n (respectively, $(\Lambda^n)'_i$) denotes the number of boxes in i -th row (respectively, i -th column) of Young diagram Λ^n .

We will also need the following more refined information about the growth of the number of rows and the number of columns in the case when some parameters in Thoma simplex are zero.

Lemma 5.2. *We keep notations from Fact 5.1.*

- Assume that $\beta = (0, 0, \dots)$ and $\gamma = 0$. Then for each $\epsilon > 0$ there exists a constant $d > 0$ such that

$$P\left(\frac{(\Lambda^n)'_1}{\sqrt{n}} > \epsilon\right) = O\left(e^{-d\sqrt{n}}\right).$$

- Assume that $\alpha = (0, 0, \dots)$ and $\gamma = 0$. Then for each $\epsilon > 0$ there exists a constant $d > 0$ such that

$$P\left(\frac{\Lambda_1^n}{\sqrt{n}} > \epsilon\right) = O\left(e^{-d\sqrt{n}}\right).$$

Proof. Without loss of generality we may assume that the tableau $(\Lambda^0 \nearrow \Lambda^1 \nearrow \dots) = Q(W_1, W_2, \dots)$ is the recording tableau of an i.i.d. sequence of random letters with a suitable probability distribution, as prescribed by Fact 3.1. Thus the claim is equivalent to Lemma 5.3 below. \square

Lemma 5.3.

- Let $\mathbb{A} = \mathbb{A}_r$ be an alphabet which consist only of row letters, equipped with a probability measure \mathcal{M} which does not have any continuous part. Let (W_1, W_2, \dots) be a sequence of independent, identically distributed elements of \mathbb{A} with distribution \mathcal{M} .

Then, for each $\epsilon > 0$ there exists some $d > 0$ such that

$$P(\text{RSK shape of } (W_1, \dots, W_n) \text{ has at least } \epsilon\sqrt{n} \text{ rows}) = O\left(e^{-d\sqrt{n}}\right).$$

- Let $\mathbb{A} = \mathbb{A}_c$ be an alphabet which consist only of column letters, equipped with a probability measure \mathcal{M} which does not have any continuous part. Let (W_1, W_2, \dots) be a sequence of independent, identically distributed elements of \mathbb{A} with distribution \mathcal{M} .

Then, for each $\epsilon > 0$ there exists some $d > 0$ such that

$$P(\text{RSK shape of } (W_1, \dots, W_n) \text{ has at least } \epsilon\sqrt{n} \text{ columns}) = O\left(e^{-d\sqrt{n}}\right).$$

Proof. We will show the first part of the lemma. Let $\alpha_1 \geq \alpha_2 \geq \dots \geq 0$ be the probabilities of the atoms of the probability measure \mathcal{M} ; clearly

$$\alpha_1 + \alpha_2 + \dots = 1.$$

Let $D > 0$ be a positive constant, we will fix its value at the end of the proof. Let m be big enough so that

$$\alpha_1 + \dots + \alpha_m > 1 - D.$$

Let $x_1, \dots, x_m \in \mathbb{A}$ be the atoms of the measure \mathcal{M} with the biggest weights.

Note that the case when $(\alpha_1, \alpha_2, \dots)$ contains only a finite number of non-zero entries will require later on some special attention; in this case we set m to be the number of such non-zero entries; thus

$$(5.1) \quad \alpha_1 + \dots + \alpha_m = 1.$$

We denote by $\mathbf{W}' = (W'_1, \dots, W'_{\ell(n)})$ the tuple (W_1, \dots, W_n) with all entries which belong to $\{x_1, \dots, x_m\}$ removed. By Greene's theorem (Fact 3.2), the number of rows of the RSK shape of (W_1, \dots, W_n) is equal to the length of the longest $<_c$ -decreasing subsequence of (W_1, \dots, W_n) . In our case, there are no column letters, so such a sequence is strictly $<$ -decreasing, hence its length is bounded from above by

$$m + (\text{length of the longest strictly decreasing subsequence of } \mathbf{W}').$$

Thus it remains to show that (with high probability) the second summand grows sufficiently slowly with n .

In the case (5.1) when $(\alpha_1, \alpha_2, \dots)$ contains only finitely many non-zero entries, the tuple \mathbf{W}' is almost surely empty and the statement of the lemma follows trivially. Thus it remains to show the lemma in the remaining case

$$\alpha_1 + \dots + \alpha_m < 1.$$

We denote by $g > 0$ any constant such that

$$\alpha_1 + \dots + \alpha_m + g < 1.$$

The distribution of the random length $\ell(n)$ of the word \mathbf{W}' is given by a binomial distribution with success probability $p := 1 - (\alpha_1 + \dots + \alpha_m)$ with $g < p < D$. Thus by elementary large deviations theory there exists some constant $h > 0$ such that

$$(5.2) \quad P\left(\frac{\ell(n)}{n} \notin (g, D)\right) = O(e^{-hn}).$$

In the following we condition over $\ell = \ell(n)$ and assume that

$$(5.3) \quad gn < \ell(n) < Dn.$$

We consider the set $Z(\mathbf{W}')$ of all permutations $\pi = (\pi_1, \dots, \pi_\ell)$ with the property that for any $1 \leq i, j \leq \ell$

$$(W'_i \neq W'_j) \implies [(\pi_i < \pi_j) \iff (W'_i < W'_j)],$$

in other words, except for repeating letters, the order of the entries of π should coincide with the order of entries of \mathbf{W}' . Any such a permutation has the property that

$$\begin{aligned} & (\text{length of the longest strictly decreasing subsequence of } \mathbf{W}') \leq \\ & (\text{length of the longest decreasing subsequence of } \pi). \end{aligned}$$

Let π be a random element of the (random) set $Z(\mathbf{W}')$ (we sample with the uniform probability). We claim that π is uniformly distributed on the symmetric group. Indeed, the natural action of the symmetric group \mathfrak{S}_ℓ on the set of words of length ℓ (by permutation of the letters) is such that each $\sigma \in \mathfrak{S}_\ell$ maps the set $Z(\mathbf{W}')$ to the set $Z(\sigma(\mathbf{W}'))$. Since the words \mathbf{W}' and $\sigma(\mathbf{W}')$ have the same probability, it follows that the probability distribution of the random permutation π coincides with the distribution of $\sigma\pi$. This invariance uniquely characterizes the uniform distribution, so the claim that π is uniformly distributed follows immediately. Therefore it remains to find a suitable bound for the length of the longest decreasing subsequence of a random permutation π , distributed uniformly on the symmetric group. This is the classical Ulam-Hammersley problem for which lot of results are available, see [Rom15]. We provide an elementary estimate below.

By Markov's inequality, the probability that π contains a decreasing sequence of length at least $r := \lceil 3\sqrt{\ell} \rceil \leq 3\sqrt{Dn} + 1$ is at most the expected number of such subsequences which is

$$(5.4) \quad \binom{\ell}{r} \frac{1}{r!} < \left(\frac{e^2 \ell}{r^2} \right)^r \leq \left(\frac{e^2}{3^2} \right)^{3\sqrt{\ell}} \leq e^{-d\sqrt{n}},$$

for some constant $d > 0$, where we used Stirling's approximation $r! > r^r e^{-r}$ and the assumption (5.3).

This shows that the unconditional probability of the event

$$\begin{aligned} & \left(\text{the number of rows of the RSK shape of } (W_1, \dots, W_n) \right) \geq \\ & m + 3\sqrt{Dn} + 1 \end{aligned}$$

is bounded from above by the sum of the right-hand sides of (5.2) and (5.4). Thus, by choosing $D > 0$ in such a way that $3\sqrt{D} < \epsilon$ we finish the proof of the first part of the Lemma.

The second part of the Lemma is completely analogous. Alternatively, one can apply the symmetry argument, as follows. We define an alphabet $\mathbb{A}' = \mathbb{A}'_r$ which consists only of row letters, and which, as a set, is equal to \mathbb{A} . The linear order on \mathbb{A}' is defined as the opposite of the linear order on \mathbb{A} . Greene's theorem (Fact 3.2) shows that the number of columns of RSK shape of (W_1, \dots, W_n) , regarded as a word in \mathbb{A} , is equal to the number of rows of the RSK shape of (W_1, \dots, W_n) , this time regarded as a word in \mathbb{A}' . Thus the first part of the Lemma implies immediately the second part. \square

5.2. Plancherel measure and Vershik-Kerov-Logan-Shepp limit shape.

The *Plancherel measure* on the set \mathbb{Y}_n of Young diagrams with n boxes is the probability measure given by

$$P(\lambda) = \frac{(\dim \lambda)^2}{n!},$$

where $\dim \lambda$ is the dimension of the irreducible representation of the symmetric group \mathfrak{S}_n corresponding to λ or, in other words, the number of Young tableaux with shape λ . Equivalently, Plancherel measure is the distribution of RSK shape associated to a random permutation in \mathfrak{S}_n with the uniform distribution.

Asymptotically, the shape of a random Plancherel-distributed Young diagram converges to a well-known limit shape discovered in the celebrated works of Logan-Shepp [LS77] and Vershik-Kerov [VK77, VK85]. Below we present this shape in a parametrization which is not the simplest one, but the most convenient for our purposes. The reason for this choice of parametrization will become obvious in Fact 5.5.

For $-2 \leq u \leq 2$ and $0 \leq w \leq 1$ we define:

$$\begin{aligned} \Omega(u) &= \frac{2}{\pi} \left(u \sin^{-1} \left(\frac{u}{2} \right) + \sqrt{4 - u^2} \right), \\ F(u) &= \frac{1}{2} + \frac{1}{\pi} \left(\frac{u\sqrt{4 - u^2}}{4} + \sin^{-1} \left(\frac{u}{2} \right) \right), \\ U(w) &= F^{-1}(w), \\ V(w) &= \Omega(U(w)), \\ X(w) &= \frac{V(w) + U(w)}{2}, \\ Y(w) &= \frac{V(w) - U(w)}{2}, \end{aligned}$$

where F^{-1} denotes the compositional inverse. See Figure 11 for an illustration.

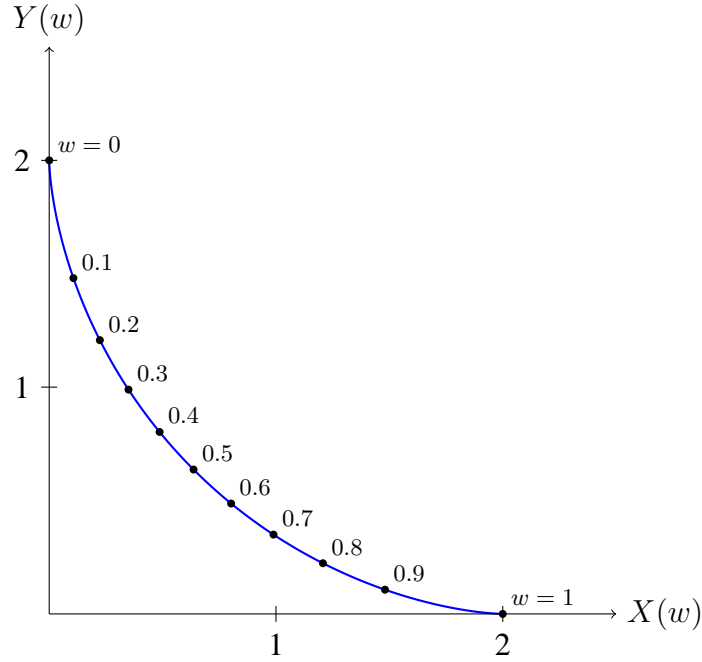


Figure 11. Vershik-Kerov-Logan-Shepp limit shape and its parametrization $(X(w), Y(w))$.

Fact 5.4 (Typical shape of random, Plancherel distributed Young diagrams). *For each $n \geq 1$ let $\Lambda^n = (\Lambda_1^n, \Lambda_2^n, \dots)$ be a random Young diagram with n boxes, distributed according to Plancherel measure. Let (y_n) be a sequence of positive integers with the property that*

$$\mathbf{y} := \lim_{n \rightarrow \infty} \frac{y_n}{\sqrt{n}} > 0.$$

Then the lengths of the rows of these Young diagrams behave asymptotically as follows:

$$\frac{\Lambda_{y_n}^n}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{P} X(Y^{-1}(\mathbf{y})),$$

where Y^{-1} denotes the compositional inverse. Furthermore, the rate of convergence is given as follows: for each $\epsilon > 0$ there exists some $d > 0$ with the property that

$$P \left(\left| \frac{\Lambda_{y_n}^n}{\sqrt{n}} - X(Y^{-1}(\mathbf{y})) \right| > \epsilon \right) = O \left(e^{-d\sqrt{n}} \right).$$

Proof. Essentially, this result is a rather straightforward reformulation of the results of Vershik and Kerov. We provide the details below.

We consider the rotated (so called, Russian) coordinate system

$$u = x - y, \quad v = x + y$$

on the plane. Figure 12 shows how a Young diagram in the Russian coordinate system can be identified with its *profile* which is just a function on the real line \mathbb{R} .

A slight variation of the results of Vershik and Kerov [VK85] (it follows from the numerical estimates in Section 3 of that paper by modifying some parameters in an obvious way; see also [Rom15, Chapter 1]) states that for each $\varepsilon > 0$ there exists some $d = d(\varepsilon) > 0$ with the property that the rescaled (by factor $\frac{1}{\sqrt{n}}$) profile of a Plancherel-random Young diagram with n boxes is (with probability at least $1 - O(e^{-d\sqrt{n}})$) contained in an ε -neighborhood of the graph of the function $v = \Omega(u)$, see Figure 13.

The diagonal solid line on Figure 13 shows the intersection of this neighborhood with the line $y = \frac{y_n}{\sqrt{n}}$; we are interested in the x -coordinates of the points from this intersection since they correspond to (scaled by a factor $\frac{1}{\sqrt{n}}$) possible values of $\Lambda_{y_n}^n$. In the following we will show that as $\varepsilon \rightarrow 0$, the length of this intersection converges to zero uniformly over $\frac{y_n}{\sqrt{n}} > C$ for arbitrary $C > 0$. This would imply that for each $\varepsilon > 0$ it is possible to choose $\varepsilon > 0$ small enough that

$$P \left(\left| \frac{\Lambda_{y_n}^n}{\sqrt{n}} - X \left(Y^{-1} \left(\frac{y_n}{\sqrt{n}} \right) \right) \right| > \varepsilon \right) = O \left(e^{-d(\varepsilon)\sqrt{n}} \right)$$

as the common point of the curve Ω and the line $y = \frac{y_n}{\sqrt{n}}$ belongs to the above intersection as well. The continuity of the function $X(Y^{-1}(\cdot))$ would finish the proof.

It remains to show that as $\varepsilon \rightarrow 0$, the length of the intersection converges to zero uniformly over $\frac{y_n}{\sqrt{n}} > C$ for arbitrary $C > 0$. Let $(u_1, v_1), (u_2, v_2)$ be the coordinates (in the Russian coordinate system) of some points on this intersection. This implies that their y -coordinates are equal:

$$2y = v_1 - u_1 = v_2 - u_2.$$

On the other hand,

$$|\Omega(u_i) - v_i| < \varepsilon$$

for each $i \in \{1, 2\}$. Thus

$$(5.5) \quad \Omega(u_2) - \Omega(u_1) > u_2 - u_1 - 2\varepsilon.$$

Suppose that $u_i > 2 - \delta$ for some $\delta > 0$. Note that y -coordinate of (u_i, v_i) fulfills

$$C < y \leq \frac{\Omega(u_i) - u_i + \varepsilon}{2};$$

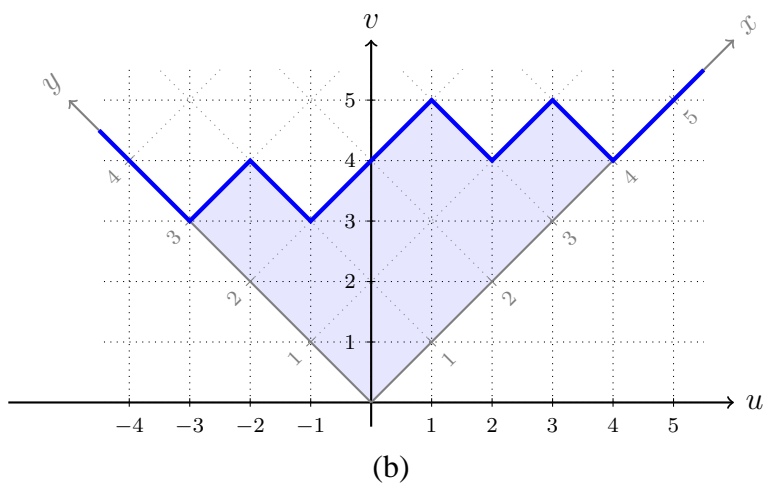
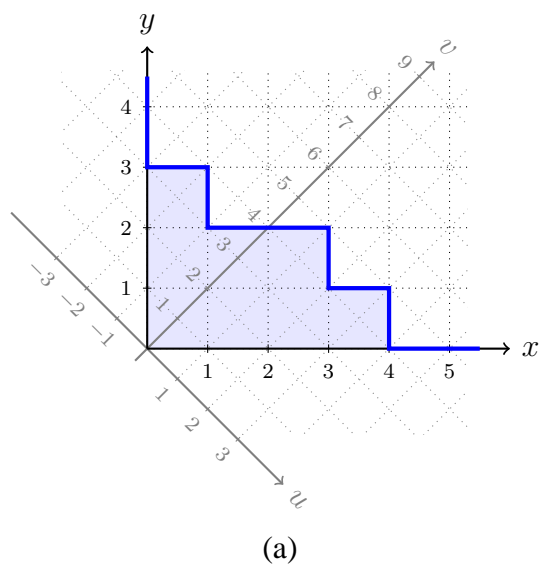


Figure 12. A Young diagram $\lambda = (4, 3, 1)$ shown in (a) the French and (b) the Russian convention. The solid line represents the *profile* of the Young diagram. The coordinate system (u, v) corresponding to the Russian convention and the coordinate system (x, y) corresponding to the French convention are shown.

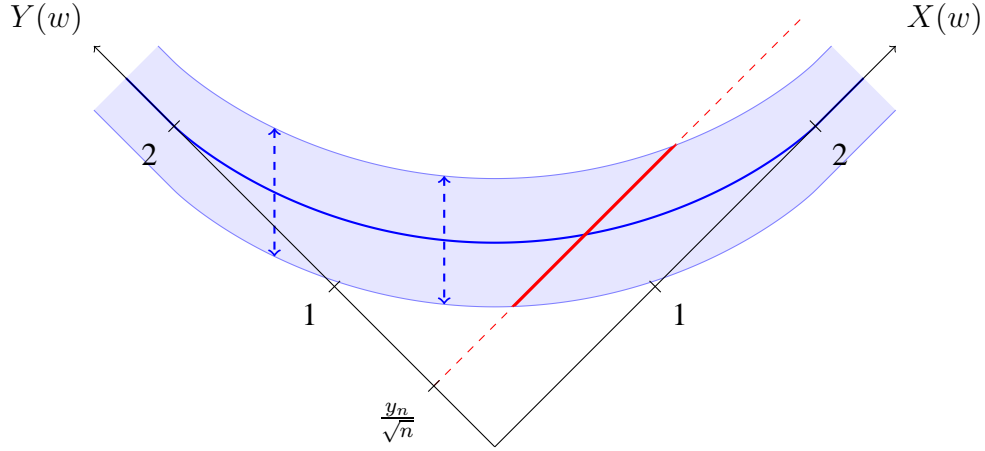


Figure 13. Logan-Shepp-Vershik-Kerov curve and its ε -neighborhood.

as $\varepsilon \rightarrow 0$ and $\delta \rightarrow 0$, the right-hand side converges to zero, which leads to a contradiction. This shows that there exists $\delta > 0$ such that $u_i < 2 - \delta$ for all $\varepsilon > 0$ which are sufficiently small.

A direct calculation of the derivative shows that there exists $c > 0$ such that $\Omega'(u) < 1 - c$ for any $u \in (-\infty, 2 - \delta)$. Thus, for any $u_1 \leq u_2$ such that $u_1, u_2 \in (-\infty, 2 - \delta)$

$$(5.6) \quad \Omega(u_2) - \Omega(u_1) \leq (u_2 - u_1)(1 - c).$$

Equations (5.5) and (5.6) show that if $\varepsilon \rightarrow 0$ then $u_2 - u_1 \rightarrow 0$ as well. This implies that the difference of the x -coordinates $\frac{\Omega(u_i) + u_i}{2}$ converges to zero as well. This concludes the proof that the length of the intersection converges to zero. \square

5.3. Asymptotic determinism of Schensted insertion for Plancherel measure. In order to show Theorem 6.4 we will need the following special case of it for $\alpha = \beta = (0, 0, \dots)$ and $\gamma = 1$ which has been proved in our previous work. It explains our parametrization of Vershik-Kerov-Logan-Shepp curve: $(X(w), Y(w))$ is just the (rescaled) typical position of the newly created box by Schensted insertion, when $w \in (0, 1)$ is inserted.

Fact 5.5 ([RS15, Theorem 5.1]). *Let W_1, W_2, \dots be the sequence of random, i.i.d. letters from the interval $(0, 1)$, taken with the uniform distribution. Let $w \in (0, 1)$ be deterministic. Let \square_n denote the location of the last box added to the recording tableau by RSK algorithm applied to the sequence*

$$(W_1, \dots, W_{n-1}, w).$$

Then

$$\frac{\square_n}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{P} (X(w), Y(w)).$$

The rate of convergence is given, for each $\epsilon > 0$, by

$$P \left\{ \left\| \frac{\square_n}{\sqrt{n}} - (X(w), Y(w)) \right\| > \epsilon \right\} = O\left(n^{-\frac{1}{4}}\right).$$

6. ASYMPTOTIC DETERMINISM OF SCHENSTED INSERTION

6.1. The insertion alphabet. The second most important example of an alphabet is $\mathbb{I} = \mathbb{I}_r \sqcup \mathbb{I}_0 \sqcup \mathbb{I}_c$ with $\mathbb{I}_r = \{\dots, -3, -2, -1\}$, $\mathbb{I}_0 = (0, 1) \subset \mathbb{R}$ and $\mathbb{I}_c = \{1, 2, 3, \dots\}$ with the linear order defined as follows: on each of the sets \mathbb{I}_r , \mathbb{I}_0 , \mathbb{I}_c we consider the natural order; we declare any element of \mathbb{I}_c smaller than any element of \mathbb{I}_0 , which is smaller than any element of \mathbb{I}_r . This linear order can be visualized as follows:

$$\underbrace{1 < 2 < 3 < \dots}_{\text{column letters}} < \underbrace{\dots < 0.1 < \dots < 0.9 < \dots}_{\text{row letters}} < \underbrace{\dots < -3 < -2 < -1}_{\text{row letters}},$$

compare with (1.1). This alphabet will be called *the insertion alphabet*.

If (α, β, γ) belongs to Thoma simplex, we define the following probability measure $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{I}}$ on \mathbb{I} :

- for $-i \in \{-1, -2, -3, \dots\} = \mathbb{I}_r$ we set $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{I}}(-i) = \alpha_i$;
- for $i \in \{1, 2, 3, \dots\} = \mathbb{I}_c$ we set $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{I}}(i) = \beta_i$;
- on $\mathbb{I}_0 = (0, 1)$ we take as $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{I}}$ the absolutely continuous measure on the unit interval $(0, 1)$ with constant density γ .

This alphabet and the measure are the ones used in Theorem 6.4.

6.2. The opposite alphabets. The alphabets \mathbb{J} and \mathbb{I} , regarded as ordered sets, are equal. However, since their decompositions $\mathbb{A} = \mathbb{A}_r \sqcup \mathbb{A}_0 \sqcup \mathbb{A}_c$ into row letters and column letters are different, this equality turns out to be not very important.

It is much more convenient to consider the bijection $\iota : \mathbb{J} \rightarrow \mathbb{I}$ defined by

$$\begin{aligned} \mathbb{J}_r &= \{1, 2, \dots\} \ni k \xrightarrow{\iota} -k \in \{-1, -2, -3, \dots\} = \mathbb{I}_r, \\ \mathbb{J}_c &= \{-1, -2, \dots\} \ni -k \xrightarrow{\iota} k \in \{1, 2, 3, \dots\} = \mathbb{I}_c, \\ \mathbb{J}_0 &= (0, 1) \ni x \xrightarrow{\iota} 1 - x \in (0, 1) = \mathbb{I}_0. \end{aligned}$$

The map ι is an anti-isomorphism of ordered sets which preserves the decomposition of the alphabets $\mathbb{A} = \mathbb{A}_r \sqcup \mathbb{A}_0 \sqcup \mathbb{A}_c$ into row letters and column letters. Furthermore, the pushforward of $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{J}}$ is equal to $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{I}}$.

6.3. Duality between jeu de taquin and Schensted insertion. The following lemma shows that Schensted insertion and jeu de taquin are closely related to each other.

Lemma 6.1. *Let $w_1, \dots, w_n \in \mathbb{J}$. We assume that each neutral letter appears at most once in this tuple. Let $Q := Q(w_1, \dots, w_n)$ be the corresponding recording tableau and let \mathbf{q}_n be the box where the finite version of jeu de taquin leaves tableau Q .*

We consider the tuple $\iota(w_n), \dots, \iota(w_1) \in \mathbb{I}$ and the corresponding recording tableau $Q' := Q(\iota(w_n), \dots, \iota(w_1))$. Let \square_n be the box with the label n in Q' (i.e., it is the box added in the last Schensted insertion step).

Then $\mathbf{q}_n = \square_n$.

Proof. Let λ_n be the RSK shape associated to (w_1, w_2, \dots, w_n) . By Greene's theorem (Fact 3.2) it follows that it is also the RSK shape associated to $(\iota(w_n), \dots, \iota(w_2), \iota(w_1))$.

Let λ_{n-1} be the RSK shape associated to the postfix (w_2, \dots, w_n) . By the same argument it follows that it is also the RSK shape associated to $(\iota(w_n), \dots, \iota(w_2))$.

By definition, λ_n is the shape of Q ; Lemma 4.1 shows that λ_{n-1} is the shape of $j(Q)$ thus

$$\{\mathbf{q}_n\} = \lambda_n \setminus \lambda_{n-1}.$$

On the other hand,

$$\{\square_n\} = \lambda_n \setminus \lambda_{n-1}$$

which finishes the proof. \square

Remark 6.2. Lemma 6.1 holds true in bigger generality with the alphabets \mathbb{J} and \mathbb{I} replaced by arbitrary alphabets \mathbb{A} , \mathbb{B} with the property that there exists anti-isomorphism of ordered sets $\iota : \mathbb{A} \rightarrow \mathbb{B}$ which preserves the decompositions $\mathbb{A} = \mathbb{A}_r \sqcup \mathbb{A}_0 \sqcup \mathbb{A}_c$.

Lemma 6.3. *Let (α, β, γ) be an element of Thoma simplex. Let W_1, W_2, \dots be the sequence of random, i.i.d. letters from the insertion alphabet \mathbb{I} , with probability distribution $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{I}}$. Let $w \in \mathbb{I}$ be a deterministic letter. Let \square_n denote the location of the last box added to the recording tableau by RSK algorithm applied to the sequence*

$$(W_1, \dots, W_{n-1}, w).$$

Then for arbitrary $k \in \{1, 2, \dots\}$

$$(6.1) \quad \left(P(\square_n \text{ is in one of the first } k \text{ rows}) \right)_n$$

is a weakly decreasing sequence.

Proof. We apply Lemma 6.1; it implies that the sequence (6.1) coincides with

$$(6.2) \quad \left(P(\mathbf{q}_n(Q_n) \text{ is in one of the first } k \text{ rows}) \right)_n,$$

where Q_n is defined as the recording tableau associated to the sequence $(\iota(w), \iota(W_{n-1}), \dots, \iota(W_2), \iota(W_1))$. It does not change the sequence (6.2) if we change the definition of Q_n to be the recording tableau associated to the sequence $(\iota(w), \iota(W_1), \iota(W_2), \dots, \iota(W_{n-1}))$. In particular, the sequence (6.2) coincides with the sequence

$$(6.3) \quad \left(P(\mathbf{q}_n(Q) \text{ is in one of the first } k \text{ rows}) \right)_n,$$

where $Q := \text{RSK}(\iota(w), \iota(W_1), \iota(W_2), \dots) \in \mathbb{T}$. Clearly, for any tableau Q , the sequence $y(\mathbf{q}_n(Q))$ of y -coordinates is weakly increasing which immediately implies that (6.3) and thus (6.1) are weakly decreasing. \square

6.4. Asymptotic determinism of Schensted insertion. The proofs of our results will be based on the following technical result, which might be interesting on its own.

Theorem 6.4 (Asymptotic determinism of Schensted insertion). *Let (α, β, γ) be an element of Thoma simplex. Let W_1, W_2, \dots be the sequence of random, i.i.d. letters from the insertion alphabet \mathbb{I} , with probability distribution $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{I}}$. Let $w \in \mathbb{I}$ be a deterministic letter. Let \square_n denote the location of the last box added to the recording tableau by RSK algorithm applied to the sequence*

$$(W_1, \dots, W_{n-1}, w).$$

(A) *In the case when $w = -k \in \{-1, -2, -3, \dots\} = \mathbb{I}_r$ we assume that $\alpha_k > 0$. Then*

$$\lim_{n \rightarrow \infty} P(\square_n \text{ is in row } k) = 1.$$

(B) *In the case when $w = k \in \{1, 2, 3, \dots\} = \mathbb{I}_c$ we assume that $\beta_k > 0$. Then*

$$\lim_{n \rightarrow \infty} P(\square_n \text{ is in column } k) = 1.$$

(C) *In the case when $w \in (0, 1) = \mathbb{I}_0$ we assume that $\gamma > 0$. Then*

$$\frac{\square_n}{\sqrt{\gamma n}} \xrightarrow[n \rightarrow \infty]{P} (X(w), Y(w)),$$

where $(X(w), Y(w))$ is the parametrization of Vershik-Kerov-Logan-Shepp curve considered in Figure 11. The rate of convergence is

given, for any $\epsilon > 0$, by

$$(6.4) \quad P \left\{ \left\| \frac{\square_n}{\sqrt{\gamma n}} - (X(w), Y(w)) \right\| > \epsilon \right\} = O \left(n^{-\frac{1}{4}} \right).$$

In each of the above three cases,

$$(6.5) \quad \|\square_n\| \xrightarrow[n \rightarrow \infty]{P} \infty.$$

Informally speaking, \square_n converges in probability (as $n \rightarrow \infty$) to the appropriate asymptote depicted in Figure 8.

Proof. We will consider each of the three cases separately.

The case (A).

The elements which are bumped from consecutive rows in a given Schensted insertion step form an $<_c$ -increasing sequence. The insertion alphabet \mathbb{I} has the property that any $<_c$ -increasing sequence of its elements which starts with $-k$ is of length (at most) k . This shows that \square_n belongs to one of the first k rows. Thus it remains to show that

$$\lim_{n \rightarrow \infty} P(\square_n \text{ is in one of the first } k-1 \text{ rows}) = 0.$$

Let $\Lambda^n = (\Lambda_1^n, \Lambda_2^n, \dots)$ be the RSK shape associated to the sequence (W_1, \dots, W_n) of i.i.d. random letters from \mathbb{I} distributed according to the probability measure $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{I}}$. We use the notational shorthand

$$[\text{condition}] = \begin{cases} 1 & \text{if condition is true,} \\ 0 & \text{otherwise.} \end{cases}$$

We clearly have

$$\begin{aligned} \Lambda_1^n + \dots + \Lambda_{k-1}^n = \\ \sum_{1 \leq m \leq n} \left[\begin{array}{l} \text{the box created in the insertion } P(W_1, \dots, W_{m-1}) \leftarrow W_m \\ \text{is in one of the first } k-1 \text{ rows} \end{array} \right] \end{aligned}$$

thus, by considering the events $W_m \in \{-1, -2, \dots, -(k-1)\}$ and $W_m = -k$, we obtain

$$\begin{aligned} \mathbb{E}(\Lambda_1^n + \dots + \Lambda_{k-1}^n) \geq \\ \sum_{1 \leq m \leq n} \left[\alpha_1 + \dots + \alpha_{k-1} + \alpha_k P(\square_m \text{ is in one of the first } k-1 \text{ rows}) \right]. \end{aligned}$$

This, together with Lemma 6.3 implies that

$$(6.6) \quad \liminf_{n \rightarrow \infty} \mathbb{E} \frac{\Lambda_1^n + \cdots + \Lambda_{k-1}^n}{n} \geq \alpha_1 + \cdots + \alpha_{k-1} + \alpha_k \lim_{n \rightarrow \infty} P(\square_n \text{ is in one of the first } k-1 \text{ rows}).$$

On the other hand, Fact 3.1 implies that $(\Lambda^0 \nearrow \Lambda^1 \nearrow \cdots)$ is a random infinite Young tableau with the distribution given by Vershik-Kerov measure $\mathfrak{M}_{\alpha, \beta, \gamma}$ thus Fact 5.1 can be applied. By Lebesgue's dominated convergence theorem

$$(6.7) \quad \lim_{n \rightarrow \infty} \mathbb{E} \frac{\Lambda_1^n + \cdots + \Lambda_{k-1}^n}{n} = \alpha_1 + \cdots + \alpha_{k-1}.$$

Eqs. (6.6) and (6.7) finish the proof.

In order to show (6.5) in this case it enough to use that $\Lambda_1^n, \dots, \Lambda_k^n$ all tend almost surely to infinity.

The case (B).

The insertion alphabet \mathbb{I} has the property that any $<_r$ -increasing sequence of its elements which ends with k is of length (at most) k . This implies that when k is inserted by Schensted insertion to an arbitrary tableau, it is inserted into one of the first k columns, thus \square_n belongs to one of the first k columns as well. Thus it remains to show that

$$\lim_{n \rightarrow \infty} P(\square_n \text{ is in one of the first } k-1 \text{ columns}) = 0.$$

The remaining part of the proof is completely analogous to the case (A) considered above; one should simply replace the notion of *rows* by *columns*, the lengths of rows $\Lambda_1^n, \Lambda_2^n, \dots$ should be replaced by the lengths of columns $(\Lambda^n)'_1, (\Lambda^n)'_2, \dots$, and one should consider the events $W_m \in \{1, 2, \dots, k-1\}$ and $W_m = k$.

The case (C).

Our goal is to find the location (x_1, y_1) of the box containing n in the recording tableau corresponding to $\mathbf{W} := (W_1, \dots, W_{n-1}, w)$. Let $\pi = (\pi_1, \dots, \pi_n)$ be the permutation given by standardization (see Section 3.7) of the sequence \mathbf{W} . By Lemma 3.3, the recording tableaux corresponding to \mathbf{W} and π are equal. The latter recording tableau is equal to the insertion tableau $P(\pi^{-1})$. In the remaining part of the proof we will be studying this insertion tableau. We use the shorthand notation $(\pi_1^{-1}, \dots, \pi_n^{-1}) := \pi^{-1}$.

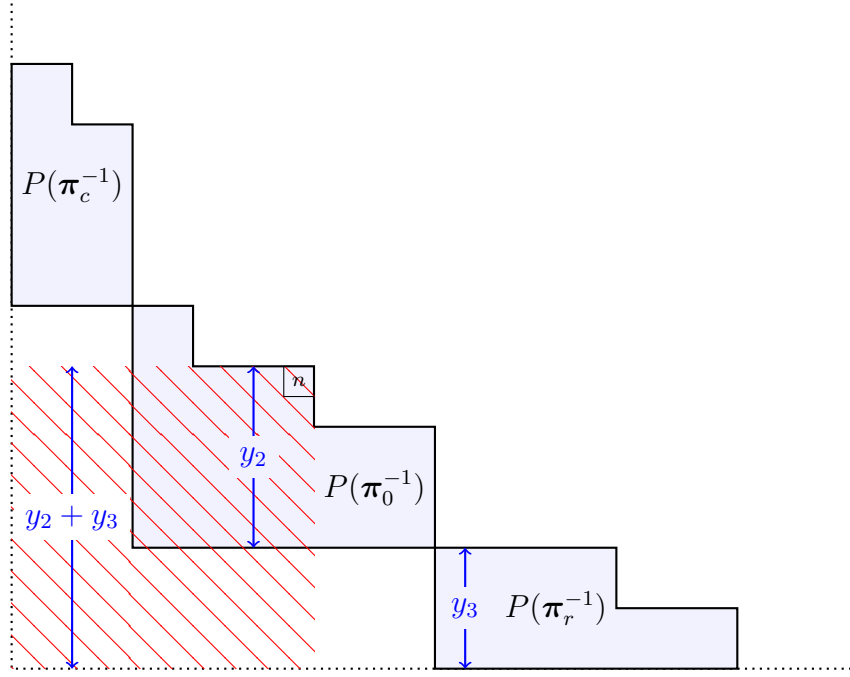


Figure 14. The way of stacking the insertion tableaux corresponding to π_c^{-1} , π_0^{-1} , π_r^{-1} . As Schützenberger’s jeu de taquin involves only slides of the boxes down and to the left, the hatched area shows possible locations of the box n in $P(\pi^{-1})$.

Let r (respectively, c) denote the number of row letters (respectively, column letters) in \mathbf{W} . We define

$$\begin{aligned}\pi_c^{-1} &:= (\pi_1^{-1}, \dots, \pi_c^{-1}), \\ \pi_0^{-1} &:= (\pi_{c+1}^{-1}, \dots, \pi_{n-r}^{-1}), \\ \pi_r^{-1} &:= (\pi_{n+1-r}^{-1}, \dots, \pi_n^{-1})\end{aligned}$$

so that π^{-1} is a concatenation of the words π_c^{-1} , π_0^{-1} , π_r^{-1} . In this way the insertion tableau $P(\pi^{-1})$ can be obtained by stacking the insertion tableaux $P(\pi_c^{-1})$, $P(\pi_0^{-1})$, $P(\pi_r^{-1})$ as shown in Figure 14 and by performing Schützenberger’s jeu de taquin.

The entries of π_c^{-1} (respectively, π_r^{-1}) are the locations in the word \mathbf{W} of the column letters (respectively, row letters); in particular n is one of the entries of π_0^{-1} .

Let (x_2, y_2) be the location of the box containing n in the insertion tableau $P(\pi_0^{-1})$; let y_3 be the number of rows of the insertion tableau $P(\pi_r^{-1})$ and

let x_4 be the number of columns of the insertion tableau $P(\pi_c^{-1})$. Thus

$$\begin{aligned} x_1 &\leq x_2 + x_4, \\ y_1 &\leq y_2 + y_3 \end{aligned}$$

(the proof of the second inequality is illustrated in Figure 14; the proof of the first inequality is analogous).

Word π^{-1} is created from the word π_0^{-1} by (i) adding a postfix π_r^{-1} and then (ii) adding a prefix π_c^{-1} ; it follows that the insertion tableau $P(\pi^{-1})$ can be created from $P(\pi_0^{-1})$ by (i) a sequence of *row insertions* of the letters forming π_r^{-1} (this part of the claim follows from the definition (3.1) of the insertion tableau), followed by (ii) a sequence of *column insertions* of the letters (in the reverse order) forming π_c^{-1} (for this part of the claim and for the definition of the column insertion see [Ful97, Section A.2]). It follows that the RSK shape corresponding to π^{-1} contains the RSK shape corresponding to π_0^{-1} .

Let $(\lambda_1, \lambda_2, \dots)$ denote the RSK shape corresponding to π_0^{-1} . As (x_1, y_1) is one of the inner corners of $P(\pi^{-1})$,

$$\begin{aligned} x_1 &\geq \lambda_{y_1} \geq \lambda_{y_2+y_3}, \\ y_1 &\geq \lambda'_{x_1} \geq \lambda'_{x_2+x_4}. \end{aligned}$$

For a moment let us condition over the value of c . The RSK shape corresponding to π_c^{-1} depends only on the relative order of its entries $(\pi_1^{-1}, \dots, \pi_c^{-1})$ which are the positions of the column letters in the tuple \mathbf{W} . This order would not change if we remove from \mathbf{W} all letters which are not column letters. It follows that the number of columns of RSK shape corresponding to π_c^{-1} has the same distribution as the number of columns of RSK shape corresponding to a sequence of length c of i.i.d. letters from \mathbb{I}_c such that the probability of the letter k is equal to $\frac{\beta_k}{\beta_1+\beta_2+\dots}$. The number of columns can only increase if we increase the length of the sequence to n ; thus, by Lemma 5.3, we have unconditional convergence

$$(6.8) \quad \frac{x_4}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{P} 0.$$

An analogous reasoning shows that

$$(6.9) \quad \frac{y_3}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{P} 0.$$

We denote by $n' := n - r - c$ the length of the sequence π_0^{-1} , which is the number of the elements of the tuple (W_1, \dots, W_{n-1}, w) which belong to \mathbb{I}_0 . By the law of large numbers,

$$(6.10) \quad \frac{n'}{n} \xrightarrow[n \rightarrow \infty]{P} \gamma,$$

in particular

$$n' \xrightarrow[n \rightarrow \infty]{P} \infty.$$

Thus by Fact 5.5

$$(6.11) \quad \frac{x_2}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{P} \sqrt{\gamma} X(w),$$

$$(6.12) \quad \frac{y_2}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{P} \sqrt{\gamma} Y(w).$$

Thus we have shown that

$$\frac{\lambda_{y_2+y_3}}{\sqrt{n}} \leq \frac{x_1}{\sqrt{n}} \leq \frac{x_2 + x_4}{\sqrt{n}};$$

the right-hand side converges in probability to $\sqrt{\gamma} X(w)$; by Fact 5.4 the left-hand side also converges in probability to the same limit. Thus we have shown that

$$\frac{x_1}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{P} \sqrt{\gamma} X(w),$$

as required. Proof of the other limit

$$\frac{y_1}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{P} \sqrt{\gamma} Y(w)$$

follows in an analogous way.

In order to show (6.4) it is enough to revisit the above proof and check the rates of convergence in Eqs. (6.8)–(6.12).

As $X(w), Y(w) > 0$ for $w \in (0, 1)$, Equation (6.5) follows immediately. \square

7. PROOFS OF THE MAIN RESULTS

7.1. Asymptotic determinism of jeu de taquin. The following result is the final tool necessary in order to show the main results of the paper.

Theorem 7.1 (Asymptotic determinism of jeu de taquin). *Let (α, β, γ) be an element of Thoma's simplex. Let W_1, W_2, \dots be a sequence of i.i.d. random letters in \mathbb{J} with the distribution $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{J}}$. Let $w \in \mathbb{J}$ be fixed. Let \mathbf{q}_n be the natural parametrization of the jeu de taquin path associated with the random infinite Young tableau*

$$\text{RSK}(w, W_1, W_2, \dots).$$

(A) *In the case when $w = k \in \{1, 2, 3, \dots\} = \mathbb{J}_r$ we assume that $\alpha_k > 0$.*

Then, almost surely, jeu de taquin trajectory stabilizes in k -th row:

$$\lim_{n \rightarrow \infty} y(\mathbf{q}_n) = k.$$

(B) In the case when $w = -k \in \{-1, -2, -3, \dots\} = \mathbb{J}_c$ we assume that $\beta_k > 0$.

Then, almost surely, jeu de taquin trajectory stabilizes in k -th column:

$$\lim_{n \rightarrow \infty} x(\mathbf{q}_n) = k.$$

(C) In the case when $w \in (0, 1) = \mathbb{J}_0$ we assume that $\gamma > 0$.

Then, almost surely,

$$(7.1) \quad \lim_{n \rightarrow \infty} \frac{\mathbf{q}_n}{\sqrt{n}} = \sqrt{\gamma} \cdot (X(1-w), Y(1-w)).$$

In all three above cases,

$$\lim_{n \rightarrow \infty} \|\mathbf{q}_n\| = \infty$$

holds almost surely.

Note that while in Theorem 6.4 the convergence holds only in the sense of convergence in probability, in the above theorem the convergence is in the almost sure sense.

Proof. The proof which we provide below is analogous to the proof of [RŚ15, Theorem 5.2].

Let \square_n be the box with the label n in

$$\text{RSK}(\iota(W_{n-1}), \iota(W_{n-2}), \dots, \iota(W_1), \iota(w)).$$

By Lemma 6.1,

$$\mathbf{q}_n = \square_n.$$

Theorem 6.4 can be applied in order to study the asymptotic behavior of the right-hand side; we will discuss the three cases separately.

The case (A). Theorem 6.4 shows that $y(\mathbf{q}_n)$ converges to k in probability. Since $y(\mathbf{q}_n)$ is a weakly increasing sequence, the limit $\lim_{n \rightarrow \infty} y(\mathbf{q}_n) \in \{1, 2, \dots, \infty\}$ exists almost surely; this implies that $\lim_{n \rightarrow \infty} y(\mathbf{q}_n) = k$ holds almost surely, as required.

The case (B). This case is analogous to the case (A) considered above.

The case (C). Setting $n_m = m^8$, from Theorem 6.4 and Borel-Cantelli lemma we show an almost sure convergence

$$\lim_{m \rightarrow \infty} \frac{\mathbf{q}_{n_m}}{\sqrt{n_m}} = \sqrt{\gamma} \cdot (X(1-w), Y(1-w))$$

along the subsequence $n = n_m$. Finally, note that $n_{m+1}/n_m \rightarrow 1$ as $m \rightarrow \infty$. It is easy to see that this, together with the fact that the path $(\mathbf{q}_n)_n$ advances monotonically in both the x and y directions, guarantees (deterministically) that convergence along the subsequence implies convergence for the entire sequence.

In all three above cases, the sequence $\|\mathbf{q}_n\|$ is weakly increasing and Theorem 6.4 guarantees that $\|\mathbf{q}_n\| \xrightarrow[n \rightarrow \infty]{P} \infty$; this implies that $\lim_{n \rightarrow \infty} \|\mathbf{q}_n\| = \infty$ holds almost surely. \square

7.2. Proof of Theorem 1.4.

Proof of Theorem 1.4. Again, without loss of generality, we can take

$$T := \text{RSK}(W_1, W_2, \dots),$$

where W_1, W_2, \dots is a sequence of i.i.d. random letters from \mathbb{J} with the probability distribution $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{J}}$. We apply Theorem 7.1; the asymptotic behavior of jeu de taquin path depends only on the value of W_1 , the first letter. Note that jeu de taquin path is parametrized in a different way in Theorem 1.4 and in Theorem 7.1; this difference, however, creates no difficulties. \square

7.3. Probability distribution of jeu de taquin asymptotic angles.

Proposition 7.2. *We keep notations from Theorem 1.4. If $\gamma > 0$, the distribution of the asymptotic angle Θ (conditioned under event that the case (C) holds true) is an absolutely continuous random variable on $(0, \pi/2)$ whose distribution has the following explicit description:*

$$(7.2) \quad \Theta \stackrel{\mathcal{D}}{=} \Pi(W),$$

where W is a random variable distributed according to the semicircle distribution \mathcal{L}_{SC} on $[-2, 2]$, i.e., having density given by

$$(7.3) \quad \mathcal{L}_{\text{SC}}(dw) = \frac{1}{2\pi} \sqrt{4 - w^2} dw \quad (|w| \leq 2),$$

and $\Pi(\cdot)$ is the function

$$\Pi(w) = \frac{\pi}{4} - \cot^{-1} \left[\frac{2}{\pi} \left(\sin^{-1} \left(\frac{w}{2} \right) + \frac{\sqrt{4 - w^2}}{w} \right) \right] \quad (-2 \leq w \leq 2).$$

Proof. Since Theorem 1.4 depends only on the distribution of the random infinite tableau T , without loss of generality we can assume, by Fact 3.1, that $T = \text{RSK}(W_1, W_2, \dots)$, where $W_1, W_2, \dots \in \mathbb{J}$ is a sequence of i.i.d. random letters with the distribution $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{J}}$.

By Theorem 7.1 it follows that (as long as $\gamma > 0$) the conditional distribution of Θ does not depend on the element (α, β, γ) of Thoma's simplex. Again, the difference of parametrizations of jeu de taquin paths creates no difficulties. In particular, this conditional distribution coincides with the unconditional distribution of Θ in the case $\alpha = \beta = (0, 0, \dots)$, $\gamma = 1$ which corresponds to Plancherel measure. The result in this special case has been proved in our previous paper [RS15, Theorem 1.1]. \square

7.4. Proof of Theorems 1.2, 1.3 and 1.5.

Proof of Theorems 1.2, 1.3 and 1.5. Theorems 1.2, 1.3 and 1.5 contain several claims:

- RSK is a homomorphism of probability spaces (this is a part of Theorem 1.3).

This has been shown by Kerov and Vershik, see Fact 3.1.

- RSK is a factor map of dynamical systems, i.e.

$$(7.4) \quad J \circ \text{RSK} = \text{RSK} \circ S$$

(this is a part of Theorem 1.3).

This follows from Lemma 4.1; for details see [RS15, Section 2.4].

- Jeu de taquin transformation J is measure-preserving (this is a part of Theorem 1.2).

We need to show that if T is a random infinite Young tableau with the distribution $\mathfrak{M}_{\alpha, \beta, \gamma}$ then $J(T)$ has the same distribution. Without loss of generality we may assume that

$$T = Q(W_1, W_2, \dots)$$

is as prescribed by Fact 3.1. Equation (7.4) implies

$$J(T) = Q(W_2, W_3, \dots).$$

Another invocation of Fact 3.1 shows that the distribution of $J(T)$ is as required.

- Map RSK^{-1} defined by (1.2) is well defined almost everywhere (this is a part of Theorem 1.5).

This follows from the facts that J is measure-preserving and Ψ is well-defined almost everywhere.

- $\text{RSK}^{-1} \circ \text{RSK} = \text{Id}$ almost everywhere, where RSK^{-1} is defined by (1.2) (this is a part of Theorem 1.5).

Let $\mathbf{W} = (W_1, W_2, \dots) \in \mathbb{J}^{\mathbb{N}}$ be an i.i.d. sequence of letters with distribution $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{J}}$ and let $(U_1, U_2, \dots) = \text{RSK}^{-1} \circ \text{RSK}(\mathbf{W})$. For any $k \geq 1$

$$(7.5) \quad U_k = \Psi \left[J^{k-1} \left(\text{RSK}(W_1, W_2, \dots) \right) \right] = \Psi \left[\text{RSK}(W_k, W_{k+1}, \dots) \right],$$

where the last equality follows from (7.4). We apply Theorem 7.1 in order to show that $U_k = W_k$ almost surely; in the case when $W_k \in \mathbb{J}_r$ or $W_k \in \mathbb{J}_c$ this is straightforward, below we present a more detailed analysis of the case when $W_k \in (0, 1) = \mathbb{J}_0$.

Concerning the right-hand side of (7.5), the value of Θ (and thus the value of Ψ as well) corresponding to (7.1) depends only on W_k and not on the element (α, β, γ) of Thoma simplex, as long as $\gamma > 0$; in particular we can take $\alpha = \beta = (0, 0, \dots)$, $\gamma = 1$ which corresponds to Plancherel measure. The result in this case has been proved in our previous work [RS15, Eq. (47)].

- $\text{RSK} \circ \text{RSK}^{-1} = \text{Id}$ almost everywhere, where RSK^{-1} is defined by (1.2) (this is a part of Theorem 1.5).

Let T be a random infinite Young tableau with the distribution $\mathfrak{M}_{\alpha, \beta, \gamma}$. Without loss of generality we may assume that $T = \text{RSK}(W_1, W_2, \dots)$, where W_1, W_2, \dots is an i.i.d. sequence of random letters with the distribution $\mathcal{M}_{\alpha, \beta, \gamma}^{\mathbb{J}}$ (Fact 3.1). Then

$$\begin{aligned} \text{RSK} \circ \text{RSK}^{-1}(T) &= \text{RSK} \circ \underbrace{\text{RSK}^{-1} \circ \text{RSK}}_{=\text{Id, as shown above}}(W_1, W_2, \dots) = \\ & \text{RSK}(W_1, W_2, \dots) = T \end{aligned}$$

holds true almost surely, as required.

- *Jeu de taquin transformation J is ergodic (this is a part of Theorem 1.2).*

By Theorem 1.3, J is isomorphic to a Bernoulli shift S which is clearly ergodic, see [Sil08].

□

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APPENDIX A. THE ‘‘COUNTEREXAMPLE’’ OF FULMAN

The work [KV86] of Kerov and Vershik has been criticized by Fulman [Ful02]. Since the current paper heavily uses the results of Kerov and Vershik, we feel obliged to respond to this criticism.

Fulman writes (all quotations are from [Ful02, p. 186–187]):

The paper [KV86] states a version of Theorem 12 in which there is also a parameter γ (their Proposition 3), but it is incorrect for $\gamma \neq 0$ as the following counterexample shows. Setting all parameters other than α and $\gamma = 1 - \alpha$ equal to 0, it follows from the definitions that the extended Schur function \tilde{s}_2 is equal to $\frac{\alpha^2+1}{2}$.

Indeed, this is the correct value of the extended Schur function.

But if Proposition 3 of [KV86] were correct, it would also equal $\alpha^2 + (1 - \alpha)\alpha = \alpha$ since the two words giving a Young tableau with 1 row of length 2 are 11 and 01.

With our notations, [KV86, Proposition 3] states that the extended Schur function \tilde{s}_2 is equal to the probability that a random filling

u	v
-----	-----

of a Young diagram (2) with letters $u, v \in \mathbb{A}$ gives a (semistandard) tableau. The probability distribution of the letters is assumed to have a unique atom (of weight α) on some row letter $L \in \mathbb{A}_r$. There are the following disjoint possibilities:

- (a) $u = v = L$;
- (b) $u = L, v \neq L, u < v$;
- (c) $u \neq L, v = L, u < v$;
- (d) $u, v \neq L, u < v$.

The event (a) occurs with probability α^2 . The union of the events (b) and (c) occurs with probability $\alpha(1 - \alpha)$. The event (d) occurs with probability $\frac{1}{2}(1 - \alpha)^2$. The sum of these probabilities gives the correct value of the extended Schur function \tilde{s}_2 .

It seems that in the calculation of Fulman the case (d) is missing. His explanation: “since the two words giving a Young tableau with 1 row of length 2 are 11 and 01” probably stems from a collision in the notation used by Kerov and Vershik and the one used by Fulman.

In fact as the 2 in the denominator of $\frac{\alpha^2+1}{2}$ shows, one can't interpret the extended Schur functions with $\gamma \neq 0$ in terms of RSK and words on a finite number of symbols.

Indeed, in order to have $\gamma > 0$ one should use an infinite alphabet and the non-atomic part of the probability distribution should be non-zero, just as claimed by Kerov and Vershik.

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