TIME-INCONSISTENT STOCHASTIC LINEAR–QUADRATIC CONTROL

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Abstract. In this paper, we formulate a general time-inconsistent stochastic linear–quadratic (LQ) control problem. The time-inconsistency arises from the presence of a quadratic term of the expected state as well as a state-dependent term in the objective functional. We define an equilibrium, instead of optimal, solution within the class of open-loop controls, and derive a sufficient condition for equilibrium controls via a flow of forward–backward stochastic differential equations. When the state is one dimensional and the coefficients in the problem are all deterministic, we find an explicit equilibrium control. As an application, we then consider a mean-variance portfolio selection model in a complete financial market where the risk-free rate is a deterministic function of time but all the other market parameters are possibly stochastic processes. Applying the general sufficient condition, we obtain explicit equilibrium strategies when the risk premium is both deterministic and stochastic.

Key words. time inconsistency, stochastic LQ control, equilibrium control, forward–backward stochastic differential equation, mean–variance portfolio selection.

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1. Introduction.

Stochastic control is now a mature and well established subject of study [8, 19]. Though not explicitly stated at most of the times, a standing assumption in the study of stochastic control is the time consistency, a fundamental property of conditional expectation with respect to a progressive filtration. As a result, an optimal control viewed from today will remain optimal viewed from tomorrow. Time-consistency provides the theoretical foundation of the dynamic programming approach including the resulting HJB equation, which is in turn a pillar of the modern stochastic control theory.

However, there are overwhelmingly more time-inconsistent problems than their time-consistent counterparts. Hyperbolic discounting [1, 14] and continuous-time mean–variance portfolio selection model [20, 2] provide two well-known examples of time-inconsistency. Probability distortion, as in behavioral finance models [11], is yet another distinctive source of time-inconsistency.

One way to get around the time-inconsistency issue is to consider only pre-committed controls (i.e., the controls are optimal only when viewed at the initial time); see, e.g., [20] and all the follow-up works to date on the Markowitz problem, as well as [11] on the behavioral portfolio choice problem. While these controls are of practical and theoretical value, they have not really addressed the time-inconsistency nor provided solutions in a dynamic sense.

Motivated by practical applications especially in mathematical finance, time-

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inconsistent control problems have recently attracted considerable research interest and efforts attempting to seek equilibrium, instead of optimal, controls. At a conceptual level, the idea is that a decision the controller makes at every instant of time is considered as a game against all the decisions the future incarnations of the controller are going to make. An "equilibrium" control is therefore one such that any deviation from it at any time instant will be worse off. Taking this game perspective, Ekeland and Lazrak [6] approach the (deterministic) time-inconsistent optimal control, and Björk and Murgoci [4] and Björk, Murgoci and Zhou [5] extend the idea to the stochastic setting, derive an (albeit very complicated) HJB equation, and apply the theory to a dynamic Markowitz problem. Yong [18] investigate a time-inconsistent deterministic linear–quadratic control problem and derive equilibrium controls via some integral equations. However, study of time-inconsistent control is, in general, still in its infancy.

In this paper we formulate a general stochastic linear–quadratic (LQ) control problem, where the objective functional includes both a quadratic term of the expected state and a state-dependent term. These non-standard terms each introduces time-inconsistency into the problem in somewhat different ways. Different from most of the existing literature [6, 4, 5, 18] where an equilibrium control is defined within the class of *feedback* controls, we define our equilibrium via open-loop controls.[1] Then we derive a general sufficient condition for equilibriums through a system of forward–backward stochastic differential equations (FBSDEs). A intriguing feature of these FBSDEs is that a time parameter is involved; so these form a *flow* of FBSDEs. When the state process is scalar valued and all the coefficients are deterministic functions of time, we are able to reduce this flow of FBSDEs into several Riccati-like ODEs, and hence obtain explicitly an equilibrium control, which turns out to be a linear feedback.

In the latter part of the paper, we study a continuous-time mean–variance portfolio selection model with state dependent trade-off between mean and variance. A similar problem was first considered in [5] in the framework of feedback controls and its solution derived via a very complicated (generalized) HJB equation. Here we allow random market parameters (hence the model and approach of [5] will not work) and consider open-loop equilibriums. Applying the general sufficient condition and working through a delicate analysis, we will solve the corresponding FBSDEs and obtain equilibrium strategies. Again, these strategies happen to be linear feedbacks. We also compare our strategies with the ones in [5] when all the market coefficients are deterministic, and find that they are generally different. This suggests that how we define equilibrium controls is critical in studying time inconsistent control problems.

The remainder of the paper is organized as follows. The next section is devoted to the formulation of our problem and the definition of equilibrium control. In Section 3, we apply the spike variation technique to derive a flow of FBSDEs and a sufficient condition of equilibrium controls. Based on this general result, we solve in Section 4 the case when the state is one dimensional and all the coefficients are deterministic. In Section 5, we formulate a continuous-time mean–variance portfolio selection model which is a special case of the general LQ model investigated, and derive explicitly its solution. Finally, some concluding remarks are given in Section 6.

2. Problem Setting. Let $T > 0$ be the end of a finite time horizon and $(W_t)_{0\le t\le T} = (W_t^1, \cdots, W_t^d)_{0\le t\le T}$ a d-dimensional Brownian motion on a probability

[1] Recall the class of feedback controls is a subset of that of open-loop ones. In standard (time-consistent) stochastic control theory, an optimal control is usually defined in the whole class of open-loops [8, 19].

space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $(\mathcal{F}_t)$ the augmented filtration generated by (W_t).

Throughout this paper, we use the following notation with l being a generic integer:

$\mathbb{S}^l$: the set of symmetric $l \times l$ real matrices.

$L^2_{\mathcal{G}}(\Omega; \mathbb{R}^l)$: the set of random variables $\xi : (\Omega, \mathcal{G}) \to (\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))$ with $\mathbb{E}\left[|\xi|^2\right] < +\infty$.

$L^\infty_{\mathcal{G}}(\Omega; \mathbb{R}^l)$: the set of essentially bounded random variables $\xi : (\Omega, \mathcal{G}) \to (\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))$.

$L^2_{\mathcal{G}}(t, T; \mathbb{R}^l)$: the set of $\{\mathcal{G}_s\}_{s\in[t,T]}$-adapted processes $f = \{f_s : t \le s \le T\}$ with $\mathbb{E}\left[\int_t^T |f_s|^2\, ds\right] < \infty$.

$L^\infty_{\mathcal{G}}(t, T; \mathbb{R}^l)$: the set of essentially bounded $\{\mathcal{G}_s\}_{s\in[t,T]}$-adapted processes.

$L^2_{\mathcal{G}}(\Omega; C(t, T; \mathbb{R}^l))$: the set of continuous $\{\mathcal{G}_t\}_{s\in[t,T]}$-adapted processes $f = \{f_s : t \le s \le T\}$ with $\mathbb{E}\left[\sup_{s\in[t,T]} |f_s|^2\right] < \infty$.

We will often use vectors and matrices in this paper, where all vectors are column vectors. For a matrix M, define

M': Transpose of a matrix M.

$|M| = \sqrt{\sum_{i,j} m_{ij}^2}$: Frobenius norm of a matrix M.

For a square matrix M, we define $\mathcal{S}(M) = \frac{1}{2}(M + M')$ as the symmetrization of M, and $\mathrm{tr}(M) = \sum_i M_{ii}$ as the trace of M. For a symmetric matrix M, we write $M \succeq 0$ if M is positive semi-definite, and $M \succ 0$ if M is positive definite.

We consider a continuous-time, n-dimensional non-homogeneous linear controlled system

$$dX_s = [A_s X_s + B_s' u_s + b_s]ds + \sum_{j=1}^{d} [C_s^j X_s + D_s^j u_s + \sigma_s^j] dW_s^j; \quad X_0 = x_0. \tag{2.1}$$

Here A is a bounded deterministic function on $[0,T]$ with value in $\mathbb{R}^{n\times n}$. The other parameters B, C^j, D^j are all essentially bounded adapted processes on $[0,T]$ with values in $\mathbb{R}^{l\times n}$, $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times l}$, respectively; b and σ^j are stochastic processes in $L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$. The process $u \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^l)$ is the control, and X is the state process valued in $\mathbb{R}^n$. Finally $x_0 \in \mathbb{R}^n$ is the initial state. It is obvious that for any control $u \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^l)$, there exists a unique solution $X \in L^2_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}^n))$.

As time evolves, we need to consider the controlled system starting from time $t \in [0,T]$ and state $x_t \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$:

$$dX_s = [A_s X_s + B_s' u_s + b_s]ds + \sum_{j=1}^{d} [C_s^j X_s + D_s^j u_s + \sigma_s^j] dW_s^j; \quad X_t = x_t. \tag{2.2}$$

For any control $u \in L^2_{\mathcal{F}}(t,T;\mathbb{R}^l)$, there exists a unique solution $X^{t,x_t,u} \in L^2_{\mathcal{F}}(\Omega; C(t,T;\mathbb{R}^n))$.

At any time t with the system state $X_t = x_t$, our aim is to minimize

$$J(t, x_t; u) \triangleq \frac{1}{2}\mathbb{E}_t \int_t^T [\langle Q_s X_s, X_s\rangle + \langle R_s u_s, u_s\rangle] ds + \frac{1}{2}\mathbb{E}_t[\langle G X_T, X_T\rangle] \\ -\frac{1}{2}\langle h\mathbb{E}_t\left[X_T\right], \mathbb{E}_t\left[X_T\right]\rangle - \langle \mu_1 x_t + \mu_2, \mathbb{E}_t\left[X_T\right]\rangle \tag{2.3}$$

over $u \in L^2_{\mathcal{F}}(t, T; \mathbb{R}^l)$, where $X = X^{t,x_t,u}$, and $\mathbb{E}_t\left[\cdot\right] = \mathbb{E}\left[\cdot|\mathcal{F}_t\right]$. Here Q and R are both given essentially bounded adapted processes on $[0, T]$ with values in $\mathbb{S}^n$ and $\mathbb{S}^l$ respectively, G, h, μ_1, μ_2 are all constants in $\mathbb{S}^n$, $\mathbb{S}^n$, $\mathbb{R}^{n\times n}$ and $\mathbb{R}^n$ respectively. Throughout this paper, we assume that $Q \succeq 0$, $R \succeq 0$ a.s.,a.e., and $G \succeq 0$.

The first two terms in the cost functional (2.3) are standard in a classical LQ control problem, whereas the last two are unconventional. Specifically, the term $-\frac{1}{2}\langle h\mathbb{E}_t[X_T], \mathbb{E}_t\left[X_T\right]\rangle$ is motivated by the variance term in a mean–variance portfolio choice model [9, 20], and the last term, $-\langle \mu_1 x_t + \mu_2, \mathbb{E}_t\left[X_T\right]\rangle$, which depends on the state x_t at time t, stems from a state-dependent utility function in economics [5].

Each of these two terms introduces time-inconsistency of the underlying model in somewhat different ways. With the time-inconsistency, the notion "optimality" needs to be defined in an appropriate way. Here we adopt the concept of equilibrium solution, which is, for any $t \in [0, T)$, optimal only for spike variation in an infinitesimal way.

Given a control u^*. For any $t \in [0, T)$, $\varepsilon > 0$ and $v \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^l)$, define

$$u_s^{t,\varepsilon,v} = u_s^* + v\mathbf{1}_{s\in[t,t+\varepsilon)}, \quad s \in [t, T]. \tag{2.4}$$

Definition 2.1. *Let $u^* \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^l)$ be a given control and X^* be the state process corresponding to u^*. The control u^* is called an equilibrium if*

$$\lim_{\varepsilon\downarrow 0} \frac{J(t, X_t^*; u^{t,\varepsilon,v}) - J(t, X_t^*; u^*)}{\varepsilon} \geq 0,$$

where $u^{t,\varepsilon,v}$ is defined by (2.4), for any $t \in [0, T)$ and $v \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^l)$.

Notice that an equilibrium control here is defined in the class of *open-loop* controls, which is different from the one in [2], [4], [5],[6] and [7] where only *feedback* controls are considered. In our definition, the perturbation of the control in $[t, t + \varepsilon)$ will not change the control process in $[t + \varepsilon, T]$, whereas it is not the case with feedback controls.

In this paper, we will characterize equilibriums in general case and identify them in some special cases including that of the mean–variance portfolio selection.

3. Sufficient Condition of Equilibrium Controls. In this section we present a general sufficient condition for equilibriums. We derive this condition by the second-order expansion in the spike variation, in the same spirit of proving the stochastic Pontryagin's maximum principle [16, 19].

Let u^* be a fixed control and X^* be the corresponding state process. For any $t \in [0, T)$, define in the time interval $[t, T]$ the processes $(p(\cdot; t), (k^j(\cdot; t))_{j=1,\cdots,d}) \in L^2_{\mathcal{F}}(t, T; \mathbb{R}^n)\times(L^2_{\mathcal{F}}(t, T; \mathbb{R}^n))^d$ and $(P(\cdot; t), (K^j(\cdot; t))_{j=1,\cdots,d}) \in L^2_{\mathcal{F}}(t, T; \mathbb{S}^n)\times(L^2_{\mathcal{F}}(t, T; \mathbb{S}^n))^d$

as the solutions to the following equations:

$$\begin{cases} dp(s;t) = -[A_s' p(s;t) + \sum_{j=1}^d (C_s^j)' k^j(s;t) + Q_s X_s^*] ds \\ \qquad\qquad + \sum_{j=1}^d k^j(s;t) dW_s^j, \quad s \in [t,T], \\ p(T;t) = G X_T^* - h\mathbb{E}_t [X_T^*] - \mu_1 X_t^* - \mu_2; \end{cases} \tag{3.1}$$

$$\begin{cases} dP(s;t) = -\Big\{ A_s' P(s;t) + P(s;t) A_s \\ \qquad\qquad + \sum_{j=1}^d [(C_s^j)' P(s;t) C_s^j + (C_s^j)' K^j(s;t) + K^j(s;t) C_s^j] + Q_s \Big\} ds \\ \qquad\qquad + \sum_{j=1}^d K^j(s;t) dW_s^j, \quad s \in [t,T], \\ P(T;t) = G. \end{cases} \tag{3.2}$$

Note that for each fixed $t \in [0,T]$, the above equations are backward stochastic differential equations (BSDEs). So these essentially form a flow of BSDEs. From the assumption that $Q \succeq 0$ and $G \succeq 0$, it follows that $P(s;t) \succeq 0$.

Proposition 3.1. *For any $t \in [0,T)$, $\varepsilon > 0$ and $v \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^l)$, define $u^{t,\varepsilon,v}$ by (2.4). Then*

$$J(t, X_t^*; u^{t,\varepsilon,v}) - J(t, X_t^*; u^*) = \mathbb{E}_t \int_t^{t+\varepsilon} \left\{ \langle \Lambda(s;t), v \rangle + \frac{1}{2} \langle H(s;t) v, v \rangle \right\} ds + o(\varepsilon) \tag{3.3}$$

where $\Lambda(s;t) \triangleq B_s p(s;t) + \sum_{j=1}^d (D_s^j)' k^j(s;t) + R_s u_s^*$ *and* $H(s;t) \triangleq R_s + \sum_{j=1}^d (D_s^j)' P(s;t) D_s^j$.

Proof. Let $X^{t,\varepsilon,v}$ be the state process corresponding to $u^{t,\varepsilon,v}$. Then by the standard perturbation approach (see, e.g., [19]), we have

$$X_s^{t,\varepsilon,v} = X_s^* + Y_s^{t,\varepsilon,v} + Z_s^{t,\varepsilon,v}, \quad s \in [t,T],$$

where $Y \equiv Y^{t,\varepsilon,v}$ and $Z \equiv Z^{t,\varepsilon,v}$ satisfy

$$\begin{cases} dY_s = A_s Y_s ds + \sum_{j=1}^d [C_s^j Y_s + D_s^j v \mathbf{1}_{s\in[t,t+\varepsilon)}] dW_s^j, \quad s \in [t,T], \\ Y_t = 0; \end{cases}$$

$$\begin{cases} dZ_s = [A_s Z_s + B_s' v \mathbf{1}_{s\in[t,t+\varepsilon)}] ds + \sum_{j=1}^d C_s^j Z_s dW_s^j, \quad s \in [t,T], \\ Z_t = 0. \end{cases}$$

Moreover

$$\mathbb{E}_t [Y_s] = 0, \quad \mathbb{E}_t \left[\sup_{s\in[t,T]} |Y_s|^2 \right] = O(\varepsilon), \quad \mathbb{E}_t \left[\sup_{s\in[t,T]} |Z_s|^2 \right] = O(\varepsilon^2).$$

By these estimates, we can calculate

$$
\begin{aligned}
&2[J(t,X_t^*;u^{t,\varepsilon,v})-J(t,X_t^*,u^*)]\\
=&\,\mathbb{E}_t\int_t^T\left[\langle Q_s(2X_s^*+Y_s+Z_s),Y_s+Z_s\rangle+\langle R_s(2u_s^*+v),v\rangle\mathbf{1}_{s\in[t,t+\varepsilon)}\right]ds\\
&+2\mathbb{E}_t\left[\langle GX_T^*,Y_T+Z_T\rangle\right]+\mathbb{E}_t\left[\langle G(Y_T+Z_T),Y_T+Z_T\rangle\right]\\
&-2\langle h\mathbb{E}_t\left[X_T^*\right]+\mu_1X_t^*+\mu_2,\mathbb{E}_t\left[Y_T+Z_T\right]\rangle-\langle h\mathbb{E}_t\left[Y_T+Z_T\right],\mathbb{E}_t\left[Y_T+Z_T\right]\rangle\\
=&\,\mathbb{E}_t\int_t^T\left[\langle Q_s(2X_s^*+Y_s+Z_s),Y_s+Z_s\rangle+\langle R_s(2u_s^*+v),v\rangle\mathbf{1}_{s\in[t,t+\varepsilon)}\right]ds\\
&+\mathbb{E}_t\left[2\langle GX_T^*-h\mathbb{E}_t[X_T^*]-\mu_1X_t^*-\mu_2,Y_T+Z_T\rangle+\langle G(Y_T+Z_T),Y_T+Z_T\rangle\right]+o(\varepsilon).
\end{aligned}
$$

Recalling that $(p(\cdot;t),k(\cdot;t))$ and $(P(\cdot;t),K(\cdot;t))$ solve respectively (3.1) and (3.2), we have

$$
\begin{aligned}
&\mathbb{E}_t\left[\langle GX_T^*-h\mathbb{E}_t\left[X_T^*\right]-\mu_1X_t^*-\mu_2,Y_T+Z_T\rangle\right]\\
=&\,\mathbb{E}_t\int_t^T\{\langle p(s;t),A_s(Y_s+Z_s)+B_s'v\mathbf{1}_{s\in[t,t+\varepsilon)}\rangle\\
&-\langle A_s'p(s;t)+\sum_{j=1}^d(C_s^j)'k^j(s;t)+Q_sX_s^*,Y_s+Z_s\rangle\\
&+\sum_{j=1}^d\langle k^j(s;t),C_s^j(Y_s+Z_s)+D_s^jv\mathbf{1}_{s\in[t,t+\varepsilon)}\rangle\}ds\\
=&\,\mathbb{E}_t\int_t^T[\langle -Q_sX_s^*,Y_s+Z_s\rangle+\langle B_sp(s;t)+\sum_{j=1}^d(D_s^j)'k^j(s;t),v\mathbf{1}_{s\in[t,t+\varepsilon)}\rangle]ds;
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}_t\left[\langle G(Y_T+Z_T),Y_T+Z_T\rangle\right]\\
=&\,\mathbb{E}_t\int_0^T\left[-\langle Q_s(Y_s+Z_s),Y_s+Z_s\rangle+\sum_{j=1}^d\langle (D_s^j)'P(s;t)D_sv,v\rangle\mathbf{1}_{s\in[t,t+\varepsilon)}\right]ds+o(\varepsilon).
\end{aligned}
$$

This proves (3.3). □

It follows from $R\succeq 0$ and $P(s;t)\succeq 0$ that $H(s;t)\succeq 0$. In view of (3.3), a sufficient condition for an equilibrium is

$$
\mathbb{E}_t\int_t^T|\Lambda(s;t)|ds<+\infty,\qquad \lim_{s\downarrow t}\mathbb{E}_t\left[\Lambda(s;t)\right]=0,\ \text{a.s.},\ \forall t\in[0,T]. \tag{3.4}
$$

Under some condition, the second equality in (3.4) is ensured by

$$
R_tu_t^*+B_tp(t;t)+\sum_{j=1}^d(D_t^j)'k^j(t;t)=0,\ \text{a.s.},\ \forall t\in[0,T]. \tag{3.5}
$$

The following is the main general result for the time-inconsistent stochastic LQ control.

Theorem 3.2. *If the following system of stochastic differential equations*

$$\begin{cases} dX^*_s = [A_s X^*_s + B'_s u^*_s + b_s]ds + \sum_{j=1}^d [C^j_s X^*_s + D^j_s u^*_s + \sigma^j_s]dW^j_s, \ s \in [0,T], \\ X^*_0 = x_0, \\ dp(s;t) = -[A'_s p(s;t) + \sum_{j=1}^d (C^j_s)' k^j(s;t) + Q_s X^*_s]ds + \sum_{j=1}^d k^j(s;t)dW^j_s, \ s \in [t,T], \\ p(T;t) = GX^*_T - h\mathbb{E}_t\left[X^*_T\right] - \mu_1 X^*_t - \mu_2 \end{cases} \tag{3.6}$$

admits a solution (u^*, X^*, p, k), for any $t \in [0,T)$, such that $\Lambda(\cdot;t) \triangleq B.p(\cdot;t) + \sum_{j=1}^d (D^j_\cdot)' k(\cdot;t)^j + R.u^*_\cdot$ satisfies condition (3.4), and $u^* \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)$, then u^* is an equilibrium control.

Proof. Given (u^*, X^*, p, k) satisfying the conditions in this theorem, at any time t, for any $v \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^l)$, define Λ and H as in Proposition 3.1. Then

$$\begin{aligned} \lim_{\varepsilon\downarrow 0} \frac{J(t, X^*_t; u^{t,\varepsilon}) - J(t, X^*_t; u^*)}{\varepsilon} &= \lim_{\varepsilon\downarrow 0} \frac{\mathbb{E}_t \int_t^{t+\varepsilon} \left\{\langle \Lambda(s;t), v\rangle + \frac{1}{2}\langle H(s;t)v, v\rangle\right\} ds}{\varepsilon} \\ &\geq \lim_{\varepsilon\downarrow 0} \frac{\int_t^{t+\varepsilon} \langle \mathbb{E}_t\left[\Lambda(s;t)\right], v\rangle ds}{\varepsilon} \\ &\geq 0, \end{aligned}$$

proving the result. □

Theorem 3.2 involves the existence of solutions to a flow of FBSDEs along with other conditions. Proving the general existence remains an outstanding open problem. In the rest of this paper we will focus on the case when $n = 1$. This case is important especially in financial applications, as will be demonstrated by the mean–variance portfolio selection model.

When $n = 1$, the state process X is a scalar-valued process evolving by the dynamics

$$dX_s = [A_s X_s + B'_s u_s + b_s]ds + [C_s X_s + D_s u_s + \sigma_s]' dW_s; \quad X_0 = x_0, \tag{3.7}$$

where A is a bounded deterministic scalar function on $[0,T]$. The other parameters B, C, D are all essentially bounded and $\mathcal{F}_t$-adapted processes on $[0,T]$ with values in $\mathbb{R}^l$, $\mathbb{R}^d$, $\mathbb{R}^{d\times l}$, respectively. Moreover, $b \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ and $\sigma \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$.

In this case, the two adjoint equations for the equilibrium become

$$\begin{cases} dp(s;t) = -[A_s p(s;t) + C'_s k(s;t) + Q_s X^*_s]ds + k(s;t)' dW_s, \quad s \in [t,T], \\ p(T;t) = GX^*_T - h\mathbb{E}_t[X^*_T] - \mu_1 X^*_t - \mu_2; \end{cases} \tag{3.8}$$

$$\begin{cases} dP(s;t) = -[(2A_s + |C_s|^2)P(s;t) + 2C'_s K(s;t) + Q_s]ds \\ \qquad\qquad + K(s;t)' dW_s, \quad s \in [t,T], \\ P(T;t) = G. \end{cases} \tag{3.9}$$

For reader's convenience, we state here the $n = 1$ version of Theorem 3.2:

THEOREM 3.3. *If the following system of stochastic differential equations*

$$\begin{cases} dX_s^* = [A_s X_s^* + B_s' u_s^* + b_s]ds + [C_s X_s^* + D_s u_s^* + \sigma_s]' dW_s, \ s \in [0,T], \\ X_0^* = x_0, \\ dp(s;t) = -[A_s p(s;t) + C_s' k(s;t) + Q_s X_s^*]ds + k(s;t)' dW_s, \ s \in [t,T], \\ p(T;t) = G X_T^* - h E_t[X_T^*] - \mu_1 X_t^* - \mu_2, \ t \in [0,T] \end{cases} \tag{3.10}$$

admits a solution (u^*, X^*, p, k), *for any* $t \in [0,T)$, *such that* $\Lambda(\cdot;t) \stackrel{\triangle}{=} p(\cdot;t)B_\cdot + D_\cdot' k(\cdot;t) + R_\cdot u_\cdot^*$ *satisfies the condition (3.4), and* $u^* \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)$, *then* u^* *is an equilibrium control.*

4. Equilibrium When Coefficients Are Deterministic. Theorem 3.3 shows that one can obtain equilibrium controls by solving the system of FBSDEs (3.10). However, the FBSDEs in (3.10) are not standard since a "flow" of unknowns $(p(\cdot;t), k(\cdot;t))$ is involved. Moreover, there is an additional constraint (3.4), which under some condition boils down to an algebraic constraint (3.5) that acts on the "diagonal" (i.e. when $s = t$) of the flow. The unique solvability of this type of equations remains a challenging open problem even for the case $n = 1$. However, we are able to solve quite thoroughly this problem when the parameters A, B, C, D, b, σ, Q and R are all deterministic functions.

Throughout this section we assume all the parameters are deterministic functions of t. In this case, the BSDE (3.9) turns out to be an ODE with solution $K \equiv 0$ and $P(s;t) = Ge^{\int_s^T (2A_u + |C_u|^2)du} + \int_s^T e^{\int_s^v (2A_u+|C_u|^2)du} Q_v dv$.

4.1. An Ansatz. As in the classical LQ control (see, e.g. [19]), we attempt to look for a linear feedback equilibrium. For this, given any $t \in [0,T]$, we consider the following Ansatz:

$$p(s;t) = M_s X_s^* - N_s \mathbb{E}_t[X_s^*] - \Gamma_s^{(1)} X_t^* + \Phi_s, \quad 0 \le t \le s \le T, \tag{4.1}$$

where $M, N, \Gamma^{(1)}, \Phi$ are deterministic differentiable functions with $\dot{M} = m, \dot{N} = n, \dot{\Gamma}^{(1)} = \gamma^{(1)}$ and $\dot{\Phi} = \phi$.

For any fixed t, applying Ito's formula to (4.1) in the time variable s , we get

$$\begin{aligned} & dp(s;t) \\ = \ & \{M_s(A_s X_s^* + B_s' u_s^* + b_s) + m_s X_s^* - N_s \mathbb{E}_t\left[A_s X_s^* + B_s' u_s^* + b_s\right] - n_s \mathbb{E}_t\left[X_s^*\right] \\ & -\gamma_s^{(1)} X_t^* + \phi_s\}ds + M_s(C_s X_s^* + D_s u_s^* + \sigma_s)' dW_s. \end{aligned} \tag{4.2}$$

Comparing the dW_s term with the dW_s term of $dp(s;t)$ in (3.10), we obtain

$$k(s;t) = M_s[C_s X_s^* + D_s u_s^* + \sigma_s], \quad s \in [t,T]. \tag{4.3}$$

Notice that $k(s;t)$ turns out to be independent of t.

Now we ignore the difference between the conditions (3.4) and (3.5), and put the above expressions of $p(s;t)$ and $k(s;t)$ into (3.5). Then we have

$$[(M_s - N_s - \Gamma_s^{(1)})X_s^* + \Phi_s]B_s + M_s D_s'[C_s X_s^* + D_s u_s^* + \sigma_s] + R_s u_s^* = 0, \quad s \in [0,T],$$

from which we formally deduce

$$u_s^* = \alpha_s X_s^* + \beta_s, \tag{4.4}$$

where

$$\begin{aligned}
\alpha_s &\triangleq -(R_s + M_s D_s' D_s)^{-1}[(M_s - N_s - \Gamma_s^{(1)})B_s + M_s D_s' C_s],\\
\beta_s &\triangleq -(R_s + M_s D_s' D_s)^{-1}(\Phi_s B_s + M_s D_s' \sigma_s).
\end{aligned}$$

Next, comparing the ds term in (4.2) with the one in (3.10) (we suppress the argument s here), we obtain

$$\begin{aligned}
0 &= mX^* + M(AX^* + B'u^* + b) - n\mathbb{E}_t[X^*] - N(A\mathbb{E}_t[X^*] + B'\mathbb{E}_t[u^*] + b) - \gamma^{(1)}X_t^* + \phi\\
&\quad + AMX^* - AN\mathbb{E}_t[X^*] - A\Gamma^{(1)}X_t^* + A\Phi + MC'[CX^* + Du^* + \sigma] + QX^*\\
&= [m + 2MA + M|C|^2 + Q + (MB' + MC'D)\alpha]X^* - [n + 2NA + NB'\alpha]\mathbb{E}_t[X^*]\\
&\quad -(\gamma^{(1)} + A\Gamma^{(1)})X_t^* + [(M-N)(B'\beta + b) + \phi + A\Phi + MC'(D\beta + \sigma)].
\end{aligned}$$

Notice in the above $X^* \equiv X_s^*$ and $\mathbb{E}_t[X^*] \equiv \mathbb{E}_t[X_s^*]$ due to the omission of s. This leads to the following equations for $M, N, \Gamma^{(1)}, \Phi$ (again the argument s is suppressed):

$$\begin{cases}
\dot{M} + (2A + |C|^2)M + Q\\
\quad -M(B' + C'D)(R + MD'D)^{-1}[(M - N - \Gamma^{(1)})B + MD'C] = 0, \quad s \in [0,T],\\
M_T = G;
\end{cases} \tag{4.5}$$

$$\begin{cases}
\dot{N} + 2AN - NB'(R + MD'D)^{-1}[(M - N - \Gamma^{(1)})B + MD'C] = 0, \quad s \in [0,T],\\
N_T = h;
\end{cases} \tag{4.6}$$

$$\begin{cases}
\dot{\Gamma}^{(1)} = -A\Gamma^{(1)}, \quad s \in [0,T],\\
\Gamma_T^{(1)} = \mu_1;
\end{cases} \tag{4.7}$$

$$\begin{cases}
\dot{\Phi} + \{A - [(M-N)B' + MC'D](R + MD'D)^{-1}B\}\Phi + (M-N)b + C'M\sigma\\
\quad -[(M-N)B' + MC'D](R + MD'D)^{-1}MD'\sigma = 0, \quad s \in [0,T],\\
\Phi_T = -\mu_2.
\end{cases} \tag{4.8}$$

The solution to equation (4.7) is $\Gamma_s^{(1)} = \mu_1 e^{\int_s^T A_t dt}$. Equations (4.5) and (4.6) form a system of coupled Riccati equations[2] for (M, N)

$$\begin{cases} \dot{M} &= - \left[2A + |C|^2 + \Gamma^{(1)} B'(R + MD'D)^{-1}(B + D'C)\right] M - Q \\ & \quad +(B + D'C)'(R + MD'D)^{-1}(B + D'C)M^2 - B'(R + MD'D)^{-1}(B + D'C)MN, \\ M_T = G; \\ \dot{N} &= - \left[2A + \Gamma^{(1)} B'(R + MD'D)^{-1}B\right] N + B'(R + MD'D)^{-1}(B + D'C)MN \\ & \quad -B'(R + MD'D)^{-1}BN^2, \\ N_T &= h. \end{cases} \tag{4.9}$$

Finally, once we get the solution for (M, N), equation (4.8) is a simple ODE. Therefore, it is crucial to solve (4.9), which will be carried out in the next subsection.

4.2. Solution to Riccati System (4.9). Formally, we define $J = \frac{M}{N}$, and study the following equation for (M, J):

$$\begin{cases} \dot{M} &= - \left[2A + |C|^2 + \Gamma^{(1)} B'(R + MD'D)^{-1}(B + D'C)\right] M - Q \\ & \quad +(B + D'C)'(R + MD'D)^{-1}(B + D'C)M^2 - B'(R + MD'D)^{-1}(B + D'C)\frac{M^2}{J}, \\ M_T = G; \\ \dot{J} &= -[|C|^2 - C'D(R + MD'D)^{-1}(B + D'C)M + \Gamma^{(1)} B'(R + MD'D)^{-1}D'C + \frac{Q}{M}]J \\ & \quad -B'(R + MD'D)^{-1}D'CM, \\ J_T &= \frac{G}{h}. \end{cases} \tag{4.10}$$

PROPOSITION 4.1. *If the system (4.10) admits a positive solution pair (M, J), then the system (4.9) admits a positive solution pair $(M, \frac{M}{J})$.*

Proof. The proof is straightforward. □

In the following two subsections, we will study the system (4.10) for two cases respectively. The main technique is the truncation method. This method involves "truncation functions" $\cdot \vee c$ for a small number $c > 0$, and $\cdot \wedge K$ for a large number K.

4.2.1. Standard case. We first consider the standard case where $R - \delta I \succeq 0$ for some $\delta > 0$.

THEOREM 4.2. *Assume that $R - \delta I \succeq 0$ for some $\delta > 0$ and $G \geq h > 0$. Then (4.10) and (4.9) admit unique positive solution pairs if $\frac{QD'D + |C|^2 R}{l} + \Gamma^{(1)} \mathcal{S}(D'CB') \succeq 0$, and either (i) there exists a constant $\lambda \geq 0$ such that $B = \lambda D'C$, or (ii) $D'D - \delta I \succeq 0$ for some $\delta > 0$.*

[2]Strictly speaking, these are not Riccati equations in the usual sense as they are not symmetric. However, we still use the term so as to see the connection and difference between time-inconsistent and time-consistent LQ control problems.

Proof. For fixed $c>0$ and $K>0$, consider the following truncated system of (4.10):

$$\begin{cases} \dot{M} = -\left[2A+|C|^2+\Gamma^{(1)}B'(R+M^+D'D)^{-1}(B+D'C)\right]M-Q \\ \quad +(B+D'C)'(R+M^+D'D)^{-1}(B+D'C)M(M^+\wedge K) \\ \quad -B'(R+M^+D'D)^{-1}(B+D'C)\frac{M(M^+\wedge K)}{J\vee c}, \\ M_T = G; \\ \dot{J} = -\lambda^{(1)}J-B'(R+M^+D'D)^{-1}D'C(M^+\wedge K), \\ J_T = \frac{G}{h} \end{cases} \tag{4.11}$$

where $M^+=\max\{M,0\}$ and

$$\lambda^{(1)} \triangleq |C|^2-C'D(R+M^+D'D)^{-1}(B+D'C)(M^+\wedge K)+\Gamma^{(1)}B'(R+M^+D'D)^{-1}D'C+\frac{Q}{M\vee c}.$$

Since $R-\delta I\succeq 0$, the above system is locally Lipschitz with linear growth, hence it admits a unique solution $(M^{c,K},J^{c,K})$. We omit the superscript (c,K) when no confusion might arise.

We are going to prove that $J\geq 1$, and $M\in[\eta,L]$ for some $\eta>0$ and $L>0$ independent of c and K appearing in the truncation functions. To this end, denote

$$\begin{aligned}\lambda^{(2)} = &(2A+|C|^2+\Gamma^{(1)}B'(R+M^+D'D)^{-1}(B+D'C)) \\ &-(B+D'C)'(R+M^+D'D)^{-1}(B+D'C)(M^+\wedge K) \\ &+B'(R+M^+D'D)^{-1}(B+D'C)\frac{M^+\wedge K}{J\vee c}.\end{aligned}$$

Then $\lambda^{(2)}$ is bounded, and M satisfies

$$\dot{M}+\lambda^{(2)}M+Q=0,\ M_T=G. \tag{4.12}$$

Hence $M>0$. As a result, the terms $R+M^+D'D$ and M^+ can be replaced by $R+MD'D$ and M respectively in (4.11) without changing their values.

Now we prove $J\geq 1$. Denote $\tilde{J}\triangleq J-1$, then $\tilde{J}$ satisfies the ODE

$$\begin{aligned}\dot{\tilde{J}} &= -\lambda^{(1)}\tilde{J}-\left[\lambda^{(1)}+B'(R+MD'D)^{-1}D'C(M\wedge K)\right] \\ &= -\lambda^{(1)}\tilde{J}-a^{(1)}\end{aligned}$$

where

$$\begin{aligned}a^{(1)} &= \lambda^{(1)}+B'(R+MD'D)^{-1}D'C(M\wedge K) \\ &= |C|^2-C'D(R+MD'D)^{-1}D'C(M\wedge K)+\Gamma^{(1)}B'(R+MD'D)^{-1}D'C+\frac{Q}{M\vee c} \\ &\geq |C|^2-C'D(R+MD'D)^{-1}D'C)M+\Gamma^{(1)}B'(R+MD'D)^{-1}D'C+\frac{Q}{M\vee c} \\ &= \operatorname{tr}\left\{(R+MD'D)^{-1}\frac{|C|^2+Q/(M\vee c)}{l}(R+MD'D)\right\} \\ &\quad -\operatorname{tr}\{(R+MD'D)^{-1}D'CC'DM\}+\operatorname{tr}\{(R+MD'D)^{-1}\Gamma^{(1)}D'CB'\} \\ &= \operatorname{tr}\left\{(R+MD'D)^{-1}H\right\}\end{aligned}$$

with $H \triangleq \frac{|C|^2+Q/(M\vee c)}{l}(R+MD'D)-D'CC'DM+\Gamma^{(1)}\mathcal{S}(D'CB')$.

When c is small enough such that $R-cD'D\succeq 0$, we have

$$\frac{Q}{M\vee c}(R+MD'D)\geq QD'D.$$

Furthermore,

$$\frac{|C|^2}{l}D'D-D'CC'D\succeq 0.$$

Hence,

$$H\succeq \frac{QD'D+|C|^2R}{l}+\Gamma^{(1)}\mathcal{S}(D'CB')\succeq 0,$$

and consequently $a^{(1)}\geq \mathrm{tr}\{(R+MD'D)^{-1}H\}\geq 0$.[3] We deduce that $\tilde{J}\geq 0$, or equivalently $J\geq 1$.

Next we prove M is bounded above by a constant $L>0$ independent of the truncation. Choosing c small enough, the equation for M turns out to be

$$\begin{cases} -\dot{M}=\left(2A+|C|^2+\Gamma^{(1)}B'(R+MD'D)^{-1}(B+D'C)\right)M+Q-kM(M\wedge K),\\ M_T=G \end{cases}$$

where

$$\begin{aligned} k&=(B+D'C)'(R+MD'D)^{-1}(B+D'C)-B'(R+MD'D)^{-1}(B+D'C)\frac{1}{J}\\ &=B'(R+MD'D)^{-1}B\left(1-\frac{1}{J}\right)+B'(R+MD'D)^{-1}D'C\left(2-\frac{1}{J}\right)\\ &\quad +C'D(R+MD'D)^{-1}D'C\\ &\geq B'(R+MD'D)^{-1}D'C\left(2-\frac{1}{J}\right). \end{aligned}$$

If $B=\lambda D'C$ for some $\lambda\geq 0$, then we have $k\geq 0$. Hence M admits an upper bound L independent of c and K.

If $D'D-\delta I\succeq 0$, then $|kM|$ admits a bound independent of c and K; hence once again M admits an upper bound L independent of c and K.

Choosing $K=L$ and examining again equation (4.12) we deduce that there exists $\eta>0$ independent of c such that $M\geq\eta$. It now suffices to take $c=\eta$ to finish the proof. □

4.2.2. Singular case. Let us now consider the singular case $R\equiv 0$. We suppose here that $D'D-\delta I\succeq 0$ for some $\delta>0$ in this subsection. Then the system of (M,J)

[3]Here we used the inequality that $\mathrm{tr}(AB)\geq 0$ for any positive semi-definite matrices A,B.

is

$$\begin{cases} \dot{M} &= -\left[2A+|C|^2-(B+D'C)'(D'D)^{-1}(B+D'C)+B'(D'D)^{-1}(B+D'C)\tfrac{1}{J}\right]M \\ & \quad -Q-\Gamma^{(1)}B'(D'D)^{-1}(B+D'C) \\ M_T &= G; \\ \dot{J} &= -[|C|^2-C'D(D'D)^{-1}(B+D'C)+(\Gamma^{(1)}B'(D'D)^{-1}D'C+Q)\tfrac{1}{M}]J \\ & \quad -B'(D'D)^{-1}D'C, \\ J_T &= \tfrac{G}{h}. \end{cases} \tag{4.13}$$

This system is even easier than the previous one. We will use the same truncation argument to prove the existence of a solution.

THEOREM 4.3. *Given $G \geq h > 0$, $R \equiv 0$ and $D'D - \delta I \succeq 0$ for some $\delta > 0$. If $Q+\Gamma^{(1)}B'(D'D)^{-1}(B+D'C) \geq 0$ and $Q+\Gamma^{(1)}B'(D'D)^{-1}D'C \geq 0$, then (4.13) and (4.9) admit positive solution pairs.*

Proof. For a fixed $c > 0$, consider the following truncated system:

$$\begin{cases} \dot{M} &= -\left[2A+|C|^2-(B+D'C)'(D'D)^{-1}(B+D'C)+B'(D'D)^{-1}(B+D'C)\tfrac{1}{J\vee c}\right]M \\ & \quad -Q-\Gamma^{(1)}B'(D'D)^{-1}(B+D'C), \\ M_T &= G; \\ \dot{J} &= -[|C|^2-C'D(D'D)^{-1}(B+D'C)+(\Gamma^{(1)}B'(D'D)^{-1}D'C+Q)\tfrac{1}{M\vee c}]J \\ & \quad -B'(D'D)^{-1}D'C, \\ J_T &= \tfrac{G}{h}. \end{cases} \tag{4.14}$$

This system is locally Lipschitz with linear growth, hence it admits a unique solution pair (M, J) depending on c.

Define $\tilde{J} = J - 1$. Then

$$\dot{\tilde{J}} = -\lambda^{(3)}\tilde{J} - a^{(3)}$$

with $\lambda^{(3)} = |C|^2 - C'D(D'D)^{-1}(B+D'C) + (\Gamma^{(1)}B'(D'D)^{-1}D'C+Q)\frac{1}{M\vee c}$ being bounded, and

$$\begin{aligned} a^{(3)} &= \lambda^{(3)} + B'(D'D)^{-1}D'C \\ &= |C|^2 - C'D(D'D)^{-1}D'C + (\Gamma^{(1)}B'(D'D)^{-1}D'C+Q)\frac{1}{M\vee c} \\ &\geq (\Gamma^{(1)}B'(D'D)^{-1}D'C+Q)\frac{1}{M\vee c} \\ &\geq 0. \end{aligned}$$

Hence $J \geq 1$. Now we choose $c \leq 1$.

Denote $\lambda^{(4)} = 2A+|C|^2-(B+D'C)'(D'D)^{-1}(B+D'C)+B'(D'D)^{-1}(B+D'C)\frac{1}{J\vee c}$, $\tilde{Q} = Q+\Gamma^{(1)}B'(D'D)^{-1}(B+D'C) \geq 0$. Then $|\lambda^{(4)}|$ admits a bound independent of c, and

$$\dot{M} + \lambda^{(4)}M + \tilde{Q} = 0,\ M_T = G.$$

Hence there exists some $\eta > 0$ (independent of c) such that $M \geq \eta$. Choosing $c = \eta$, we conclude the proof. ◻

4.3. Equilibrium Controls. We now present the main result of this section.

THEOREM 4.4. *Suppose $G \geq h > 0$, The system of the Riccati equations (4.9) admits a unique positive solution pair (M, N) in the following three cases:*

(i) $R - \delta I \succeq 0$ for some $\delta > 0$, $\frac{QD'D+|C|^2R}{l} + \Gamma^{(1)}\mathcal{S}(D'CB') \succeq 0$ and $B = \lambda D'C$ for some $\lambda \geq 0$;

(ii) $R - \delta I \succeq 0$ for some $\delta > 0$, $\frac{QD'D+|C|^2R}{l} + \Gamma^{(1)}\mathcal{S}(D'CB') \succeq 0$ and $D'D - \delta I \succeq 0$ for some $\delta > 0$;

(iii) $R \equiv 0$, $D'D - \delta I \succeq 0$ for some $\delta > 0$, $Q + \Gamma^{(1)}B'(D'D)^{-1}(B + D'C) \geq 0$, $Q + \Gamma^{(1)}B'(D'D)^{-1}D'C \geq 0$.

Moreover, let Φ be a solution of ODE (4.8). Then $u^*(\cdot)$ given by (4.4) is an equilibrium.

Proof. Define $p(\cdot;\cdot)$ and $k(\cdot;\cdot)$ by (4.1) and (4.3) respectively. It is straightforward to check that $(u^*_\cdot, X^*_\cdot, p(\cdot;\cdot), k(\cdot;\cdot))$ satisfies the system of SDEs (3.10).

In all the three cases, we can check that α_s and β_s in (4.4) are both uniformly bounded, hence $u^* \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^l)$ and $X^*_s \in L^2(\Omega; C(0, T; \mathbb{R}))$.

Finally, denote $\Lambda(s;t) = R_s u^*_s + p(s;t)B + D'_s k(s;t)$. By plug p, k, u^* defined in (4.1), (4.3) and (4.4) into Λ, we have

$$\begin{aligned}
\Lambda(s;t) &= R_s u^*_s + (M_s X^*_s - N_s \mathbb{E}_t[X^*_s] - \Gamma^{(1)}_s X^*_t + \Phi_s)B_s + M_s D'_s[C_s X^*_s + D_s u^*_s + \sigma_s] \\
&= (R_s + M_s D'_s D_s)u^*_s + (B_s + D'_s C_s)M_s X^*_s - N_s \mathbb{E}_t\left[X^*_s\right] B_s - \Gamma^{(1)}_s X^*_t B_s \\
&\quad + (\Phi_s B_s + M_s D'_s \sigma_s) \\
&= -[(M_s - N_s - \Gamma^{(1)}_s)B_s + M_s D'_s C_s]X^*_s - \Phi_s B_s - M_s D'_s \sigma_s \\
&\quad + (B_s + D'_s C_s)M_s X^*_s - N_s \mathbb{E}_t\left[X^*_s\right] B_s - \Gamma^{(1)}_s X^*_t B_s + (\Phi_s B_s + M_s D'_s \sigma_s) \\
&= (N_s + \Gamma^{(1)}_s)X^*_s B_s - N_s \mathbb{E}_t\left[X^*_s\right] B_s - \Gamma^{(1)}_s X^*_t B_s \\
&= N_s[X^*_s - \mathbb{E}_t\left[X^*_s\right]]B_s + \Gamma^{(1)}_s (X^*_s - X^*_t)B_s.
\end{aligned}$$

Clearly Λ satisfies the first condition in (3.4). Furthermore, we have

$$\lim_{s\downarrow t} \mathbb{E}_t\left[|X^*_s - \mathbb{E}_t\left[X^*_s\right]|\right] = 0, \qquad \text{and} \lim_{s\downarrow t} \mathbb{E}_t\left[|X^*_s - X^*_t|\right] = 0;$$

hence Λ satisfies the second condition in (3.4).

By Theorem 3.3, u^* is an equilibrium. ◻

REMARK 4.5. If $\mu_1 \geq 0$ (e.g. in the mean–variance model to be studied subsequently), then $\Gamma^{(1)}_t = \mu_1 e^{\int_t^T A_s ds} \geq 0$. With this condition, the first case and the third case in Theorem 4.4 can be simplified as

(i') $R - \delta I \succeq 0$ for some $\delta > 0$, and $B = \lambda D'C$ for some $\lambda \geq 0$;

(iii') $R \equiv 0$, $D'D - \delta I \succeq 0$ for some $\delta > 0$, and $Q + \Gamma^{(1)}B'(D'D)^{-1}D'C \geq 0$.

5. Mean-Variance Equilibrium Strategies in Complete Market. In this section, we study the continuous-time Markowitz's mean–variance portfolio selection model in a complete market. The problem is inherently time inconsistent due to the variance term. Moreover, as in [5] we consider a state-dependent mean expectation. Hence there are two different sources of time inconsistency. The definition of equilibrium strategies is in the sense of open-loop, which is different from the feedback one in [4, 5].

The model is mathematically a special case of the general LQ problem formulated earlier in this paper, with $n = 1$ naturally. However, some coefficients are allowed to be random; so it is not a direct application of the previous section. Indeed the analysis in this section is much more involved due to the randomness of the coefficients.

For each $t \in [0, T)$, consider a wealth-portfolio process (X_t, π_t) satisfying the wealth equation

$$\begin{cases} dX_s = r_s X_s ds + (\mu_s - r_s \mathbf{1})'\pi_s ds + \pi_s' \sigma_s dW_s, \qquad s \in [t, T], \\ X_t = x_t, \end{cases} \tag{5.1}$$

where $r \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R})$ is the interest rate process, $\mu \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^d)$ and $\sigma \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^{d\times d})$ are the drift rate vector and volatility processes of risky assets respectively. We assume throughout that $\sigma_s \sigma_s' - \varepsilon I \succeq 0$ for some $\varepsilon > 0$ to ensure the completeness of the market .

Denote $\theta_t = \sigma_t^{-1}(\mu_t - r_t \mathbf{1}), u_t = \sigma_t' \pi_t$. Then the wealth equation is equivalent to the equation of (X_t, u_t)

$$\begin{cases} dX_s = r_s X_s ds + \theta_s' u_s ds + u_s' dW_s, \qquad s \in [t, T], \\ X_t = x_t. \end{cases} \tag{5.2}$$

We interchangeably call π and u as (trading) strategies. It follows from our assumptions on θ that $\pi \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$ if and only if $u \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$. The objective of a mean-variance portfolio choice model at time $t \in [0, T)$ is to achieve a balance between conditional variance and conditional expectation of terminal wealth; namely, to choose a strategy u so as to minimize

$$\begin{aligned} J(t, x_t; u) &\triangleq \frac{1}{2} Var_t(X_T) - (\mu_1 x_t + \mu_2)\mathbb{E}_t[X_T] \\ &= \frac{1}{2}\left(\mathbb{E}_t[X_T^2] - (\mathbb{E}_t[X_T])^2\right) - (\mu_1 x_t + \mu_2)\mathbb{E}_t[X_T] \end{aligned} \tag{5.3}$$

with $\mu_1 \geq 0$. Here we insist that the weight between the conditional variance (as a risk measure) and the conditional expectation should depend on the current wealth level, the reason having been elaborated in [5].

When the market parameters r and θ are both deterministic, the problem is a special case of the one studied in Section 4. In this section, we will find the equilibrium strategies for the model where the interest rate r is deterministic but θ is allowed to be random.

The problem (5.1) – (5.3) is clearly a special case of LQ problem (2.2) – (2.3) with $n = 1$. The FBSDE (3.10) specializes to

$$\begin{cases} dX_s^* = [r_s X_s^* + \theta_s' u_s^*]ds + (u_s^*)' dW_s, \quad X_0^* = x_0, \\ dp(s; t) = -r_s p(s; t)ds + k(s; t)' dW_s, \\ p(T; t) = X_T^* - \mathbb{E}_t[X_T^*] - \mu_1 X_t^* - \mu_2, \end{cases} \tag{5.4}$$

and the process $\Lambda(s; t)$ in condition (3.4) is

$$\Lambda(s; t) = p(s; t)\theta_s + k(s; t).$$

5.1. Formal Derivation. As before, let us look for a solution in the form

$$p(s;t) = M_s X_s^* - \Gamma_s^{(1)} X_t^* + \Gamma_s^{(2)} - \mathbb{E}_t[N_s X_s^* + \Gamma_s^{(3)}], \tag{5.5}$$

where (M, U), (N, V), $(\Gamma^{(1)}, \gamma^{(1)})$, $(\Gamma^{(2)}, \gamma^{(2)})$ and $(\Gamma^{(3)}, \gamma^{(3)})$ are solutions of the following BSDEs:

$$\begin{cases} dM_s &= -F_{M,U} ds + U_s' dW_s, \quad M_T = 1; \\ dN_s &= -F_{N,V} ds + V_s' dW_s, \quad N_T = 1; \\ d\Gamma_s^{(1)} &= -F^{(1)} ds + (\gamma_s^{(1)})' dW_s, \quad \Gamma_T^{(1)} = \mu_1; \\ d\Gamma_s^{(2)} &= -F^{(2)} ds + (\gamma_s^{(2)})' dW_s, \quad \Gamma_T^{(2)} = -\mu_2; \\ d\Gamma_s^{(3)} &= -F^{(3)} ds + (\gamma_s^{(3)})' dW_s, \quad \Gamma_T^{(3)} = 0. \end{cases} \tag{5.6}$$

It is an easy exercise to obtain

$$\begin{aligned} d[N_s X_s^*] &= [rNX^* + N\theta' u^* - X^* F_{N,V} + V' u^*] ds + [Nu^* + X^* V]' dW_s, \\ d\mathbb{E}_t[N_s X_s^*] &= \mathbb{E}_t[rNX^* + N\theta' u^* - X^* F_{N,V} + V' u^*] ds, \\ d[M_s X_s^*] &= [rMX^* + M\theta' u^* - X^* F_{M,U} + U' u^*] ds + [Mu^* + X^* U]' dW_s. \end{aligned}$$

Applying Ito's formula to $p(s;t) = M_s X_s^* + \Gamma_s^{(2)} - \mathbb{E}_t[N_s X_s + \Gamma_s^{(3)}] - \Gamma_s^{(1)} X_t^*$ and comparing the dW_s term in the second equation of (5.4), we get

$$k(s;t) = X_s^* U_s + M_s u_s^* + \gamma_s^{(2)} - \gamma_s^{(1)} X_t^*. \tag{5.7}$$

Putting the expressions of p and k into the formal condition $\Lambda(s;s) = 0$, we obtain

$$\begin{aligned} u_s^* &= -M_s^{-1} \left[\left(\theta_s (M_s - N_s - \Gamma_s^{(1)}) + U_s - \gamma_s^{(1)} \right) X_s^* + \theta_s (\Gamma_s^{(2)} - \Gamma_s^{(3)}) + \gamma_s^{(2)} \right] \\ &= \alpha_s X_s^* + \beta_s, \end{aligned}$$

where

$$\alpha_s \triangleq -M_s^{-1} \left(\theta_s (M_s - N_s - \Gamma_s^{(1)}) + U_s - \gamma_s^{(1)}) \right), \quad \beta_s \triangleq -M_s^{-1} \left(\theta_s (\Gamma_s^{(2)} - \Gamma_s^{(3)}) + \gamma_s^{(2)} \right).$$

Applying again Ito's formula to p and using the above expression of u, we deduce

$$\begin{aligned} dp(s;t) = {} & [-F_{M,U} X_s^* + r_s M_s X_s^* + (\theta_s M_s + U_s)(\alpha X_s^* + \beta_s) - F^{(2)} + X_t^* F^{(1)}] ds \\ & + \mathbb{E}_t[F_{N,V} X_s^* - r_s N_s X_s^* - (\theta_s N_s + V_s)(\alpha X_s + \beta_s) + F^{(3)}] ds + k(s,t)' dW_s, \end{aligned}$$

while the second equation in (5.4) gives

$$dp(s;t) = \{-r_s M_s X_s^* + r_s \Gamma_s^{(1)} X_t^* - r_s \Gamma_s^{(2)} + r_s \mathbb{E}_t[N_s X_s^* + \Gamma_s^{(3)}]\} ds + k(s;t)' dW_s.$$

Comparing the corresponding terms, we obtain (again we supress the subscripts $s \in [t, T]$):

$$\begin{aligned} F_{M,U} &= 2rM + (\theta M + U)' \alpha; \\ F_{N,V} &= 2rN + (\theta N + V)' \alpha; \\ F^{(1)} &= r\Gamma^{(1)}; \\ F^{(2)} &= r\Gamma^{(2)} + (\theta M + U)' \beta; \\ F^{(3)} &= r\Gamma^{(3)} + (\theta N + V)' \beta. \end{aligned}$$

5.2. Solution to the BSDEs (5.6) . It now suffices to solve the BSDEs (5.6). Its third equation can be easily solved, whose solution is

$$\Gamma_t^{(1)} = \mu_1 e^{\int_t^T r_s ds}, \ \gamma_t^{(1)} = 0.$$

Noting that the first two equations are identical, we conclude that

$$M = N, \quad U = V.$$

Then

$$F^{(2)} - F^{(3)} = r(\Gamma^{(2)} - \Gamma^{(3)}).$$

By the last two equations in (5.6), we have

$$\Gamma_s^{(2)} - \Gamma_s^{(3)} = -\mu_2 e^{\int_s^T r_t dt} \triangleq \Gamma_s.$$

To proceed, let us recall some facts about BMO martingales; see Kazamaki [12]. The process $Z \cdot W \triangleq \int_0^\cdot Z_s' dW_s$ is a BMO martingale if and only if there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\int_\tau^T |Z_s|^2 ds \Big| \mathcal{F}_\tau\right] \leq C$$

for all stopping times $\tau \leq T$. For every such Z, the stochastic exponential of $Z \cdot W$ denoted by $\mathcal{E}(Z \cdot W)$ is a positive martingale; and for any $p > 1$, there exists a constant $C_p > 0$ such that $\mathbb{E}\left[\left(\int_\tau^T |Z_s|^2 ds\right)^p \Big| \mathcal{F}_\tau\right] \leq C_p$ for any stopping time $\tau \leq T$. Moreover, if $Z \cdot W$ and $V \cdot W$ are both BMO martingales, then under the probability measure $\mathbb{Q}$ defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}_T(V \cdot W)$, $W_t^{\mathbb{Q}} \triangleq W_t - \int_0^t V_s ds$ is a standard Brownian motion, and $Z \cdot W^{\mathbb{Q}}$ is a BMO martingale.

Now plug the definition of α into the first equation in (5.6), we get the BSDE satisfied by (M, U):

$$\begin{cases} dM_s = -(2r_s M_s - U_s'\theta_s + \Gamma_s^{(1)}|\theta_s|^2 - M_s^{-1}|U_s|^2 + \Gamma_s^{(1)} M_s^{-1} U_s'\theta_s)ds + U_s' dW_s, \\ M_T = 1. \end{cases} \tag{5.8}$$

This is a type of indefinite stochastic Riccati equation due to the presence of M^{-1} in the driver; however it is different from the one studied in [10].

PROPOSITION 5.1. *BSDE (5.8) admits a unique solution $(M, U) \in L_{\mathcal{F}}^\infty(0, T; \mathbb{R}) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^d)$ satisfying $M \geq c$ for some constant $c > 0$. Moreover, $U \cdot W$ is a BMO martingale.*

Proof. Once again, we will prove the existence by a truncation argument. Let $c > 0$ be a given number to be chosen later. Consider the following quadratic BSDE:

$$\begin{cases} dM_s = -\left[2r_s M_s - U_s'\theta_s + \Gamma_s^{(1)}|\theta_s|^2 - \frac{|U_s|^2}{M_s \vee c} + \Gamma_s^{(1)} \frac{U_s'\theta}{M_s \vee c}\right] ds + U_s' dW_s, \\ M_T = 1. \end{cases} \tag{5.9}$$

This BSDE is a standard quadratic BSDE. Hence there exists a solution $(M^c, U^c) \in L_{\mathcal{F}}^\infty(0, T; \mathbb{R}) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^d)$ and $U^c \cdot W$ is a BMO martingale; see [13] and [15].

We can rewrite the above BSDE as:

$$\begin{cases} dM_s = -(2r_sM_s + \Gamma_s^{(1)}|\theta_s|^2)ds + U_s'[dW_s - (\Gamma_s^{(1)}\frac{1}{M_s\vee c}\theta_s - \theta_s - \frac{1}{M_s\vee c}U_s)ds], \\ M_T = 1. \end{cases} \tag{5.10}$$

As $(\Gamma^{(1)}\frac{1}{M^c\vee c}\theta - \theta - \frac{1}{M^c\vee c}U^c)\cdot W$ is a BMO martingale, there exists a new probability measure $\mathbb{Q}$ such that

$$W_t^{\mathbb{Q}} = W_t - \int_0^t \left(\Gamma_s^{(1)}\frac{1}{M_s^c\vee c}\theta_s - \theta_s - \frac{1}{M_s^c\vee c}U_s^c\right)ds$$

is a Brownian motion under $\mathbb{Q}$.

Hence,

$$M_s^c = \mathbb{E}_s^{\mathbb{Q}}\left[e^{2\int_s^T r_t dt} + \int_s^T \Gamma_v^{(1)} e^{2\int_s^v r_t dt}|\theta_v|^2 dv\right],$$

from which we deduce that there exists a constant $\eta > 0$ independent of c such that $M \geq \eta$. Taking $c = \eta$, we obtain a solution.

Let us now prove the uniqueness. First we note that if $(M,U) \in L_{\mathcal{F}}^\infty(0,T;\mathbb{R}) \times L_{\mathcal{F}}^2(0,T;\mathbb{R}^d)$ is a solution and there exists $c > 0$ such that $M \geq c$, then $U\cdot W$ is a BMO martingale. Let us define

$$Y_s = M_s^{-1}, \quad Z_s = -M_s^{-2}U_s.$$

Then (Y,Z) is a solution in $L_{\mathcal{F}}^\infty(0,T;\mathbb{R}) \times L_{\mathcal{F}}^2(0,T;\mathbb{R}^d)$ of the following BSDE

$$\begin{cases} dY_s = -[-2rY_s - Z_s'\theta_s - \Gamma_s^{(1)}|\theta_s|^2Y_s^2 + \Gamma_s^{(1)}Y_sZ_s'\theta]ds + Z_s'dW_s, \\ Y_T = 1. \end{cases} \tag{5.11}$$

Moreover, $Z\cdot W$ is a BMO martingale.

It suffices to prove uniqueness of solution to BSDE (5.11). For this, let $(Y^{(1)}, Z^{(1)})$ and $(Y^{(2)}, Z^{(2)})$ be two solutions in $L_{\mathcal{F}}^\infty(0,T;\mathbb{R}) \times L_{\mathcal{F}}^2(0,T;\mathbb{R}^d)$ such that $Z^{(1)}\cdot W$ and $Z^{(2)}\cdot W$ are BMO martingales. Set

$$\bar{Y} = Y^{(1)} - Y^{(2)}, \quad \bar{Z} = Z^{(1)} - Z^{(2)}.$$

Then

$$\begin{cases} d\bar{Y}_s = -[-2r_s\bar{Y}_s - \bar{Z}_s'\theta_s - \Gamma_s^{(1)}|\theta_s|^2(Y_s^{(1)} + Y_s^{(2)})\bar{Y}_s + \Gamma_s^{(1)}\theta_s'(\bar{Y}_sZ_s^{(1)} + Y_s^{(2)}\bar{Z}_s)]ds + \bar{Z}_s'dW_s, \\ \bar{Y}_T = 0. \end{cases} \tag{5.12}$$

Applying Ito's formula to $|\bar{Y}_s|^2$ and taking conditional expectation, we deduce (where $C > 0$ is a constant which may change from line to line).

$$
\begin{aligned}
&|\bar{Y}_s|^2 + \mathbb{E}_s \left[\int_s^T |\bar{Z}_r|^2 dr \right] \\
&\leq C\mathbb{E}_s \left[\int_s^T |\bar{Y}_r|(|\bar{Y}_r| + |\bar{Z}_r| + |Z_r^{(1)}||\bar{Y}_r|)dr \right] \\
&\leq C\mathbb{E}_s \left[\int_s^T |\bar{Y}_r|^2 dr \right] + \frac{1}{2}\mathbb{E}_s \left[\int_s^T |\bar{Z}_r|^2 dr \right] + C\sqrt{\mathbb{E}_s \left[\int_s^T |Z_r^{(1)}|^2 dr \right] \mathbb{E}_s \left[\int_s^T |\bar{Y}_r|^4 dr \right]} \\
&\leq C\mathbb{E}_s \left[\int_s^T |\bar{Y}_r|^2 dr \right] + \frac{1}{2}\mathbb{E}_s \left[\int_s^T |\bar{Z}_r|^2 dr \right] + C\sqrt{\mathbb{E}_s \left[\int_s^T |\bar{Y}_r|^4 dr \right]}.
\end{aligned}
$$

Let us assume that $s \in [T-\delta, T]$. Then by setting

$$
\bar{Y}^*_{T-\delta,T} = \|\bar{Y}_\cdot\|_{L^\infty_{\mathcal{F}}(T-\delta,T;\mathbb{R})},
$$

we obtain

$$
|\bar{Y}_s|^2 \leq C(\delta + \delta^{1/2})|\bar{Y}^*_{T-\delta,T}|^2.
$$

Hence,

$$
|\bar{Y}^*_{T-\delta,T}|^2 \leq C\delta^{1/2}|\bar{Y}^*_{T-\delta,T}|^2.
$$

By taking δ sufficiently small, we deduce that $\bar{Y}^*_{T-\delta,T} = 0$. We conclude the proof of uniqueness by continuing on $[T-2\delta, T-\delta], \ldots,$ until time 0 is reached. □

Then we consider the BSDE satisfied by $(\Gamma^{(2)}, \gamma^{(2)})$:

$$
\begin{cases}
d\Gamma_t^{(2)} = -\left[r_t\Gamma_t^{(2)} - \left(\theta_t + \frac{U_t}{M_t}\right)' \gamma_t^{(2)} - \left(|\theta_t|^2 + \frac{U_t'\theta_t}{M_t}\right)\Gamma_t \right] dt + (\gamma_t^{(2)})'dW_t, \\
\Gamma_T^{(2)} = -\mu_2.
\end{cases}
\tag{5.13}
$$

Proposition 5.2. *BSDE (5.13) admits a unique solution* $(\Gamma^{(2)}, \gamma^{(2)}) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$. *Moreover,* $\gamma^{(2)} \cdot W$ *is a BMO martingale.*

Proof. As $-(\theta + \frac{U}{M}) \cdot W)$ is a BMO martingale, it suffices to apply the result of Section 3 in [3] to deduce that BSDE (5.13) admits a unique solution $(\Gamma^{(2)}, \gamma^{(2)}) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$. Let $\mathbb{Q}$ be the probability measure defined by $\frac{d\mathbb{Q}}{dP} = \mathcal{E}_T(-(\theta + \frac{U}{M}) \cdot W)$. Then under $\mathbb{Q}$,

$$
W_t^{\mathbb{Q}} = W_t + \int_0^t (\theta_s + M_s^{-1}U_s)ds
$$

is a Brownian motion and $U \cdot W^{\mathbb{Q}}$ is a BMO martingale. Furthermore,

$$
d\Gamma_t^{(2)} = -\left[r_t\Gamma_t^{(2)} - \left(|\theta_t|^2 + \frac{U_t'\theta_t}{M_t}\right)\Gamma_t \right] dt + (\gamma_t^{(2)})'dW_t^{\mathbb{Q}}, \qquad \Gamma_T^{(2)} = -\mu_2.
$$

Hence

$$
\Gamma_t^{(2)} = \mathbb{E}_t^{\mathbb{Q}} \left[-e^{\int_t^T r_v dv}\mu_2 - \int_t^T e^{\int_t^s r_v dv}\Gamma_s \left(|\theta_s|^2 + \frac{U_s'\theta_s}{M_s}\right) ds \right].
$$

From this we deduce that $\Gamma^{(2)}$ is a bounded process. Moreover, from (5.13),

$$\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T |\gamma_s^{(2)}|^2 ds\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\left|\int_t^T (\gamma_s^2)' dW_s^{\mathbb{Q}}\right|^2\right]$$
$$= \mathbb{E}_t^{\mathbb{Q}}\left[\left|\Gamma_T^{(2)} - \Gamma_t^{(2)} + \int_t^T \left[r_s\Gamma_s^{(2)} - \Gamma_s\left(|\theta_s|^2 + \frac{U_s'\theta_s}{M_s}\right)\right] ds\right|^2\right].$$

Hence from the last equality, $\gamma^{(2)} \cdot W^{\mathbb{Q}}$ is a BMO martingale under $\mathbb{Q}$ and then $\gamma^{(2)} \cdot W$ is a BMO martingale under $\mathbb{P}$. ▯

With $M, U, \gamma^{(2)}$ obtained, we can construct a (feedback) strategy

$$u_s^* = \alpha_s X_s^* + \beta_s \tag{5.14}$$

where

$$\alpha_s \triangleq \frac{\Gamma_s^{(1)}\theta_s - U_s}{M_s}, \quad \beta_s \triangleq -\frac{\Gamma_s\theta_s + \gamma_s^{(2)}}{M_s}.$$

In order to confirm that the above is indeed an *admissible* feedback strategy, we need to prove the following technical result. Its proof is intriguing in its own right.

PROPOSITION 5.3. *Let X^* be the solution to the first equation of (5.4) where u^* is substituted by (5.14). Then $X^* \in L^2_{\mathcal{F}}(0,T;C(0,T;\mathbb{R}))$ and $u^* \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$.*

Proof. Plug the feedback strategy u^* into the wealth equation (5.2), we get

$$X_t^* = \rho_t\Big(x_0 - \int_0^t \rho_s^{-1}\alpha_s'\beta_s ds + \int_0^t \rho_s^{-1}\beta_s dW_s^\theta\Big), \tag{5.15}$$

with $W_t^\theta = W_t + \int_0^t \theta_s ds$ and $\rho_t = e^{\int_0^t r_s ds}\mathcal{E}_t(\alpha \cdot W^\theta)$.

On the one hand,

$$\mathcal{E}_t(\alpha \cdot W^\theta) = e^{-\int_0^t \frac{|\alpha_s|^2}{2} ds + \int_0^t \alpha_s'(dW_s + \theta_s ds)}$$
$$= e^{-\int_0^t \frac{|\alpha_s|^2}{2} ds - \int_0^t \frac{U_s'}{M_s} dW_s^\theta + \int_0^t \frac{\Gamma_s^{(1)}|\theta_s|^2}{M_s} ds + \int_0^t \frac{\Gamma_s^{(1)}\theta_s'}{M_s} dW_s}$$
$$= e^{-\int_0^t \left[\frac{|\alpha_s|^2}{2} - \frac{1}{2}\left(\frac{\Gamma_s^{(1)}\theta_s}{M}\right)^2 - \frac{\Gamma_s^{(1)}|\theta_s|^2}{M_s}\right] ds - \int_0^t \frac{U_s'}{M_s} dW_s^\theta} \mathcal{E}_t\left(\frac{\Gamma^{(1)}\theta}{M} \cdot W\right).$$

Applying Ito's formula to $\ln(M)$, we get

$$d\ln(M_s) = [-2r_s + \frac{U_s'\theta_s}{M_s} - \Gamma_s^{(1)}\frac{|\theta_s|^2}{M_s} + \frac{1}{2}\frac{|U_s|^2}{M_s^2} - \Gamma_s^{(1)}\frac{U_s'\theta_s}{M_s^2}]ds + \frac{U_s'}{M_s} dW_s$$
$$= \left[-2r_s + \frac{|\alpha_s|^2}{2} - \frac{1}{2}\left|\frac{\Gamma_s^{(1)}\theta_s}{M_s}\right|^2 - \Gamma_s^{(1)}\frac{|\theta_s|^2}{M_s}\right] ds + \frac{U_s'}{M_s} dW_s^\theta.$$

Combining the above equations, we obtain

$$\mathcal{E}_t(\alpha \cdot W^\theta) = \frac{M_0}{M_t}\mathcal{E}_t\left(\frac{\Gamma^{(1)}\theta}{M} \cdot W\right) e^{-2\int_0^t r_s ds}$$

or

$$\rho_t = \frac{M_0}{M_t}\mathcal{E}_t\left(\frac{\Gamma^{(1)}\theta}{M}\cdot W\right)e^{-\int_0^t r_s ds}.$$

By the fact that M and $\frac{1}{M}$ are both bounded and $\mathbb{E}\left[\sup_{t\in[0,T]}\left|\mathcal{E}_t\left(\frac{\Gamma^{(1)}\theta}{M}\cdot W\right)\right|^p\right] < +\infty$ for any $p \in \mathbb{R}$, we have $\mathbb{E}\left[\sup_{t\in[0,T]}\rho_t^p\right] < +\infty$ for any $p \in \mathbb{R}$.

Now we validate $X^* \in L^2_{\mathcal{F}}(\Omega, C(0,T;\mathbb{R}))$ using (5.15). For any $p > 1$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \rho_s^{-1}\alpha_s'\beta_s ds\right|^p\right] \\
&\le \mathbb{E}\left[\sup_{t\in[0,T]}\rho_t^{-p}\left(\int_0^T|\alpha_s|^2ds + \int_0^T|\beta_s|^2ds\right)^p\right] \\
&\le c_p\sqrt{\mathbb{E}\left[\sup_{t\in[0,T]}\rho_t^{-2p}\right]\left(\mathbb{E}\left[\left(\int_0^T|\alpha_s|^2ds\right)^{2p}\right] + \mathbb{E}\left[\left(\int_0^T|\beta_s|^2ds\right)^{2p}\right]\right)} \\
&< +\infty.
\end{aligned}$$

Similarly we have $\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \rho_s^{-1}\theta_s'\beta_s ds\right|^p\right] < +\infty$. Also we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \rho_s^{-1}\beta_s dW_s\right|^{2p}\right] &\le \bar{c}_p\mathbb{E}\left[\left(\int_0^T \rho_s^{-2}|\beta_s|^2ds\right)^p\right] \\
&\le \bar{c}_p\mathbb{E}\left[\sup_{t\in[0,T]}\rho_t^{-2p}\left(\int_0^T|\beta_s|^2ds\right)^p\right] \\
&< +\infty,
\end{aligned}$$

where $c_p, \bar{c}_p$ are both constants only depending on p. These two inequalities lead to $X^* \in L^2_{\mathcal{F}}(\Omega; C(0,T;\mathbb{R}))$.

Finally, regarding $(X_\cdot^*, u_\cdot^*)$ as the solution to the BSDE

$$\begin{cases} dX_s = r_sX_sds + \theta_s'u_sds + u_s'dW_s, & s\in[0,T], \\ X_T = X_T^*. \end{cases} \tag{5.16}$$

By the standard estimates for Lipschitz BSDE, $u^* \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ as soon as $X^* \in L^2_{\mathcal{F}}(\Omega, C(0,T;\mathbb{R}))$. □

5.3. Equilibrium Strategy. Summarizing the preceding analysis, we obtain finally the main result of this section.

THEOREM 5.4. *Let (M,U) and $(\Gamma^{(2)},\gamma^{(2)})$ be the solutions to BSDEs (5.8) and (5.13) respectively, and $\Gamma_s = -\mu_2 e^{\int_s^T r_t dt}$. Then*

$$u_s^* = -M_s^{-1}\left[(U_s - \theta_s\mu_1 e^{\int_s^T r_v dv})X_s^* + \Gamma_s\theta_s + \gamma_s^{(2)}\right]$$

is an equilibrium strategy.

Proof. Define p, k by (5.5) and (5.7) (recall that $N = M$ and $V = U$). It is easy to check that u^*, X^*, p, k satisfies (5.4). Furthermore, Λ in the condition (3.4) is

$$\begin{aligned}\Lambda(s;t) &= p(s;t)\theta_s + k(s;t)\\ &= (M_s X_s^* - \Gamma_s^{(1)} X_t^* + \Gamma_s^{(2)} - \mathbb{E}_t\left[M_s X_s^* + \Gamma_s^{(3)}\right])\theta_s + X_s^* U_s + M_s u_s^* + \gamma_s^{(2)} - \gamma_s^{(1)} X_t^*\\ &= (M_s X_s^* + \Gamma_s^{(3)} - \mathbb{E}_t\left[M_s X_s^* + \Gamma_s^{(3)}\right])\theta_s + \Gamma_s^{(1)}(X_s^* - X_t^*)\theta_s.\end{aligned}$$

Since $M, \theta, \Gamma^{(3)}, \Gamma^{(1)}$ are all essentially bounded, $\mathbb{E}_t\left[\sup_{s\in[t,T]}(X_s^*)^2\right] < +\infty$, we deduce that Λ meets condition (3.4). It follows from Theorem 3.3 that u^* is an equilibrium. ∎

5.4. Examples. Equilibrium strategies for mean–variance models have been studied in [2, 4, 5] among others in different frameworks. In this subsection, we will compare our results with some existing ones in literature.

5.4.1. Deterministic risk premium. Let us first consider the case when the risk premium is deterministic function of time. Then $U = 0$, $\gamma^{(2)} = 0$, and

$$M_t = e^{2\int_t^T r_v dv}\left(1 + \mu_1 \int_t^T e^{-\int_s^T r_v dv}|\theta_s|^2 ds\right).$$

The equilibrium strategy is given by

$$u_t^* = \frac{\mu_1 e^{\int_t^T r_v dv}}{M_t}\theta_t X_t^* + \frac{\mu_2 e^{\int_t^T r_v dv}}{M_t}\theta_t.$$

Case 1: $\mu_1 = 0$**.**

When $\mu_1 = 0$, the objective is exactly the same as in [2] and [4], in which the equilibrium is however defined within the class of (deterministic) feedback controls.

By Theorem 5.4,

$$u_t^* = e^{-\int_t^T r_v dv}\mu_2\theta_t$$

is a mean-variance equilibrium strategy. This equilibrium coincides with the one obtained in [2] and [4] although the definitions of equilibrium are different. The *ex-post* reason is that the feedback part of our equilibrium is absent, and so is the gap between the two definitions.

Case 2: $\mu_2 = 0$**.**

When $\mu_2 = 0$, the objective is equivalent to the one in [5]. In this case, our equilibrium is, explicitly,

$$u_t^* = \frac{\mu_1 e^{\int_t^T r_v dv}}{M_t}\theta_t X_t^*.$$

In [5], the equilibrium is defined for the class of feedback controls as in [4]. Therein the equilibrium strategy is derived in a linear feedback form $u_t^* = c_t X_t^*$ with c_t uniquely determined by an integral equation (whose unique solvability is established). We can easily show that the linear coefficient of our equilibrium above does not satisfy the integral equation in [5]. This, in turn, indicates the difference between the two definitions of equilibriums (open-loop and feedback).

5.4.2. Stochastic risk premium. When the risk premium of the market is a stochastic process, the PDE (HJB equation) approach employed by [4] or [5], where the definition of equilibrium is in the class of feedback controls, does no longer work. To our best knowledge, our result is the first attempt to formulate and find equilibrium with random market parameters.

Case 1: $\mu_1 = 0$**.**

When $\mu_1 = 0$, $U = 0$, $M_t = e^{2\int_t^T r_v dv}$, and our equilibrium is

$$u_t^* = e^{-\int_t^T r_v dv}\mu_2\theta_t - e^{-2\int_t^T r_v dv}\gamma_t^{(2)}.$$

This strategy consists of two parts. The first part is in the same form as that in the deterministic risk premium case, and the second part is to hedge the uncertainty arising from the randomness of θ.

Case 2: $\mu_2 = 0$**.**

When $\mu_2 = 0$, $\gamma^{(2)} = 0$, and our equilibrium is

$$u_t^* = \left(\frac{\mu_1 e^{\int_t^T r_v dv}}{M_t}\theta_t - \frac{U_t}{M_t} \right) X_t^*.$$

The linear feedback coefficient in this equilibrium also consists of two parts. The first part is formally the same as its deterministic counterpart, whereas the second part is for the randomness of the parameter θ.

6. Concluding Remarks. This paper, we believe, has posed more questions than answers. The flow of FBSDEs (3.6) is an interesting class of equations, whose general solvability begs for systematic investigations. How to adapt the generalized HJB approach of [4, 5] to our open-loop control framework, even when all the coefficients are deterministic, warrants a careful study (but notice the fundamental difference in the definitions of equilibrium). Extension beyond the realm of LQ may open up an entirely new avenue for stochastic control. Finally, how our game theoretic formulation may be extended to other types of time-inconsistency, e.g., that caused by probability distortion, promises to be an equally exciting research topic. The research on the last problem is in progress and will appear in a forthcoming paper.

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