On Borwein-Wiersma Decompositions of Monotone Linear Relations

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Abstract

Monotone operators are of basic importance in optimization as they generalize simultaneously subdifferential operators of convex functions and positive semidefinite (not necessarily symmetric) matrices. In 1970, Asplund studied the additive decomposition of a maximal monotone operator as the sum of a subdifferential operator and an "irreducible" monotone operator. In 2007, Borwein and Wiersma [*SIAM J. Optim.* 18 (2007), pp. 946–960] introduced another additive decomposition, where the maximal monotone operator is written as the sum of a subdifferential operator and a "skew" monotone operator. Both decompositions are variants of the well-known additive decomposition of a matrix via its symmetric and skew part.

This paper presents a detailed study of the Borwein-Wiersma decomposition of a maximal monotone linear relation. We give sufficient conditions and characterizations for a maximal monotone linear relation to be Borwein-Wiersma decomposable, and show that Borwein-Wiersma decomposability implies Asplund decomposability. We exhibit irreducible linear maximal monotone operators without full domain, thus answering one of the questions raised by Borwein and Wiersma. The Borwein-Wiersma decomposition of any maximal monotone linear relation is made quite explicit in Hilbert space.

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1 Introduction

Monotone operators play important roles in convex analysis and optimization [18, 24, 19, 16, 30, 23, 9, 6]. In the current literature, there are two decompositions for maximal monotone operators: the first was introduced by Asplund in 1970 [1] and the second by Borwein and Wiersma in 2007 [7]. These decompositions express a maximal monotone operator as the sum of the subdifferential operator of a convex function and a singular

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part (either irreducible or skew), and they can be viewed as analogues of the well known decomposition of a matrix into the sum of a symmetric and a skew part. They provide intrinsic insight into the structure of monotone operators and they have the potential to be employed in numerical algorithms (such as proximal point algorithms [10, 20]). It is instructive to study these decompositions for monotone linear relations to test the general theory and include linear monotone operators as interesting special cases [17, 2]. Our goal in this paper is to study the Borwein-Wiersma decomposition of a maximal monotone linear relation. It turns out that a complete and elegant characterization of Borwein-Wiersma decomposability exists and that the Borwein-Wiersma decomposition can be made quite explicit (see Theorem 5.1 and Example 6.4).

The paper is organized as follows. After presenting auxiliary results in Sections 2, we show in Section 3 that Borwein-Wiersma decomposability always implies Asplund decomposability, and we present some sufficient conditions for a maximal monotone linear relation to be Borwein-Wiersma decomposable. Section 4 is devoted to the uniqueness of the Borwein-Wiersma decomposition, and we characterize those linear relations that are subdifferential operators of proper lower semicontinuous convex functions. In Section 5, it is shown that a maximal monotone linear relation A is Borwein-Wiersma decomposable if and only if the domain of A is a subset of the domain of its adjoint A^*. This is followed by examples illustrating neither A nor A^* may be Borwein-Wiersma decomposable. Moreover, it can happen that A is Borwein-Wiersma decomposable, whereas A^* is not. Residing in a Hilbert space either ℓ^2 or $L^2[0,1]$, our examples are irreducible linear maximal monotone operators without full domain, and they are utilized to provide an answer to Borwein and Wiersma's [7, Question (4) in Section 7]. In Section 6, we give more explicit Borwein-Wiersma decompositions in Hilbert spaces. The paper is concluded by a summary in Section 7.

We start with some definitions and terminology. Throughout this paper, we assume that

$$X \text{ is a reflexive real Banach space, with topological dual space } X^*, \text{ and pairing } \langle \cdot, \cdot \rangle.$$

Let A be a set-valued operator from X to X^*. Then A is *monotone* if

$$\big(\forall (x,x^*) \in \operatorname{gra} A\big)\big(\forall (y,y^*) \in \operatorname{gra} A\big) \quad \langle x-y, x^*-y^* \rangle \geq 0,$$

where $\operatorname{gra} A := \big\{(x,x^*) \in X \times X^* \mid x^* \in Ax\big\}$; A is said to be *maximal monotone* if no proper enlargement (in the sense of graph inclusion) of A is monotone. The *inverse operator* $A^{-1}\colon X^* \rightrightarrows X$ is given by $\operatorname{gra} A^{-1} := \big\{(x^*,x) \in X^* \times X \mid x^* \in Ax\big\}$; the *domain* of A is $\operatorname{dom} A := \big\{x \in X \mid Ax \neq \varnothing\big\}$, and its *range* is $\operatorname{ran} A := A(X)$. Note that A is said to be a *linear relation* if $\operatorname{gra} A$ is a linear subspace of $X \times X^*$ (see [12]). We say A is a *maximal monotone linear relation* if A is a maximal monotone operator and $\operatorname{gra} A$ is a linear subspace of $X \times X^*$. The *adjoint* of A, written A^*, is defined by

$$\operatorname{gra} A^* := \big\{(x,x^*) \in X \times X^* \mid (x^*, -x) \in (\operatorname{gra} A)^\perp\big\},$$

where, for any subset S of a reflexive Banach space Z with continuous dual space Z^*, $S^\perp := \big\{z^* \in Z^* \mid z^*|_S \equiv 0\big\}$. Let A be a linear relation from X to X^*. We say that A is *skew* if $\langle x, x^* \rangle = 0$, $\forall (x,x^*) \in \operatorname{gra} A$; equivalently, if $\operatorname{gra} A \subseteq \operatorname{gra}(-A^*)$. Furthermore, A is *symmetric* if $\operatorname{gra} A \subseteq \operatorname{gra} A^*$; equivalently, if $\langle x, y^* \rangle = \langle y, x^* \rangle$, $\forall (x,x^*), (y,y^*) \in \operatorname{gra} A$. By saying $A : X \rightrightarrows X^*$ *at most single-valued*, we mean that for every $x \in X$, Ax is either a singleton or empty. In this case, we follow a slight but common abuse of notation and write $A\colon \operatorname{dom} A \to X^*$. Conversely, if $T\colon D \to X^*$, we may identify T with $A : X \rightrightarrows X^*$, where A is at most single-valued with $\operatorname{dom} A = D$. We define the *symmetric part* and the *skew part* of A via

$$A_+ := \tfrac{1}{2}A + \tfrac{1}{2}A^* \quad \text{and} \quad A_\circ := \tfrac{1}{2}A - \tfrac{1}{2}A^*, \tag{1}$$

respectively. It is easy to check that A_+ is symmetric and that $A_\circ$ is skew.

Let $x \in X$ and $C^* \subseteq X^*$. We write $\langle x, C^* \rangle := \{\langle x, c^* \rangle \mid c^* \in C^*\}$. If $\langle x, C^* \rangle = \{a\}$ for some constant $a \in \mathbb{R}$, then we write $\langle x, C^* \rangle = a$ for convenience. For a monotone linear relation $A\colon X \rightrightarrows X^*$ it will be very useful to define the extended-valued quadratic function (which is actually a special case of *Fitzpatrick's last function* [6] for the linear relation A)

$$q_A\colon x \mapsto \begin{cases} \frac{1}{2}\langle x, Ax\rangle, & \text{if } x \in \operatorname{dom} A; \\ +\infty, & \text{otherwise.} \end{cases} \tag{2}$$

When A is linear and single-valued with full domain, we shall use the well known fact (see, e.g., [17]) that

$$\nabla q_A = A_+. \tag{3}$$

For $f\colon X \to \left]-\infty, +\infty\right]$, set $\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}$ and let $f^*\colon X^* \to [-\infty, +\infty]\colon x^* \mapsto \sup_{x\in X}(\langle x, x^*\rangle - f(x))$ be the *Fenchel conjugate* of f. We denote by $\overline{f}$ the *lower semicontinuous* hull of f. Recall that f is said to be *proper* if $\operatorname{dom} f \neq \varnothing$. If f is convex, $\partial f\colon X \rightrightarrows X^*\colon x \mapsto \big\{x^* \in X^* \mid (\forall y \in X)\ \langle y - x, x^*\rangle + f(x) \leq f(y)\big\}$ is the *subdifferential operator* of f. For a subset C of X, $\overline{C}$ stands for the closure of C in X. Write ι_C for the *indicator function* of C, i.e., $\iota_C(x) = 0$, if $x \in C$; and $\iota_C(x) = +\infty$, otherwise. It will be convenient to work with the *indicator mapping* $\mathbb{I}_C\colon X \to X^*$, defined by $\mathbb{I}_C(x) = \{0\}$, if $x \in C$; $\mathbb{I}_C(x) = \varnothing$, otherwise.

The central goal of this paper is to provide a detailed analysis of the following notion in the context of maximal monotone linear relations.

Definition 1.1 (Borwein-Wiersma decomposition [7]) *The set-valued operator* $A : X \rightrightarrows X^*$ *is* Borwein-Wiersma decomposable *if*

$$A = \partial f + S, \tag{4}$$

where $f : X \to \left]-\infty, +\infty\right]$ *is proper lower semicontinuous and convex, and where* $S : X \rightrightarrows X^*$ *is skew and at most single-valued. The right side of* (4) *is a* Borwein-Wiersma decomposition *of* A.

Note that every single-valued linear monotone operator A with full domain is Borwein-Wiersma decomposable, with Borwein-Wiersma decomposition

$$A = A_+ + A_\circ = \nabla q_A + A_\circ. \tag{5}$$

Definition 1.2 (Asplund irreducibility [1]) *The set-valued operator* $A\colon X \rightrightarrows X^*$ *is* irreducible *(sometimes termed "acyclic"* [7]*) if whenever*

$$A = \partial f + S,$$

with $f : X \to \left]-\infty, +\infty\right]$ *proper lower semicontinuous and convex, and* $S : X \rightrightarrows X^*$ *monotone, then necessarily* $\operatorname{ran}(\partial f)|_{\operatorname{dom} A}$ *is a singleton.*

As we shall see in Section 3, the following decomposition is less restrictive.

Definition 1.3 (Asplund decomposition [1]) *The set-valued operator* $A\colon X \rightrightarrows X^*$ *is* Asplund decomposable *if*

$$A = \partial f + S, \tag{6}$$

where $f\colon X \to \left]-\infty, +\infty\right]$ *is proper, lower semicontinuous, and convex, and where* S *is irreducible. The right side of* (6) *is an* Asplund decomposition *of* A.

2 Auxiliary results on monotone linear relations

In this section, we gather some basic properties about monotone linear relations, and conditions for them to be maximal monotone. These results are used frequently in the sequel. We start with properties for general linear relations.

Fact 2.1 (Cross) *Let $A: X \rightrightarrows X^*$ be a linear relation. Then the following hold.*

(i) *$A0$ is a linear subspace of X^*.*

(ii) *$Ax = x^* + A0, \quad \forall x^* \in Ax$.*

(iii) *$(\forall(\alpha, \beta) \in \mathbb{R}^2 \smallsetminus \{(0,0)\})$ $(\forall x, y \in \operatorname{dom} A)$ $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$.*

(iv) *$\operatorname{dom} A^* = \big\{x \in X \mid \langle x, A(\cdot)\rangle$ is single-valued and continuous on $\operatorname{dom} A\big\}$.*

(v) *$(A^*)^{-1} = (A^{-1})^*$.*

(vi) *$(\forall x \in \operatorname{dom} A^*)(\forall y \in \operatorname{dom} A)$ $\langle A^*x, y\rangle = \langle x, Ay\rangle$ is a singleton.*

(vii) *If $\operatorname{gra} A$ is closed, then $A^{**} = A$.*

(viii) *If $\operatorname{dom} A$ is closed, then $\operatorname{dom} A^*$ is closed.*

Proof. (i): See [12, Corollary I.2.4]. (ii): See [12, Proposition I.2.8(a)]. (iii): See [12, Corollary I.2.5]. (iv): See [12, Proposition III.1.2]. (v): See [12, Proposition III.1.3(b)]. (vi): See [12, Proposition III.1.2]. (vii): See [12, Exercise VIII.1.12]. (viii): See [12, Corollary III.4.3(a), Proposition III.4.9(i)(ii), Theorem III.4.2(a) and Corollary III.4.5]. ■

Additional information is available when dealing with *monotone* linear relations.

Fact 2.2 *Let $A: X \rightrightarrows X^*$ be a monotone linear relation. Then the following hold.*

(i) *$\operatorname{dom} A \subseteq (A0)^\perp$ and $A0 \subseteq (\operatorname{dom} A)^\perp$.*

(ii) *The function $\operatorname{dom} A \to \mathbb{R}\colon y \mapsto \langle y, Ay\rangle$ is well defined and convex.*

(iii) *For every $x \in (A0)^\perp$, the function $\operatorname{dom} A \to \mathbb{R}\colon y \mapsto \langle x, Ay\rangle$ is well defined and linear.*

(iv) *If A is maximal monotone, then $\overline{\operatorname{dom} A^*} = \overline{\operatorname{dom} A} = (A0)^\perp$ and $A0 = A^*0 = A_+0 = A_\circ 0 = (\operatorname{dom} A)^\perp$.*

(v) *If $\operatorname{dom} A$ is closed, then: A is maximal monotone $\Leftrightarrow (\operatorname{dom} A)^\perp = A0$.*

(vi) *If A is maximal monotone and $\operatorname{dom} A$ is closed, then $\operatorname{dom} A^* = \operatorname{dom} A$.*

(vii) *If A is maximal monotone and $\operatorname{dom} A \subseteq \operatorname{dom} A^*$, then $A = A_+ + A_\circ$, $A_+ = A - A_\circ$, and $A_\circ = A - A_+$.*

Proof. (i): See [3, Proposition 2.2(i)]. (ii): See [3, Proposition 2.3]. (iii): See [3, Proposition 2.2(iii)]. (iv): By [3, Theorem 3.2], we have $A0 = A^*0 = (\operatorname{dom} A)^\perp = (\operatorname{dom} A^*)^\perp$ and $\overline{\operatorname{dom} A} = \overline{\operatorname{dom} A^*}$. Hence $(A0)^\perp = (\operatorname{dom} A)^{\perp\perp} = \overline{\operatorname{dom} A}$. By Fact 2.1(i), $A0$ is a linear subspace of X^*. Hence $A_+0 = (A0 + A^*0)/2 = (A0 + A0)/2 = A0$ and similarly $A_\circ 0 = A0$. (v): See [3, Corollary 6.6]. (vi): Combine (iv) with Fact 2.1(viii). (vii): We show only the proof of $A = A_+ + A_\circ$ as the other two proofs are analogous. Clearly,

$\operatorname{dom} A_+ = \operatorname{dom} A_\circ = \operatorname{dom} A \cap \operatorname{dom} A^* = \operatorname{dom} A$. Let $x \in \operatorname{dom} A$, and $x^* \in Ax$ and $y^* \in A^*x$. We write $x^* = \frac{x^*+y^*}{2} + \frac{x^*-y^*}{2} \in (A_+ + A_\circ)x$. Then, by Fact 2.1(ii), $Ax = x^* + A0 = x^* + (A_+ + A_\circ)0 = (A_+ + A_\circ)x$. Therefore, $A = A_+ + A_\circ$. ■

Proposition 2.3 *Let $S\colon X \rightrightarrows X^*$ be a linear relation such that S is at most single-valued. Then S is skew if and only if $\langle Sx, y\rangle = -\langle Sy, x\rangle$, $\forall x, y \in \operatorname{dom} S$.*

Proof. "⇒": Let $x, y \in \operatorname{dom} S$. Then $0 = \langle S(x+y), x+y\rangle = \langle Sx, x\rangle + \langle Sy, y\rangle + \langle Sx, y\rangle + \langle Sy, x\rangle = \langle Sx, y\rangle + \langle Sy, x\rangle$. Hence $\langle Sx, y\rangle = -\langle Sy, x\rangle$. "⇐": Indeed, for $x \in \operatorname{dom} S$, we have $\langle Sx, x\rangle = -\langle Sx, x\rangle$ and so $\langle Sx, x\rangle = 0$. ■

Fact 2.4 (Brézis-Browder) (See [8, Theorem 2].) *Let $A\colon X \rightrightarrows X^*$ be a monotone linear relation such that $\operatorname{gra} A$ is closed. Then the following are equivalent.*

(i) *A is maximal monotone.*

(ii) *A^* is maximal monotone.*

(iii) *A^* is monotone.*

Fact 2.5 (Phelps-Simons) (See [17, Corollary 2.6 and Proposition 3.2(h)].) *Let $A\colon X \to X^*$ be monotone and linear. Then A is maximal monotone and continuous.*

Remark 2.6 Fact 2.5 also holds in locally convex spaces, see [28, Proposition 23].

Proposition 2.7 *Let $A\colon X \rightrightarrows X^*$ be a maximal monotone linear relation. Then A is symmetric ⇔ $A = A^*$.*

Proof. "⇒": Assume that A is symmetric, i.e., $\operatorname{gra} A \subseteq \operatorname{gra} A^*$. Since A is maximal monotone, so is A^* by Fact 2.4. Therefore, $A = A^*$. "⇐": Obvious. ■

Fact 2.2(v) provides a characterization of maximal monotonicity for certain monotone linear relations. More can be said in finite-dimensional spaces. We require the following lemma, where $\dim F$ stands for the dimension of a subspace F of X.

Lemma 2.8 *Suppose that X is finite-dimensional and let $A\colon X \rightrightarrows X^*$ be a linear relation. Then $\dim(\operatorname{gra} A) = \dim(\operatorname{dom} A) + \dim A0$.*

Proof. We shall construct a basis of $\operatorname{gra} A$. By Fact 2.1(i), $A0$ is a linear subspace. Let $\{x_1^*, \ldots, x_k^*\}$ be a basis of $A0$, and let $\{x_{k+1}, \ldots, x_l\}$ be a basis of $\operatorname{dom} A$. From Fact 2.1(ii), it is easy to show $\{(0, x_1^*), \ldots, (0, x_k^*), (x_{k+1}, x_{k+1}^*), \ldots, (x_l, x_l^*)\}$ is a basis of $\operatorname{gra} A$, where $x_i^* \in Ax_i$, $i \in \{k+1, \ldots, l\}$. Thus $\dim(\operatorname{gra} A) = l = \dim(\operatorname{dom} A) + \dim A0$. ■

Lemma 2.8 allows us to get a satisfactory characterization of maximal monotonicity of linear relations in finite-dimensional spaces.

Proposition 2.9 *Suppose that X is finite-dimensional, set $n = \dim X$, and let $A\colon X \rightrightarrows X^*$ be a monotone linear relation. Then A is maximal monotone if and only if $\dim \operatorname{gra} A = n$.*

Proof. Since linear subspaces of X are closed, we see from Fact 2.2(v) that

$$A \text{ is maximal monotone} \Leftrightarrow \operatorname{dom} A = (A0)^{\perp}. \tag{7}$$

Assume first that A is maximal monotone. Then $\operatorname{dom} A = (A0)^{\perp}$. By Lemma 2.8, $\dim(\operatorname{gra} A) = \dim(\operatorname{dom} A) + \dim(A0) = \dim((A0)^{\perp}) + \dim(A0) = n$. Conversely, let $\dim(\operatorname{gra} A) = n$. By Lemma 2.8, we have that $\dim(\operatorname{dom} A) = n - \dim(A0)$. As $\dim((A0)^{\perp}) = n - \dim(A0)$ and $\operatorname{dom} A \subseteq (A0)^{\perp}$ by Fact 2.2(i), we have that $\operatorname{dom} A = (A0)^{\perp}$. By (7), A is maximal monotone. ■

3 Borwein-Wiersma decompositions

The following fact, due to Censor, Iusem and Zenios [11, 15], was previously known in $\mathbb{R}^n$. Here we give a different proof and extend the result to Banach spaces.

Fact 3.1 (Censor, Iusem and Zenios) *The subdifferential operator of a proper lower semicontinuous convex function $f\colon X \to\ ]-\infty, +\infty]$ is* paramonotone*, i.e., if*

$$x^* \in \partial f(x), \quad y^* \in \partial f(y), \tag{8}$$

and

$$\langle x^* - y^*, x - y\rangle = 0, \tag{9}$$

then $x^* \in \partial f(y)$ and $y^* \in \partial f(x)$.

Proof. By (9),

$$\langle x^*, x\rangle + \langle y^*, y\rangle = \langle x^*, y\rangle + \langle y^*, x\rangle. \tag{10}$$

By (8),

$$f^*(x^*) + f(x) = \langle x^*, x\rangle, \quad f^*(y^*) + f(y) = \langle y^*, y\rangle.$$

Adding them, followed by using (10), yields

$$f^*(x^*) + f(y) + f^*(y^*) + f(x) = \langle x^*, y\rangle + \langle y^*, x\rangle,$$

$$[f^*(x^*) + f(y) - \langle x^*, y\rangle] + [f^*(y^*) + f(x) - \langle y^*, x\rangle] = 0.$$

Since each bracketed term is nonnegative, we must have $f^*(x^*) + f(y) = \langle x^*, y\rangle$ and $f^*(y^*) + f(x) = \langle y^*, x\rangle$. It follows that $x^* \in \partial f(y)$ and that $y^* \in \partial f(x)$. ■

The following result provides a powerful criterion for determining whether a given operator is irreducible and hence Asplund decomposable.

Theorem 3.2 *Let $A : X \rightrightarrows X^*$ be monotone and at most single-valued. Suppose that there exists a dense subset D of* $\operatorname{dom} A$ *such that*

$$\langle Ax - Ay, x - y\rangle = 0 \quad \forall x, y \in D.$$

Then A is irreducible and hence Asplund decomposable.

Proof. Let $a \in D$ and $D' := D - \{a\}$. Define $A' : \operatorname{dom} A - \{a\} \to A(\cdot + a)$. Then A is irreducible if and only if A' is irreducible. Now we show A' is irreducible. By assumptions, $0 \in D'$ and

$$\langle A'x - A'y, x - y\rangle = 0 \quad \forall x, y \in D'.$$

Let $A' = \partial f + R$, where f is proper lower semicontinuous and convex, and R is monotone. Since A' is single-valued on $\operatorname{dom} A'$, we have that ∂f and R are single-valued on $\operatorname{dom} A'$ and that

$$R = A' - \partial f \quad \text{ on } \operatorname{dom} A'.$$

By taking $x_0^* \in \partial f(0)$, rewriting $A' = (\partial f - x_0^*) + (x_0^* + R)$, we can and do suppose $\partial f(0) = \{0\}$. For $x, y \in D'$ we have $\langle A'x - A'y, x - y\rangle = 0$. Then for $x, y \in D'$

$$0 \leq \langle R(x) - R(y), x - y\rangle = \langle A'x - A'y, x - y\rangle - \langle \partial f(x) - \partial f(y), x - y\rangle = -\langle \partial f(x) - \partial f(y), x - y\rangle.$$

On the other hand, ∂f is monotone, thus,

$$\langle \partial f(x) - \partial f(y), x - y\rangle = 0, \quad \forall x, y \in D'. \tag{11}$$

Using $\partial f(0) = \{0\}$,

$$\langle \partial f(x) - 0, x - 0\rangle = 0, \quad \forall x \in D'. \tag{12}$$

As ∂f is paramonotone by Fact 3.1, $\partial f(x) = \{0\}$ so that $x \in \operatorname{argmin} f$. This implies that $D' \subseteq \operatorname{argmin} f$ since $x \in D'$ was chosen arbitrarily. As f is lower semicontinuous, $\operatorname{argmin} f$ is closed. Using that D' is dense in $\operatorname{dom} A'$, it follows that $\operatorname{dom} A' \subseteq \overline{D'} \subseteq \operatorname{argmin} f$. Since ∂f is single-valued on $\operatorname{dom} A'$, $\partial f(x) = \{0\}$, $\forall x \in \operatorname{dom} A'$. Hence A' is irreducible, and so is A. ■

Remark 3.3 In Theorem 3.2, the assumption that A be at most single-valued is important: indeed, let L be a proper subspace of $\mathbb{R}^n$. Then $\partial\iota_L$ is a linear relation and skew, yet $\partial\iota_L = \partial\iota_L + 0$ is not irreducible.

Theorem 3.2 and the definitions of the two decomposabilities now yield the following.

Corollary 3.4 *Let $A : X \rightrightarrows X^*$ be maximal monotone such that A is Borwein-Wiersma decomposable. Then A is Asplund decomposable.*

We proceed to give a few sufficient conditions for a maximal monotone linear relation to be Borwein-Wiersma decomposable. The following simple observation will be needed.

Lemma 3.5 *Let $A : X \rightrightarrows X^*$ be a monotone linear relation such that A is Borwein-Wiersma decomposable, say $A = \partial f + S$, where $f\colon X \to \left]-\infty, +\infty\right]$ is proper, lower semicontinuous, and convex, and where $S\colon X \rightrightarrows X^*$ is at most single-valued and skew. Then the following hold.*

(i) $\partial f + \mathbb{I}_{\operatorname{dom} A}\colon x \mapsto \begin{cases} \partial f(x), & \text{if } x \in \operatorname{dom} A;\\ \varnothing, & \text{otherwise}\end{cases}$ *is a monotone linear relation.*

(ii) $\operatorname{dom} A \subseteq \operatorname{dom}\partial f \subseteq \operatorname{dom} f \subseteq (A0)^{\perp}$.

(iii) *If A is maximal monotone, then* $\operatorname{dom} A \subseteq \operatorname{dom}\partial f \subseteq \operatorname{dom} f \subseteq \overline{\operatorname{dom} A}$.

(iv) *If A is maximal monotone and* $\operatorname{dom} A$ *is closed, then* $\operatorname{dom}\partial f = \operatorname{dom} A = \operatorname{dom} f$.

Proof. (i): Indeed, on $\operatorname{dom} A$, we see that $\partial f = A - S$ is the difference of two linear relations.

(ii): Clearly $\operatorname{dom} A \subseteq \operatorname{dom} \partial f$. As $S0 = 0$, we have $A0 = \partial f(0)$. Thus, $\forall x^* \in A0$, $x \in X$,

$$\langle x^*, x\rangle \leq f(x) - f(0).$$

Then $\sigma_{A0}(x) \leq f(x) - f(0)$, where σ_{A0} is the support function of $A0$. If $x \notin (A0)^{\perp}$, then $\sigma_{A0}(x) = +\infty$ since $A0$ is a linear subspace, so $f(x) = +\infty$, $\forall x \notin (A0)^{\perp}$. Therefore, $\operatorname{dom} f \subseteq (A0)^{\perp}$. Altogether, (ii) holds.

(iii): Combine (ii) with Fact 2.2(iv). (iv): This is clear from (iii). ■

Fact 3.6 (See [29, Proposition 3.3].) *Let $A : X \rightrightarrows X^*$ be a monotone linear relation such that A is symmetric. Then the following hold.*

(i) *q_A is convex and $\overline{q_A} + \iota_{\operatorname{dom} A} = q_A$.*

(ii) $\operatorname{gra} A \subseteq \operatorname{gra} \partial \overline{q_A}$.

(iii) *If A is maximal monotone, then $A = \partial \overline{q_A}$.*

Theorem 3.7 *Let $A\colon X \rightrightarrows X^*$ be a maximal monotone linear relation such that $\operatorname{dom} A \subseteq \operatorname{dom} A^*$. Then A is Borwein-Wiersma decomposable via*

$$A = \partial \overline{q_A} + S,$$

where S is an arbitrary linear single-valued selection of $A_\circ$. Moreover, $\partial \overline{q_A} = A_+$ on $\operatorname{dom} A$.

Proof. From Fact 2.4, A^* is monotone, so A_+ is monotone. By Fact 2.1(vi), $q_{A_+} = q_A$, using Fact 3.6(ii), $\operatorname{gra} A_+ \subseteq \operatorname{gra} \partial \overline{q_{A_+}} = \operatorname{gra} \partial \overline{q_A}$. Let $S : \operatorname{dom} A \to X^*$ be a linear selection of $A_\circ$ (the existence of which is guaranteed by a standard Zorn's lemma argument). By Fact 2.1(vi), S is skew. Then, by Fact 2.2(vii), we have $A = A_+ + S \subseteq \partial \overline{q_A} + S$. Since A is maximal monotone, $A = \partial \overline{q_A} + S$, which is the announced Borwein-Wiersma decomposition. Moreover, on $\operatorname{dom} A$, we have $\partial \overline{q_A} = A - S = A_+$. ■

Corollary 3.8 *Let $A : X \rightrightarrows X^*$ be a maximal monotone linear relation such that A is symmetric. Then A and A^{-1} are Borwein-Wiersma decomposable, with decompositions $A = \partial \overline{q_A} + 0$ and $A^{-1} = \partial q_A^* + 0$, respectively.*

Proof. Using Proposition 2.7 and Fact 2.1(v), we obtain $A = A^*$ and $A^{-1} = (A^*)^{-1} = (A^{-1})^*$. Hence, Theorem 3.7 applies; in fact, $A = \partial \overline{q_A}$ and $A^{-1} = \partial \overline{q_{A^{-1}}} = \partial q_A^*$. ■

Corollary 3.9 *Let $A\colon X \rightrightarrows X^*$ be a maximal monotone linear relation such that $\operatorname{dom} A$ is closed, and let S be a single-valued linear selection of $A_\circ$. Then $q_A = \overline{q_A}$, $A_+ = \partial q_A$ is maximal monotone, and A and A^* are Borwein-Wiersma decomposable, with decompositions $A = A_+ + S$ and $A^* = A_+ - S$, respectively.*

Proof. Fact 2.1(vii) and Fact 2.2(vi) imply that $A^{**} = A$ and that $\operatorname{dom} A^* = \operatorname{dom} A$. By Fact 2.4, A^* is maximal monotone. In view of Fact 2.2(vii), $A = A_+ + A_\circ$ and $A^* = (A^*)_+ + (A^*)_\circ = A_+ - A_\circ$. Theorem 3.7 yields the Borwein-Wiersma decomposition $A = \partial \overline{q_A} + S$. Hence $\operatorname{dom} A \subseteq \operatorname{dom} \partial \overline{q_A} \subseteq \operatorname{dom} \overline{q_A} \subseteq \overline{\operatorname{dom} A} = \operatorname{dom} A$. In turn, since $\operatorname{dom} A = \operatorname{dom} A_+$ and $q_A = q_{A_+}$, this implies that $\operatorname{dom} A_+ = \operatorname{dom} \partial \overline{q_{A_+}} = \operatorname{dom} \overline{q_{A_+}}$. In view of Fact 3.6(i)&(ii), $q_{A_+} = \overline{q_{A_+}}$ and $\operatorname{gra} A_+ \subseteq \operatorname{gra} \partial \overline{q_{A_+}}$. By Theorem 3.7, $A_+ = \partial \overline{q_A}$ on $\operatorname{dom} A$. Since $\operatorname{dom} A = \operatorname{dom} A_+ = \operatorname{dom} \partial \overline{q_A}$ and $q_A = q_{A_+} = \overline{q_{A_+}} = \overline{q_A}$, this implies that $A_+ = \partial q_A = \partial \overline{q_A}$ *everywhere*. Therefore, A_+ is maximal monotone. Since $A_+ = (A^*)_+$ and $-S$ is a single-valued linear section of $(A^*)_\circ = -A_\circ$, we obtain similarly the Borwein-Wiersma decomposition $A^* = A_+ - S$. ■

Theorem 3.10 *Let $A : X \rightrightarrows X^*$ be a maximal monotone linear relation such that A is skew, and let S be a single-valued linear selection of A. Then A is Borwein-Wiersma decomposable via $\partial\iota_{\overline{\operatorname{dom} A}} + S$.*

Proof. Clearly, S is skew. Fact 2.1(ii) and Fact 2.2(iv) imply that $A = A0+S = (\operatorname{dom} A)^\perp + S = \partial\iota_{\overline{\operatorname{dom} A}} + S$, as announced. Alternatively, by [26, Lemma 2.2], $\operatorname{dom} A \subseteq \operatorname{dom} A^*$ and now apply Theorem 3.7. ■

Under a mild constraint qualification, the sum of two Borwein-Wiersma decomposable operators is also Borwein-Wiersma decomposable and the decomposition of the sum is the corresponding sum of the decompositions.

Proposition 3.11 (sum rule) *Let A_1 and A_2 be maximal monotone linear relations from X to X^*. Suppose that A_1 and A_2 are Borwein-Wiersma decomposable via $A_1 = \partial f_1 + S_1$ and $A_2 = \partial f_2 + S_2$, respectively. Suppose that* $\operatorname{dom} A_1 - \operatorname{dom} A_2$ *is closed. Then $A_1 + A_2$ is Borwein-Wiersma decomposable via $A_1 + A_2 = \partial(f_1 + f_2) + (S_1 + S_2)$.*

Proof. By Lemma 3.5(iii), $\operatorname{dom} A_1 \subseteq \operatorname{dom} f_1 \subseteq \overline{\operatorname{dom} A_1}$ and $\operatorname{dom} A_2 \subseteq \operatorname{dom} f_2 \subseteq \overline{\operatorname{dom} A_2}$. Hence $\operatorname{dom} A_1 - \operatorname{dom} A_2 \subseteq \operatorname{dom} f_1 - \operatorname{dom} f_2 \subseteq \overline{\operatorname{dom} A_1} - \overline{\operatorname{dom} A_2} \subseteq \overline{\operatorname{dom} A_1 - \operatorname{dom} A_2} = \operatorname{dom} A_1 - \operatorname{dom} A_2$. Thus, $\operatorname{dom} f_1 - \operatorname{dom} f_2 = \operatorname{dom} A_1 - \operatorname{dom} A_2$ is a closed subspace of X. By [24, Theorem 18.2], $\partial f_1 + \partial f_2 = \partial(f_1 + f_2)$; furthermore, $S_1 + S_2$ is clearly skew. The result thus follows. ■

4 Uniqueness results

The main result in this section (Theorem 4.8) states that if a maximal monotone linear relation A is Borwein-Wiersma decomposable, then the subdifferential part of its decomposition is unique on $\operatorname{dom} A$. We start by showing that subdifferential operators that are monotone linear relations are actually symmetric, which is a variant of a well known result from Calculus.

Lemma 4.1 *Let $f : X \to \left]-\infty, +\infty\right]$ be proper, lower semicontinuous, and convex. Suppose that the maximal monotone operator ∂f is a linear relation with closed domain. Then $\partial f = (\partial f)^*$.*

Proof. Set $A := \partial f$ and $Y := \operatorname{dom} f$. Since $\operatorname{dom} A$ is closed, [24, Theorem 18.6] implies that $\operatorname{dom} f = Y = \operatorname{dom} A$. By Fact 2.2(vi), $\operatorname{dom} A^* = \operatorname{dom} A$. Let $x \in Y$ and consider the directional derivative $g = f'(x; \cdot)$, i.e.,

$$g \colon X \to [-\infty, +\infty] : y \mapsto \lim_{t \downarrow 0} \frac{f(x + ty) - f(x)}{t}.$$

By [30, Theorem 2.1.14], $\operatorname{dom} g = \bigcup_{r \geq 0} r \cdot (\operatorname{dom} f - x) = Y$. On the other hand, f is lower semicontinuous on X. Thus, since $Y = \operatorname{dom} f$ is a Banach space, $f|_Y$ is continuous by [30, Theorem 2.2.20(b)]. Altogether, in view of [30, Theorem 2.4.9], $g|_Y$ is continuous. Hence g is lower semicontinuous. Using [30, Corollary 2.4.15] and Fact 2.1(vi), we now deduce that $(\forall y \in Y)\ g(y) = \sup\langle \partial f(x), y\rangle = \langle Ax, y\rangle = \langle x, A^* y\rangle$. We thus have verified that

$$(\forall x \in Y)(\forall y \in Y) \quad f'(x; y) = \langle Ax, y\rangle = \langle x, A^* y\rangle. \tag{13}$$

In particular, $f|_Y$ is *differentiable*. Now fix x, y, z in Y. Then, using (13), we see that

$$\begin{aligned}\text{(14)}\qquad \langle Az, y\rangle &= \lim_{s\downarrow 0}\frac{\langle A(x+sz), y\rangle - \langle Ax, y\rangle}{s} = \lim_{s\downarrow 0}\frac{f'(x+sz;y) - f'(x;y)}{s}\\ &= \lim_{s\downarrow 0}\lim_{t\downarrow 0}\Big(\frac{f(x+sz+ty)-f(x+sz)}{st} - \frac{f(x+ty)-f(x)}{st}\Big).\end{aligned}$$

Set $h\colon \mathbb{R}\to\mathbb{R}\colon s\mapsto f(x+sz+ty)-f(x+sz)$. Since $f|_Y$ is differentiable, so is h. For $s>0$, the Mean Value Theorem thus yields $r_{s,t}\in\,]0,s[$ such that

$$\begin{aligned}\text{(15)}\qquad \frac{f(x+sz+ty)-f(x+sz)}{s} - \frac{f(x+ty)-f(x)}{s} &= \frac{h(s)}{s} - \frac{h(0)}{s} = h'(r_{s,t})\\ &= f'(x+r_{s,t}z+ty;z) - f'(x+r_{s,t}z;z)\\ &= t\langle Ay, z\rangle.\end{aligned}$$

Combining (14) with (15), we deduce that $\langle Az, y\rangle = \langle Ay, z\rangle$. Thus, A is symmetric. The result now follows from Proposition 2.7. ■

To improve Lemma 4.1, we need the following "shrink and dilate" technique.

Lemma 4.2 *Let $A : X \rightrightarrows X^*$ be a monotone linear relation, and let Z be a closed subspace of* $\operatorname{dom} A$. *Set $B = (A + \mathbb{I}_Z) + Z^\perp$. Then B is maximal monotone and* $\operatorname{dom} B = Z$.

Proof. Since $Z \subseteq \operatorname{dom} A$ and $B = A + \partial\iota_Z$ it is clear that B is a monotone linear relation with $\operatorname{dom} B = Z$. By Fact 2.2 (i), we have

$$Z^\perp \subseteq B0 = A0 + Z^\perp \subseteq (\operatorname{dom} A)^\perp + Z^\perp \subseteq Z^\perp + Z^\perp = Z^\perp.$$

Hence $B0 = Z^\perp = (\operatorname{dom} B)^\perp$. Therefore, by Fact 2.2(v), B is maximal monotone. ■

Theorem 4.3 *Let $f : X \to\,]-\infty, +\infty]$ be proper, lower semicontinuous, and convex, and let Y be a linear subspace of X. Suppose that $\partial f + \mathbb{I}_Y$ is a linear relation. Then $\partial f + \mathbb{I}_Y$ is symmetric.*

Proof. Put $A = \partial f + \mathbb{I}_Y$. Assume that $(x, x^*), (y, y^*) \in \operatorname{gra} A$. Set $Z = \operatorname{span}\{x, y\}$. Let $B : X \rightrightarrows X^*$ be defined as in Lemma 4.2. Clearly, $\operatorname{gra} B \subseteq \operatorname{gra} \partial(f + \iota_Z)$. In view of the maximal monotonicity of B, we see that $B = \partial(f + \iota_Z)$. Since $\operatorname{dom} B = Z$ is closed, it follows from Lemma 4.1 that $B = B^*$. In particular, we obtain that $\langle x^*, y\rangle = \langle y^*, x\rangle$. Hence, $\langle \partial f(x), y\rangle = \langle \partial f(y), x\rangle$ and therefore $\partial f + \mathbb{I}_Y$ is symmetric. ■

Lemma 4.4 *Let $A\colon X \rightrightarrows X^*$ be a maximal monotone linear relation such that A is Borwein-Wiersma decomposable. Then* $\operatorname{dom} A \subseteq \operatorname{dom} A^*$.

Proof. By hypothesis, there exists a proper lower semicontinuous and convex function $f\colon X \to\,]-\infty, +\infty]$ and an at most single-valued skew operator S such that $A = \partial f + S$. Hence $\operatorname{dom} A \subseteq \operatorname{dom} S$, and Theorem 4.3 implies that $(A - S) + \mathbb{I}_{\operatorname{dom} A}$ is symmetric. Let x and y be in $\operatorname{dom} A$.

$$\begin{aligned}\langle Ax - 2Sx, y\rangle &= \langle Ax - Sx, y\rangle - \langle Sx, y\rangle = \langle Ay - Sy, x\rangle - \langle Sx, y\rangle\\ &= \langle Ay, x\rangle - \langle Sy, x\rangle - \langle Sx, y\rangle = \langle Ay, x\rangle,\end{aligned}$$

which implies that $(A - 2S)x \subseteq A^*x$. Therefore, $\operatorname{dom} A = \operatorname{dom}(A - 2S) \subseteq \operatorname{dom} A^*$. ■

Remark 4.5 We can now derive part of the conclusion of of Proposition 3.11 differently as follows. Since $\operatorname{dom} A_1 - \operatorname{dom} A_2$ is closed, [25, Theorem 5.5] or [27] implies that $A_1 + A_2$ is maximal monotone; moreover, [5, Theorem 7.4] yields $(A_1 + A_2)^* = A_1^* + A_2^*$. Using Lemma 4.4, we thus obtain $\operatorname{dom}(A_1 + A_2) = \operatorname{dom} A_1 \cap \operatorname{dom} A_2 \subseteq \operatorname{dom} A_1^* \cap \operatorname{dom} A_2^* = \operatorname{dom}(A_1^* + A_2^*) = \operatorname{dom}(A_1 + A_2)^*$. Therefore, $A_1 + A_2$ is Borwein-Wiersma decomposable by Theorem 3.7.

Theorem 4.6 (characterization of subdifferential operators) *Let $A : X \rightrightarrows X^*$ be a monotone linear relation. Then A is maximal monotone and symmetric $\Leftrightarrow$ there exists a proper lower semicontinuous convex function $f \colon X \to\,]-\infty, +\infty]$ such that $A = \partial f$.*

Proof. "$\Rightarrow$": Fact 3.6(iii). "$\Leftarrow$": Apply Theorem 4.3 with $Y = X$. ■

Remark 4.7 Theorem 4.6 generalizes [17, Theorem 5.1] of Simons and Phelps.

Theorem 4.8 (uniqueness of the subdifferential part) *Let $A : X \rightrightarrows X^*$ be a maximal monotone linear relation such that A is Borwein-Wiersma decomposable. Then on $\operatorname{dom} A$, the subdifferential part in the decomposition is unique and the skew part must be a linear selection of $A_\circ$.*

Proof. Let f_1 and f_2 be proper lower semicontinuous convex functions from X to $]-\infty, +\infty]$, and let S_1 and S_2 be at most single-valued skew operators from X to X^* such that

$$A = \partial f_1 + S_1 = \partial f_2 + S_2. \tag{16}$$

Set $D = \operatorname{dom} A$. Since S_1 and S_2 are single-valued on D, we have $A - S_1 = \partial f_1$ and $A - S_2 = \partial f_2$ on D. Hence $\partial f_1 + \mathbb{I}_D$ and $\partial f_2 + \mathbb{I}_D$ are monotone linear relations with

$$(\partial f_1 + \mathbb{I}_D)(0) = (\partial f_2 + \mathbb{I}_D)(0) = A0. \tag{17}$$

By Theorem 4.3, $\partial f_1 + \mathbb{I}_D$ and $\partial f_2 + \mathbb{I}_D$ are symmetric, i.e.,

$$(\forall x \in D)(\forall y \in D) \quad \langle \partial f_1(x), y\rangle = \langle \partial f_1(y), x\rangle \quad \text{and} \quad \langle \partial f_2(x), y\rangle = \langle \partial f_2(y), x\rangle.$$

Thus,

$$(\forall x \in D)(\forall y \in D) \quad \langle \partial f_2(x) - \partial f_1(x), y\rangle = \langle \partial f_2(y) - \partial f_1(y), x\rangle. \tag{18}$$

On the other hand, by (16), $(\forall x \in D)\ S_1 x - S_2 x \in \partial f_2(x) - \partial f_1(x)$. Then by Fact 2.2(iii) and Proposition 2.3,

$$\begin{aligned}(\forall x \in D)(\forall y \in D) \quad \langle \partial f_2(x) - \partial f_1(x), y\rangle &= \langle S_1 x - S_2 x, y\rangle \\ &= -\langle S_1 y - S_2 y, x\rangle \\ &= -\langle \partial f_2(y) - \partial f_1(y), x\rangle.\end{aligned} \tag{19}$$

Now fix $x \in D$. Combining (18) and (19), we get $(\forall y \in D)\ \langle \partial f_2(x) - \partial f_1(x), y\rangle = 0$. Using Fact 2.2(iv), we see that

$$\partial f_2(x) - \partial f_1(x) \subseteq D^\perp = (\operatorname{dom} A)^\perp = A0.$$

Hence, in view of Lemma 3.5(i), (17), and Fact 2.1(ii),

$$\partial f_1 + \mathbb{I}_D = \partial f_2 + \mathbb{I}_D.$$

Furthermore, combining (18) and (19) gives $(\forall y \in D)\ \langle S_2 x - S_1 x, y\rangle = 0$; thus,

$$S_2 x - S_1 x \in D^\perp = (\operatorname{dom} A)^\perp = A0.$$

Now Lemma 4.4 implies that $\operatorname{dom} A \subseteq \operatorname{dom} A^*$. In turn, Theorem 3.7 allows us to consider the case when S_1 is a linear selection of $A_\circ$ on $\operatorname{dom} A_\circ = \operatorname{dom} A$. Using Fact 2.2(iv), we obtain Then $S_2 x \in S_1 x + A0 = S_1 x + A_\circ 0 = A_\circ x$. Therefore, S_2 must be a linear selection of $A_\circ$ on $\operatorname{dom} A$ as well. ■

Remark 4.9 In a Borwein-Wiersma decomposition, the skew part need not be unique: indeed, assume that $X = \mathbb{R}^2$, set $Y := \mathbb{R} \times \{0\}$, and let S be given by $\operatorname{gra} S = \big\{\big((x,0),(0,x)\big) \mid x \in \mathbb{R}\big\}$. Then S is skew and the maximal monotone linear relation $\partial\iota_Y$ has two distinct Borwein-Wiersma decompositions, namely $\partial\iota_Y + 0$ and $\partial\iota_Y + S$.

Proposition 4.10 *Let $A : X \rightrightarrows X^*$ be a maximal monotone linear relation. Suppose that A is Borwein-Wiersma decomposable, with subdifferential part ∂f, where $f\colon X \to \left]-\infty,+\infty\right]$ is proper, lower semicontinuous and convex. Then there exists a constant $\alpha \in \mathbb{R}$ such that the following hold.*

(i) $f = \overline{q_A} + \alpha$ *on* $\operatorname{dom} A$.

(ii) *If* $\operatorname{dom} A$ *is closed, then* $f = \overline{q_A} + \alpha = q_A + \alpha$ *on* X.

Proof. Let S be a linear single-valued selection of $A_\circ$. By Lemma 4.4, $\operatorname{dom} A \subseteq \operatorname{dom} A^*$. In turn, Theorem 3.7 yields

$$A = \partial\overline{q_A} + S.$$

Let $\{x,y\} \subset \operatorname{dom} A$. By Theorem 4.8, $\partial f + \mathbb{I}_{\operatorname{dom} A} = \partial\overline{q_A} + \mathbb{I}_{\operatorname{dom} A}$. Now set $Z = \operatorname{span}\{x,y\}$, apply Lemma 4.2 to the monotone linear relation $\partial f + \mathbb{I}_{\operatorname{dom} A} = \partial\overline{q_A} + \mathbb{I}_{\operatorname{dom} A}$, and let B be as in Lemma 4.2. Note that $\operatorname{gra} B = \operatorname{gra}(\partial\overline{q_A} + \partial\iota_Z) \subseteq \operatorname{gra}\partial(\overline{q_A} + \iota_Z)$ and that $\operatorname{gra} B = \operatorname{gra}(\partial f + \partial\iota_Z) \subseteq \operatorname{gra}\partial(f + \iota_Z)$. By maximal monotonicity of B, we conclude that $B = \partial(\overline{q_A} + \iota_Z) = \partial(f + \iota_Z)$. By [22, Theorem B], there exists $\alpha \in \mathbb{R}$ such that $f + \iota_Z = \overline{q_A} + \iota_Z + \alpha$. Hence $\alpha = f(x) - \overline{q_A}(x) = f(y) - \overline{q_A}(y)$ and repeating this argument with $y \in (\operatorname{dom} A) \smallsetminus \{x\}$, we see that

$$f = \overline{q_A} + \alpha \quad \text{on } \operatorname{dom} A \tag{20}$$

and (i) is thus verified. Now assume in addition that $\operatorname{dom} A$ is closed. Applying Lemma 3.5(iv) with both ∂f and $\partial\overline{q_A}$, we obtain

$$\operatorname{dom}\overline{q_A} = \operatorname{dom}\partial\overline{q_A} = \operatorname{dom} A = \operatorname{dom}\partial f = \operatorname{dom} f.$$

Consequently, (20) now yields $f = \overline{q_A} + \alpha$. Finally, Corollary 3.9 implies that $q_A = \overline{q_A}$. ■

5 Characterizations and examples

The following characterization of Borwein-Wiersma decomposability of a maximal monotone linear relation is quite pleasing.

Theorem 5.1 (characterization of Borwein-Wiersma decomposability) *Let $A\colon X \rightrightarrows X^*$ be a maximal monotone linear relation. Then the following are equivalent.*

(i) *A is Borwein-Wiersma decomposable.*

(ii) $\operatorname{dom} A \subseteq \operatorname{dom} A^*$.

(iii) $A = A_+ + A_\circ$.

Proof. "(i)⇒(ii)": Lemma 4.4. "(i)⇐(ii)": Theorem 3.7. "(ii)⇒(iii)": Fact 2.2(vii). "(ii)⇐(iii)": This is clear. ■

Corollary 5.2 *Let $A\colon X \rightrightarrows X^*$ be a maximal monotone linear relation. Then both A and A^* are Borwein-Wiersma decomposable if and only if* $\operatorname{dom} A = \operatorname{dom} A^*$.

Proof. Combine Theorem 5.1, Fact 2.4, and Fact 2.1(vii). ■

We shall now provide two examples of a linear relation S to illustrate that the following do occur:

- S is Borwein-Wiersma decomposable, but S^* is not.
- Neither S nor S^* is Borwein-Wiersma decomposable.
- S is not Borwein-Wiersma decomposable, but S^{-1} is.

Example 5.3 (See [4].) Suppose that X is the Hilbert space ℓ^2, and set

$$S\colon \operatorname{dom} S \to X\colon y \mapsto \Big(\tfrac{1}{2}y_n + \sum_{i<n} y_i\Big), \tag{21}$$

with

$$\operatorname{dom} S := \Big\{ y = (y_n) \in X \;\Big|\; \sum_{i\geq 1} y_i = 0, \Big(\sum_{i\leq n} y_i\Big) \in X \Big\}.$$

Then

$$S^*\colon \operatorname{dom} S^* \to X\colon y \mapsto \Big(\tfrac{1}{2}y_n + \sum_{i>n} y_i\Big) \tag{22}$$

where

$$\operatorname{dom} S^* = \Big\{ y = (y_n) \in X \;\Big|\; \Big(\sum_{i>n} y_i\Big) \in X \Big\}.$$

Then S can be identified with an at most single-valued linear relation such that the following hold. (See [17, Theorem 2.5] and [4, Proposition 3.2, Proposition 3.5, Proposition 3.6, and Theorem 3.9].)

(i) S is maximal monotone and skew.

(ii) S^* is maximal monotone but not skew.

(iii) $\operatorname{dom} S$ is dense in ℓ^2, and $\operatorname{dom} S \subsetneqq \operatorname{dom} S^*$.

(iv) $S^* = -S$ on $\operatorname{dom} S$.

In view of Theorem 5.1, S is Borwein-Wiersma decomposable while S^* is not. However, both S and S^* are irreducible and Asplund decomposable by Theorem 3.2. Because S^* is irreducible but not skew, we see that the class of irreducible operators is strictly larger than the class of skew operators.

Example 5.4 (inverse Volterra operator) (See [4, Example 4.4 and Theorem 4.5].) Suppose that X is the Hilbert space $L^2[0,1]$, and consider the *Volterra integration operator* (see, e.g., [14, Problem 148]), which is defined by

$$V\colon X \to X\colon x \mapsto Vx, \quad \text{where} \quad Vx\colon [0,1] \to \mathbb{R}\colon t \mapsto \int_0^t x, \tag{23}$$

and set $A = V^{-1}$. Then

$$V^*: X \to X: x \mapsto V^*x, \quad \text{where} \quad V^*x: [0,1] \to \mathbb{R}: t \mapsto \int_t^1 x,$$

and the following hold.

(i) $\operatorname{dom} A = \big\{x \in X \bigm| x \text{ is absolutely continuous, } x(0) = 0, \text{ and } x' \in X\big\}$ and

$$A\colon \operatorname{dom} A \to X : x \mapsto x'.$$

(ii) $\operatorname{dom} A^* = \big\{x \in X \bigm| x \text{ is absolutely continuous, } x(1) = 0, \text{ and } x' \in X\big\}$ and

$$A^*\colon \operatorname{dom} A^* \to X : x \mapsto -x'.$$

(iii) Both A and A^* are maximal monotone linear operators.

(iv) Neither A nor A^* is symmetric.

(v) Neither A nor A^* is skew.

(vi) $\operatorname{dom} A \not\subseteq \operatorname{dom} A^*$, and $\operatorname{dom} A^* \not\subseteq \operatorname{dom} A$.

(vii) $Y := \operatorname{dom} A \cap \operatorname{dom} A^*$ is dense in X.

(viii) Both $A + \mathbb{I}_Y$ and $A^* + \mathbb{I}_Y$ are skew.

By Theorem 3.2, both A and A^* are irreducible and Asplund decomposable. On the other hand, by Theorem 5.1, neither A nor A^* is Borwein-Wiersma decomposable. Finally, $A^{-1} = V$ and $(A^*)^{-1} = V^*$ are Borwein-Wiersma decomposable since they are continuous linear operators with full domain.

Remark 5.5 (an answer to a question posed by Borwein and Wiersma) The operators S, S^*, A, and A^* defined in this section are all irreducible and Asplund decomposable, but none of them has full domain. This provides an answer to [7, Question (4) in Section 7].

6 When X is a Hilbert space

Throughout this short section, we suppose that X is a Hilbert space. Recall (see, e.g., [13, Chapter 5] for basic properties) that if C is a nonempty closed convex subset of X, then the *(nearest point) projector* P_C is well defined and continuous. If Y is a closed subspace of X, then P_Y is linear and $P_Y = P_Y^*$.

Definition 6.1 *Let $A\colon X \rightrightarrows X$ be a maximal monotone linear relation. We define Q_A by*

$$Q_A : \operatorname{dom} A \to X : x \mapsto P_{Ax}x.$$

Note that Q_A is monotone and a single-valued selection of A because $(\forall x \in \operatorname{dom} A)$ Ax is a nonempty closed convex subset of X.

Proposition 6.2 (linear selection) *Let $A\colon X \rightrightarrows X$ be a maximal monotone linear relation. Then the following hold.*

(i) $(\forall x \in \operatorname{dom} A)$ $Q_A x = P_{(A0)^\perp}(Ax)$, *and* $Q_A x \in Ax$.

(ii) Q_A *is monotone and linear.*

(iii) $A = Q_A + A0$.

Proof. Let $x \in \operatorname{dom} A = \operatorname{dom} Q_A$ and let $x^* \in Ax$. Using Fact 2.1(ii) and Fact 2.2(i), we see that

$$\begin{aligned} Q_A x &= P_{Ax} x = P_{x^*+A0} x = x^* + P_{A0}(x - x^*) = x^* + P_{A0}x - P_{A0}x^* = P_{A0}x + P_{(A0)^\perp}x^* \\ &= P_{(A0)^\perp}x^*. \end{aligned}$$

Since $x^* \in Ax$ is arbitrary, we have thus verified (i). Now let x and y be in $\operatorname{dom} A$, and let α and β be in $\mathbb{R}$. If $\alpha = \beta = 0$, then, by Fact 2.1(i), we have $Q_A(\alpha x + \beta y) = Q_A 0 = P_{A0}0 = 0 = \alpha Q_A x + \beta Q_A y$. Now assume that $\alpha \neq 0$ or $\beta \neq 0$. By (i) and Fact 2.1(iii), we have

$$Q_A(\alpha x + \beta y) = P_{(A0)^\perp} A(\alpha x + \beta y) = \alpha P_{(A0)^\perp}(Ax) + \beta P_{(A0)^\perp}(Ay) = \alpha Q_A x + \beta Q_A y.$$

Hence Q_A is a linear selection of A and (ii) holds. Finally, (iii) follows from Fact 2.1(ii). ■

Example 6.3 Let $A\colon X \rightrightarrows X$ be maximal monotone and skew. Then $A = \partial\iota_{\overline{\operatorname{dom} A}} + Q_A$ is a Borwein-Wiersma decomposition.

Proof. By Proposition 6.2(ii), Q_A is a linear selection of A. Now apply Theorem 3.10. ■

Example 6.4 Let $A\colon X \rightrightarrows X$ be a maximal monotone linear relation such that $\operatorname{dom} A$ is closed. Set $B := P_{\operatorname{dom} A} Q_A P_{\operatorname{dom} A}$ and $f := q_B + \iota_{\operatorname{dom} A}$. Then the following hold.

(i) $B\colon X \to X$ is continuous, linear, and maximal monotone.

(ii) $f\colon X \to \left]-\infty, +\infty\right]$ is convex, lower semicontinuous, and proper.

(iii) $A = \partial\iota_{\operatorname{dom} A} + B$.

(iv) $\partial f + B_\circ$ is a Borwein-Wiersma decomposition of A.

Proof. (i): By Proposition 6.2(ii), Q_A is monotone and a linear selection of A. Hence, $B\colon X \to X$ is linear; moreover, $(\forall x \in X)$ $\langle x, Bx\rangle = \langle x,\ P_{\operatorname{dom} A} Q_A P_{\operatorname{dom} A} x\rangle = \langle P_{\operatorname{dom} A} x,\ Q_A P_{\operatorname{dom} A} x\rangle \geq 0$. Altogether, $B\colon X \to X$ is linear and monotone. By Fact 2.5, B is continuous and maximal monotone.

(ii): By (i), q_B is thus convex and continuous; in turn, f is convex, lower semicontinuous, and proper.

(iii): Using Proposition 6.2(i) and Fact 2.2(iv), we have $(\forall x \in X)$ $(Q_A P_{\operatorname{dom} A})x \in (A0)^\perp = \overline{\operatorname{dom} A} = \operatorname{dom} A$. Hence, $(\forall x \in \operatorname{dom} A)$ $Bx = (P_{\operatorname{dom} A} Q_A P_{\operatorname{dom} A})x = Q_A x \in Ax$. Thus, $B + \mathbb{I}_{\operatorname{dom} A} = Q_A$. In view of Proposition 6.2(iii) and Fact 2.2(iv), we now obtain $A = B + \mathbb{I}_{\operatorname{dom} A} + A0 = B + \partial\iota_{\operatorname{dom} A}$.

(iv): It follows from (iii) and (5) that $A = B + \partial\iota_{\operatorname{dom} A} = \nabla q_B + \partial\iota_{\operatorname{dom} A} + B_\circ = \partial(q_B + \iota_{\operatorname{dom} A}) + B_\circ = \partial f + B_\circ$. ■

Proposition 6.5 *Let* $A : X \rightrightarrows X$ *be such that* $\operatorname{dom} A$ *is a closed subspace of* X*. Then* A *is a maximal monotone linear relation* $\Leftrightarrow$ $A = \partial\iota_{\operatorname{dom} A} + B$*, where* $B : X \to X$ *is linear and monotone.*

Proof. "$\Rightarrow$": This is clear from Example 6.4(i)&(iii). "$\Leftarrow$": Clearly, A is a linear relation. By Fact 2.5, B is continuous and maximal monotone. Using Rockafellar's sum theorem [21], we conclude that $\partial\iota_{\operatorname{dom} A} + B$ is maximal monotone. ■

7 Conclusion

The original papers by Asplund [1] and by Borwein and Wiersma [7] concerned the additive decomposition of a maximal monotone operator whose domain has *nonempty interior*. In this paper, we focused on maximal monotone *linear relations* and we specifically allowed for domains with empty interior. All maximal monotone linear relations on finite-dimensional spaces are Borwein-Wiersma decomposable; however, this fails in infinite-dimensional settings. We presented characterizations of Borwein-Wiersma decomposability of maximal monotone linear relations in reflexive Banach spaces and provided a more explicit decomposition in Hilbert spaces.

The characterization of Asplund decomposability and the corresponding construction of an Asplund decomposition remain interesting unresolved topics for future explorations, even for maximal monotone linear operators whose domains are proper dense subspaces of infinite-dimensional Hilbert spaces.

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