STUDY OF FULL IMPLICIT PETROLEUM ENGINEERING FINITE VOLUME SCHEME FOR COMPRESSIBLE TWO PHASE FLOW IN POROUS MEDIA

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Abstract. An industrial scheme, to simulate the two compressible phase flow in porous media, consists in a finite volume method together with a phase-by-phase upstream scheme. The implicit finite volume scheme satisfies industrial constraints of robustness. We show that the proposed scheme satisfy the maximum principle for the saturation, a discrete energy estimate on the pressures and a function of the saturation that denote capillary terms. These stabilities results allow us to derive the convergence of a subsequence to a weak solution of the continuous equations as the size of the discretization tends to zero. The proof is given for the complete system when the density of the each phase depends on the own pressure.

1. Introduction

A rigorous mathematical study of a petroleum engineering schemes takes an important place in oil recovery engineering for production of hydrocarbons from petroleum reservoirs. This important problem renews the mathematical interest in the equations describing the multi-phase flows through porous media. The derivation of the mathematical equations describing this phenomenon may be found in [6], [10]. The differential equations describing the flow of two incompressible, immiscible fluids in porous media have been studied in the past decades. Existence of weak solutions to these equations has been shown under various assumptions on physical data [4, 10, 11, 12, 13, 17, 18, 24, 25].

The numerical discretization of the two-phase incompressible immiscible flows has been the object of several studies, the description of the numerical treatment by finite difference scheme may be found in the books [5], [27].

The finite volume methods have been proved to be well adapted to discretize conservative equations and have been used in industry because they are cheap, simple to code and robust. The porous media problems are one of the privileged field of applications. This success induced us to study and prove the mathematical convergence of a classical finite volume method for a model of two-phase flow in porous media.

For the two-phase incompressible immiscible flows, the convergence of a cell-centered finite volume scheme to a weak solution is studied in [26], and for a cell-centered finite volume scheme, using a "phase by phase" upstream choice for computations of the fluxes have been treated in [16] and in [8]. The authors give an iterative method to calculate explicitly the phase by phase upwind scheme in the case where the flow is driven by gravitational forces and the capillary pressure is neglected. An introduction of the cell-centered finite volume can be found in [15].

For the convergence analysis of an approximation to miscible fluid flows in porous media by combining mixed finite element and finite volume methods, we refer to [2], [3].

Key words and phrases. Finite volume scheme, degenerate problem.

Pioneers works have been done recently by C. Galusinski and M. Saad in a serie of articles about "Degenerate parabolic system for compressible, immiscible, two-phase flows in porous media" ([19], [20], [21]) when the densities depend on the global pressure , and by Z. Khalil and M. Saad in ([22], [23]) for the general case where the density of each phase depends on its own pressure. And for the two compressible, partially miscible flow in porous media, we refer to [9], [28]. For the convergence analysis of a finite volume scheme for a degenerate compressible and immiscible flow in porous media with the feature of global pressure, we refer to [7].

In this paper, we consider a two-phase flow model where the fluids are immiscible. The medium is saturated by a two compressible phase flows. The model is treated without simplified assumptions on the density of each phase, we consider that the density of each phase depends on its corresponding pressure. It is well known that equations arising from multiphase flow in porous media are degenerated. The first type of degeneracy derives from the behavior of relative permeability of each phase which vanishes when his saturation goes to zero. The second type of degeneracy is due to the time derivative term when the saturation of each phase vanishes.

This paper deals with construction and convergence analysis of a finite volume scheme for two compressible and immiscible flow in porous media without simplified assumptions on the state law of the density of each phase.

The goal of this paper is to show that the approximate solution obtained with the proposed upwind finite volume scheme (3.8)–(3.9) converges as the mesh size tends to zero, to a solution of system (2.1) in an appropriate sense defined in section 2. In section 3, we introduce some notations for the finite volume method and we present our numerical scheme and the main theorem of convergence.

In section 4, we derive three preliminary fundamental lemmas. In fact, we will see that we can't control the discrete gradient of pressure since the mobility of each phase vanishes in the region where the phase is missing. So we are going to use the feature of global pressure. We show that the control of velocities ensures the control of the global pressure and a dissipative term on saturation in the whole domain regardless of the presence or the disappearance of the phases.
Section 5 is devoted to a maximum principle on saturation and a well posedness of the scheme which inspired from H.W. Alt, S. Luckhaus [1]. Section 7 is devoted to a space-time L^1 compactness of sequences of approximate solutions.
Finally, the passage to the limit on the scheme and convergence analysis are performed in section 8. Some numerical results are stated in the last section 9.

2. Mathematical formulation of the continuous problem

Let us state the physical model describing the immiscible displacement of two compressible fluids in porous media. Let $T > 0$ be the final time fixed, and let be Ω a bounded open subset of $\mathbb{R}^\ell$ ($\ell \geq 1$). We set $Q_T = (0,T) \times \Omega$, $\Sigma_T = (0,T) \times \partial\Omega$. The mass conservation of each phase is given in Q_T

$$\phi(x)\partial_t(\rho_\alpha(p_\alpha)s_\alpha)(t,x) + \operatorname{div}(\rho_\alpha(p_\alpha)\mathbf{V}_\alpha)(t,x) + \rho_\alpha(p_\alpha)s_\alpha f_P(t,x) = \rho_\alpha(p_\alpha)s_\alpha^I f_I(t,x), \tag{2.1}$$

where ϕ, ρ_α and s_α are respectively the porosity of the medium, the density of the α phase and the saturation of the α phase. Here the functions f_I and f_P are respectively the injection and production terms. Note that in equation (2.1) the injection term is multiplied by a known saturation s_α^I corresponding to the known injected fluid, whereas the production term

is multiplied by the unknown saturation s_α corresponding to the produced fluid.
The velocity of each fluid $\mathbf{V}_\alpha$ is given by the Darcy law:

$$\mathbf{V}_\alpha = -\mathbf{K}\frac{k_{r_\alpha}(s_\alpha)}{\mu_\alpha}\big(\nabla p_\alpha - \rho_\alpha(p_\alpha)\mathbf{g}\big), \qquad \alpha = l, g. \tag{2.2}$$

where $\mathbf{K}$ is the permeability tensor of the porous medium, k_{r_α} the relative permeability of the α phase, μ_α the constant α-phase's viscosity, p_α the α-phase's pressure and $\mathbf{g}$ is the gravity term. Assuming that the phases occupy the whole pore space, the phase saturations satisfy

$$s_l + s_g = 1. \tag{2.3}$$

The curvature of the contact surface between the two fluids links the jump of pressure of the two phases to the saturation by the capillary pressure law in order to close the system (2.1)-(2.3)

$$p_c(s_l(t,x)) = p_g(t,x) - p_l(t,x). \tag{2.4}$$

With the arbitrary choice of (2.4) (the jump of pressure is a function of s_l), the application $s_l \mapsto p_c(s_l)$ is non-increasing, ($\frac{\mathrm{d}p_c}{\mathrm{d}s_l}(s_l) < 0$, for all $s_l \in [0,1]$), and usually $p_c(s_l = 1) = 0$ when the wetting fluid is at its maximum saturation.

2.1. **Assumptions and main result.** The model is treated without simplified assumptions on the density of each phase, we consider that the density of each phase depends on its corresponding pressure. The main point is to handle a priori estimates on the approximate solution. The studied system represents two kinds of degeneracy: the degeneracy for evolution terms $\partial_t(\rho_\alpha s_\alpha)$ and the degeneracy for dissipative terms $\operatorname{div}(\rho_\alpha M_\alpha \nabla p_\alpha)$ when the saturation vanishes. We will see in the section 5 that we can't control the discrete gradient of pressure since the mobility of each phase vanishes in the region where the phase is missing. So, we are going to use the feature of global pressure to obtain uniform estimates on the discrete gradient of the global pressure and the discrete gradient of the capillary term $\mathcal{B}$ (defined on (2.7)) to treat the degeneracy of this system.

Let us summarize some useful notations in the sequel. We recall the conception of the global pressure as describe in [10]

$$M(s_l)\nabla p = M_l(s_l)\nabla p_l + M_g(s_g)\nabla p_g,$$

with the α-phase's mobility M_α and the total mobility are defined by

$$M_\alpha(s_\alpha) = k_{r_\alpha}(s_\alpha)/\mu_\alpha, \quad M(s_l) = M_l(s_l) + M_g(s_g).$$

This global pressure p can be written as

$$p = p_g + \tilde{p}(s_l) = p_l + \bar{p}(s_l), \tag{2.5}$$

or the artificial pressures are denoted by $\bar{p}$ and $\tilde{p}$ defined by:

$$\tilde{p}(s_l) = -\int_0^{s_l} \frac{M_l(z)}{M(z)} p_c^{'}(z)\mathrm{d}z \text{ and } \bar{p}(s_l) = \int_0^{s_l} \frac{M_g(z)}{M(z)} p_c^{'}(z)\mathrm{d}z \tag{2.6}$$

We also define the capillary terms by

$$\gamma(s_l) = -\frac{M_l(s_l)M_g(s_g)}{M(s_l)}\frac{\mathrm{d}p_c}{\mathrm{d}s_l}(s_l) \ge 0,$$

and let us finally define the function $\mathcal{B}$ from $[0,1]$ to $\mathbb{R}$ by:

$$\begin{aligned}\mathcal{B}(s_l) &= \int_0^{s_l} \gamma(z)\mathrm{d}z = -\int_0^{s_l} \frac{M_l(z)M_g(z)}{M(z)}\frac{\mathrm{d}p_c}{\mathrm{d}s_l}(z)\mathrm{d}z \\ &= -\int_0^{s_g} M_l(z)\frac{\mathrm{d}\bar{p}}{\mathrm{d}s_l}(z)\mathrm{d}z = \int_0^{s_l} M_g(z)\frac{\mathrm{d}\tilde{p}}{\mathrm{d}s_l}(z)\mathrm{d}z.\end{aligned} \tag{2.7}$$

We complete the description of the model (2.1) by introducing boundary conditions and initial conditions. To the system (2.1)–(2.4) we add the following mixed boundary conditions. We consider the boundary $\partial\Omega = \Gamma_l \cup \Gamma_{imp}$, where Γ_l denotes the water injection boundary and Γ_{imp} the impervious one.

$$\begin{cases} p_l(t,x) = p_g(t,x) = 0 \text{ on } (0,T)\times\Gamma_l, \\ \rho_l \mathbf{V}_l\cdot\mathbf{n} = \rho_g\mathbf{V}_g\cdot\mathbf{n} = 0 \text{ on } (0,T)\times\Gamma_{imp}, \end{cases} \tag{2.8}$$

where $\mathbf{n}$ is the outward normal to Γ_{imp}.

The initial conditions are defined on pressures

$$p_\alpha(t=0) = p_\alpha^0 \text{ for } \alpha = l,g \text{ in } \Omega \tag{2.9}$$

We are going to construct a finite volume scheme on orthogonal admissible mesh, we treat here the case where

$$K = k\mathcal{I}_d$$

where k is a constant positive. For clarity, we take $k=1$ which equivalent to change the scale in time.

Next we introduce some physically relevant assumptions on the coefficients of the system.

$(H1)$ There is two positive constants ϕ_0 and ϕ_1 such that $\phi_0 \le \phi(x) \le \phi_1$ almost everywhere $x\in\Omega$.

$(H2)$ The functions M_l and M_g belongs to $\mathcal{C}^0([0,1],\mathbb{R}^+)$, $M_\alpha(s_\alpha=0)=0$. In addition, there is a positive constant $m_0>0$ such that for all $s_l\in[0,1]$,

$$M_l(s_l) + M_g(s_g) \ge m_0.$$

$(H3)$ $(f_P, f_I) \in (L^2(Q_T))^2$, $f_P(t,x)$, $f_I(t,x) \ge 0$ almost everywhere $(t,x)\in Q_T$.

$(H4)$ The density ρ_α is $\mathcal{C}^1(\mathbb{R})$, increasing and there exist two positive constants $\rho_m>0$ and $\rho_M>0$ such that $0<\rho_m\le\rho_\alpha(p_g)\le\rho_M$.

$(H5)$ The capillary pressure fonction $p_c\in\mathcal{C}^1([0,1];\mathbb{R}^+)$, decreasing and there exists $\underline{p_c}>0$ such that $0<\underline{p_c}\le|\frac{\mathrm{d}p_c}{\mathrm{d}s_l}|$.

$(H6)$ The function $\gamma\in C^1([0,1];\mathbb{R}^+)$ satisfies $\gamma(s_l)>0$ for $0<s_l<1$ and $\gamma(s_l=1)=\gamma(s_l=0)=0$. We assume that $\mathcal{B}^{-1}$ (the inverse of $\mathcal{B}(s_l)=\int_0^{s_l}\gamma(z)\mathrm{d}z$) is an Hölder[1] function of order θ, with $0<\theta\le1$, on $[0,\mathcal{B}(1)]$.

The assumptions (H1)–(H6) are classical for porous media. Note that, due to the boundedness of the capillary pressure function, the functions $\tilde{p}$ and $\bar{p}$ defined in (2.6) are bounded on $[0,1]$.

Let us define the following Sobolev space

$$H^1_{\Gamma_l}(\Omega) = \{u\in H^1(\Omega); u=0 \ sur\ \Gamma_l\},$$

this is an Hilbert space with the norm $\|u\|_{H^1_{\Gamma_l}(\Omega)} = \|\nabla u\|_{(L^2(\Omega))^\ell}$.

[1]This means that there exists a positive constant c such that for all $a,b\in[0,\mathcal{B}(1)]$, one has $|\mathcal{B}^{-1}(a)-\mathcal{B}^{-1}(b)|\le c|a-b|^\theta$.

Definition 1. (*Weak solutions*). *Under assumptions (H1)-(H6) and definitions* (2.5)-(2.9) *with the fact that* p_l^0, p_g^0 *belongs to* $L^2(\Omega)$ *and* s_α^0 *satisfies* $0 \leq s_\alpha^0 \leq 1$ *almost everywhere in* Ω*, then the pair* (p_l, p_g) *is a weak solution of problem* (2.1) *satisfying :*

$$p_\alpha \in L^2(0,T;L^2(\Omega)),\ \sqrt{M_\alpha(s_\alpha)}\nabla p_\alpha \in (L^2(0,T;L^2(\Omega)))^\ell, \tag{2.10}$$

$$0 \leq s_\alpha(t,x) \leq 1 \ a.e \ in \ Q_T \ (\alpha = l,g),\ \mathcal{B}(s_l) \in L^2(0,T;H^1_{\Gamma_l}(\Omega)), \tag{2.11}$$

$$\phi\partial_t(\rho_\alpha(p_\alpha)s_\alpha) \in L^2(0,T;(H^1_{\Gamma_l}(\Omega))') \in L^2(0,T;(H^1_{\Gamma_l}(\Omega))'), \tag{2.12}$$

such that for all φ, $\psi \in \mathbb{C}^1([0,T];H^1_{\Gamma_l}(\Omega))$ *with* $\varphi(T,\cdot) = \psi(T,\cdot) = 0$,

$$\begin{aligned} &-\int_{Q_T} \phi\rho_l(p_l)s_l\partial_t\varphi \mathrm{d}x\mathrm{d}t - \int_\Omega \phi(x)\rho_l(p_l^0(x))s_l^0(x)\varphi(0,x)\,dx \\ &+\int_{Q_T} M_l(s_l)\rho_l(p_l)\nabla p_l\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t - \int_{Q_T} M_l(s_l)\rho_l^2(p_l)\mathbf{g}\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t \\ &+\int_{Q_T} \rho_l(p_l)s_lf_P\varphi\mathrm{d}x\mathrm{d}t = \int_{Q_T} \rho_l(p_l)s_l^If_I\varphi\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{2.13}$$

$$\begin{aligned} &-\int_{Q_T} \phi\rho_g(p_g)s_g\partial_t\psi \mathrm{d}x\mathrm{d}t - \int_\Omega \phi(x)\rho_g(p_g^0(x))s_g^0(x)\psi(0,x)\,dx \\ &+\int_{Q_T} M_g(s_g)\rho_g(p_g)\nabla p_g\cdot\nabla\psi\mathrm{d}x\mathrm{d}t - \int_{Q_T} M_g(s_g)\rho_g^2(p_g)\mathbf{g}\cdot\nabla\psi\mathrm{d}x\mathrm{d}t \\ &+\int_{Q_T} \rho_g(p_g)s_gf_P\psi\mathrm{d}x\mathrm{d}t = \int_{Q_T} \rho_g(p_g)s_g^If_I\psi\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.14}$$

3. The finite volume scheme

3.1. **Finite volume definitions and notations.**

Following [15], let us define a finite volume discretization of $\Omega \times (0,T)$.

Definition 2. (*Admissible mesh of* Ω). *An admissible mesh* $\mathcal{T}$ *of* Ω *is given by a set of open bounded polygonal convex subsets of* Ω *called control volumes and a family of points (the "centers" of control volumes) satisfying the following properties:*

(1) The closure of the union of all control volumes is $\overline{\Omega}$*. We denote by* $|K|$ *the measure of* K*, and define*

$$h = size(\mathcal{T}) = max\{diam(K), K \in \mathcal{T}\}.$$

(2) For any $(K,L) \in \mathcal{T}^2$ *with* $K \neq L$*, then* $K \cap L = \emptyset$*. One denotes by* $\mathcal{E} \subset \mathcal{T}^2$ *the set of* (K,L) *such that the* $d-1$*-Lebesgue measure of* $\overline{K}\cap\overline{L}$ *is positive. For* $(K,L) \in \mathcal{E}$*, one denotes* $\sigma_{K|L} = \overline{K}\cap\overline{L}$ *and* $|\sigma_{K|L}|$ *the* $d-1$*-Lebesgue measure of* $\sigma_{K|L}$*. And one denotes* $\eta_{K|L}$ *the unit normal vector to* $\sigma_{K|L}$ *outward to* K

(3) For any $K \in \mathcal{T}$*, one defines* $N(K) = \{L \in \mathcal{T}, (K,L) \in \mathcal{E}\}$ *and one assumes that* $\partial K = \overline{K}\backslash K = (\overline{K}\cap\partial\Omega) \cup (\cup_{L\in N(K)}\sigma_{K|L})$.

(4) The family of points $(x_K)_{K\in\mathcal{T}}$ *is such that* $x_K \in K$ *(for all* $K \in \mathcal{T}$*) and, if* $L \in N(K)$*, it is assumed that the straight line* (x_K, x_L) *is orthogonal to* $\sigma_{K|L}$*. We set* $d_{K|L} = d(x_K,x_L)$ *the distance between the points* x_K *and* x_L*, and* $\tau_{K|L} = \frac{|\sigma_{K|L}|}{d_{K|L}}$*, that is sometimes called the "transmissivity" through* $\sigma_{K|L}$ *(see Figure 1).*

(5) Let $\xi > 0$. We assume the following regularity of the mesh :

$$\forall K \in \mathcal{T}, \sum_{L \in N(K)} |\sigma_{K|L}| d_{K|L} \leq \xi |K| \tag{3.1}$$

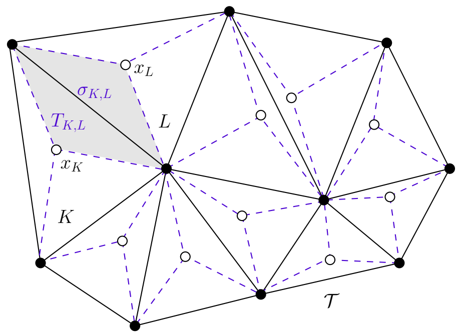

FIGURE 1. Control volumes, centers and diamonds

We denote by $H_h(\Omega) \subset L^2(\Omega)$ the space of functions which are piecewise constant on each control volume $K \in \mathcal{T}$. For all $u_h \in H_h(\Omega)$ and for all $K \in \mathcal{T}$, we denote by u_K the constant value of u_h in K. For $(u_h, v_h) \in (H_h(\Omega))^2$, we define the following inner product:

$$\langle u_h, v_h \rangle_{H_h} = \ell \sum_{K \in \mathcal{T}} \sum_{L \in N(K)} \frac{|\sigma_{K|L}|}{d_{K|L}} (u_L - u_K)(v_L - v_K),$$

and the norm in $H_h(\Omega)$ by

$$\|u_h\|_{H_h(\Omega)} = (\langle u_h, u_h \rangle_{H_h})^{1/2}.$$

Finally, we define $L_h(\Omega) \subset L^2(\Omega)$ the space of functions which are piecewise constant on each control volume $K \in \mathcal{T}$ with the associated norm

$$(u_h, v_h)_{L_h(\Omega)} = \sum_{K \in \mathcal{T}} |K| \, u_K v_K, \qquad \|u_h\|^2_{L_h(\Omega)} = \sum_{K \in \mathcal{T}} |K| \, |u_K|^2,$$

for $(u_h, v_h) \in (L_h(\Omega))^2$. Further, a diamond $T_{K|L}$ is constructed upon the interface $\sigma_{K|L}$, having x_K, x_L for vertices (see Figure 1) and the ℓ-dimensional mesure $|T_{K|L}|$ of $T_{K|L}$ equals to $\frac{1}{\ell} |\sigma_{K|L}| \, d_{K|L}$.

The discrete gradient $\nabla_h u_h$ of a constant per control volume function u_h is defined as the constant per diamond $T_{K|L}$ $\mathbb{R}^\ell$-valued function with values

$$\nabla_h u_h(x) = \begin{cases} \ell \frac{u_L - u_K}{d_{K|L}} \eta_{K|L} \text{ if } x \in T_{K|L}, \\ \ell \frac{u_\sigma - u_K}{d_{K,\sigma}} \eta_{K|\sigma} \text{ if } x \in T^{\text{ext}}_{K|\sigma}. \end{cases}$$

And the semi-norm $\|u_h\|_{H_h}$ coincides with the $L^2(\Omega)$ norm of $\nabla_h u_h$, in fact

$$\begin{aligned} \|\nabla_h u_h\|^2_{L^2(\Omega)} = \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \int_{T_{K|L}} |\nabla_h u_h|^2 \, dx &= \ell^2 \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \left|T_{K|L}\right| \frac{|u_L - u_K|^2}{|d_{K|L}|^2} \\ &= \ell \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{\left|\sigma_{K|L}\right|}{d_{K|L}} |u_L - u_K|^2 := \|u_h\|^2_{H_h(\Omega)} . \end{aligned}$$

We assimilate a discrete field $(\vec{F}_{K|L})$ on Ω to the piecewise constant vector-function

$$\vec{F}_h = \sum_{\sigma_{K|L}\in\mathcal{E}} \vec{F}_{K|L} 1\!\!1_{T_{K|L}}.$$

The discrete divergence of the field $\vec{F}_h$ is defined as the discrete function $w_h = div_h \vec{F}_h$ with the entires

$$div_K \vec{F}_h := \frac{1}{|K|} \sum_{L\in N(K)} |\sigma_{K|L}| \vec{F}_{K|L} \cdot \eta_{K|L}. \tag{3.2}$$

The problem under consideration is time-dependent, hence we also need to discretize the time interval $(0, T)$.

Definition 3. (*Time discretization*). *A time discretization of $(0, T)$ is given by an integer value N and by a strictly increasing sequence of real values $(t^n)_{n\in[0,N+1]}$ with $t^0 = 0$ and $t^{N+1} = T$. Without restriction, we consider a uniform step time $\delta t = t^{n+1} - t^n$, for $n \in [0, N]$.*

We may then define a discretization of the whole domain $\Omega \times (0, T)$ in the following way:

Definition 4. (*Discretization of $\Omega \times (0, T)$*). *A finite volume discretization $\mathcal{D}$ of $\Omega \times (0, T)$ is defined by*

$$\mathcal{D} = \Big(\mathcal{T}, \mathcal{E}, (x_K)_{K\in\mathcal{T}}, N, (t^n)_{n\in[0,N]}\Big),$$

where $\mathcal{T}, \mathcal{E}, (x_K)_{K\in\mathcal{T}}$ is an admissible mesh of Ω in the sense of Definition 2 and $N, (t^n)_{n\in[0,N]}$ is a time discretization of $(0, T)$ in the sense of Definition 3. One then sets

$$size(\mathcal{D}) = max(size(\mathcal{T}), \delta t).$$

Definition 5. (*Discrete functions and notations*). *Let $\mathcal{D}$ be a discretization of $\Omega \times (0, T)$ in the sense of Definition 4. We denote any function from $\mathcal{T} \times [0, N + 1]$ to $\mathbb{R}$ by using the subscript $\mathcal{D}$, ($s_{\alpha,\mathcal{D}}$ and $p_{\alpha,\mathcal{D}}$ for instance) and we denote its value at the point (x_K, t^n) using the subscript K and the superscript n ($s^n_{\alpha,K}$ for instance, we then denote $s_{\alpha,\mathcal{D}} = (s^n_{\alpha,K})_{K\in\mathcal{T},n\in[0,N+1]}$). To any discrete function $u_\mathcal{D}$ corresponds an approximate function defined almost everywhere on $\Omega \times (0, T)$ by:*

$$u_\mathcal{D}(t, x) = u^{n+1}_K, \text{ for a.e. } (t, x) \in (t^n, t^{n+1}) \times K, \forall K \in \mathcal{T}, \forall n \in [0, N].$$

For any continuous function $f : \mathbb{R} \to \mathbb{R}$, $f(u_{\mathcal{D}})$ *denotes the discrete function* $(K,n) \to f(u_K^{n+1})$. *if* $L \in N(K)$, *and* $u_{\mathcal{D}}$ *is a discrete function, we denote by* $\delta_{K|L}^{n+1}(u) = u_L^{n+1} - u_K^{n+1}$. *For example,* $\delta_{K|L}^{n+1}(f(u)) = f(u_L^{n+1}) - f(u_K^{n+1})$.

Let us recall the following two lemmas :

Lemma 1. (*Discrete Poincaré inequality*) [15]. *Let* Ω *be an open bounded polygonal subset of* $\mathbb{R}^\ell$, $\ell = 2$ *or* 3. *Let* $\mathcal{T}$ *be a finite volume discretization of* Ω *in the sense of Definition 2, and let* u *be a function which is constant on each cell* $K \in \mathcal{T}$, *that is,* $u(x) = u_K$ *if* $x \in K$, *then*

$$\|u\|_{L^2(\Omega)} \le diam(\Omega) \, \|u\|_{H_h(\Omega)} ,$$

where $\|\cdot\|_{H_h(\Omega)}$ *is the discrete* H_0^1 *norm.*

Remark 1. (*Dirichlet condition on part of the boundary*). *The lemma 1 gives a discrete Poincaré inequality for Dirichlet boundary conditions on the boundary* $\partial\Omega$. *In the case of Dirichlet condition on part of the boundary only, it is still possible to prove a discrete Poincaré inequality provided that the polygonal bounded open set* Ω *is connected.*

Lemma 2. (*Discrete integration by parts formula*). *Let* $F_{K/L}$, $K \in \mathcal{T}$ *and* $L \in N(K)$ *be a value in* $\mathbb{R}$ *depends on* K *and* L *such that* $F_{K/L} = -F_{L/K}$ *and let* φ *be a function which is constant on each cell* $K \in \mathcal{T}$, *that is,* $\varphi(x) = \varphi_K$ *if* $x \in K$, *then*

$$\sum_{K\in\mathcal{T}} \sum_{L\in N(K)} F_{K/L}\varphi_K = -\frac{1}{2} \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} F_{K/L}(\varphi_L - \varphi_K) \tag{3.3}$$

Consequently, if $F_{K/L} = a_{K/L}(b_L - b_K)$, *with* $a_{K/L} = a_{L/K}$, *then*

$$\sum_{K\in\mathcal{T}} \sum_{L\in N(K)} a_{K/L}(b_L - b_K)\varphi_K = -\frac{1}{2} \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} a_{K/L}(b_L - b_K)(\varphi_L - \varphi_K) \tag{3.4}$$

3.2. **The coupled finite volume scheme.** The finite volume scheme is obtained by writing the balance equations of the fluxes on each control volume. Let $\mathcal{D}$ be a discretization of $\Omega \times (0,T)$ in the sense of Definition 4. Let us integrate equations (2.1) over each control volume K. By using the Green formula, if Φ is a vector field, the integral of $\mathrm{div}(\Phi)$ on a control volume K is equal to the sum of the normal fluxes of Φ on the edges (3.2). Here we apply this formula to approximate $M_\alpha(s_\alpha)\nabla p_\alpha \cdot \eta_{K|L}$, $(\alpha = l, g)$ by means of the values $s_{\alpha,K}, s_{\alpha,L}$ and $p_{\alpha,K}, p_{\alpha,L}$ that are available in the neighborhood of the interface $\sigma_{K|L}$. To do this, let us use some function G_α of $(a,b,c) \in \mathbb{R}^3$. The numerical convection flux functions $G_\alpha \in C(\mathbb{R}^3, \mathbb{R})$, are required to satisfy the properties:

$$\begin{cases} \text{(a) } G_\alpha(\cdot, b, c) \text{ is non-decreasing for all } b, c \in \mathbb{R}, \\ \quad \text{and } G_\alpha(a, \cdot, c) \text{ is non-increasing for all } a, c \in \mathbb{R}; \\ \text{(b) } G_\alpha(a, a, c) = -M_\alpha(a)\, c \text{ for all } a, c \in \mathbb{R}; \\ \text{(c) } G_\alpha(a, b, c) = -G_\alpha(b, a, -c) \text{ and there exists } C > 0 \text{ such that} \\ \quad |G_\alpha(a, b, c)| \le C\,\big(|a| + |b|\big)|c| \text{ for all } a, b, c \in \mathbb{R}. \end{cases} \tag{3.5}$$

Note that the assumptions (a), (b) and (c) are standard and they respectively ensure the maximum principle on saturation, the consistency of the numerical flux and the conservation of the numerical flux on each interface. Practical examples of numerical convective flux functions can be found in [15].

In our context, we consider an upwind scheme, the numerical flux G_α satisfying (3.5) defined by

$$G_\alpha(a,b,c) = -M_\alpha(b)\,c^+ + M_\alpha(a)\,c^- \tag{3.6}$$

where $c^+ = \max(c,0)$ and $c^- = \max(-c,0)$. Note that the function $s_\alpha \mapsto M_\alpha(s_\alpha)$ is non-decreasing, which lead to the monotony property of the function G_α.

The resulting equation is discretized with a implicit Euler scheme in time; the normal gradients are discretized with a centered finite difference scheme.
Denote by $p_{\alpha,\mathcal{D}} = (p^{n+1}_{\alpha,K})_{K\in\mathcal{T},n\in[0,N]}$ and $s_{\alpha,\mathcal{D}} = (s^{n+1}_{\alpha,K})_{K\in\mathcal{T},n\in[0,N]}$ the discrete unknowns corresponding to p_α and s_α. The finite volume scheme is the following set of equations :

$$p^0_{\alpha,K} = \frac{1}{|K|}\int_K p^0_\alpha(x)\mathrm{d}x,\ s^0_{\alpha,K} = \frac{1}{|K|}\int_K s^0_\alpha(x)\mathrm{d}x,\ \text{for all } K\in\mathcal{T}, \tag{3.7}$$

$$\begin{aligned}|K|\,\phi_K\frac{\rho_l(p^{n+1}_{l,K})s^{n+1}_{l,K} - \rho_l(p^n_{l,K})s^n_{l,K}}{\delta t} &+ \sum_{L\in N(K)} \tau_{K|L}\rho^{n+1}_{l,K|L}G_l(s^{n+1}_{l,K}, s^{n+1}_{l,L}; \delta^{n+1}_{K|L}(p_l)) \\ &+ F^{n+1}_{l,K} + |K|\,\rho_l(p^{n+1}_{l,K})s^{n+1}_{l,K}f^{n+1}_{P,K} = |K|\,\rho_l(p^{n+1}_{l,K})(s^I_{l,K})^{n+1}f^{n+1}_{I,K},\end{aligned} \tag{3.8}$$

$$\begin{aligned}|K|\,\phi_K\frac{\rho_g(p^{n+1}_{g,K})s^{n+1}_{g,K} - \rho_g(p^n_{g,K})s^n_{g,K}}{\delta t} &+ \sum_{L\in N(K)} \tau_{K|L}\rho^{n+1}_{g,K|L}G_g(s^{n+1}_{g,K}, s^{n+1}_{g,L}; \delta^{n+1}_{K|L}(p_g)) \\ &+ F^{n+1}_{g,K} + |K|\,\rho_g(p^{n+1}_{g,K})s^{n+1}_{g,K}f^{n+1}_{P,K} = |K|\,\rho_g(p^{n+1}_{g,K})(s^I_{g,K})^{n+1}f^{n+1}_{I,K},\end{aligned} \tag{3.9}$$

$$p_c(s^{n+1}_{g,K}) = p^{n+1}_{l,K} - p^{n+1}_{g,K}, \tag{3.10}$$

where $F^{n+1}_{\alpha,K}$ ($\alpha = l,\ g$) the approximation of $\displaystyle\int_{\partial K}\rho^2_\alpha(p^{n+1}_\alpha)M_\alpha(s^{n+1}_\alpha)\mathbf{g}\cdot\eta_{K|L}\,\mathrm{d}\Gamma(x)$ by an upwind scheme:

$$F^{n+1}_{\alpha,K} = \sum_{L\in N(K)} F^{n+1}_{\alpha,K|L} = \sum_{L\in N(K)} |\sigma_{K|L}|(\rho^{n+1}_{\alpha,K|L})^2\Big(M_\alpha(s^{n+1}_{\alpha,K})\mathbf{g}^+_{K|L} - M_\alpha(s^{n+1}_{\alpha,L})\mathbf{g}^-_{K|L}\Big), \tag{3.11}$$

with $\mathbf{g}^+_{K|L} := (\mathbf{g}\cdot\eta_{K|L})^+$ and $\mathbf{g}^-_{K|L} := (\mathbf{g}\cdot\eta_{K|L})^-$. Notice that the source terms are, for $n\in\{0,\dots,N-1\}$

$$f^{n+1}_{P,K} := \frac{1}{\delta t\,|K|}\int_{t^n}^{t^{n+1}}\int_K f_P(t,x)\,dxdt,\quad f^{n+1}_{I,K} := \frac{1}{\delta t\,|K|}\int_{t^n}^{t^{n+1}}\int_K f_I(t,x)\,dxdt$$

The mean value of the density of each phase on interfaces is not classical since it is given as

$$\frac{1}{\rho^{n+1}_{\alpha,K|L}} = \begin{cases} \frac{1}{p^{n+1}_{\alpha,L}-p^{n+1}_{\alpha,K}}\int_{p^{n+1}_{\alpha,K}}^{p^{n+1}_{\alpha,L}}\frac{1}{\rho_\alpha(\zeta)}\,d\zeta & \text{if } p^{n+1}_{\alpha,K}\neq p^{n+1}_{\alpha,L}, \\ \frac{1}{\rho^{n+1}_{\alpha,K}} & \text{otherwise.}\end{cases} \tag{3.12}$$

This choice is crucial to obtain estimates on discrete pressures.

Note that the numerical fluxes to approach the gravity terms F_α are nondecreasing with respect to $s_{\alpha,K}$ and nonincreasing with respect to $s_{\alpha,L}$.

The upwind fluxes (3.6) can be rewritten in the equivalent form

$$G_\alpha(s^{n+1}_{\alpha,K}, s^{n+1}_{\alpha,L}; \delta^{n+1}_{K|L}(p_\alpha)) = -M_\alpha(s^{n+1}_{\alpha,K|L})\, \delta^{n+1}_{K|L}(p_\alpha), \tag{3.13}$$

where $M_\alpha(s^{n+1}_{\alpha,K|L})$ denote the upwind discretization of $M_\alpha(s_\alpha)$ on the interface $\sigma_{K|L}$ and

$$s^{n+1}_{\alpha,K|L} = \begin{cases} s^{n+1}_{\alpha,K} \text{ if } (K,L) \in \mathcal{E}^{n+1}_\alpha, \\ s^{n+1}_{\alpha,L} \text{ otherwise,} \end{cases} \tag{3.14}$$

with the set $\mathcal{E}^{n+1}_\alpha$ is subset of $\mathcal{E}$ such that

$$\mathcal{E}^{n+1}_\alpha = \{(K,L) \in \mathcal{E}, \delta^{n+1}_{K|L}(p_\alpha) = p^{n+1}_{\alpha,L} - p^{n+1}_{\alpha,K} \le 0\}. \tag{3.15}$$

We extend the mobility functions $s_\alpha \mapsto M_\alpha(s_\alpha)$ outside $[0,1]$ by continuous constant functions. We show below (see Prop. 2) that there exists at least one solution to this scheme. From this discrete solution, we build an approximation solution $p_{\alpha,\mathcal{D}}$ defined almost everywhere on Q_T by (see Definition 5):

$$p_{\alpha,\mathcal{D}}(t,x) = p^{n+1}_{\alpha,K},\ \forall x \in K, \forall t \in (t^n, t^{n+1}). \tag{3.16}$$

The main result of this paper is the following theorem.

Theorem 1. *Assume hypothesis (H1)-(H6) hold. Let $\{\mathcal{D}_m\}_{m\in\mathbb{N}}$ be a sequence of discretization of Q_T in the sense of definition 4 such that $\lim_{m\to+\infty} size(\mathcal{D}_m) = 0$. Let $(p^0_\alpha, s^0_\alpha) \in L^2(\Omega,\mathbb{R}) \times L^\infty(\Omega,\mathbb{R})$. Then there exists an approximate solutions $(p_{\alpha,\mathcal{D}_m})_{m\in\mathbb{N}}$ corresponding to the system (3.8)-(3.9), which converges (up to a subsequence) to a weak solution p_α of* (2.1) *in the sense of the Definition 1.*

4. Preliminary fundamental lemmas

The mobility of each phase vanishes in the region where the phase is missing. Therefore, if we control the quantities $M_\alpha \nabla p_\alpha$ in the L^2-norm, this does not permit the control of the gradient of pressure of each phase. In the continuous case, we have the following relationship between the global pressure, capillary pressure and the pressure of each phase

$$M|\nabla p|^2 + \frac{M_l M_g}{M}|\nabla p_c|^2 = M_l|\nabla p_l|^2 + M_g|\nabla p_g|^2. \tag{4.1}$$

This relationship, means that, the control of the velocities ensures the control of the global pressure and the capillary terms $\mathcal{B}$ in the whole domain regardless of the presence or the disappearance of the phases. This estimates (of the global pressure and the capillary terms $\mathcal{B}$) has a major role in the analysis, to treat the degeneracy of the dissipative terms $\operatorname{div}(\rho_\alpha M_\alpha \nabla p_\alpha)$.

In the discrete case, these relationship, are not obtained in a straightforward way. This equality is replaced by three discrete inequalities which we state in the following three lemmas.

We derive in the next lemma the preliminary step to proof the estimates of the global pressure and the capillary terms given in Proposition 1 and Corollary 1. These lemmas are first used to prove a compactness lemma and then used for the convergence result.

Lemma 3. *(Total mobility and global pressure* [16]*). Under the assumptions* $(H1)-(H6)$ *and the notations* (2.5)*. Let* $\mathcal{D}$ *be a finite volume discretization of* $\Omega \times (0,T)$ *in the sense of Definition 4. Then for all* $(K,L) \in \mathcal{E}$ *and for all* $n \in [0,N]$ *the following inequalities hold:*

$$M^{n+1}_{l,K|L} + M^{n+1}_{g,K|L} \geq m_0, \tag{4.2}$$

and

$$m_0\Big(\delta^{n+1}_{K|L}(p)\Big)^2 \leq M^{n+1}_{l,K|L}\Big(\delta^{n+1}_{K|L}(p_l)\Big)^2 + M^{n+1}_{g,K|L}\Big(\delta^{n+1}_{K|L}(p_g)\Big)^2. \tag{4.3}$$

The proof of this lemma is made by R. Eymard and al. in [16]. The proof of this result can be applied for compressible flow since the proof use only the definition of the global pressure.

Lemma 4. (*Capillary term* $\mathcal{B}$). *Under the assumptions* $(H1)-(H6)$ *and the notations* (2.5)*. Let* $\mathcal{D}$ *be a finite volume discretization of* $\Omega \times (0,T)$ *in the sense of Definition 4. Then there exists a constant* $C > 0$ *such that for all* $(K,L) \in \mathcal{E}$ *and* $n \in [0,N]$ *:*

$$(\delta^{n+1}_{K|L}(\mathcal{B}(s_l)))^2 \leq M^{n+1}_{l,K|L}\Big(\delta^{n+1}_{K|L}(p_l)\Big)^2 + M^{n+1}_{g,K|L}\Big(\delta^{n+1}_{K|L}(p_g)\Big)^2. \tag{4.4}$$

In the incompressible case (see [16]) this kind of estimate is obtained by using the mass conservation equation and under hypotheses ont the relative permeability of the α phase, whereas, the compressibility add more difficulties, our approach use only the definition of the function $\mathcal{B}$ and consequently this lemma can be used for compressible and incompressible degenerate flows.

Proof. We take the same decomposition of the interface as that proposed by R. Eymard and al. in [16], namely the different possible cases $(K,L) \in \mathcal{E}^{n+1}_l \cap \mathcal{E}^{n+1}_g$, $(K,L) \notin \mathcal{E}^{n+1}_l \cup \mathcal{E}^{n+1}_g$, $(K,L) \in \mathcal{E}^{n+1}_l$ and $(K,L) \notin \mathcal{E}^{n+1}_g$, and the last case $(K,L) \notin \mathcal{E}^{n+1}_l$ and $(K,L) \in \mathcal{E}^{n+1}_g$; where the sets $\mathcal{E}^{n+1}_l$ and $\mathcal{E}^{n+1}_g$ are defined in (3.15). We establish for the four cases.

•**First case.** If $(K,L) \notin \mathcal{E}_l$ and $(K,L) \in \mathcal{E}_g$. We may notice that if the upwind choice is different for the two equations, we have

$$M^{n+1}_{\alpha,K|L} = \max_{[s_{l,K},s_{l,L}]} M_\alpha.$$

By definition of $\mathcal{B}$ in (2.7), there exists some $a \in [s_{l,K}, s_{l,L}]$ such that

$$\delta^{n+1}_{K|L}(\mathcal{B}(s_l)) = -\frac{M_l(a)M_g(a)}{M_l(a)+M_g(a)}\delta^{n+1}_{K|L}(p_c(s_l)),$$

we then get

$$\begin{aligned}(\delta^{n+1}_{K|L}(\mathcal{B}(s_l)))^2 &\leq M^{n+1}_{l,K|L}M^{n+1}_{g,K|L}(\delta^{n+1}_{K|L}(p_c(s_l)))^2\\ &\leq C_1 M^{n+1}_{l,K|L}M^{n+1}_{g,K|L}\Big((\delta^{n+1}_{K|L}(p_g))^2 + (\delta^{n+1}_{K|L}(p_l))^2\Big)\\ &\leq C_2\Big(M^{n+1}_{l,K|L}(\delta^{n+1}_{K|L}(p_l))^2 + M^{n+1}_{g,K|L}(\delta^{n+1}_{K|L}(p_g))^2\Big).\end{aligned}$$

•**Second case:** The case $(K,L) \in \mathcal{E}_l$ and $(K,L) \notin \mathcal{E}_g$ is similar.

•**Third case:** The case $(K,L) \in \mathcal{E}_l$ and $(K,L) \in \mathcal{E}_g$. We have

$$
\begin{aligned}
&M_{l,K|L}^{n+1}(\delta_{K|L}^{n+1}(p_l))^2 + M_{g,K|L}^{n+1}(\delta_{K|L}^{n+1}(p_g))^2 \\
&= M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(p_l))^2 + M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(p_g))^2 \\
&= \Big(M_l(s_{l,K}^{n+1}) + M_g(s_{g,K}^{n+1})\Big)(\delta_{K|L}^{n+1}(p))^2 \\
&+ M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(\tilde{p}(s_l)))^2 + M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(\bar{p}(s_l)))^2 \\
&- 2M_g(s_{g,K}^{n+1})\delta_{K|L}^{n+1}(p)\delta_{K|L}^{n+1}(\tilde{p}(s_l)) - 2M_l(s_{l,K}^{n+1})\delta_{K|L}^{n+1}(p)\delta_{K|L}^{n+1}(\bar{p}(s_l)).
\end{aligned}
\tag{4.5}
$$

We will distinguish the case $s_{l,K}^{n+1} \leq s_{l,L}^{n+1}$ and the case $s_{l,K}^{n+1} \geq s_{l,L}^{n+1}$.

(1) If we assume that $s_{l,K}^{n+1} \leq s_{l,L}^{n+1}$, we deduce that
   (a) $\delta_{K|L}^{n+1}(\bar{p}(s_l)) \leq 0$ since $\bar{p}(s_l)$ is nonincreasing,
   (b) $\delta_{K|L}^{n+1}(\tilde{p}(s_l)) \geq 0$ since $\tilde{p}(s_l)$ is nondecreasing,
   (c) $\delta_{K|L}^{n+1}(p) = \delta_{K|L}^{n+1}(p_l) + \delta_{K|L}^{n+1}(\bar{p}(s_l)) \leq 0$.

   One then gets from (4.5) that:

$$
\begin{aligned}
&M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(p_l))^2 + M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(p_g))^2 \\
&\geq \Big(M_l(s_{l,K}^{n+1}) + M_g(s_{g,K}^{n+1})\Big)(\delta_{K|L}^{n+1}(p))^2 \\
&+ M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(\tilde{p}(s_l))^2) + M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(\bar{p}(s_l)))^2 \\
&- 2M_l(s_{l,K}^{n+1})\delta_{K|L}^{n+1}(p)\delta_{K|L}^{n+1}(\bar{p}(s_l)).
\end{aligned}
$$

   The previous inequality gives:

$$
\begin{aligned}
&\Big(M_l(s_{l,K}^{n+1}) + M_g(s_{g,K}^{n+1})\Big)(\delta_{K|L}^{n+1}(p))^2 \\
&+ M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(\tilde{p}(s_l)))^2 + M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(\bar{p}(s_l)))^2 \\
&\leq M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(p_g))^2 + M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(p_l))^2 + 2M_l(s_{l,K}^{n+1})\delta_{K|L}^{n+1}(p)\delta_{K|L}^{n+1}(\bar{p}(s_l)) \\
&\leq M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(p_l))^2 + M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(p_g))^2 \\
&+ M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(p))^2 + M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(\bar{p}(s_l)))^2,
\end{aligned}
$$

   which implies the inequality:

$$
\begin{aligned}
&M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(p))^2 + M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(\tilde{p}(s_l)))^2 \\
&\leq M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(p_l))^2 + M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(p_g))^2.
\end{aligned}
\tag{4.6}
$$

   Or, by definition of $\mathcal{B}$ (2.7), there exists some $a \in [s_{l,K}, s_{l,L}]$ such that $\delta_{K|L}^{n+1}(\mathcal{B}(s_l)) = M_g(a)\delta_{K|L}^{n+1}(\tilde{p}(s_l))$, we get then

$$
\begin{aligned}
(\delta_{K|L}^{n+1}(\mathcal{B}(s_l)))^2 &\leq M_g(s_{g,K})(\delta_{K|L}^{n+1}(\tilde{p}(s_l)))^2 \\
&\leq M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(p_l))^2 + M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(p_g))^2.
\end{aligned}
$$

   which is (4) in that case.

(2) If we assume that $s_{l,L}^{n+1} \leq s_{l,K}^{n+1}$, we get that

(a) $\delta_{K|L}^{n+1}(\bar{p}(s_l)) \geq 0$ since $\bar{p}(s_l)$ is nonincreasing,
(b) $\delta_{K|L}^{n+1}(\tilde{p}(s_l)) \leq 0$ since $\tilde{p}(s_l)$ is nondecreasing,
(c) $\delta_{K|L}^{n+1}(p) = \delta_{K|L}^{n+1}(p_g) + \delta_{K|L}^{n+1}(\tilde{p}(s_l)) \leq 0$.

One then gets from (4.5) that:

$$\begin{aligned}
&M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(p_l))^2 + M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(p_g))^2 \\
&\geq \Big(M_g(s_{g,K}^{n+1}) + M_l(s_{l,K}^{n+1})\Big)(\delta_{K|L}^{n+1}(p))^2 \\
&+ M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(\tilde{p}(s_l))^2) + M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(\bar{p}(s_l)))^2 \\
&- 2M_g(s_{g,K}^{n+1})\delta_{K|L}^{n+1}(p)\delta_{K|L}^{n+1}(\tilde{p}(s_l)).
\end{aligned}$$

The previous inequality gives:

$$\begin{aligned}
&\Big(M_l(s_{l,K}^{n+1}) + M_g(s_{g,K}^{n+1})\Big)(\delta_{K|L}^{n+1}(p))^2 \\
&+ M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(\tilde{p}(s_l))^2) + M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(\bar{p}(s_l)))^2 \\
&\leq M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(p_l))^2 + M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(p_g))^2 + 2M_g(s_{g,K}^{n+1})\delta_{K|L}^{n+1}(p)\delta_{K|L}^{n+1}(\tilde{p}(s_l)) \\
&\leq M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(p_l))^2 + M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(p_g))^2 \\
&+ M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(p))^2 + M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(\tilde{p}(s_l)))^2,
\end{aligned}$$

which implies the inequality:

$$\begin{aligned}
&M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(p))^2 + M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(\bar{p}(s_l)))^2 \\
&\leq M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(p_l))^2 + M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(p_g))^2.
\end{aligned} \tag{4.7}$$

Or, by definition of $\mathcal{B}$ (2.7) there exists some $a \in [s_{l,K}, s_{l,L}]$ such that $\delta_{K|L}^{n+1}(\mathcal{B}(s_l)) = -M_l(a)\delta_{K|L}^{n+1}(\bar{p}(s_l))$, we get then

$$\begin{aligned}
(\delta_{K|L}^{n+1}(\mathcal{B}(s_l)))^2 &\leq M_l(s_{l,K})(\delta_{K|L}^{n+1}(\bar{p}(s_l)))^2 \\
&\leq M_l(s_{l,K}^{n+1})(\delta_{K|L}^{n+1}(p_l))^2 + M_g(s_{g,K}^{n+1})(\delta_{K|L}^{n+1}(p_g))^2,
\end{aligned}$$

which is (4) in that case.

•**Fourth case:** The case $(K, L) \notin \mathcal{E}_l$ and $(K, L) \notin \mathcal{E}_g$ is similar of the third case. □

Lemma 5. (*Dissipative terms*). *Under the assumptions* $(H1)-(H6)$ *and the notations* (2.5). *Let $\mathcal{D}$ be a finite volume discretization of $\Omega \times (0, T)$ in the sense of Definition 4. Then there exists a constant $C > 0$ such that for all $(K, L) \in \mathcal{E}$ and $n \in [0, N]$*

$$M_{l,K|L}^{n+1}(\delta_{K|L}^{n+1}(\bar{p}(s_l)))^2 \leq M_{l,K|L}^{n+1}\Big(\delta_{K|L}^{n+1}(p_l)\Big)^2 + M_{g,K|L}^{n+1}\Big(\delta_{K|L}^{n+1}(p_g)\Big)^2, \tag{4.8}$$

and

$$M_{g,K|L}^{n+1}(\delta_{K|L}^{n+1}(\tilde{p}(s_l)))^2 \leq M_{l,K|L}^{n+1}\Big(\delta_{K|L}^{n+1}(p_l)\Big)^2 + M_{g,K|L}^{n+1}\Big(\delta_{K|L}^{n+1}(p_g)\Big)^2. \tag{4.9}$$

Proof. In order to prove (4.8) and (4.9), we consider the exclusive cases $(K, L) \in \mathcal{E}_l^{n+1} \cap \mathcal{E}_g^{n+1}$, $(K, L) \notin \mathcal{E}_l^{n+1} \cup \mathcal{E}_g^{n+1}$, $(K, L) \notin \mathcal{E}_l^{n+1}$ and $(K, L) \in \mathcal{E}_g^{n+1}$ and the last case $(K, L) \in \mathcal{E}_l^{n+1}$ and $(K, L) \notin \mathcal{E}_g^{n+1}$.

First case. If $(K,L) \notin \mathcal{E}_l$ and $(K,L) \in \mathcal{E}_g$. We have

$$M_{\alpha,K|L}^{n+1} = \max_{[s_{l,K},s_{l,L}]} M_\alpha,$$

and by definition of $\bar{p}$ there exists some $a \in [s_{l,K}, s_{l,L}]$ such that $\delta_{K|L}^{n+1}(\bar{p}(s_l)) = \frac{M_g(a)}{M_g(a)+M_l(a)} \delta_{K|L}^{n+1}(p_c(s_l))$, we get then

$$\begin{aligned} M_{l,K|L}^{n+1}(\delta_{K|L}^{n+1}(\bar{p}(s_l)))^2 &\leq M_{l,K|L}^{n+1} M_{g,K|L}^{n+1}(\delta_{K|L}^{n+1}(p_c(s_l)))^2 \\ &\leq C_1 M_{l,K|L}^{n+1} M_{g,K|L}^{n+1}\Big((\delta_{K|L}^{n+1}(p_g))^2 + (\delta_{K|L}^{n+1}(p_l))^2\Big) \\ &\leq C_2\Big(M_{l,K|L}^{n+1}(\delta_{K|L}^{n+1}(p_l))^2 + M_{g,K|L}^{n+1}(\delta_{K|L}^{n+1}(p_g))^2\Big), \end{aligned}$$

which gives (4.8). For the discrete estimate (4.9) and by definition of $\tilde{p}$ there exists some $b \in [s_{l,K}, s_{l,L}]$ such that $\delta_{K|L}^{n+1}(\tilde{p}(s_l)) = -\frac{M_l(b)}{M_g(b)+M_l(b)} \delta_{K|L}^{n+1}(p_c(s_l))$, we get then

$$\begin{aligned} M_{g,K|L}^{n+1}(\delta_{K|L}^{n+1}(\tilde{p}(s_l)))^2 &\leq M_{g,K|L}^{n+1} M_{l,K|L}^{n+1}(\delta_{K|L}^{n+1}(p_c(s_l)))^2 \\ &\leq C_1 M_{g,K|L}^{n+1} M_{l,K|L}^{n+1}\Big((\delta_{K|L}^{n+1}(p_g))^2 + (\delta_{K|L}^{n+1}(p_l))^2\Big) \\ &\leq C_2\Big(M_{l,K|L}^{n+1}(\delta_{K|L}^{n+1}(p_l))^2 + M_{g,K|L}^{n+1}(\delta_{K|L}^{n+1}(p_g))^2\Big), \end{aligned}$$

which gives (4.9).

Second case. The case $(K,L) \in \mathcal{E}_l$ and $(K,L) \notin E_g$ is similar.

The **third case** and the **fourth case** can be treated as the cases in the lemma 4. □

5. A priori estimates and existence of the approximate solution

We derive new energy estimates on the discrete velocities $M_\alpha(s_{\alpha,K|L}^{n+1})\delta_{K|L}^{n+1}(p_\alpha)$. Nevertheless, these estimates are degenerate in the sense that they do not permit the control of $\delta_{K|L}^{n+1}(p_\alpha)$, especially when a phase is missing. So, the global pressure has a major role in the analysis, we will show that the control of the discrete velocities $M_\alpha(s_{\alpha,K|L}^{n+1})\delta_{K|L}^{n+1}(p_\alpha)$ ensures the control of the discrete gradient of the global pressure and the discrete gradient of the capillary term $\mathcal{B}$ in the whole domain regardless of the presence or the disappearance of the phases.

The following section gives us some necessary energy estimates to prove the theorem 1.

5.1. **The maximum principle.**

Let us show in the following Lemma that the phase by phase upstream choice yields the L^∞ stability of the scheme which is a basis to the analysis that we are going to perform.

Lemma 6. (*Maximum principe*). *Under assumptions (H1)-(H6). Let* $(s_{\alpha,K}^0)_{K\in\mathcal{T}} \in [0,1]$ *and let* $\mathcal{D} = \Big(\mathcal{T}, \mathcal{E}, (x_K)_{K\in\mathcal{T}}, N, (t^n)_{n\in[0,N]}\Big)$ *be a discretization of* $\Omega \times (0,T)$ *in the sense of Definition 4 and assume that* $(p_{\alpha,\mathcal{D}})$ *is a solution of the finite volume* (3.7)-(3.10). *Then, the saturation* $(s_{\alpha,K}^n)_{K\in\mathcal{T},n\in\{0,\ldots,N\}}$ *remains in* $[0,1]$.

Proof. Let us show by induction in n that for all $K \in \mathcal{T}$, $s_{\alpha,K}^n \geq 0$ where $\alpha = l, g$. For $\alpha = l$, the claim is true for $n = 0$ and for all $K \in \mathcal{T}$. We argue by induction that for all $K \in \mathcal{T}$, the claim is true up to order n. We consider the control volume K such that

$s_{l,K}^{n+1} = \min\{s_{l,L}^{n+1}\}_{L\in\mathcal{T}}$ and we seek that $s_{l,K}^{n+1} \geq 0$.
For the above mentioned purpose, multiply the equation in (3.8) by $-(s_{l,K}^{n+1})^-$, we obtain

$$\begin{aligned}(5.1)\quad & -|K|\,\phi_K \frac{\rho_l(p_{l,K}^{n+1})s_{l,K}^{n+1} - \rho_l(p_{l,K}^{n})s_{l,K}^{n}}{\delta t}(s_{l,K}^{n+1})^- \\ & - \sum_{L\in N(K)} \tau_{K|L}\rho_{l,K|L}^{n+1} G_l(s_{l,K}^{n+1}, s_{l,L}^{n+1}; \delta_{K|L}^{n+1}(p_l))(s_{l,K}^{n+1})^- - F_{l,K}^{(n+1)}(s_{l,K}^{n+1})^- \\ & - |K|\,\rho_l(p_{l,K}^{n+1})s_{l,K}^{n+1} f_{P,K}^{n+1}(s_{l,K}^{n+1})^- = -|K|\,\rho_l(p_{l,K}^{n+1})(s_{g,K}^{I})^{n+1} f_{I,K}^{n+1}(s_{l,K}^{n+1})^- \leq 0.\end{aligned}$$

The numerical flux G_l is nonincreasing with respect to $s_{l,L}^{n+1}$ (see (a) in (3.5)), and consistence (see (c) in (3.5)), we get

$$\begin{aligned} & G_l(s_{l,K}^{n+1}, s_{l,L}^{n+1}; \delta_{K|L}^{n+1}(p_l))\,(s_{l,K}^{n+1})^- \leq G_l(s_{l,K}^{n+1}, s_{l,K}^{n+1}; \delta_{K|L}^{n+1}(p_l))\,(s_{l,K}^{n+1})^- \\ (5.2)\quad & \qquad = -\delta_{K|L}^{n+1}(p_l)\,M_l(s_{l,K}^{n+1})\,(s_{l,K}^{n+1})^- = 0.\end{aligned}$$

Using the identity $s_{l,K}^{n+1} = (s_{l,K}^{n+1})^+ - (s_{l,K}^{n+1})^-$, and the mobility M_l extended by zero on $]-\infty, 0]$, then $M_l(s_{l,K}^{n+1})(s_{l,K}^{n+1})^- = 0$ and

$$\begin{aligned}(5.3)\quad & -F_{l,K}^{(n+1)}(s_{l,K}^{n+1})^- - |K|\,\rho_l(p_{l,K}^{n+1})s_{l,K}^{n+1} f_{P,K}^{n+1}(s_{l,K}^{n+1})^- \\ & = \sum_{L\in N(K)} (\rho_{l,K|L}^{n+1})^2 M_l(s_{l,L}^{n+1})\mathbf{g}_{L,K}(s_{l,K}^{n+1})^- + |K|\,\rho_l(p_{l,K}^{n+1}) f_{P,K}^{n+1}((s_{l,K}^{n+1})^-)^2 \geq 0.\end{aligned}$$

Then, we deduce from (5.1) that

$$\rho_l(p_{l,K}^{n+1})|(s_{l,K}^{n+1})^-|^2 + \rho_l(p_{l,K}^{n})s_{l,K}^{n}(s_{l,K}^{n+1})^- \leq 0,$$

and from the nonnegativity of $s_{l,K}^{n}$, we obtain $(s_{l,K}^{n+1})^- = 0$. This implies that $s_{l,K}^{n+1} \geq 0$ and

$$0 \leq s_{l,K}^{n+1} \leq s_{l,L}^{n+1} \text{ for all } n \in [0, N-1] \text{ and } L \in \mathcal{T}.$$

In the same way, we prove $s_{g,K}^{n+1} \geq 0$. □

5.2. Estimations on the pressures.

Proposition 1. *Let $p_{\alpha,\mathcal{D}}$ be a solution of (3.7)-(3.10). Then, there exists a constant $C > 0$, which only depends on M_α, Ω, T, p_α^0, s_α^0, s_α^I, f_P, f_I and not on $\mathcal{D}$, such that the following discrete $L^2(0,T;H^1(\Omega))$ estimates hold:*

$$(5.4)\quad \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \tau_{K|L} M_\alpha(s_{\alpha,K|L}^{n+1})|p_{\alpha,L}^{n+1} - p_{\alpha,K}^{n+1}|^2 \leq C,$$

and

$$(5.5)\quad \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \tau_{K|L}|p_{L}^{n+1} - p_{K}^{n+1}|^2 \leq C.$$

Proof. We define the function $\mathcal{H}_\alpha(p_\alpha) := \rho_\alpha(p_\alpha)g_\alpha(p_\alpha) - p_\alpha$, $\mathcal{P}_c(s_l) := \int_0^{s_l} p_c(z)\mathrm{d}z$ and $g_\alpha(p_\alpha) = \int_0^{p_\alpha} \frac{1}{\rho_\alpha(z)}\mathrm{d}z$. In the following proof, we denote by C_i various real values which only depend on M_α, Ω, T, p^0_α, s^0_α, s^I_α, f_P, f_I and not on $\mathcal{D}$. To prove the estimate (5.4), we multiply (3.8) and (3.9) respectively by $g_l(p_{l,K})$, $g_g(p_{g,K})$ and adding them, then summing the resulting equation over K and n. We thus get:

$$E_1 + E_2 + E_3 + E_4 = 0, \tag{5.6}$$

where

$$E_1 = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}} |K|\,\phi_K\Big(\big(\rho_l(p^{n+1}_{l,K})s^{n+1}_{l,K} - \rho_l(p^n_{l,K})s^n_{l,K}\big)\, g_l(p^{n+1}_{l,K}) + \big(\rho_g(p^{n+1}_{g,K})s^{n+1}_{g,K} - \rho_g(p^n_{g,K})s^n_{g,K}\big)\, g_g(p^{n+1}_{g,K})\Big),$$

$$E_2 = \sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \tau_{K|L}\Big(\rho^{n+1}_{l,K|L}G_l(s^{n+1}_{l,K}, s^{n+1}_{l,L}; \delta^{n+1}_{K|L}(p_l))\, g_l(p^{n+1}_{l,K}) + G_g(s^{n+1}_{g,K}, s^{n+1}_{g,L}; \delta^{n+1}_{K|L}(p_g))\, g_g(p^{n+1}_{g,K})\Big),$$

$$E_3 = \sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\Big(F^{(n+1)}_{l,K|L}\; g_l(p^{n+1}_{l,K}) + F^{(n+1)}_{g,K|L}\; g_g(p^{n+1}_{g,K})\Big),$$

$$E_4 = \sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}|K|\,\Big(\rho_l(p^{n+1}_{l,K})s^{n+1}_{l,K}f^{n+1}_{P,K}g_l(p^{n+1}_{l,K}) - \rho_l(p^{n+1}_{l,K})(s^I_{g,K})^{n+1}f^{n+1}_{I,K}g_l(p^{n+1}_{l,K}) + \rho_g(p^{n+1}_{g,K})s^{n+1}_{g,K}f^{n+1}_{P,K}g_g(p^{n+1}_{g,K}) - \rho_g(p^{n+1}_{g,K})(s^I_{g,K})^{n+1}f^{n+1}_{I,K}g_g(p^{n+1}_{g,K})\Big).$$

To handle the first term of the equality (5.6). Let us forget the exponent $n+1$ and let note with the exponent $*$ the physical quantities at time t^n. In [22] the authors prove that : for all $s_\alpha \geq 0$ and $s^\star_\alpha \geq 0$ such that $s_l + s_g = s^\star_l + s^\star_g = 1$,

$$\big(\rho_l(p_l)s_l - \rho_l(p^\star_l)s^\star_l\big)g_l(p_l) + \big(\rho_g(p_g)s_g - \rho_g(p^\star_g)s^\star_g\big)g_g(p_g) \geq \mathcal{H}_l(p_l)s_l - \mathcal{H}_l(p^\star_l)s^\star_l + \mathcal{H}_g(p_g)s_g - \mathcal{H}_g(p^\star_g)s^\star_g - \mathcal{P}_c(s_l) + \mathcal{P}_c(s^\star_l). \tag{5.7}$$

The proof of (5.7) is based on the concavity property of g_α and $\mathcal{P}_c$. So, this yields to

$$E_1 \geq \sum_{K\in\mathcal{T}}\phi_K\,|K|\,\Big(s^N_{l,K}\mathcal{H}(p^N_{l,K}) - s^0_{l,K}\mathcal{H}(p^0_{l,K}) + s^N_{g,K}\mathcal{H}(p^N_{g,K}) - s^0_{g,K}\mathcal{H}(p^0_{g,K})\Big) - \sum_{K\in\mathcal{T}}\phi_K\,|K|\,\mathcal{P}_c(s^N_{l,K}) + \sum_{K\in\mathcal{T}}\phi_K\,|K|\,\mathcal{P}_c(s^0_{l,K}). \tag{5.8}$$

Using the fact that the numerical fluxes G_l and G_g are conservative in the sense of (c) in (3.5), we obtain by discrete integration by parts (see Lemma 2)

$$E_2 = \frac{1}{2}\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\tau_{K|L}\Big(\rho^{n+1}_{l,K|L}G_l(s^{n+1}_{l,K}, s^{n+1}_{l,L}; \delta^{n+1}_{K|L}(p_l))(g_l(p^{n+1}_{l,K}) - g_l(p^{n+1}_{l,L})) + \rho^{n+1}_{g,K|L}G_g(s^{n+1}_{g,K}, s^{n+1}_{g,L}; \delta^{n+1}_{K|L}(p_g))(g_g(p^{n+1}_{g,K}) - g_g(p^{n+1}_{g,L}))\Big),$$

and due to the correct choice of the density of the phase α on each interface,

$$\rho^{n+1}_{\alpha,K|L}(g_\alpha(p^{n+1}_{\alpha,K}) - g_\alpha(p^{n+1}_{\alpha,L})) = p^{n+1}_{\alpha,K} - p^{n+1}_{\alpha,L}, \tag{5.9}$$

we obtain

$$E_2 = \frac{1}{2}\sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \tau_{K|L}\Big(G_l(s^{n+1}_{l,K}, s^{n+1}_{l,L}; \delta^{n+1}_{K|L}(p_l))(p^{n+1}_{l,K} - p^{n+1}_{l,L}) + G_g(s^{n+1}_{g,K}, s^{n+1}_{g,L}; \delta^{n+1}_{K|L}(p_g))(p^{n+1}_{g,K} - p^{n+1}_{g,L})\Big).$$

The definition of the upwind fluxes in (3.13) implies

$$G_l(s^{n+1}_{l,K}, s^{n+1}_{l,L}; \delta^{n+1}_{K|L}(p_l))(p^{n+1}_{l,K} - p^{n+1}_{l,L}) + G_g(s^{n+1}_{g,K}, s^{n+1}_{g,L}; \delta^{n+1}_{K|L}(p_g))(p^{n+1}_{g,K} - p^{n+1}_{g,L}) = M_l(s^{n+1}_{l,K|L})(\delta^{n+1}_{K|L}(p_l))^2 + M_g(s^{n+1}_{g,K|L})(\delta^{n+1}_{K|L}(p_g))^2.$$

Then, we obtain the following equality

$$E_2 = \frac{1}{2}\sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \tau_{K|L}\Big(M_l(s^{n+1}_{l,K|L})(\delta^{n+1}_{K|L}(p_l))^2 + M_g(s^{n+1}_{g,K|L})(\delta^{n+1}_{K|L}(p_g)^2\Big). \tag{5.10}$$

To handle the other terms of the equality (5.6), firstly let us remark that the numerical fluxes of gravity term are conservative which satisfy $F^{n+1}_{l,K|L} = -F^{n+1}_{l,L,K}$ and $F^{n+1}_{g,K|L} = -F^{n+1}_{g,L,K}$, so we integrate by parts and we obtain

$$E_3 = \frac{1}{2}\sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} |\sigma_{K|L}|\Big(F^{(n+1)}_{l,K|L}(g_l(p^{n+1}_{l,K}) - g_l(p^{n+1}_{l,L})) + F^{(n+1)}_{g,K|L}(g_g(p^{n+1}_{g,K}) - g_g(p^{n+1}_{g,L}))\Big).$$

According to the choice of the density of the phase α on each interface (5.9) and the definition (3.11) we obtain

$$E_3 = -\frac{1}{2}\sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \left|\sigma_{K|L}\right| \rho^{n+1}_{l,K|L}[M_l(s^{n+1}_{l,K})\mathbf{g}^+_{K|L} - M_l(s^{n+1}_{l,L})\mathbf{g}^-_{K|L}](\delta^{n+1}_{K|L}(p_l)) - \frac{1}{2}\sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \left|\sigma_{K|L}\right| \rho^{n+1}_{g,K|L}[M_g(s^{n+1}_{g,K})\mathbf{g}^+_{K|L} - M_g(s^{n+1}_{g,L})\mathbf{g}^-_{K|L}](\delta^{n+1}_{K|L}(p_g)).$$

Recall the truncations of $\delta^{n+1}_{K|L}(p_\alpha)$

$$(\delta^{n+1}_{K|L}(p_\alpha))^+ = max\{\delta^{n+1}_{K|L}(p_\alpha), 0\}, \quad (\delta^{n+1}_{K|L}(p_\alpha))^- = max\{-\delta^{n+1}_{K|L}(p_\alpha), 0\},$$

with $\delta^{n+1}_{K|L}(p_\alpha) = (\delta^{n+1}_{K|L}(p_\alpha))^+ - (\delta^{n+1}_{K|L}(p_\alpha))^-$. So we obtain

$$E_3 \le \frac{1}{2}\sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \left|\sigma_{K|L}\right| \rho^{n+1}_{l,K|L} M_l(s^{n+1}_{l,K})\mathbf{g}^+_{K|L}(\delta^{n+1}_{K|L}(p_l))^- + \frac{1}{2}\sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \left|\sigma_{K|L}\right| \rho^{n+1}_{l,K|L} M_l(s^{n+1}_{l,L})\mathbf{g}^-_{K|L}(\delta^{n+1}_{K|L}(p_l))^+$$

$$+ \frac{1}{2}\sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \left|\sigma_{K|L}\right| \rho_{g,K|L}^{n+1} M_g(s_{g,K}^{n+1})\mathbf{g}_{K|L}^{+}(\delta_{K|L}^{n+1}(p_g))^{-}$$
$$+ \frac{1}{2}\sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \left|\sigma_{K|L}\right| \rho_{g,K|L}^{n+1} M_g(s_{g,L}^{n+1})\mathbf{g}_{K|L}^{-}(\delta_{K|L}^{n+1}(p_g))^{+}.$$

From the following equality $\left|\sigma_{K|L}\right| = (d_{K|L}|\sigma_{K|L}|)^{\frac{1}{2}}\tau_{K|L}^{\frac{1}{2}}$ and apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
E_3 \leq & 2C\sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} d_{K|L}|\sigma_{K|L}| \\
& + \frac{1}{4}\sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \tau_{K|L}\Big(M_l(s_{l,L}^{n+1})\ ((\delta_{K|L}^{n+1}(p_l))^{+})^2 + M_l(s_{l,K}^{n+1})\ ((\delta_{K|L}^{n+1}(p_l))^{-})^2 \\
& \qquad + M_g(s_{g,L}^{n+1})\ ((\delta_{K|L}^{n+1}(p_g))^{+})^2 + M_g(s_{g,K}^{n+1})\ ((\delta_{K|L}^{n+1}(p_g))^{-})^2\Big) \\
\leq & \ 2CT|\Omega| \\
& + \frac{1}{4}\sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \tau_{K|L}\Big(M_l(s_{l,L}^{n+1})\ ((\delta_{K|L}^{n+1}(p_l))^{+})^2 + M_g(s_{l,K}^{n+1})\ ((\delta_{K|L}^{n+1}(p_l))^{-})^2 \\
& \qquad + M_g(s_{g,L}^{n+1})\ ((\delta_{K|L}^{n+1}(p_g))^{+})^2 + M_g(s_{g,K}^{n+1})\ ((\delta_{K|L}^{n+1}(p_g))^{-})^2\Big).
\end{aligned}$$

From the definition of the truncations of $\delta_{K|L}^{n+1}(p_\alpha)$, we obtain

$$\begin{aligned}
E_3 \leq 2CT|\Omega| + \frac{1}{4}\sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \tau_{K|L}\Big(& M_l(s_{l,K|L}^{n+1})(\delta_{K|L}^{n+1}(p_l))^2 \\
& + M_g(s_{g,K|L}^{n+1})(\delta_{K|L}^{n+1}(p_g))^2\Big).
\end{aligned} \tag{5.11}$$

The last term will be absorbed by the terms on pressures from the estimate (5.10).
In order to estimate E_4, using the fact that the densities are bounded and the map g_α is sublinear (a.e.$|g(p_\alpha)| \leq C|p_\alpha|$), we have

$$|E_4| \leq C_1 \sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}} |K|\,(f_{P,K}^{n+1} + f_{I,K}^{n+1})(|p_{l,K}^{n+1}| + |p_{g,K}^{n+1}|),$$

then

$$|E_4| \leq C_1 \sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}} |K|\,(f_{P,K}^{n+1} + f_{I,K}^{n+1})(2|p_K^{n+1}| + |\bar{p}_K^{n+1}| + |\tilde{p}_K^{n+1}|).$$

Hence, by the Hölder inequality, we get that

$$|E_4| \leq C_2 \left\|f_P + f_I\right\|_{L^2(Q_T)} \Big(\sum_{n=0}^{N-1}\delta t \left\|p_h^{n+1}\right\|_{L^2(\Omega)}^2\Big)^{\frac{1}{2}},$$

and, from the discrete Poincaré inequality lemma 1, we get

$$|E_4| \leq C_3\Big(\sum_{n=0}^{N-1} \delta t \left\|p_h^{n+1}\right\|_{H_h}^2\Big)^{\frac{1}{2}} + C_4. \tag{5.12}$$

The equality (5.6) with the inequalities (5.8), (5.10), (5.11), (5.12) give (5.4). Then we deduce (5.5) from (4.3). □

We now state the following corollary, which is essential for the compactness and limit study.

Corollary 1. *From the previous Proposition, we deduce the following estimations:*

$$\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \tau_{K|L}(\delta_{K|L}^{n+1}(\mathcal{B}(s_l)))^2 \leq C, \tag{5.13}$$

$$\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \tau_{K|L} M_{l,K|L}^{n+1}(\delta_{K|L}^{n+1}(\bar{p}(s_l)))^2 \leq C, \tag{5.14}$$

and

$$\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \tau_{K|L} M_{g,K|L}^{n+1}(\delta_{K|L}^{n+1}(\tilde{p}(s_l)))^2 \leq C. \tag{5.15}$$

Proof. The prove of the estimates (5.13), (5.14) and (5.15) are a direct consequence of the inequality (4), (4.8), (4.9) and the Proposition 1. □

6. Existence of the finite volume scheme

We start with a technical assertion to characterize the zeros of a vector field which stated and proved in [14].

Lemma 7. *([14], p. 529) Assume the continuous function $v : \mathbb{R}^n \to \mathbb{R}^n$ satisfies*

$$v(z)\cdot z \geq 0 \text{ if } \|z\| = r,$$

for some $r > 0$. Then there exists a point z with $\|z\| \leq r$ such that

$$v(z) = 0.$$

Proposition 2. *The problem (3.8)-(3.9) admits at least one solution $(p_{l,K}^n, p_{g,K}^n)_{(K,n)\in\mathcal{D}}$.*

Proof. At the beginning of the proof, we set the following notations;

$$\mathcal{M} := Card(\mathcal{T}),$$
$$p_{l,\mathcal{M}} := \{p_{l,K}^{n+1}\}_{K\in\mathcal{T}} \in \mathbb{R}^{\mathcal{M}},$$
$$p_{g,\mathcal{M}} := \{p_{g,K}^{n+1}\}_{K\in\mathcal{T}} \in \mathbb{R}^{\mathcal{M}}.$$

We define the map $\mathcal{T}_h : \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}} \longrightarrow \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}}$,

$$\mathcal{T}_h(p_{l,\mathcal{M}}, p_{g,\mathcal{M}}) = (\{\mathcal{T}_{l,K}\}_{K\in\mathcal{T}}, \{\mathcal{T}_{g,K}\}_{K\in\mathcal{T}}) \text{ where,}$$

$$\mathcal{T}_{l,K} = |K|\,\phi_K \frac{\rho_l(p_{l,K}^{n+1})s_{l,K}^{n+1} - \rho_l(p_{l,K}^n)s_{l,K}^n}{\delta t} + \sum_{L\in N(K)} \tau_{K|L}\rho_{l,K|L}^{n+1} G_l(s_{l,K}^{n+1}, s_{l,L}^{n+1}; \delta_{K|L}^{n+1}(p_l))$$

$$+ F_{l,K}^{n+1} + |K|\,\rho_l(p_{l,K}^{n+1})\big(s_{l,K}^{n+1} f_{P,K}^{n+1} - (s_{l,K}^I)^{n+1} f_{I,K}^{n+1}\big), \tag{6.1}$$

$$\mathcal{T}_{g,K} = |K|\,\phi_K \frac{\rho_g(p_{g,K}^{n+1})s_{g,K}^{n+1} - \rho_g(p_{g,K}^n)s_{g,K}^n}{\delta t} + \sum_{L\in N(K)} \tau_{K|L}\rho_{g,K|L}^{n+1} G_g(s_{g,K}^{n+1}, s_{g,L}^{n+1}; \delta_{K|L}^{n+1}(p_g))$$

$$+ F_{g,K}^{n+1} + |K|\,\rho_g(p_{g,K}^{n+1})\big(s_{g,K}^{n+1} f_{P,K}^{n+1} - (s_{g,K}^I)^{n+1} f_{I,K}^{n+1}\big). \tag{6.2}$$

Note that $\mathcal{T}_h$ is well defined as a continuous function. Also we define the following homeomorphism $\mathcal{F} : \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}} \mapsto \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}}$ such that,

$$\mathcal{F}(p_{l,\mathcal{M}}, p_{g,\mathcal{M}}) = (v_{l,\mathcal{M}}, v_{g,\mathcal{M}})$$

where $v_{\alpha,\mathcal{M}} = \{g_\alpha(p_{\alpha,K}^{n+1})\}_{K\in\mathcal{T}}$.
Now let us consider the following continuous mapping $\mathcal{P}_h$ defined as

$$\mathcal{P}_h(v_{l,\mathcal{M}}, v_{g,\mathcal{M}}) = \mathcal{T}_h \circ \mathcal{F}^{-1}(v_{l,\mathcal{M}}, v_{g,\mathcal{M}}) = \mathcal{T}_h(p_{l,\mathcal{M}}, p_{g,\mathcal{M}}).$$

According to Lemma 7, our goal now is to show that

$$\mathcal{P}_h(v_{l,\mathcal{M}}, v_{g,\mathcal{M}}) \cdot (v_{l,\mathcal{M}}, v_{g,\mathcal{M}}) > 0, \quad \text{for } \|(v_{l,\mathcal{M}}, v_{g,\mathcal{M}})\|_{\mathbb{R}^{2\mathcal{M}}} = r > 0, \tag{6.3}$$

and for a sufficiently large r.
We observe that

$$\begin{aligned}\mathcal{P}_h(v_{l,\mathcal{M}}, v_{g,\mathcal{M}}) \cdot (v_{l,\mathcal{M}}, v_{g,\mathcal{M}}) \geq &\frac{1}{\delta t}\sum_{K\in\mathcal{T}} \phi_K\,|K| \Big(s_{l,K}^{n+1}\mathcal{H}(p_{l,K}^{n+1}) - s_{l,K}^n \mathcal{H}(p_{l,K}^n) \\ &\qquad + s_{g,K}^{n+1}\mathcal{H}(p_{g,K}^{n+1}) - s_{g,K}^n\mathcal{H}(p_{g,K}^n)\Big) \\ &- \frac{1}{\delta t}\mathcal{P}_c(s_{l,K}^{n+1}) + \frac{1}{\delta t}\mathcal{P}_c(s_{l,K}^n) + C\,\|p_h^{n+1}\|_{H_h(\Omega)}^2 - C,\end{aligned}$$

for some constants $C > 0$. This implies that

$$\begin{aligned}\mathcal{P}_h(v_{l,\mathcal{M}}, v_{g,\mathcal{M}}) \cdot (v_{l,\mathcal{M}}, v_{g,\mathcal{M}}) \geq &-\frac{1}{\delta t}\sum_{K\in\mathcal{T}} \phi_K\,|K| \Big(s_{l,K}^n\mathcal{H}(p_{l,K}^n) + s_{g,K}^n \mathcal{H}(p_{g,K}^n)\Big) \\ &- \frac{1}{\delta t}\mathcal{P}_c(s_{l,K}^{n+1}) + C\,\|p_h^{n+1}\|_{H_h(\Omega)}^2 - C',\end{aligned} \tag{6.4}$$

for some constants $C, C' > 0$. Finally using the fact that g_α is a Lipschitz function, then there exists a constant $C > 0$ such that

$$\begin{aligned}\Big\|(\{g_l(p_{l,K}^{n+1})\}_{K\in\mathcal{T}}, \{g_g(p_{g,K}^{n+1}))\}_{K\in\mathcal{T}})\Big\|_{\mathbb{R}^{2\mathcal{M}}} &\leq C\Big(\Big\|p_{l,h}^{n+1}\Big\|_{L^2(\Omega)} + \Big\|p_{g,h}^{n+1}\Big\|_{L^2(\Omega)}\Big) \\ &\leq 2C\Big(\|p_h^{n+1}\|_{L^2(\Omega)} + \|\bar{p}_h^{n+1}\|_{L^2(\Omega)} + \|\tilde{p}_h^{n+1}\|_{L^2(\Omega)}\Big) \\ &\leq 2C\big(\|p_h^{n+1}\|_{H_h(\Omega)} + C_1\big).\end{aligned}$$

Using this to deduce from (6.4) that (6.3) holds for r large enough. Hence, we obtain the existence of at least one solution to the scheme (3.8)-(3.9). □

7. Compactness properties

In this section we derive estimates on differences of space and time translates of the function $\phi_{\mathcal{D}}\rho_\alpha(p_{\alpha,\mathcal{D}})s_{\alpha,\mathcal{D}}$ which imply that the sequence $\phi_{\mathcal{D}}\rho_\alpha(p_{\alpha,\mathcal{D}})s_{\alpha,\mathcal{D}}$ is relatively compact in $L^1(Q_T)$.

We replace the study of discrete functions $U_{\alpha,\mathcal{D}} = \phi_{\mathcal{D}}\rho_\alpha(p_{\alpha,\mathcal{D}})s_{\alpha,\mathcal{D}}$ (constant per cylinder $Q_K^n := (t^n, t^{n+1}) \times K$) by the study of functions $\bar{U}_{\alpha,\mathcal{D}} = \phi_{\mathcal{D}}\rho_\alpha(\bar{p}_{\alpha,\mathcal{D}})\bar{s}_{\alpha,\mathcal{D}}$ piecewise continuous in t for all x, constant in x for all volume K, defined as

$$\bar{U}_{\alpha,\mathcal{D}}(t,x) = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \frac{1}{\delta t}\Big(\ (t - n\delta t)U_{\alpha,K}^{n+1}\ +\ ((n+1)\delta t - t)U_{\alpha,K}^n\ \Big)\ \mathbb{1}_{Q_K^n}(t,x).$$

One may deduce from the estimates (5.5) and (5.13) the following property.

Lemma 8. $\big($*Space translate of* $\bar{U}_{\alpha,\mathcal{D}}\big)$. *Under the assumptions* $(H1)-(H6)$. *Let* $\mathcal{D}$ *be a finite volume discretization of* $\Omega\times(0,T)$ *in the sense of Definition 4 and let* $p_{\alpha,\mathcal{D}}$ *be a solution of* (3.7)-(3.10). *Then, the following inequality hold:*

$$\int_{\Omega'\times(0,T)} \left|\bar{U}_{\alpha,\mathcal{D}}(t,x+y) - \bar{U}_{\alpha,\mathcal{D}}(t,x)\right| \mathrm{d}x\mathrm{d}t \le \omega(|y|), \tag{7.1}$$

for all $y\in\mathbb{R}^\ell$ *with* $\Omega' = \{x\in\Omega,\ [x,x+y]\subset\Omega\}$ *and* $\omega(|y|)\to 0$ *when* $|y|\to 0$.

Proof. For $\alpha = l$ and from the definition of $U_{l,\mathcal{D}}$, one gets

$$\begin{aligned}
&\int_{(0,T)\times\Omega'} |U_{l,\mathcal{D}}(t,x+y) - U_{l,\mathcal{D}}(t,x)|\,\mathrm{d}x\mathrm{d}t\\
&= \int_{(0,T)\times\Omega'} \left|\Big(\rho_l(p_{l,\mathcal{D}})s_{l,\mathcal{D}}\Big)(t,x+y) - \Big(\rho_l(p_{l,\mathcal{D}})s_{l,\mathcal{D}}\Big)(t,x)\right| \mathrm{d}x\mathrm{d}t\\
&\le \int_{(0,T)\times\Omega'} \left|s_{l,\mathcal{D}}(t,x+y)\Big(\rho_l(p_{l,\mathcal{D}}(t,x+y)) - \rho_l(p_{l,\mathcal{D}}(t,x))\Big)\right| \mathrm{d}x\mathrm{d}t\\
&\quad + \int_{(0,T)\times\Omega'} \left|\rho_l(p_{l,\mathcal{D}})(t,x)\Big(s_{l,\mathcal{D}}(t,x+y) - s_{l,\mathcal{D}}(t,x)\Big)\right| \mathrm{d}x\mathrm{d}t\\
&\le E_1 + E_2
\end{aligned}$$

where E_1 and E_2 defined as follows

$$E_1 = \rho_M \int_{(0,T)\times\Omega'} |s_{l,\mathcal{D}}(t,x+y) - s_{l,\mathcal{D}}(t,x)|\,\mathrm{d}x\mathrm{d}t, \tag{7.2}$$

$$E_2 = \int_{(0,T)\times\Omega'} |\rho_l(p_{l,\mathcal{D}}(t,x+y)) - \rho_l(p_{l,\mathcal{D}}(t,x))|\,\mathrm{d}x\mathrm{d}t. \tag{7.3}$$

To handle with the space translation on saturation, we use the fact that $\mathcal{B}^{-1}$ is an hölder function, then

$$E_1 \le \rho_M C \int_{(0,T)\times\Omega'} |\mathcal{B}(s_{l,\mathcal{D}}(t,x+y)) - \mathcal{B}(s_{l,\mathcal{D}}(t,x))|^\theta\,\mathrm{d}x\mathrm{d}t$$

and by application of the Cauchy-Schwarz inequality, we deduce

$$E_1 \le C\Big(\int_{(0,T)\times\Omega'} |\mathcal{B}(s_{l,\mathcal{D}}(t,x+y)) - \mathcal{B}(s_{l,\mathcal{D}}(t,x))|\,\mathrm{d}x\mathrm{d}t\Big)^\theta.$$

According to [15]), let $y \in \mathbb{R}^\ell$, $x \in \Omega'$, and $L \in N(K)$. We set

$$\beta_{\sigma_{K|L}} = \begin{cases} 1, & \text{if the line segment } [x, x+y] \text{ intersects } \sigma_{K|L},\ K \text{ and } L, \\ 0, & \text{otherwise.} \end{cases}$$

We observe that (see for more details [15])

$$\int_{\Omega'} \beta_{\sigma_{K|L}}(x)\, dx \le |\sigma_{K|L}|\, |y| \,. \tag{7.4}$$

To simplify the notation, we write $\sum\limits_{\sigma_{K|L}}$ instead of $\sum\limits_{\{(K,L)\in\mathcal{T}^2,\, K\neq L,\, |\sigma_{K|L}|\neq 0\}}$.

Now, denote that

$$\begin{aligned} E_1 \le & C\Big(\sum_{n=0}^{N-1} \delta t \sum_{\sigma_{K,L}} \Big|\mathcal{B}(s_{l,L}) - \mathcal{B}(s_{l,K})\Big| \int_{\Omega'} \beta_{\sigma_{K|L}}(x)\mathrm{d}x\Big)^\theta \\ & \le C\Big(|y| \sum_{n=0}^{N-1} \delta t \sum_{\sigma_{K,L}} |\sigma_{K|L}|\, \Big|\mathcal{B}(s_{l,L}) - \mathcal{B}(s_{l,K})\Big|\Big)^\theta . \end{aligned}$$

Let us again write $|\sigma_{K,L}| = (d_{K,L}|\sigma_{K,L}|)^{\frac{1}{2}} \tau_{K|L}^{\frac{1}{2}}$, applying again the Cauchy-Schwarz inequality and using the fact that the discrete gradient of the function $\mathcal{B}$ is bounded (5.13) to obtain

$$E_1 \le C\, |y|^\theta \,. \tag{7.5}$$

To treat the space translate of E_2, we use the fact that the map ρ_l' is bounded and the relationship between the gas pressure and the global pressure, namely : $p_l = p - \bar{p}$ defined in (2.5), then we have

$$\begin{aligned} E_2 &\le \max_{\mathbb{R}} |\rho_l'| \int_{(0,T)\times\Omega'} |p_{l,\mathcal{D}}(t, x+y) - p_{l,\mathcal{D}}(t,x)|\, \mathrm{d}x\mathrm{d}t \\ &\le \max_{\mathbb{R}} |\rho_l'| \int_{(0,T)\times\Omega'} |p_{\mathcal{D}}(t, x+y) - p_{\mathcal{D}}(t,x)|\, \mathrm{d}x\mathrm{d}t \\ &\quad + \max_{\mathbb{R}} |\rho_l'| \int_{(0,T)\times\Omega'} |\bar{p}(s_{l,\mathcal{D}}(t, x+y)) - \bar{p}(s_{l,\mathcal{D}}(t,x))|\, \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{7.6}$$

furthermore one can easily show that $\bar{p}$ is a $C^1([0,1];\mathbb{R})$, it follows, there exists a positive constant $C > 0$ such that

$$\begin{aligned} E_2 \le C &\int_{(0,T)\times\Omega'} |p_{\mathcal{D}}(t, x+y) - p_{\mathcal{D}}(t,x)|\mathrm{d}x\mathrm{d}t \\ &+ C \int_{(0,T)\times\Omega'} |s_{l,\mathcal{D}}(t, x+y) - s_{l,\mathcal{D}}(t,x)|\mathrm{d}x\mathrm{d}t. \end{aligned}$$

The last term in the previous inequality is proportional to E_1, and consequently it remains to show that the space translate on the global pressure is small with y. In fact

$$\begin{aligned}\int_{(0,T)\times\Omega'} |p_{\mathcal{D}}(t,x+y)-p_{\mathcal{D}}(t,x)|\mathrm{d}x\mathrm{d}t &\leq \sum_{n=0}^{N-1}\delta t\sum_{\sigma_{K,L}}|p_L^{n+1}-p_K^{n+1}|\int_{\Omega'}\beta_{\sigma_{K|L}}(x)\mathrm{d}x\\ &\leq |y|\sum_{n=0}^{N-1}\delta t\sum_{\sigma_{K,L}}\left|\sigma_{K|L}\right||p_L^{n+1}-p_K^{n+1}|.\end{aligned}$$

Finally, using the fact that the discrete gradient of global pressure is bounded (5.5), we deduce that

$$\int_{(0,T)\times\Omega'} |U_{l,\mathcal{D}}(t,x+y)-U_{l,\mathcal{D}}(t,x)|\,\mathrm{d}x \leq C(|y|+|y|^{\theta}), \tag{7.7}$$

for some constant $C>0$.
In addition, we have

$$\begin{aligned}\int_0^{+\infty}\int_{\Omega'}|\bar{U}_{l,\mathcal{D}}(t,x+\mathrm{d}x)-\bar{U}_{l,\mathcal{D}}(t,x)|\mathrm{d}x\mathrm{d}t \leq& 2\int_0^T\int_{\Omega'}|U_{l,\mathcal{D}}(t,x+\mathrm{d}x)-U_{l,\mathcal{D}}(t,x)|\mathrm{d}x\mathrm{d}t\\ &+2\delta t\int_{\Omega'_\delta}|U^0_{l,\mathcal{D}}(x)|\,\mathrm{d}x\end{aligned}$$

where $U_l^0=\rho_l(p_l^0)s_l^0$ and $\Omega'_\delta=\{x\in\Omega,\ \mathrm{dist}(x,\Omega')<|\delta|\}$. By (7.7), the assumption $\delta t\to 0$ as $\mathrm{size}(\mathcal{D})\to 0$ and the boundedness of $(U^0_{l,h})_h$ in $L^1(\Omega'_\delta)$, then the space translates of $\bar{U}_{l,\mathcal{D}}$ on Ω' are estimated uniformly for all sequence $\mathrm{size}(\mathcal{D}_m)_m$ tend to zero.
In the same way, we prove the space translate for $\alpha=g$. □

We state the following lemma on time translate of $\bar{U}_{\alpha,\mathcal{D}}$.

Lemma 9. $\big($*Time translate of* $\bar{U}_{\alpha,\mathcal{D}}\big)$. *Under the assumptions* $(H1)-(H6)$. *Let* $\mathcal{D}$ *be a finite volume discretization of* $\Omega\times(0,T)$ *in the sense of Definition 4 and let* $p_{\alpha,\mathcal{D}}$ *be a solution of* (3.7)-(3.10). *Then, there exists a positive constant* $C>0$ *depending on* Ω, T *such that the following inequality hold:*

$$\int_{\Omega\times(0,T-\tau)}\left|\bar{U}_{\alpha,\mathcal{D}}(t+\tau,x)-\bar{U}_{\alpha,\mathcal{D}}(t,x)\right|^2\,dx\,dt\leq\tilde{\omega}(\tau), \tag{7.8}$$

for all $\tau\in(0,T)$. *Here* $\tilde{\omega}:\mathbb{R}^+\to\mathbb{R}^+$ *is a modulus of continuity, i.e.* $\lim_{\tau\to 0}\tilde{\omega}(\tau)=0$.

We state without proof the following lemma on time translate of $\bar{U}_{\alpha,\mathcal{D}}$. Following the lemma **??**, the proof is a direct consequence of the estimations (5.5) and (5.13), then we omit it.

8. Study of the limit

Proposition 3. *Let* $(\mathcal{D}_m)_m$ *be a sequence of finite volume discretizations of* $\Omega\times(0,T)$ *such that* $\lim_{m\to+\infty}\mathrm{size}(\mathcal{D}_m)=0$. *Then there exists subsequences, still denoted* $(s_{\alpha,\mathcal{D}_m})_{m\in\mathbb{N}}$, $(p_{\alpha,\mathcal{D}_m})_{m\in\mathbb{N}}$ *verify the following convergences*

$$\|U_{\alpha,\mathcal{D}_m}-\bar{U}_{\alpha,\mathcal{D}_m}\|_{L^1(\Omega')}\longrightarrow 0, \tag{8.1}$$

$$U_{\alpha,\mathcal{D}_m}\longrightarrow U_\alpha \qquad \textit{strongly in } L^p(Q_T) \textit{ and a.e. in } Q_T \textit{ for all } p\geq 1, \tag{8.2}$$

(8.3) $\nabla_{\mathcal{D}_m}\mathcal{B}(s_{l,\mathcal{D}_m})\longrightarrow\nabla\mathcal{B}(s_l)$ *weakly in* $(L^2(Q_T))^\ell$,

(8.4) $\nabla_{\mathcal{D}_m}p_{\mathcal{D}_m}\longrightarrow\nabla p$ *weakly in* $(L^2(Q_T))^\ell$,

(8.5) $s_{\alpha,\mathcal{D}_m}\longrightarrow s_\alpha$ *almost everywhere in* Q_T,

(8.6) $p_{\alpha,\mathcal{D}_m}\longrightarrow p_\alpha$ *almost everywhere in* Q_T.

Furthermore,

$$0\le s_\alpha\le 1 \text{ a.e. in } Q_T, \tag{8.7}$$

$$U_\alpha=\phi\rho_\alpha(p_\alpha)s_\alpha \text{ a.e. in } Q_T. \tag{8.8}$$

Proof. For the first convergence (8.1) it is useful to introduce the following inequality, for all $a,b\in\mathbb{R}$,

$$\int_0^1|\theta a+(1-\theta)b|\,d\theta\ge\frac{1}{2}(|a|+|b|).$$

Applying this inequality to $a=U^{n+1}_{\alpha,\mathcal{D}_m}-U^n_{\alpha,\mathcal{D}_m}$, $b=U^n_{\alpha,\mathcal{D}_m}-U^{n-1}_{\alpha,\mathcal{D}_m}$, from the definition of $\bar U_{\alpha,\mathcal{D}_m}$ we deduce

$$\int_0^T\int_{\Omega'}|U_{\alpha,\mathcal{D}_m}(t,x)-\bar U_{\alpha,\mathcal{D}_m}(t,x)|\mathrm{d}x\mathrm{d}t\le 2\int_0^{T+\delta t}\int_{\Omega'}|\bar U_{\alpha,\mathcal{D}_m}(t+\delta t,x)-\bar U_{\alpha,\mathcal{D}_m}(t,x)|\mathrm{d}x\mathrm{d}t.$$

Since δt tends to zero as size$(\mathcal{D}_m)\to0$, estimate (7.8) in Lemma 9 implies that the right-hand side of the above inequality converges to zero as size$(\mathcal{D}_m)$ tends to zero, and this established (8.1).

By the Riesz-Frechet-Kolmogorov compactness criterion, the relative compactness of $(\bar U_{\alpha,\mathcal{D}_m})_{m\in\mathbb{N}}$ in $L^1(Q_T)$ is a consequence of the Lemmas 8 and 9. Now, the convergence (8.2) in $L^1(Q_T)$ and a.e in Q_T becomes a consequence of (8.1). Due to the fact that $U_{\alpha,\mathcal{D}_m}$ is bounded, we establish the convergence in $L^1(Q_T)$. This ensures the following strong convergences

$$\rho_\alpha(p_{\alpha,\mathcal{D}_m})s_{\alpha,\mathcal{D}_m}\longrightarrow l_\alpha \quad \text{in } L^1(Q_T) \text{ and a.e. in } Q_T\ .$$

Denote by $u_\alpha=\rho_\alpha(p_\alpha)s_\alpha$. Define the map $\mathbb{H}:\mathbb{R}^+\times\mathbb{R}^+\mapsto\mathbb{R}^+\times[0,\mathcal{B}(1)]$ defined by

$$\mathbb{H}(u_l,u_g)=(p,\mathcal{B}(s_l)) \tag{8.9}$$

where u_α are solutions of the system

$$u_l(p,\mathcal{B}(s_l))=\rho_l(p-\bar p(\mathcal{B}^{-1}(\mathcal{B}(s_l))))\mathcal{B}^{-1}(\mathcal{B}(s_l))$$
$$u_g(p,\mathcal{B}(s_l))=\rho_g(p-\tilde p(\mathcal{B}^{-1}(\mathcal{B}(s_l))))(1-\mathcal{B}^{-1}(\mathcal{B}(s_l)).$$

Note that $\mathbb{H}$ is well defined as a diffeomorphism, since

$$\begin{aligned}
\frac{\partial u_l}{\partial p} &= \rho_1'(p-\bar p(\mathcal{B}^{-1}(\mathcal{B}(s_1))))\mathcal{B}^{-1}(\mathcal{B}(s_1))\ge0\\
\frac{\partial u_l}{\partial\mathcal{B}} &= \rho_1'(p-\bar p(\mathcal{B}^{-1}(\mathcal{B}(s_1))))[-\bar p'(\mathcal{B}^{-1}(\mathcal{B}(s_1)))(\mathcal{B}^{-1'}(\mathcal{B}(s_1)))]\mathcal{B}^{-1}(\mathcal{B}(s_1))\\
&+ \rho_1(p-\bar p(\mathcal{B}^{-1}(\mathcal{B}(s_1))))\mathcal{B}^{-1'}(\mathcal{B}(s_1))\ge0\\
\frac{\partial u_g}{\partial p} &= -\rho_2'(p-\tilde p(\mathcal{B}^{-1}(\mathcal{B}(s_1))))(1-\mathcal{B}^{-1}(\mathcal{B}(s_1)))\ge0\\
\frac{\partial u_g}{\partial\mathcal{B}} &= \rho_2'(p-\tilde p(\mathcal{B}^{-1}(\mathcal{B}(s_1))))[-\tilde p'(\mathcal{B}^{-1}(\mathcal{B}(s_1)))(\mathcal{B}^{-1'}(\mathcal{B}(s_1)))][1-\mathcal{B}^{-1}(\mathcal{B}(s_1))]\\
&- \rho_2(p-\tilde p(\mathcal{B}^{-1}(\mathcal{B}(s_1))))\mathcal{B}^{-1'}(\mathcal{B}(s_1))\le0,
\end{aligned}$$

and if one of the saturations is zero the other one is one, this conserves that the jacobian determinant of the map $\mathbb{H}^{-1}$ is strictly negative.

As the map $\mathbb{H}$ defined in (8.9) is continuous, we deduce

$$p_{\mathcal{D}_m} \longrightarrow p \quad \text{a.e. in } Q_T,$$
$$\mathcal{B}(s_{l,\mathcal{D}_m}) \longrightarrow \mathcal{B}^* \quad \text{a.e. in } Q_T.$$

Then, as $\mathcal{B}^{-1}$ is continuous, we deduce

$$s_{l,\mathcal{D}_m} \longrightarrow s_l = \mathcal{B}^{-1}(\mathcal{B}^*) \quad \text{a.e. in } Q_T,$$

and the convergences (8.5) hold.

Consequently and due to the relationship between the pressure of each phase and the global pressure defined in (2.5), then the convergences (8.6) hold

$$p_{\alpha,\mathcal{D}_m} \longrightarrow p_\alpha \quad \text{a.e. in } Q_T.$$

It follows from Proposition 1 that, the sequence $(\nabla_{\mathcal{D}_m} p_{\mathcal{D}_m})_{m\in\mathbb{N}}$ is bounded in $(L^2(Q_T))^\ell$, and as a consequence of the discrete Poincaré inequality, the sequence $(p_{\mathcal{D}_m})_{m\in\mathbb{N}}$ is bounded in $L^2(Q_T)$. Therefore there exist two functions $p \in L^2(Q_T)$ and $\psi \in (L^2(Q_T))^\ell$ such that (8.4) holds and

$$\nabla_{\mathcal{D}_m} p_{\mathcal{D}_m} \longrightarrow \psi \text{ weakly in } (L^2(Q_T))^\ell.$$

It remains to identify ∇p by ψ in the sense of distributions. For that, it is enough to show as $m \to +\infty$:

$$E_m := \int\int_{Q_T} \nabla_{\mathcal{D}_m} p_{\mathcal{D}_m} \cdot \varphi \,\mathrm{d}x\mathrm{d}t + \int\int_{Q_T} p_{\mathcal{D}_m} \mathrm{div}\varphi \,\mathrm{d}x\mathrm{d}t \longrightarrow 0, \quad \forall \varphi \in D(Q_T).$$

Let $\mathcal{D}_m$ be small enough such that φ vanishes in $T^{\mathrm{ext}}_{K,\sigma}$ for all $K \in \mathcal{T}$, then

$$\int_\Omega p_{\mathcal{D}_m} \mathrm{div}\varphi(t,x)\,dx = \sum_{K\in\mathcal{T}} \int_K p_{\mathcal{D}_m} \mathrm{div}\varphi(t,x)\,dx$$
$$= \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} p_K^n \int_{\sigma_{K|L}} \varphi(t,x)\cdot\eta_{K|L}\,\mathrm{d}\Gamma = \frac{1}{2}\sum_{K\in\mathcal{T}} \sum_{L\in N(K)} (p_K^n - p_L^n) \int_{\sigma_{K|L}} \varphi(t,x)\cdot\eta_{K|L}\,\mathrm{d}\Gamma.$$

Now, from the definition of the discrete gradient,

$$\int_\Omega \nabla_{\mathcal{D}_m} p_{\mathcal{D}_m} \varphi(t,x)\,dx = \frac{1}{2}\sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \int_{T_{K|L}} \nabla_{\mathcal{D}_m} p_{\mathcal{D}_m} \varphi(t,x)\,dx$$
$$= \frac{1}{2}\sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{\ell}{d_{K|L}} (p_L^n - p_K^n) \int_{T_{K|L}} \varphi(t,x)\cdot\eta_{K|L}\,dx$$

Then,

$$E_m = \frac{1}{2}\sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \sigma_{K|L}(p_L^n - p_K^n)\Big(\frac{1}{|\sigma_{K|L}|}\int_{\sigma_{K|L}} \varphi(t,x)\cdot\eta_{K|L}\mathrm{d}\Gamma - \frac{1}{|T_{K|L}|}\int_{T_{K|L}} \varphi(t,x)\cdot\eta_{K|L}\,dx\Big)$$

Due to the smoothness of φ, one gets

$$\Big|\frac{1}{|\sigma_{K|L}|}\int_{\sigma_{K|L}}\varphi(t,x)\cdot\eta_{K|L}\mathrm{d}\Gamma-\frac{1}{|T_{K|L}|}\int_{T_{K|L}}\varphi(t,x)\cdot\eta_{K|L}\,dx\Big|\le C\,h,$$

and the Cauchy-Scharwz inequality with the estimate (5.4) in Proposition 1 yield

$$|E_m|\le Ch\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|\sigma_{K|L}||p_L^n-p_K^n|\le Ch\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|\sigma_{K|L}|d_{K|L}\le Ch|\Omega|T.$$

The identification of the limit in (8.8) follows from the previous convergence. □

8.1. **Proof of theorem 1.** Let T be a fixed positive constant and $\varphi\in D([0,T)\times\overline{\Omega})$. Set $\varphi_K^n:=\varphi(t^n,x_K)$ for all $K\in\mathcal{T}$ and $n\in[0,N]$.
For the discrete liquid equation, we multiply the equation (3.8) by $\delta t\varphi_K^{n+1}$ and sum over $K\in\mathcal{T}$ and $n\in\{0,...,N\}$. This yields

$$S_1^m+S_2^m+S_3^m+S_4^m=0,$$

where

$$S_1^m=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|\,\phi_K\left(\rho_l(p_{l,K}^{n+1})s_{l,K}^{n+1}-\rho_l(p_{l,K}^n)s_{l,K}^n\right)\varphi_K^{n+1},$$

$$S_2^m=\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\tau_{K|L}\rho_{l,K|L}^{n+1}G_l(s_{l,K}^{n+1},s_{l,L}^{n+1};\delta_{K|L}^{n+1}(p_l))\varphi_K^{n+1},$$

$$S_3^m=\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|\sigma_{K|L}|\Big((\rho_{l,K|L}^{n+1})^2M_l(s_{l,K}^{n+1})(\mathbf{g}_{K|L})^+-(\rho_{l,K|L}^{n+1})^2M_l(s_{l,L}^{n+1})(\mathbf{g}_{K|L})^-\Big)\varphi_K^{n+1},$$

$$S_4^m=\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|K|\left(\rho_l(p_{l,K}^{n+1})s_{l,K}^{n+1}f_{P,K}^{n+1}\varphi_K^{n+1}-\rho_l(p_{l,K}^{n+1})(s_{l,K}^I)^{n+1}f_{I,K}^{n+1}\varphi_K^{n+1}\right).$$

Making summation by parts in time and keeping in mind that $\varphi(T,x_K)=\varphi_K^{N+1}=0$. For all $K\in\mathcal{T}$, we get

$$\begin{aligned}S_1^m&=-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|\,\phi_K\rho_l(p_{l,K}^{n+1})s_{l,K}^{n+1}\left(\varphi_K^{n+1}-\varphi_K^n\right)-\sum_{K\in\mathcal{T}_h}|K|\,\phi_K\rho_l(p_{l,K}^0)s_{l,K}^0\varphi_K^0\\&=-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\int_{t^n}^{t^{n+1}}\int_K\phi_K\rho_l(p_{l,K}^{n+1})s_{l,K}^{n+1}\partial_t\varphi(t,x_K)\mathrm{d}x\mathrm{d}t-\sum_{K\in\mathcal{T}_h}\int_K\phi_K\rho_l(p_{l,K}^0)s_{l,K}^0\varphi(0,x_K)\mathrm{d}x.\end{aligned}$$

Since $\phi_{\mathcal{D}_m}\rho_l(p_{l,\mathcal{D}_m})s_{l,\mathcal{D}_m}$ and $\phi_{\mathcal{D}_m}\rho_l(p_{l,\mathcal{D}_m}^0)s_{l,\mathcal{D}_m}^0$ converge almost everywhere respectively to $\phi\rho_l(p_l)s_l$ and $\phi\rho_l(p_l^0)s_l^0$, and as a consequence of Lebesgue dominated convergence theorem, we get

$$\lim_{m\to+\infty}S_1^m=\int_{Q_T}\phi\rho_l(p_l)s_l\partial_t\varphi(t,x)\mathrm{d}x\mathrm{d}t-\int_\Omega\phi\rho_l(p_l^0)s_l^0\varphi(0,x)\mathrm{d}x.$$

Now, let us focus on convergence of the degenerate diffusive term to show

$$\lim_{m\to+\infty}S_2^m=-\int_{Q_T}\rho_l(p_l)M_l(s_l)\nabla p_l\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t. \tag{8.10}$$

Since the discrete gradient of each phase is not bounded, it is not possible to justify the pass to the limit in a straightforward way. To do this, we use the feature of global pressure and the auxiliary pressures defined in (2.5) and the discrete energy estimates in proposition 1 and corollary 1.

Gathering by edges, the term S_2^m can be rewritten as:

$$\begin{aligned} S_2^m &= -\frac{1}{2}\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\tau_{K|L}\rho_{l,K|L}^{n+1}G_l(s_{l,K}^{n+1},s_{l,L}^{n+1};\delta_{K|L}^{n+1}(p_l))\left(\varphi(t^{n+1},x_L)-\varphi(t^{n+1},x_K)\right)\\ &= \frac{1}{2}\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\tau_{K|L}\rho_{l,K|L}^{n+1}M_l(s_{l,K|L}^{n+1})\delta_{K|L}^{n+1}(p_l)\left(\varphi(t^{n+1},x_L)-\varphi(t^{n+1},x_K)\right)\\ &= A_1^m + A_2^m, \end{aligned}$$

with, by using the definition (2.5),

$$\begin{aligned} A_1^m &= \frac{1}{2}\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\tau_{K|L}\rho_{l,K|L}^{n+1}M_l(s_{l,K|L}^{n+1})\delta_{K|L}^{n+1}(p)\left(\varphi(t^{n+1},x_L)-\varphi(t^{n+1},x_K)\right),\\ A_2^m &= -\frac{1}{2}\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\tau_{K|L}\rho_{l,K|L}^{n+1}M_l(s_{l,K|L}^{n+1})\delta_{K|L}^{n+1}(\bar{p}(s_l))\left(\varphi(t^{n+1},x_L)-\varphi(t^{n+1},x_K)\right). \end{aligned}$$

Let us show that

$$\lim_{m\to+\infty} A_1^m = \int_{Q_T}\rho_l(p_l)M_l(s_l)\nabla p\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t. \tag{8.11}$$

For each couple of neighbours K and L we denote $s_{l,min}^{n+1}$ the minimum of $s_{l,K}^{n+1}$ and $s_{l,L}^{n+1}$ and we introduce

$$A_1^{m,*} = \frac{1}{2}\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\tau_{K|L}\rho_{l,K|L}^{n+1}M_l(s_{l,min}^{n+1})\delta_{K|L}^{n+1}(p)\left(\varphi(t^{n+1},x_L)-\varphi(t^{n+1},x_K)\right)$$

Remark that

$$\begin{aligned} A_1^{m,*} &= \frac{1}{2}\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\ell|T_{K|L}|\rho_{l,K|L}^{n+1}M_l(s_{l,min}^{n+1})\frac{p_L-p_K}{d_{K|L}}\frac{\varphi(t^{n+1},x_L)-\varphi(t^{n+1},x_K)}{d_{K|L}}\\ &= \frac{1}{2}\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|T_{K|L}|\rho_{l,K|L}^{n+1}M_l(s_{l,min}^{n+1})\nabla_{K|L}p_{\mathcal{D}_m}\cdot\eta_{K|L}\nabla\varphi(t^{n+1},x_{K|L})\cdot\eta_{K|L}, \end{aligned}$$

where $x_{K|L} = \theta x_K + (1-\theta)x_L$, $0<\theta<1$, is some point on the segment $]x_K, x_L[$. Recall that the value of $\nabla_{K|L}$ is directed by $\eta_{K|L}$, so

$$\nabla_{K|L}p_{\mathcal{D}_m}\cdot\eta_{K|L}\nabla\varphi(t^{n+1},x_{K|L})\cdot\eta_{K|L} = \nabla_{K|L}p_{\mathcal{D}_m}\cdot\nabla\varphi(t^{n+1},x_{K|L})$$

Define $\overline{s}_{\alpha,\mathcal{D}_m}$ and $\underline{s}_{\alpha,\mathcal{D}_m}$ by

$$\overline{s}_{\alpha,\mathcal{D}_m}|_{(t^n,t^{n+1}]\times T_{K|L}} := \max\{s_{\alpha,K}, s_{\alpha,L}\},\quad \underline{s}_{\alpha,\mathcal{D}_m}|_{(t^n,t^{n+1}]\times T_{K|L}} := \min\{s_{\alpha,K}, s_{\alpha,L}\}$$

Now, $A_1^{m,*}$ can be written under the following continues form

$$A_1^{m,*} = \int_0^T \int_\Omega \rho_l(p_{l,\mathcal{D}_m}) M_l(\underline{s}_{l,\mathcal{D}_m}) \nabla_{\mathcal{D}_m} p_{\mathcal{D}_m} \cdot (\nabla\varphi)_{\mathcal{D}_m} \mathrm{d}x\mathrm{d}t.$$

By the monotonicity of $\mathcal{B}$ and thanks to the estimate (5.13), we have

$$\begin{aligned}\int_0^T \int_\Omega \left|\mathcal{B}(\overline{s}_{l,\mathcal{D}_m}) - \mathcal{B}(\underline{s}_{l,\mathcal{D}_m})\right|^2 \mathrm{d}x\mathrm{d}t &\leq \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} |T_{K|L}| \left(\mathcal{B}(s_{l,L}^{n+1}) - \mathcal{B}(s_{l,K}^{n+1})\right)^2 \\ &\leq C\mathrm{size}(\mathcal{T})^2 \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K|L}|}{d_{K|L}} \left|\mathcal{B}(s_{l,L}^{n+1}) - \mathcal{B}(s_{l,K}^{n+1})\right|^2 \\ &\leq C\mathrm{size}(\mathcal{T})^2.\end{aligned}$$

Since $\mathcal{B}^{-1}$ is continuous, we deduce up to a subsequence

$$\left|\underline{s}_{\alpha,\mathcal{D}_m} - \overline{s}_{\alpha,\mathcal{D}_m}\right| \to 0 \text{ a.e. on } Q_T. \tag{8.12}$$

Moreover, we have $\underline{s}_{\alpha,\mathcal{D}_m} \leq s_{\alpha,\mathcal{D}_m} \leq \overline{s}_{\alpha,\mathcal{D}_m}$ and $s_{\alpha,\mathcal{D}_m} \to s_\alpha$ a.e. on Q_T. Consequently, and due to the continuity of the mobility function M_l we have $M_l(\underline{s}_{l,\mathcal{D}_m}) \to M_l(s_l)$ a.e on Q_T and in $L^p(Q_T)$ for $p < +\infty$.

As consequence of the convergence (8.6) and by the Lebesgue dominated convergence theorem we get

$$\rho_l(p_{l,\mathcal{D}_m}) M_l(\underline{s}_{l,\mathcal{D}_m}) (\nabla\varphi)_{\mathcal{D}_m} \to \rho_l(p_l) M_l(s_l) \nabla\varphi \text{ strongly in } (L^2(Q_T))^\ell.$$

And as consequence of the weak convergence on global pressure (8.4), we obtain that

$$\lim_{m\to+\infty} A_1^{m,*} = \int_{Q_T} \rho_l(p_l) M_l(s_l) \nabla p \cdot \nabla\varphi \,\mathrm{d}x\mathrm{d}t.$$

It remains to show that

$$\lim_{m\to+\infty} \left|A_1^m - A_1^{m,*}\right| = 0. \tag{8.13}$$

Remark that

$$\left|M_l(s_{l,K|L}^{n+1}) \delta_{K|L}^{n+1}(p) - M_l(s_{l,min}^{n+1}) \delta_{K|L}^{n+1}(p)\right| \leq C \left|s_{l,L}^{n+1} - s_{l,K}^{n+1}\right| \left|\delta_{K|L}^{n+1}(p)\right|.$$

Consequently

$$\left|A_1^m - A_1^{m,*}\right| \leq C \int_{Q_T} \left|s_{l,L}^{n+1} - s_{l,K}^{n+1}\right| \nabla_{\mathcal{D}_m} p_{\mathcal{D}_m} \cdot (\nabla\varphi)_{\mathcal{D}_m} \mathrm{d}x\mathrm{d}t.$$

Applying the Cauchy-Schwarz inequality, and thanks to the uniform bound on $\nabla_{\mathcal{D}_m} p_{\mathcal{D}_m}$ and the convergence (8.12), we establish (8.13).

To prove the pass to limit of A_2^m, we need to prove firstly that

$$\|\delta_{K|L}^{n+1}(\Gamma(s_l)) - \sqrt{M_l(s_{l,K|L}^{n+1})} \delta_{K|L}^{n+1}(\bar{p}(s_l))\|_{L^2(Q_T)} \to 0 \text{ as size}(\mathcal{T}) \to 0,$$

where $\Gamma(s_l) = \int_0^{s_l} \sqrt{M_l(z)} \frac{\mathrm{d}\bar{p}}{\mathrm{d}s_l}(z)\mathrm{d}z$.
In fact. Remark that there exist $a \in [s_{l,K}, s_{l,L}]$ such as:

$$|\delta_{K|L}^{n+1}(\Gamma(s_l)) - \sqrt{M_l(s_{l,K,L}^{n+1})} \delta_{K|L}^{n+1}(\bar{p}(s_l))| = |\sqrt{M_l(a)} - \sqrt{M_l(s_{l,K,L}^{n+1})}| |\delta_{K|L}^{n+1}(\bar{p}(s_l))|$$

$$\leq C|\delta^{n+1}_{K|L}(\bar{p}(s_l))| \leq C\left|s^{n+1}_{l,L} - s^{n+1}_{l,K}\right|$$
$$\leq C\left|\mathcal{B}(s^{n+1}_{l,L}) - \mathcal{B}(s^{n+1}_{l,K})\right|^{\theta},$$

since $\mathcal{B}^{-1}$ is an Hölder function. Thus we get,

$$\|\delta^{n+1}_{K|L}(\Gamma(s_l)) - \sqrt{M_l(s^{n+1}_{l,K,L})}\delta^{n+1}_{K|L}(\bar{p}(s_l))\|^2_{L^2(Q_T)}$$
$$= \sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|T_{K,L}||\delta^{n+1}_{K|L}(\Gamma(s_l)) - \sqrt{M_l(s^{n+1}_{l,K,L})}\delta^{n+1}_{K|L}(\bar{p}(s_l))|^2$$
$$\leq \sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|T_{K,L}|^{1-\theta}|T_{K,L}|^{\theta}\left|\mathcal{B}(s^{n+1}_{l,L}) - \mathcal{B}(s^{n+1}_{l,K})\right|^{2\theta},$$

and using the Cauchy-Schwarz inequality and the estimate , we deduce

$$\|\delta^{n+1}_{K|L}(\Gamma(s_l)) - \sqrt{M_l(s^{n+1}_{l,K,L})}\delta^{n+1}_{K|L}(\bar{p}(s_l))\|^2_{L^2(Q_T)}$$
$$\leq \left(\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|T_{K,L}|\right)^{1-\theta}\left(\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|T_{K,L}|\left|\mathcal{B}(s^{n+1}_{l,L}) - \mathcal{B}(s^{n+1}_{l,K})\right|^2\right)^{\theta}$$
$$\leq C(\mathrm{size}(\mathcal{T}))^{2\theta}\left(\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K|L}}\left|\mathcal{B}(s^{n+1}_{l,L}) - \mathcal{B}(s^{n+1}_{l,K})\right|\right)^{\theta}$$

which shows that $\|\delta^{n+1}_{K|L}(\Gamma(s_l)) - \sqrt{M_l(s^{n+1}_{l,K|L})}\delta^{n+1}_{K|L}(\bar{p}(s_l))\|^2_{L^2(Q_T)} \to 0$ as $\mathrm{size}(\mathcal{T}) \to 0$. And from (5.14) in corollary 1, we deduce that there exists a constant $C > 0$ where the following inequalities hold:

$$\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\tau_{K|L}(\delta^{n+1}_{K|L}(\Gamma(s_l)))^2 \leq C. \tag{8.14}$$

That prove

$$\nabla_{\mathcal{D}_m}\Gamma(s_{l,\mathcal{D}_m}) \to \nabla\Gamma(s_l) \text{ weakly in } (L^2(Q_T))^{\ell}. \tag{8.15}$$

As consequence

$$\sqrt{M_l(s_{l,\mathcal{D}_m})}\nabla_{\mathcal{D}_m}\bar{p}(s_{l,\mathcal{D}_m}) \to \nabla\Gamma(s_l) \text{ weakly in } (L^2(Q_T))^{\ell}. \tag{8.16}$$

Rearranging A^m_2 to write

$$A^m_2 = -\frac{1}{2}\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\left|T_{K|L}\right|\rho^{n+1}_{l,K|L}M_l(s^{n+1}_{l,K|L})\nabla_{K|L}\bar{p}(s_{l,\mathcal{D}_m})\cdot\eta_{K|L}\nabla\varphi(t^{n+1},x_{K|L})\cdot\eta_{K|L},$$

where $x_{K|L} = \theta x_K + (1-\theta)x_L$, $0 < \theta < 1$, is some point on the segment $]x_K, x_L[$. using again that the mesh is orthogonal, we can write

$$A^m_2 = -\int_0^T\int_\Omega \rho_l(p_{l,\mathcal{D}_m})M_l(s_{l,\mathcal{D}_m})\nabla_{\mathcal{D}_m}\bar{p}(s_{l,\mathcal{D}_m})\cdot(\nabla\varphi)_{\mathcal{D}_m}\mathrm{d}x\mathrm{d}t.$$

As a consequence of the convergences (8.5), (8.6) and by the Lebesgue theorem we get

$$\rho_l(p_{l,\mathcal{D}_m})\sqrt{M_l(s_{l,\mathcal{D}_m})}(\nabla\varphi)_{\mathcal{D}_m} \to \rho_l(p_l)\sqrt{M_l(s_l)}\nabla\varphi \text{ strongly in } (L^2(Q_T))^{\ell}.$$

And as consequence of (8.16),

$$\lim_{m\to+\infty} A_2^m = -\int_0^T \int_\Omega \rho_l(p_l)\sqrt{M_l(s_l)}\nabla\Gamma(s_l)\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t \tag{8.17}$$

$$= -\int_0^T \int_\Omega \rho_l(p_l)M_l(s_l)\nabla\bar{p}(s_l)\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t. \tag{8.18}$$

Now, we treat the convergence of the gravity term

$$\lim_{m\to+\infty} S_3^m = -\int_0^T \int_\Omega \rho_l(p_l)M_l(s_l)\mathbf{g}\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t. \tag{8.19}$$

Perform integration by parts (3.3)

$$\begin{aligned} S_3^m &= \sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} F_{l,K|L}^{n+1}\varphi(t^{n+1},x_K) \\ &= -\frac{1}{2}\sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} F_{l,K|L}^{n+1}\left(\varphi(t^{n+1},x_L)-\varphi(t^{n+1},x_K)\right). \end{aligned}$$

Note that the numerical flux $F_{l,K|L}^{n+1}$ is independent of the gradient of pressures and the pass to the limit on S_3^m is mush simple then the term $A_1^{m,*}$ since the discrete gradient of global pressure is replaced by the gravity vector $\mathbf{g}$. We omit this proof of (8.19).

Finally, S_4^m can be written equivalently

$$\begin{aligned} S_4^m =& \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\int_{t^n}^{t^{n+1}}\int_K \rho_l(p_{l,K}^{n+1})s_{l,K}^{n+1}f_P(t,x)\varphi(t^{n+1},x_K)\mathrm{d}x\mathrm{d}t \\ &-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\int_{t^n}^{t^{n+1}}\int_K \rho_l(p_{l,K}^{n+1})(s_{l,K}^{I})^{n+1}f_I(t,x)\varphi(t^{n+1},x_K)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

From the convergences (8.5), (8.6) and by the Lebesgue dominated convergence theorem, we get

$$\lim_{m\to+\infty} S_4^{m,*} = \int_{Q_T}\rho_l(p_l)s_lf_P(t,x)\varphi(t,x)\mathrm{d}x\mathrm{d}t - \int_{Q_T}\rho_l(p_l)s_l^If_I(t,x)\varphi(t,x)\mathrm{d}x\mathrm{d}t,$$

which completes the proof of the theorem 1.

9. Numerical results

In this section we show some numerical experiments simulating the five spot problem in petroleum engineering. A Newton algorithm is implemented to approach the solution of nonlinear system (3.8)-(3.9) coupled with a bigradient method to solve linear system arising from the Newton algorithm process.

We will provide two tests made on a nonuniform admissible grid.

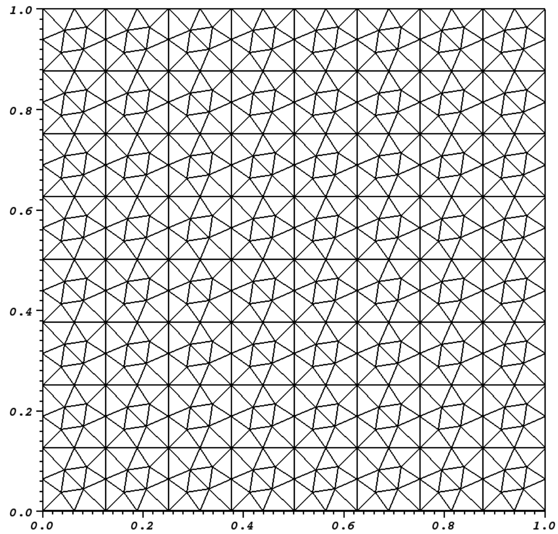

FIGURE 2. Mesh with 896 triangles

Datas used for the numerical tests are the following :

$k_1(s_1) = s_1^2$, $k_2(s_2) = s_2^2$
$\mathbf{K} = 0.15 10^{-10}\mathrm{m}^2$, $\phi = 0.206$,
$\mu_2 = 10^{-3}$ Pa.s(water viscosity), $\mu_1 = 9 10^{-5}$ Pa.s(gas viscosity),
$\rho(p) = \rho_{ref}(1 + c_{ref}(p - p_{ref}))$, with $\rho_{ref} = 400$ Kg m^{-3}, $c_{ref} = 10^{-6}$Pa^{-1}, $p_{ref} = 1.013\,10^5$ Pa,
$L_x = 1$m, $L_y = 1$m (the length and the width of the domain)
$P_c(s) = P_{max}(1 - s)$, with $P_{max} = 10^5$Pa.

Initial conditions. Initially the saturation of gas is considered to be equal to 0.9 in the whole domain and the gas pressure is considered to be $1.013\,10^5$ Pa.

Boundary conditions. The wetting fluid (water) is injected in the left-down corner in the region $([0, 0.1] \times \{0\}) \cup (\{0\} \times [0, 0.1])$ with a constant pressure equal to $4.026\,10^5$ Pa. The right-top corner where $([0.9, 1] \times \{1\}) \cup (\{1\} \times [0.9, 1])$ keeps fluids flow freely at atmospheric pressure where as the rest of the boundary is assumed to be impervious (zero fluxes are imposed). The influence of boundary conditions can be seen in all figures.

Meshes. The domain is recovered by 896 admissible triangles see figure 2.

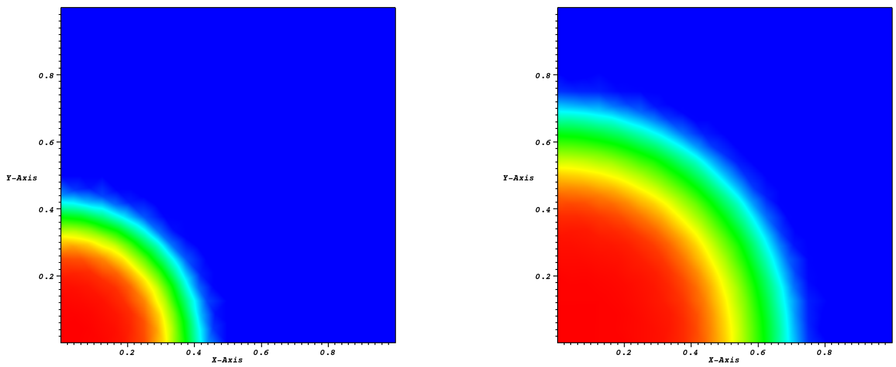

FIGURE 3. Water field including capillary effect at time $T = 6$s (left) and at time $T = 20$s with $0.1 \leq s \leq 1$.

Figures 3 - 6 show the diffusive effects of the capillary terms, notably the dissipation of chocs due to the hyperbolic operator Fig. 6. In fact, during the stage of the displacement saturation shock propagate through rock for flows where capillarity terms are neglected, see figure 6. This shock, where capillarity effects are signifiant, it is diffused. However, a part of the the shock wave maintains its sharp front.

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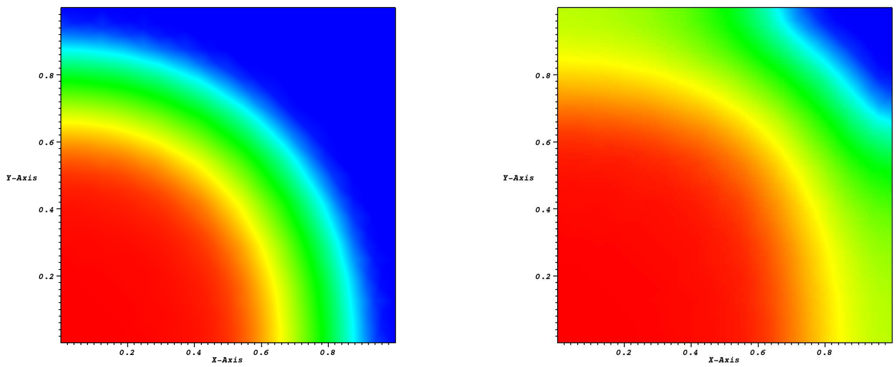

FIGURE 4. Water field including capillary effect at time $T = 35$s (left) and at time $T = 60$s with $0.1 \leq s \leq 1$.

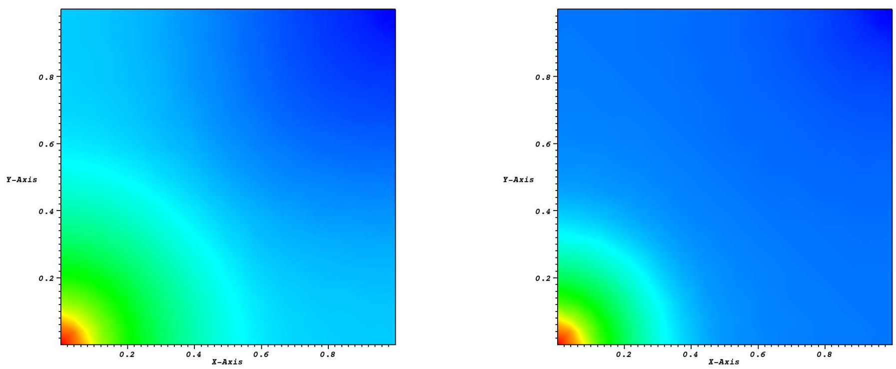

FIGURE 5. Gas pressure field including capillary effect at time $T = 35$ (left) and at time $T = 60$ with $1.013 \times 10^5 \leq P \leq 4 \times 10^5$ Pa.

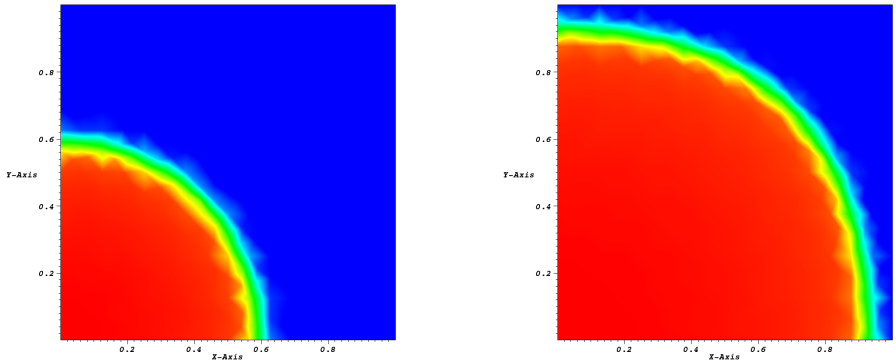

FIGURE 6. Water field without capillary terms at time $T = 35$ (left) and at time $T = 60$ with $0.1 \leq s \leq 1$

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