

ON EXPLICIT RECURSIVE FORMULAS IN THE SPECTRAL PERTURBATION ANALYSIS OF A JORDAN BLOCK*

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Abstract. Let $A(\varepsilon)$ be an analytic square matrix and λ_0 an eigenvalue of $A(0)$ of algebraic multiplicity $m \geq 1$. Then under the condition, $\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon))|_{(\varepsilon, \lambda)=(0, \lambda_0)} \neq 0$, we prove that the Jordan normal form of $A(0)$ corresponding to the eigenvalue λ_0 consists of a single $m \times m$ Jordan block, the perturbed eigenvalues near λ_0 and their corresponding eigenvectors can be represented by a single convergent Puiseux series containing only powers of $\varepsilon^{1/m}$, and there are explicit recursive formulas to compute all the Puiseux series coefficients from just the derivatives of $A(\varepsilon)$ at the origin. Using these recursive formulas we calculate the series coefficients up to the second order and list them for quick reference. This paper gives, under a generic condition, explicit recursive formulas to compute the perturbed eigenvalues and eigenvectors for non-selfadjoint analytic perturbations of matrices with non-derogatory eigenvalues.

Key words. Matrix Perturbation Theory, Degenerate Eigenvalue, Jordan Block, Perturbation of Eigenvalues and Eigenvectors, Puiseux Series, Recursive Formula

AMS subject classification. 15A15, 15A18, 15A21, 41A58, 47A55, 47A56, 65F15, 65F40

1. Introduction. Consider an analytic square matrix $A(\varepsilon)$ and its unperturbed matrix $A(0)$ with a degenerate eigenvalue λ_0 . A fundamental problem in the analytic perturbation theory of non-selfadjoint matrices is the determination of the perturbed eigenvalues near λ_0 along with their corresponding eigenvectors of the matrix $A(\varepsilon)$ near $\varepsilon = 0$. More specifically, let $A(\varepsilon)$ be a matrix-valued function having a range in $\mathbb{C}^{n \times n}$, the set of $n \times n$ matrices with complex entries, such that its matrix elements are analytic functions of ε in a neighborhood of the origin. Let λ_0 be an eigenvalue of the matrix $A(0)$ with algebraic multiplicity $m \geq 1$. Then in this situation, it is well known [1, §6.1.7], [2, §II.1.8] that for sufficiently small ε all the perturbed eigenvalues near λ_0 , called the λ_0 -group, and their corresponding eigenvectors may be represented as a collection of convergent Puiseux series, i.e., convergent Taylor series in a fractional power of ε . What is not well known, however, is how we compute these Puiseux series when $A(\varepsilon)$ is a non-selfadjoint analytic perturbation and λ_0 is a defective eigenvalue of $A(0)$. There are sources on the subject like [1, §7.4], [3], [4], [5, §32], and [6] but it was found that they lacked explicit formulas, recursive or otherwise, to compute the series coefficients beyond the first order terms. Thus the fundamental problem that this paper addresses is to find explicit recursive formulas to determine the Puiseux series coefficients for the λ_0 -group and their eigenvectors.

This problem is of applied and theoretic importance, for example, in studying the spectral properties of dispersive media such as photonic crystals. In particular, this is especially true in the study of slow light [7]–[9], where the characteristic equation, $\det(\lambda I - A(\varepsilon)) = 0$, represents implicitly the dispersion relation for Bloch waves in the periodic crystal. In this setting ε represents a small change in frequency, $A(\varepsilon)$ is the Transfer matrix of a unit cell, and its eigenpairs, $(\lambda(\varepsilon), x(\varepsilon))$, correspond to the Bloch waves. From a practical and theoretical point of view, condition (1.1) on the dispersion relation or its equivalent formulation in Theorem 2.1.ii of this paper regarding the group velocity for this setting, arises naturally in the study of slow light

*This work was supported by the Air Force Office of Scientific Research under grant #FA9550-08-1-0103.

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where the Jordan normal form of the unperturbed Transfer matrix, $A(0)$, and the perturbation expansions of the eigenpairs of the Transfer matrix play a central role in the analysis of slow light waves.

Main Results. In this paper under the *generic condition*,

$$\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon)) \Big|_{(\varepsilon, \lambda) = (0, \lambda_0)} \neq 0, \quad (1.1)$$

we show that λ_0 is a non-derogatory eigenvalue of $A(0)$ and the fundamental problem mentioned above can be solved. In particular, we prove Theorem 2.1 and Theorem 3.1 which together state that when condition (1.1) is true then the Jordan normal form of $A(0)$ corresponding to the eigenvalue λ_0 consists of a single $m \times m$ Jordan block, the λ_0 -group and their corresponding eigenvectors can each be represented by a single convergent Puiseux series whose branches are given by

$$\begin{aligned} \lambda_h(\varepsilon) &= \lambda_0 + \sum_{k=1}^{\infty} \alpha_k \left(\zeta^h \varepsilon^{\frac{1}{m}} \right)^k \\ x_h(\varepsilon) &= \beta_0 + \sum_{k=1}^{\infty} \beta_k \left(\zeta^h \varepsilon^{\frac{1}{m}} \right)^k \end{aligned}$$

for $h = 0, \dots, m-1$ and any fixed branch of $\varepsilon^{\frac{1}{m}}$, where $\zeta = e^{\frac{2\pi}{m}i}$, $\{\alpha_k\}_{k=1}^{\infty} \subseteq \mathbb{C}$, $\{\beta_k\}_{k=0}^{\infty} \subseteq \mathbb{C}^{n \times 1}$, $\alpha_1 \neq 0$, and β_0 is an eigenvector of $A(0)$ corresponding to the eigenvalue λ_0 . More importantly though, Theorem 3.1 gives explicit recursive formulas that allows us to determine the Puiseux series coefficients, $\{\alpha_k\}_{k=1}^{\infty}$ and $\{\beta_k\}_{k=0}^{\infty}$, from just the derivatives of $A(\varepsilon)$ at $\varepsilon = 0$. Using these recursive formulas, we compute the leading Puiseux series coefficients up to the second order and list them in Corollary 3.3.

The key to all of our results is the study of the characteristic equation for the analytic matrix $A(\varepsilon)$ under the generic condition (1.1). By an application of the implicit function theorem, we are able to derive the functional relation between the eigenvalues and the perturbation parameter. This leads to the implication that the Jordan normal form of the unperturbed matrix $A(0)$ corresponding to the eigenvalue λ_0 is a single $m \times m$ Jordan block. From this, we are able to use the method of undetermined coefficients along with a careful combinatorial analysis to get explicit recursive formulas for determining the Puiseux series coefficients.

We want to take a moment here to show how the results of this paper can be used to determine the Puiseux series coefficients up to the second order for the case in which the non-derogatory eigenvalue λ_0 has algebraic multiplicity $m \geq 2$. We start by putting $A(0)$ into the Jordan normal form [10, §6.5: The Jordan Theorem]

$$U^{-1}A(0)U = \left[\begin{array}{c|c} J_m(\lambda_0) & \\ \hline & W_0 \end{array} \right], \quad (1.2)$$

where (see notations at end of §1) $J_m(\lambda_0)$ is an $m \times m$ Jordan block corresponding to the eigenvalue λ_0 and W_0 is the Jordan normal form for the rest of the spectrum. Next, define the vectors u_1, \dots, u_m , as the first m columns of the matrix U ,

$$u_i := Ue_i, \quad 1 \leq i \leq m \quad (1.3)$$

(These vectors have the properties that u_1 is an eigenvector of $A(0)$ corresponding to the eigenvalue λ_0 , they form a Jordan chain with generator u_m , and are a basis

for the algebraic eigenspace of $A(0)$ corresponding to the eigenvalue λ_0). We then partition the matrix $U^{-1}A'(0)U$ conformally to the blocks $J_m(\lambda_0)$ and W_0 of the matrix $U^{-1}A(0)U$ as such

$$U^{-1}A'(0)U = \left[\begin{array}{cccc|cccc} * & * & * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * & * & \cdots & * \\ a_{m-1,1} & * & * & \cdots & * & * & \cdots & * \\ a_{m,1} & a_{m,2} & * & \cdots & * & * & \cdots & * \\ \hline * & * & * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * & * & \cdots & * \end{array} \right]. \quad (1.4)$$

Now, by Theorem 2.1 and Theorem 3.1, it follows that

$$a_{m,1} = -\frac{\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon))|_{(\varepsilon, \lambda)=(0, \lambda_0)}}{\left(\frac{\frac{\partial^m}{\partial \lambda^m} \det(\lambda I - A(\varepsilon))|_{(\varepsilon, \lambda)=(0, \lambda_0)}}{m!} \right)}. \quad (1.5)$$

And hence the generic condition is true if and only if $a_{m,1} \neq 0$. This gives us an alternative method to determine whether the generic condition (1.1) is true or not.

Lets now assume that $a_{m,1} \neq 0$ and hence that the generic condition is true. Define $f(\varepsilon, \lambda) := \det(\lambda I - A(\varepsilon))$. Then by Theorem 3.1 and Corollary 3.3 there is exactly one convergent Puiseux series for the perturbed eigenvalues near λ_0 and one for their corresponding eigenvectors whose branches are given by

$$\lambda_h(\varepsilon) = \lambda_0 + \alpha_1 \left(\zeta^h \varepsilon^{\frac{1}{m}} \right) + \alpha_2 \left(\zeta^h \varepsilon^{\frac{1}{m}} \right)^2 + \sum_{k=3}^{\infty} \alpha_k \left(\zeta^h \varepsilon^{\frac{1}{m}} \right)^k \quad (1.6)$$

$$x_h(\varepsilon) = x_0 + \beta_1 \left(\zeta^h \varepsilon^{\frac{1}{m}} \right) + \beta_2 \left(\zeta^h \varepsilon^{\frac{1}{m}} \right)^2 + \sum_{k=3}^{\infty} \beta_k \left(\zeta^h \varepsilon^{\frac{1}{m}} \right)^k \quad (1.7)$$

for $h = 0, \dots, m-1$ and any fixed branch of $\varepsilon^{\frac{1}{m}}$, where $\zeta = e^{\frac{2\pi}{m}i}$. Furthermore, the series coefficients up to second order may be given by

$$\alpha_1 = a_{m,1}^{1/m} = \left(-\frac{\frac{\partial f}{\partial \varepsilon}(0, \lambda_0)}{\frac{1}{m!} \frac{\partial^m f}{\partial \lambda^m}(0, \lambda_0)} \right)^{1/m} \neq 0, \quad (1.8)$$

$$\alpha_2 = \frac{a_{m-1,1} + a_{m,2}}{m\alpha_1^{m-2}} = \frac{-\left(\alpha_1^{m+1} \frac{1}{(m+1)!} \frac{\partial^{m+1} f}{\partial \lambda^{m+1}}(0, \lambda_0) + \alpha_1 \frac{\partial^2 f}{\partial \lambda \partial \varepsilon}(0, \lambda_0) \right)}{m\alpha_1^{m-1} \left(\frac{1}{m!} \frac{\partial^m f}{\partial \lambda^m}(0, \lambda_0) \right)}, \quad (1.9)$$

$$\beta_0 = u_1, \beta_1 = \alpha_1 u_2, \beta_2 = \begin{cases} -\Lambda A'(0)u_1 + \alpha_2 u_2, & \text{if } m = 2 \\ \alpha_2 u_2 + \alpha_1^2 u_3, & \text{if } m > 2 \end{cases} \quad (1.10)$$

for any choice of the m th root of $a_{m,1}$ and where Λ is given in (3.4).

The explicit recursive formulas for computing higher order terms, α_k, β_k , are given by (3.13) and (3.14) in Theorem 3.1. The steps which should be used to determine these higher order terms are discussed in Remark 3.4 and an example showing how to calculating α_3, β_3 using these steps, when $m \geq 3$, is provided.

Example. The following example may help to give a better idea of these results. Consider

$$A(\varepsilon) := \begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -1 & 1 & 1 \end{bmatrix} + \varepsilon \begin{bmatrix} 2 & 0 & -1 \\ 2 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}. \quad (1.11)$$

Here $\lambda_0 = 0$ is a non-derogatory eigenvalue of $A(0)$ of algebraic multiplicity $m = 2$. We put $A(0)$ into the Jordan normal form

$$U^{-1}A(0)U = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1/2 \end{array} \right], U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, U^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix},$$

so that $W_0 = 1/2$. We next define the vectors u_1, u_2 , as the first two columns of the matrix U ,

$$u_1 := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, u_2 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Next we partition the matrix $U^{-1}A'(0)U$ conformally to the blocks $J_m(\lambda_0)$ and W_0 of the matrix $U^{-1}A(0)U$ as such

$$U^{-1}A'(0)U = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 2 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & * & * \\ 1 & 1 & * \\ \hline * & * & * \end{array} \right].$$

Here $a_{2,1} = 1$, $a_{1,1} = 0$, and $a_{2,2} = 1$. Then

$$1 = a_{2,1} = -\frac{\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon))|_{(\varepsilon, \lambda) = (0, \lambda_0)}}{\left(\frac{\frac{\partial^2}{\partial \lambda^2} \det(\lambda I - A(\varepsilon))|_{(\varepsilon, \lambda) = (0, \lambda_0)}}{2!} \right)},$$

implying that the generic condition (1.1) is true. Define $f(\varepsilon, \lambda) := \det(\lambda I - A(\varepsilon)) = \lambda^3 - 2\lambda^2\varepsilon - \frac{1}{2}\lambda^2 + \lambda\varepsilon^2 - \frac{1}{2}\lambda\varepsilon + \varepsilon^2 + \frac{1}{2}\varepsilon$. Then there is exactly one convergent Puiseux series for the perturbed eigenvalues near $\lambda_0 = 0$ and one for their corresponding eigenvectors whose branches are given by

$$\lambda_h(\varepsilon) = \lambda_0 + \alpha_1 \left((-1)^h \varepsilon^{\frac{1}{2}} \right) + \alpha_2 \left((-1)^h \varepsilon^{\frac{1}{2}} \right)^2 + \sum_{k=3}^{\infty} \alpha_k \left((-1)^h \varepsilon^{\frac{1}{2}} \right)^k$$

$$x_h(\varepsilon) = \beta_0 + \beta_1 \left((-1)^h \varepsilon^{\frac{1}{2}} \right) + \beta_2 \left((-1)^h \varepsilon^{\frac{1}{2}} \right)^2 + \sum_{k=3}^{\infty} \beta_k \left((-1)^h \varepsilon^{\frac{1}{2}} \right)^k$$

for $h = 0, 1$ and any fixed branch of $\varepsilon^{\frac{1}{2}}$. Furthermore, the series coefficients up to second order may be given by

$$\alpha_1 = 1 = \sqrt{1} = \sqrt{a_{2,1}} = \sqrt{\left(-\frac{\frac{\partial f}{\partial \varepsilon}(0, \lambda_0)}{\frac{1}{2!} \frac{\partial^2 f}{\partial \lambda^2}(0, \lambda_0)} \right)} \neq 0,$$

$$\alpha_2 = \frac{1}{2} = \frac{a_{1,1} + a_{2,2}}{2} = \frac{-\left(\alpha_1^3 \frac{1}{3!} \frac{\partial^3 f}{\partial \lambda^3} (0, \lambda_0) + \alpha_1 \frac{\partial^2 f}{\partial \lambda \partial \varepsilon} (0, \lambda_0)\right)}{\alpha_1 \left(\frac{1}{2!} \frac{\partial^2 f}{\partial \lambda^2} (0, \lambda_0)\right)},$$

$$\beta_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \beta_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \beta_2 = -\Lambda A'(0)u_1 + \alpha_2 u_2$$

by choosing the positive square root of $a_{2,1} = 1$ and where Λ is given in (3.4). Here

$$\begin{aligned} \Lambda &= U \left[\begin{array}{c|c} J_m(0)^* & \\ \hline & (W_0 - \lambda_0 I_{n-m})^{-1} \end{array} \right] U^{-1} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \\ 0 & 0 \end{array} \middle| \begin{array}{c} 0 \\ 0 \\ (1/2)^{-1} \end{array} \right] \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -2 \\ 3 & -1 & -2 \\ 1 & -1 & 0 \end{bmatrix} \\ \beta_2 &= -\Lambda A'(0)u_1 + \alpha_2 u_2 \\ &= - \begin{bmatrix} 3 & -1 & -2 \\ 3 & -1 & -2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 2 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \end{aligned}$$

Now compare this to the actual perturbed eigenvalues of our example (1.11) near $\lambda_0 = 0$ and their corresponding eigenvectors

$$\begin{aligned} \lambda_h(\varepsilon) &= \frac{1}{2}\varepsilon + (-1)^h \frac{1}{2}\varepsilon^{\frac{1}{2}}(\varepsilon + 4)^{\frac{1}{2}} \\ &= \left((-1)^h \varepsilon^{\frac{1}{2}}\right) + \frac{1}{2} \left((-1)^h \varepsilon^{\frac{1}{2}}\right)^2 + \sum_{k=3}^{\infty} \alpha_k \left((-1)^h \varepsilon^{\frac{1}{2}}\right)^k \\ x_h(\varepsilon) &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \lambda_h(\varepsilon) \\ &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \left((-1)^h \varepsilon^{\frac{1}{2}}\right) + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \left((-1)^h \varepsilon^{\frac{1}{2}}\right)^2 + \sum_{k=3}^{\infty} \beta_k \left((-1)^h \varepsilon^{\frac{1}{2}}\right)^k \end{aligned}$$

for $h = 0, 1$ and any fixed branch of $\varepsilon^{\frac{1}{2}}$. We see that indeed our formulas for the Puiseux series coefficients are correct up to the second order.

Comparison to Known Results. There is a fairly large amount of literature on eigenpair perturbation expansions for analytic perturbations of non-selfadjoint matrices with degenerate eigenvalues (e.g. [1]–[6], [11]–[24]). However, most of the literature (e.g. [3], [4], [11], [12], [14], [16]–[21], [23], [24]) contains results only on the first order expansions of the Puiseux series or considers higher order terms only in the case of simple or semisimple eigenvalues. For those works that do address higher order terms for defective eigenvalues (e.g. [1], [2], [5], [6], [13], [15], [22]), it was found that there did not exist explicit recursive formulas for all the Puiseux coefficients when the matrix perturbations were non-linear. One of the purposes and achievements of this paper are the explicit recursive formulas (3.12)–(3.14) in Theorem 3.1 which give all the higher order terms in the important case of degenerate eigenvalues which are non-derogatory, that is, the case in which a degenerate eigenvalue of the unperturbed matrix has a single Jordan block for its corresponding Jordan structure.

Our theorem generalizes and extends the results of [1, pp. 315–317, (4.96) & (4.97)], [5, pp. 415–418], and [6, pp. 17–20] to non-linear analytic matrix perturbations and makes explicit the recursive formulas for calculating the perturbed eigenpair Puiseux expansions. Furthermore, in Proposition B.2 we give an explicit recursive formula for calculating the polynomials $\{r_l\}_{l \in \mathbb{N}}$. These polynomials must be calculated in order to determine the higher order terms in the eigenpair Puiseux series expansions (see (3.14) in Theorem 3.1 and Remark 3.4). These polynomials appear in [1, p. 315, (4.95)], [5, p. 414, (32.24)], and [6, p. 19, (34)] under different notation (compare with Proposition B.1.ii) but no method is given to calculate them. As such, Proposition B.2 is an important contribution in the explicit recursive calculation of the higher order terms in the eigenpair Puiseux series expansions.

Another purpose of this paper is to give, in the case of degenerate non-derogatory eigenvalues, an easily accessible and quickly referenced list of first and second order terms for the Puiseux series expansions of the perturbed eigenpairs. When the generic condition (1.1) is satisfied, Corollary 3.3 gives this list. Now for first order terms there are quite a few papers on formulas for determining them, see for example [21] which gives a good survey of first order perturbation theory. But for second order terms, it was difficult to find any results in the literature similar to and as explicit as Corollary 3.3 for the case of degenerate non-derogatory eigenvalues with arbitrary algebraic multiplicity and non-linear analytic perturbations. Results comparable to ours can be found in [1, p. 316], [5, pp. 415–418], [6, pp. 17–20], and [22, pp. 37–38, 50–54, 125–128], although it should be noted that in [5, p. 417] the formula for the second order term of the perturbed eigenvalues contains a misprint.

Overview. Section 2 deals with the generic condition (1.1). We give conditions that are equivalent to the generic condition in Theorem 2.1. In §3 we give the main results of this paper in Theorem 3.1, on the determination of the Puiseux series with the explicit recursive formulas for calculating the series coefficients. As a corollary we give the exact leading order terms, up to the second order, for the Puiseux series coefficients. Section 4 contains the proofs of the results in §2 and §3.

Notation. Let $\mathbb{C}^{n \times n}$ be the set of all $n \times n$ matrices with complex entries and $\mathbb{C}^{n \times 1}$ the set of all $n \times 1$ column vectors with complex entries. For $a \in \mathbb{C}$, $A \in \mathbb{C}^{n \times n}$, and $x = [a_{i,1}]_{i=1}^n \in \mathbb{C}^{n \times 1}$ we denote by a^* , A^* , and x^* , the complex conjugate of a , the conjugate transpose of A , and the $1 \times n$ row vector $x^* := [a_{1,1}^* \ \cdots \ a_{n,1}^*]$. For $x, y \in \mathbb{C}^{n \times 1}$ we let $(x, y) := x^*y$ be the standard inner product. The matrix $I \in \mathbb{C}^{n \times n}$ is the identity matrix and its j th column is $e_j \in \mathbb{C}^{n \times 1}$. The matrix I_{n-m} is the $(n-m) \times (n-m)$ identity matrix. Define an $m \times m$ Jordan block with eigenvalue λ to be

$$J_m(\lambda) := \begin{bmatrix} \lambda & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & \lambda \end{bmatrix}.$$

When the matrix $A(\varepsilon) \in \mathbb{C}^{n \times n}$ is analytic at $\varepsilon = 0$ we define $A'(0) := \frac{dA}{d\varepsilon}(0)$ and $A_k := \frac{1}{k!} \frac{d^k A}{d\varepsilon^k}(0)$. Let $\zeta := e^{i\frac{2\pi}{m}}$.

2. The Generic Condition. The following theorem, which is proved in §4, gives conditions which are equivalent to the generic one (1.1).

THEOREM 2.1. *Let $A(\varepsilon)$ be a matrix-valued function having a range in $\mathbb{C}^{n \times n}$ such that its matrix elements are analytic functions of ε in a neighborhood of the origin. Let λ_0 be an eigenvalue of the unperturbed matrix $A(0)$ and denote by m its algebraic multiplicity. Then the following statements are equivalent:*

(i) *The characteristic polynomial $\det(\lambda I - A(\varepsilon))$ has a simple zero with respect to ε at $\lambda = \lambda_0$ and $\varepsilon = 0$, i.e.,*

$$\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon)) \Big|_{(\varepsilon, \lambda) = (0, \lambda_0)} \neq 0.$$

(ii) *The characteristic equation, $\det(\lambda I - A(\varepsilon)) = 0$, has a unique solution, $\varepsilon(\lambda)$, in a neighborhood of $\lambda = \lambda_0$ with $\varepsilon(\lambda_0) = 0$. This solution is an analytic function with a zero of order m at $\lambda = \lambda_0$, i.e.,*

$$\frac{d^0 \varepsilon(\lambda)}{d\lambda^0} \Big|_{\lambda = \lambda_0} = \cdots = \frac{d^{m-1} \varepsilon(\lambda)}{d\lambda^{m-1}} \Big|_{\lambda = \lambda_0} = 0, \quad \frac{d^m \varepsilon(\lambda)}{d\lambda^m} \Big|_{\lambda = \lambda_0} \neq 0.$$

(iii) *There exists a convergent Puiseux series whose branches are given by*

$$\lambda_h(\varepsilon) = \lambda_0 + \alpha_1 \zeta^h \varepsilon^{\frac{1}{m}} + \sum_{k=2}^{\infty} \alpha_k \left(\zeta^h \varepsilon^{\frac{1}{m}} \right)^k,$$

for $h = 0, \dots, m-1$ and any fixed branch of $\varepsilon^{\frac{1}{m}}$, where $\zeta = e^{\frac{2\pi}{m}i}$, such that the values of the branches give all the solutions of the characteristic equation, for sufficiently small ε and λ sufficiently near λ_0 . Furthermore, the first order term is nonzero, i.e.,

$$\alpha_1 \neq 0.$$

(iv) *The Jordan normal form of $A(0)$ corresponding to the eigenvalue λ_0 consists of a single $m \times m$ Jordan block and there exists an eigenvector u_0 of $A(0)$ corresponding to the eigenvalue λ_0 and an eigenvector v_0 of $A(0)^*$ corresponding to the eigenvalue λ_0^* such that*

$$(v_0, A'(0)u_0) \neq 0.$$

3. Determination of the Puiseux Series and the Explicit Recursive Formulas for Calculating the Series. This section contains the main results of this paper presented below in Theorem 3.1. To begin we give some preliminaries that are needed to set up the theorem. Suppose that $A(\varepsilon)$ is a matrix-valued function having a range in $\mathbb{C}^{n \times n}$ with matrix elements that are analytic functions of ε in a neighborhood of the origin and λ_0 is an eigenvalue of the unperturbed matrix $A(0)$ with algebraic multiplicity m . Assume that the generic condition

$$\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon)) \Big|_{(\varepsilon, \lambda) = (0, \lambda_0)} \neq 0,$$

is true.

Now, by these assumptions, we may appeal to Theorem 2.1.iv and conclude that the Jordan canonical form of $A(0)$ has only one $m \times m$ Jordan block associated with λ_0 . Hence there exists an invertible matrix $U \in \mathbb{C}^{n \times n}$ such that

$$U^{-1}A(0)U = \left[\begin{array}{c|c} J_m(\lambda_0) & \\ \hline & W_0 \end{array} \right], \quad (3.1)$$

where W_0 is a $(n-m) \times (n-m)$ matrix such that λ_0 is not one of its eigenvalues [10, §6.5: The Jordan Theorem].

We define the vectors $u_1, \dots, u_m, v_1, \dots, v_m \in \mathbb{C}^{n \times 1}$ as the first m columns of the matrix U and $(U^{-1})^*$, respectively, i.e.,

$$u_i := U e_i, \quad 1 \leq i \leq m, \quad (3.2)$$

$$v_i := (U^{-1})^* e_i, \quad 1 \leq i \leq m. \quad (3.3)$$

And define the matrix $\Lambda \in \mathbb{C}^{n \times n}$ by

$$\Lambda := U \left[\begin{array}{c|c} J_m(0)^* & \\ \hline & (W_0 - \lambda_0 I_{n-m})^{-1} \end{array} \right] U^{-1}, \quad (3.4)$$

where $(W_0 - \lambda_0 I_{n-m})^{-1}$ exists since λ_0 is not an eigenvalue of W_0 (for the important properties of the matrix Λ see Appendix A).

Next, we introduce the polynomials $p_{j,i} = p_{j,i}(\alpha_1, \dots, \alpha_{j-i+1})$ in $\alpha_1, \dots, \alpha_{j-i+1}$, for $j \geq i \geq 0$, as the expressions

$$\left. \begin{aligned} p_{0,0} &:= 1, \quad p_{j,0} := 0, \quad \text{for } j > 0, \\ p_{j,i}(\alpha_1, \dots, \alpha_{j-i+1}) &:= \sum_{\substack{s_1 + \dots + s_i = j \\ 1 \leq s_\varrho \leq j-i+1}} \prod_{\varrho=1}^i \alpha_{s_\varrho}, \quad \text{for } j \geq i > 0 \end{aligned} \right\} \quad (3.5)$$

and the polynomials $r_l = r_l(\alpha_1, \dots, \alpha_l)$ in $\alpha_1, \dots, \alpha_l$, for $l \geq 1$, as the expressions

$$r_1 := 0, \quad r_l(\alpha_1, \dots, \alpha_l) := \sum_{\substack{s_1 + \dots + s_m = m+l \\ 1 \leq s_\varrho \leq l}} \prod_{\varrho=1}^m \alpha_{s_\varrho}, \quad \text{for } l > 1 \quad (3.6)$$

(see Appendix B for more details on these polynomials including recursive formulas for their calculation).

With these preliminaries we can now state the main results of this paper. Proofs of these results are contained in the next section.

THEOREM 3.1. *Let $A(\varepsilon)$ be a matrix-valued function having a range in $\mathbb{C}^{n \times n}$ such that its matrix elements are analytic functions of ε in a neighborhood of the origin. Let λ_0 be an eigenvalue of the unperturbed matrix $A(0)$ and denote by m its algebraic multiplicity. Suppose that the generic condition*

$$\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon)) \Big|_{(\varepsilon, \lambda) = (0, \lambda_0)} \neq 0, \quad (3.7)$$

is true. Then there is exactly one convergent Puiseux series for the λ_0 -group and one for their corresponding eigenvectors whose branches are given by

$$\lambda_h(\varepsilon) = \lambda_0 + \sum_{k=1}^{\infty} \alpha_k \left(\zeta^h \varepsilon^{\frac{1}{m}} \right)^k \quad (3.8)$$

$$x_h(\varepsilon) = \beta_0 + \sum_{k=1}^{\infty} \beta_k \left(\zeta^h \varepsilon^{\frac{1}{m}} \right)^k \quad (3.9)$$

for $h = 0, \dots, m-1$ and any fixed branch of $\varepsilon^{\frac{1}{m}}$, where $\zeta = e^{\frac{2\pi}{m}i}$ with

$$\alpha_1^m = (v_m, A_1 u_1) = - \frac{\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon)) \Big|_{(\varepsilon, \lambda) = (0, \lambda_0)}}{\left(\frac{\frac{\partial^m}{\partial \lambda^m} \det(\lambda I - A(\varepsilon)) \Big|_{(\varepsilon, \lambda) = (0, \lambda_0)}}{m!} \right)} \neq 0$$

(Here A_1 denotes $\frac{dA}{d\varepsilon}(0)$ and the vectors u_1 and v_m are defined in (3.2) and (3.3). Furthermore, we can choose

$$\alpha_1 = (v_m, A_1 u_1)^{1/m}, \quad (3.10)$$

for any fixed m th root of $(v_m, A_1 u_1)$ and the eigenvectors to satisfy the normalization conditions

$$(v_1, x_h(\varepsilon)) = 1, \quad h = 0, \dots, m-1. \quad (3.11)$$

Consequently, under these conditions $\alpha_1, \alpha_2, \dots$ and β_0, β_1, \dots are uniquely determined and are given by the recursive formulas

$$\alpha_1 = (v_m, A_1 u_1)^{1/m} = \left(-\frac{\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon))|_{(\varepsilon, \lambda)=(0, \lambda_0)}}{\left(\frac{\partial^m \det(\lambda I - A(\varepsilon))|_{(\varepsilon, \lambda)=(0, \lambda_0)}}{m!} \right)} \right)^{1/m} \quad (3.12)$$

$$\alpha_s = \frac{-r_{s-1} + \sum_{i=0}^{\min\{s, m\}-1} \sum_{j=i}^{s-1} p_{j,i} \left(v_{m-i}, \sum_{k=1}^{\lfloor \frac{m+s-1-j}{m} \rfloor} A_k \beta_{m+s-1-j-km} \right)}{m \alpha_1^{m-1}} \quad (3.13)$$

$$\beta_s = \begin{cases} \sum_{i=0}^s p_{s,i} u_{i+1}, & \text{if } 0 \leq s \leq m-1 \\ \sum_{i=0}^{m-1} p_{s,i} u_{i+1} - \sum_{j=0}^{s-m} \sum_{k=0}^j \sum_{l=1}^{\lfloor \frac{s-j}{m} \rfloor} p_{j,k} \Lambda^{k+1} A_l \beta_{s-j-lm}, & \text{if } s \geq m \end{cases} \quad (3.14)$$

where u_i and v_i are the vectors defined in (3.2) and (3.3), $p_{j,i}$ and r_l are the polynomials defined in (3.5) and (3.6), $\lfloor \cdot \rfloor$ denotes the floor function, A_k denotes the matrix $\frac{1}{k!} \frac{d^k A}{d\varepsilon^k}(0)$, and Λ is the matrix defined in (3.4).

COROLLARY 3.2. *The calculation of the k th order terms, α_k and β_k , requires only the matrices $A_0, \dots, A_{\lfloor \frac{m+k-1}{m} \rfloor}$.*

COROLLARY 3.3. *The coefficients of those Puiseux series up to second order are given by*

$$\begin{aligned} \alpha_1 &= \left(-\frac{\frac{\partial f}{\partial \varepsilon}(0, \lambda_0)}{\frac{1}{m!} \frac{\partial^m f}{\partial \lambda^m}(0, \lambda_0)} \right)^{1/m} = (v_m, A_1 u_1)^{1/m}, \\ \alpha_2 &= \begin{cases} \frac{-\left(\alpha_1^{m+1} \frac{1}{(m+1)!} \frac{\partial^{m+1} f}{\partial \lambda^{m+1}}(0, \lambda_0) + \alpha_1 \frac{\partial^2 f}{\partial \lambda \partial \varepsilon}(0, \lambda_0) + \frac{1}{2} \frac{\partial^2 f}{\partial \varepsilon^2}(0, \lambda_0) \right)}{m \alpha_1^{m-1} \left(\frac{1}{m!} \frac{\partial^m f}{\partial \lambda^m}(0, \lambda_0) \right)}, & \text{if } m = 1 \\ \frac{-\left(\alpha_1^{m+1} \frac{1}{(m+1)!} \frac{\partial^{m+1} f}{\partial \lambda^{m+1}}(0, \lambda_0) + \alpha_1 \frac{\partial^2 f}{\partial \lambda \partial \varepsilon}(0, \lambda_0) \right)}{m \alpha_1^{m-1} \left(\frac{1}{m!} \frac{\partial^m f}{\partial \lambda^m}(0, \lambda_0) \right)}, & \text{if } m > 1 \end{cases} \\ &= \begin{cases} (v_1, (A_2 - A_1 \Lambda A_1) u_1), & \text{if } m = 1 \\ \frac{(v_{m-1}, A_1 u_1) + (v_m, A_1 u_2)}{m \alpha_1^{m-2}}, & \text{if } m > 1 \end{cases}, \\ \beta_0 &= u_1, \\ \beta_1 &= \begin{cases} -\Lambda A_1 u_1, & \text{if } m = 1 \\ \alpha_1 u_2, & \text{if } m > 1 \end{cases}, \\ \beta_2 &= \begin{cases} \left(-\Lambda A_2 + (\Lambda A_1)^2 - \alpha_1 \Lambda^2 A_1 \right) u_1, & \text{if } m = 1 \\ -\Lambda A_1 u_1 + \alpha_2 u_2, & \text{if } m = 2 \\ \alpha_2 u_2 + \alpha_1^2 u_3, & \text{if } m > 2 \end{cases}, \end{aligned}$$

where $f(\varepsilon, \lambda) := \det(\lambda I - A(\varepsilon))$.

REMARK 3.4. Suppose we want to calculate the terms $\alpha_{k+1}, \beta_{k+1}$, where $k \geq 2$, using the explicit recursive formulas given in the theorem. We may assume we already known or have calculated

$$A_0, \dots, A_{\lfloor \frac{m+k}{m} \rfloor}, \{r_j\}_{j=1}^{k-1}, \{\alpha_j\}_{j=1}^k, \{\beta_j\}_{j=0}^k, \{\{p_{j,i}\}_{j=i}^k\}_{i=0}^k. \quad (3.15)$$

We need these to calculate $\alpha_{k+1}, \beta_{k+1}$ and the steps to do this are indicated by the following arrow diagram:

$$(3.15) \xrightarrow{(B.5)} r_k \xrightarrow{(3.13)} \alpha_{k+1} \xrightarrow{(B.4)} \{p_{k+1,i}\}_{i=0}^{k+1} \xrightarrow{(3.14)} \beta_{k+1}. \quad (3.16)$$

After we have followed these steps we not only will have calculated $\alpha_{k+1}, \beta_{k+1}$ but we will now know

$$A_0, \dots, A_{\lfloor \frac{m+k+1}{m} \rfloor}, \{r_j\}_{j=1}^k, \{\alpha_j\}_{j=1}^{k+1}, \{\beta_j\}_{j=0}^{k+1}, \{\{p_{j,i}\}_{j=i}^{k+1}\}_{i=0}^{k+1} \quad (3.17)$$

as well. But these are the terms in (3.15) for $k+1$ and so we may repeat the steps indicated above to calculate $\alpha_{k+2}, \beta_{k+2}$.

It is in this way we see how all the higher order terms can be calculated using the results of this paper.

Example. In order to illustrate these steps we give the following example which recursively calculates the third order terms for $m \geq 3$.

The goal is to determine α_3, β_3 . To do this we follow the steps indicated in the above remark with $k = 2$. The first step is to collect the terms in (3.15). Assuming A_0, A_1 are known then by (3.5), (3.6), Corollary 3.3, and Proposition B.1 we have

$$\begin{aligned} A_0, A_1, r_1 = 0, \alpha_1 &= (v_m, A_1 u_1)^{1/m}, \alpha_2 = \frac{(v_{m-1}, A_1 u_1) + (v_m, A_1 u_2)}{m \alpha_1^{m-2}}, \\ \beta_0 &= u_1, \beta_1 = \alpha_1 u_2, \beta_2 = \alpha_2 u_2 + \alpha_1^2 u_3, \\ p_{0,0} &= 1, p_{1,0} = 0, p_{1,1} = \alpha_1, p_{2,0} = 0, p_{2,1} = \alpha_2, p_{2,2} = \alpha_1^2. \end{aligned}$$

The next step is to determine r_2 using the recursive formula for the r_l 's given in (B.5). We find that

$$\begin{aligned} r_2 &= \frac{1}{2\alpha_1} \sum_{j=1}^1 [(3-j)m - (m+j)] \alpha_{3-j} r_j + \frac{m}{2} \alpha_1^{m-2} \sum_{j=1}^1 [(3-j)m - (m+j)] \alpha_{3-j} \alpha_{j+1} \\ &= \frac{m(m-1)}{2} \alpha_1^{m-2} \alpha_2^2. \end{aligned}$$

Now, since r_2 is determined, we can use the recursive formula in (3.13) for the α_s 's to calculate α_3 . In doing so we find that

$$\begin{aligned} \alpha_3 &= \frac{-r_2 + \sum_{i=0}^{\min\{3,m\}-1} \sum_{j=i}^2 p_{j,i} \left(v_{m-i}, \sum_{k=1}^{\lfloor \frac{m+2-j}{m} \rfloor} A_k \beta_{m+2-j-km} \right)}{m \alpha_1^{m-1}} \\ &= \frac{-r_2 + p_{2,1} (v_{m-1}, A_1 \beta_0) + p_{0,0} (v_m, A_1 \beta_2)}{m \alpha_1^{m-1}} + \\ &\quad \frac{p_{2,2} (v_{m-2}, A_1 \beta_0) + p_{1,1} (v_{m-1}, A_1 \beta_1)}{m \alpha_1^{m-1}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{-\frac{m(m-1)}{2}\alpha_1^{m-2}\alpha_2^2 + \alpha_2(v_{m-1}, A_1 u_1) + (v_m, A_1(\alpha_2 u_2 + \alpha_1^2 u_3))}{m\alpha_1^{m-1}} + \\
 &\quad \frac{\alpha_1^2(v_{m-2}, A_1 u_1) + \alpha_1(v_{m-1}, A_1 \alpha_1 u_2)}{m\alpha_1^{m-1}} \\
 &= \left(\frac{3-m}{2}\right)\alpha_1^{-1}\alpha_2^2 + \frac{(v_{m-2}, A_1 u_1) + (v_{m-1}, A_1 u_2) + (v_m, A_1 u_3)}{m\alpha_1^{m-3}}.
 \end{aligned}$$

Next, since α_3 is determined, we can use (B.4) to calculate $\{p_{3,i}\}_{i=0}^3$. In this case though it suffices to use Proposition B.1 and in doing so we find that

$$p_{3,0} = 0, p_{3,1} = \alpha_3, p_{3,2} = 2\alpha_1\alpha_2, p_{3,3} = \alpha_1^3.$$

Finally, we can compute β_3 using the recursive formula in (3.14) for the β_s 's. In doing so we find that

$$\begin{aligned}
 \beta_3 &= \begin{cases} \sum_{i=0}^3 p_{3,i}u_{i+1}, & \text{if } m > 3 \\ \sum_{i=0}^{m-1} p_{3,i}u_{i+1} - \sum_{j=0}^{3-m} \sum_{k=0}^j \sum_{l=1}^{\lfloor \frac{3-j}{m} \rfloor} p_{j,k}\Lambda^{k+1}A_l\beta_{3-j-lm}, & \text{if } m = 3 \end{cases} \\
 &= \begin{cases} p_{3,1}u_2 + p_{3,2}u_3 + p_{3,3}u_4, & \text{if } m > 3 \\ \sum_{i=0}^2 p_{3,i}u_{i+1} - \Lambda A_1\beta_0, & \text{if } m = 3 \end{cases} \\
 &= \begin{cases} \alpha_3 u_2 + 2\alpha_1\alpha_2 u_3 + \alpha_1^3 u_4, & \text{if } m > 3 \\ \alpha_3 u_2 + 2\alpha_1\alpha_2 u_3 - \Lambda A_1 u_1, & \text{if } m = 3. \end{cases}
 \end{aligned}$$

This completes the calculation of the third order terms, α_3, β_3 , when $m \geq 3$.

4. Proofs. This section contains the proofs of the results of this paper. We begin by proving Theorem 2.1 of §2 on conditions equivalent to the generic condition. We next follow this up with the proof of the main result of this paper Theorem 3.1. We finish by proving the Corollaries 3.2 and 3.3.

4.1. Proof of Theorem 2.1. To prove this theorem we will prove the following chain of statements (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i).

We begin by proving (i) \Rightarrow (ii). Define $f(\varepsilon, \lambda) := \det(\lambda I - A(\varepsilon))$ and suppose (i) is true. Then f is an analytic function of (ε, λ) near $(0, \lambda_0)$ since the matrix elements of $A(\varepsilon)$ are analytic functions of ε in a neighborhood of the origin and the determinant of a matrix is a polynomial in its matrix elements. Also we have $f(0, \lambda_0) = 0$ and $\frac{\partial f}{\partial \varepsilon}(0, \lambda_0) \neq 0$. Hence by the holomorphic implicit function theorem [25, §1.4 Theorem 1.4.11] there exists a unique solution, $\varepsilon(\lambda)$, in a neighborhood of $\lambda = \lambda_0$ with $\varepsilon(\lambda_0) = 0$ to the equation $f(\varepsilon, \lambda) = 0$, which is analytic at $\lambda = \lambda_0$. We now show that $\varepsilon(\lambda)$ has a zero there of order m at $\lambda = \lambda_0$. First, the properties of $\varepsilon(\lambda)$ imply there exists $\varepsilon_q \neq 0$ and $q \in \mathbb{N}$ such that $\varepsilon(\lambda) = \varepsilon_q(\lambda - \lambda_0)^q + O((\lambda - \lambda_0)^{q+1})$, for $|\lambda - \lambda_0| \ll 1$. Next, by hypothesis λ_0 is an eigenvalue of $A(0)$ of algebraic multiplicity m hence $\partial^i f \setminus \partial \lambda^i(0, \lambda_0) = 0$ for $0 \leq i \leq m-1$ but $\partial^m f \setminus \partial \lambda^m(0, \lambda_0) \neq 0$. Combining this with the fact that $f(0, \lambda_0) = 0$ and $\frac{\partial f}{\partial \varepsilon}(0, \lambda_0) \neq 0$ we have

$$f(\varepsilon, \lambda) = a_{10}\varepsilon + a_{0m}(\lambda - \lambda_0)^m + \sum_{\substack{i+j \geq 2, i, j \in \mathbb{N} \\ (i,j) \notin \{(0,j): j \leq m\}}} a_{ij}\varepsilon^i(\lambda - \lambda_0)^j \quad (4.1)$$

for $|\varepsilon| + |\lambda - \lambda_0| \ll 1$, where $a_{10} = \frac{\partial f}{\partial \varepsilon}(0, \lambda_0) \neq 0$ and $a_{0m} = \frac{1}{m!} \frac{\partial^m f}{\partial \lambda^m}(0, \lambda_0) \neq 0$. Then using the expansions of $f(\varepsilon, \lambda)$ and $\varepsilon(\lambda)$ together with the identity $f(\varepsilon(\lambda), \lambda) = 0$ for $|\lambda - \lambda_0| \ll 1$, we find that $q = m$ and

$$\varepsilon_m = -\frac{a_{0m}}{a_{10}} = -\frac{\frac{1}{m!} \frac{\partial^m \det(\lambda I - A(\varepsilon))}{\partial \lambda^m} \Big|_{(\lambda, \varepsilon) = (\lambda_0, 0)}}{\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon)) \Big|_{(\lambda, \varepsilon) = (\lambda_0, 0)}}. \quad (4.2)$$

Therefore we conclude that $\varepsilon(\lambda)$ has a zero of order m at $\lambda = \lambda_0$, which proves (ii).

Next, we prove (ii) \Rightarrow (iii). Suppose (ii) is true. The first part of proving (iii) involves inverting $\varepsilon(\lambda)$ near $\varepsilon = 0$ and $\lambda = \lambda_0$. To do this we expand $\varepsilon(\lambda)$ in a power series about $\lambda = \lambda_0$ and find that $\varepsilon(\lambda) = g(\lambda)^m$ where

$$g(\lambda) = (\lambda - \lambda_0) \left(\varepsilon_m + \sum_{k=m+1}^{\infty} \varepsilon_k (\lambda - \lambda_0)^{k-m} \right)^{1/m}$$

and we are taking any fixed branch of the m th root that is analytic at ε_m . Notice that, for λ in a small enough neighborhood of λ_0 , g is an analytic function, $g(\lambda_0) = 0$, and $\frac{dg}{d\lambda}(\lambda_0) = \varepsilon_m^{1/m} \neq 0$. This implies, by the inverse function theorem for analytic functions, that for λ in a small enough neighborhood of λ_0 the analytic function $g(\lambda)$ has an analytic inverse $g^{-1}(\varepsilon)$ in a neighborhood of $\varepsilon = 0$ with $g^{-1}(0) = \lambda_0$. Define a multivalued function $\lambda(\varepsilon)$, for sufficiently small ε , by $\lambda(\varepsilon) := g^{-1}\left(\varepsilon^{\frac{1}{m}}\right)$ where by $\varepsilon^{\frac{1}{m}}$ we mean all branches of the m th root of ε . We know that g^{-1} is analytic at $\varepsilon = 0$ so that for sufficiently small ε the multivalued function $\lambda(\varepsilon)$ is a Puiseux series. And since $\frac{dg^{-1}}{d\varepsilon}(0) = \left[\frac{dg}{d\lambda}(\lambda_0)\right]^{-1} \neq 0$ we have an expansion

$$\lambda(\varepsilon) = g^{-1}\left(\varepsilon^{\frac{1}{m}}\right) = \lambda_0 + \alpha_1 \varepsilon^{\frac{1}{m}} + \sum_{k=2}^{\infty} \alpha_k \left(\varepsilon^{\frac{1}{m}}\right)^k.$$

Now suppose for fixed λ sufficiently near λ_0 and for sufficiently small ε we have $\det(\lambda I - A(\varepsilon)) = 0$. We want to show this implies $\lambda = \lambda(\varepsilon)$ for one of the branches of the m th root. We know by hypothesis we must have $\varepsilon = \varepsilon(\lambda)$. But as we know this implies that $\varepsilon = \varepsilon(\lambda) = g(\lambda)^m$ hence for some branch of the m th root, $b_m(\cdot)$, we have $b_m(\varepsilon) = b_m(g(\lambda)^m) = g(\lambda)$. But λ is near enough to λ_0 and ε is sufficiently small that we may apply the g^{-1} to both sides yielding $\lambda = g^{-1}(g(\lambda)) = g^{-1}(b_m(\varepsilon)) = \lambda(\varepsilon)$, as desired. Furthermore, all the m branches $\lambda_h(\varepsilon)$, $h = 0, \dots, m-1$ of $\lambda(\varepsilon)$ are given by taking all branches of the m th root of ε so that

$$\lambda_h(\varepsilon) = \lambda_0 + \alpha_1 \zeta^h \varepsilon^{\frac{1}{m}} + \sum_{k=2}^{\infty} \alpha_k \left(\zeta^h \varepsilon^{\frac{1}{m}}\right)^k$$

for any fixed branch of $\varepsilon^{\frac{1}{m}}$, where $\zeta = e^{\frac{2\pi}{m}i}$ and

$$\alpha_1 = \frac{dg^{-1}}{d\varepsilon}(0) = \left[\frac{dg}{d\lambda}(\lambda_0)\right]^{-1} = \varepsilon_m^{-1/m} \neq 0, \quad (4.3)$$

which proves (iii).

Next, we prove (iii) \Rightarrow (iv). Suppose (iii) is true. Define the function $y(\varepsilon) := \lambda_0(\varepsilon^m)$. Then y is analytic at $\varepsilon = 0$ and $\frac{dy}{d\varepsilon}(0) = \lambda_1 \neq 0$. Also we have for ε

sufficiently small $\det(y(\varepsilon)I - A(\varepsilon^m)) = 0$. Consider the inverse of $y(\varepsilon)$, $y^{-1}(\lambda)$. It satisfies $0 = \det(y(y^{-1}(\lambda))I - A([y^{-1}(\lambda)]^m)) = \det(\lambda I - A([y^{-1}(\lambda)]^m))$ with $y^{-1}(\lambda_0) = 0$, $\frac{dy^{-1}}{d\lambda}(\lambda_0) = \alpha_1^{-1}$. Define $g(\lambda) := [y^{-1}(\lambda)]^m$. Then g has a zero of order m at λ_0 and $\det(\lambda I - A(g(\lambda))) = 0$ for λ in a neighborhood of λ_0 .

Now we consider the analytic matrix $A(g(\lambda)) - \lambda I$ in a neighborhood of $\lambda = \lambda_0$ with the constant eigenvalue 0. Because 0 is an analytic eigenvalue of it then there exists an analytic eigenvector, $x(\lambda)$, of $A(g(\lambda)) - \lambda I$ corresponding to the eigenvalue 0 in a neighborhood of λ_0 such that $x(\lambda_0) \neq 0$. Hence for λ near λ_0 we have

$$\begin{aligned} 0 &= (A(g(\lambda)) - \lambda I)x(\lambda) \\ &= \left(A \left(\alpha_1^{-m} (\lambda - \lambda_0)^m + O((\lambda - \lambda_0)^{m+1}) \right) - (\lambda - \lambda_0)I - \lambda_0 I \right) x(\lambda) \\ &= (A(0) - \lambda_0 I)x(\lambda_0) + \left((A(0) - \lambda_0 I) \frac{dx}{d\lambda}(\lambda_0) - x(\lambda_0) \right) (\lambda - \lambda_0) + \cdots \\ &\quad + \left((A(0) - \lambda_0 I) \frac{d^{m-1}x}{d\lambda^{m-1}}(\lambda_0) - \frac{d^{m-2}x}{d\lambda^{m-2}}(\lambda_0) \right) (\lambda - \lambda_0)^{m-1} \\ &\quad + \left((A(0) - \lambda_0 I) \frac{d^m x}{d\lambda^m}(\lambda_0) - \frac{d^{m-1}x}{d\lambda^{m-1}}(\lambda_0) + \alpha_1^{-m} A'(0)x(\lambda_0) \right) (\lambda - \lambda_0)^m \\ &\quad + O((\lambda - \lambda_0)^{m+1}). \end{aligned}$$

This implies that

$$\begin{aligned} (A(0) - \lambda_0 I)x(\lambda_0) &= 0, (A(0) - \lambda_0 I) \frac{d^j x}{d\lambda^j}(\lambda_0) = \frac{d^{j-1}x}{d\lambda^{j-1}}(\lambda_0), \text{ for } j = 1, \dots, m-1, \\ (A(0) - \lambda_0 I) \frac{d^m x}{d\lambda^m}(\lambda_0) &= \frac{d^{m-1}x}{d\lambda^{m-1}}(\lambda_0) - \alpha_1^{-m} A'(0)x(\lambda_0). \end{aligned} \quad (4.4)$$

The first m equations imply that $x(\lambda_0), \frac{dx}{d\lambda}(\lambda_0), \dots, \frac{d^{m-1}x}{d\lambda^{m-1}}(\lambda_0)$ is a Jordan chain of length m generated by $\frac{d^{m-1}x}{d\lambda^{m-1}}(\lambda_0)$. Since the algebraic multiplicity of λ_0 for $A(0)$ is m this implies that there is a single $m \times m$ Jordan block corresponding to the eigenvalue λ_0 where we can take $x(\lambda_0), \frac{dx}{d\lambda}(\lambda_0), \dots, \frac{d^{m-1}x}{d\lambda^{m-1}}(\lambda_0)$ as a Jordan basis. It follows from basic properties of Jordan chains that there exists an eigenvector v of $A(0)^*$ corresponding to the eigenvalue λ_0^* such that $\left(v, \frac{d^{m-1}x}{d\lambda^{m-1}}(\lambda_0) \right) = 1$. Hence

$$0 = \left((A(0) - \lambda_0 I)^* v, \frac{d^m x}{d\lambda^m}(\lambda_0) \right) \stackrel{(4.4)}{=} 1 - \alpha_1^{-m} (v, A'(0)x(\lambda_0))$$

implying that $(v, \frac{dA}{d\varepsilon}(0)x(\lambda_0)) = \alpha_1^m \neq 0$. Therefore we have shown that the Jordan normal form of $A(0)$ corresponding to the eigenvalue λ_0 consists of a single $m \times m$ Jordan block and there exists an eigenvector u of $A(0)$ corresponding to the eigenvalue λ_0 and an eigenvector v of $A(0)^*$ corresponding to the eigenvalue λ_0^* such that $(v, A'(0)u) \neq 0$. This proves (iv).

Finally, we show (iv) \Rightarrow (i). Suppose (iv) is true. We begin by noting that since

$$\det(\lambda_0 I - A(\varepsilon)) = (-1)^n \det((A(0) - \lambda_0 I) + A'(0)\varepsilon) + o(\varepsilon)$$

it suffices to show that

$$S_{n-1} := \frac{d}{d\varepsilon} \det((A(0) - \lambda_0 I) + A'(0)\varepsilon) \Big|_{\varepsilon=0} \neq 0. \quad (4.5)$$

We will use the result from [26, Theorem 2.16] to prove (4.5). Let $A(0) - \lambda_0 I = Y \Sigma X^*$ be a singular-value decomposition of the matrix $A(0) - \lambda_0 I$ where X, Y are unitary matrices and $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_{n-1}, \sigma_n)$ with $\sigma_1 \geq \dots \geq \sigma_{n-1} \geq \sigma_n \geq 0$ (see [10, §5.7, Theorem 2]). Now since the Jordan normal form of $A(0)$ corresponding to the eigenvalue λ_0 consists of a single Jordan block this implies that rank of $A(0) - \lambda_0 I$ is $n-1$. This implies that $\sigma_1 \geq \dots \geq \sigma_{n-1} > \sigma_n = 0$, $u = X e_n$ is an eigenvector of $A(0)$ corresponding to the eigenvalue λ_0 , $v = Y e_n$ is an eigenvector of $A(0)$ corresponding to the eigenvalue λ_0^* , and there exist nonzero constants c_1, c_2 such that $u = c_1 u_0$ and $v = c_2 v_0$.

Now using the result of [26, Theorem 2.16] for (4.5) we find that

$$S_{n-1} = \det(YX^*) \sum_{1 \leq i_1 < \dots < i_{n-1} \leq n} \sigma_{i_1} \cdots \sigma_{i_{n-1}} \det\left((Y^* A'(0) X)_{i_1 \dots i_{n-1}}\right),$$

where $(Y^* A'(0) X)_{i_1 \dots i_{n-1}}$ is the matrix obtained from $Y^* A'(0) X$ by removing rows and columns $i_1 \dots i_{n-1}$. But since $\sigma_n = 0$ and

$$(Y^* A'(0) X)_{1 \dots (n-1)} = e_n^* Y^* A'(0) X e_n = (v, A'(0) u) = c_2^* c_1 (v_0, A'(0) u_0) \neq 0$$

then $S_{n-1} = \det(YX^*) \prod_{j=1}^{n-1} \sigma_j c_2^* c_1 (v_0, A'(0) u_0) \neq 0$. This completes the proof. \square

4.2. Proof of Theorem 3.1. We begin by noting that our hypotheses imply that statements (ii), (iii), and (iv) of Theorem 2.1 are true. In particular, statement (iii) implies that there is exactly one convergent Puiseux series for the λ_0 -group whose branches are given by

$$\lambda_h(\varepsilon) = \lambda_0 + \alpha_1 \zeta^h \varepsilon^{\frac{1}{m}} + \sum_{k=2}^{\infty} \alpha_k \left(\zeta^h \varepsilon^{\frac{1}{m}}\right)^k,$$

for $h = 0, \dots, m-1$ and any fixed branch of $\varepsilon^{\frac{1}{m}}$, where $\zeta = e^{\frac{2\pi}{m}i}$ and $\alpha_1 \neq 0$. Then by well known results [1, §6.1.7, Theorem 2], [2, §II.1.8] there exists a convergent Puiseux series for the corresponding eigenvectors whose branches are given by

$$x_h(\varepsilon) = \beta_0 + \sum_{k=1}^{\infty} \beta_k \left(\zeta^h \varepsilon^{\frac{1}{m}}\right)^k,$$

for $h = 0, \dots, m-1$, where β_0 is an eigenvector of $A_0 = A(0)$ corresponding to the eigenvalue λ_0 . Now if we examine the proof of (ii) \Rightarrow (iii) in Theorem 2.1 we see by equation (4.3) that $\alpha_1^m = \varepsilon_m^{-1}$, where ε_m is given in equation (4.2) in the proof of (i) \Rightarrow (iii) for Theorem 2.1. Thus we can conclude that

$$\alpha_1^m = -\frac{\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon))|_{(\varepsilon, \lambda)=(0, \lambda_0)}}{\left(\frac{\frac{\partial^m}{\partial \lambda^m} \det(\lambda I - A(\varepsilon))|_{(\varepsilon, \lambda)=(0, \lambda_0)}}{m!}\right)} \neq 0. \quad (4.6)$$

Choose any m th root of $(v_m, A_1 u_1)$ and denote it by $(v_m, A_1 u_1)^{1/m}$. By (4.6) we can just reindexing the Puiseux series (3.8) and (3.9) and assume that

$$\alpha_1 = \left(-\frac{\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon))|_{(\varepsilon, \lambda)=(0, \lambda_0)}}{\left(\frac{\frac{\partial^m}{\partial \lambda^m} \det(\lambda I - A(\varepsilon))|_{(\varepsilon, \lambda)=(0, \lambda_0)}}{m!}\right)} \right)^{1/m}.$$

Next, we wish to prove that we can choose the perturbed eigenvectors (3.9) to satisfy the normalization conditions (3.11). But this follows by Theorem 2.1 (iv) and the fact β_0 is an eigenvector of $A(0)$ corresponding to the eigenvalue λ_0 since then $(v_1, \beta_0) \neq 0$ and so we may take $\frac{x_h(\varepsilon)}{(v_1, x_h(\varepsilon))}$, for $h = 0, \dots, m-1$, to be the perturbed eigenvectors in (3.9) that satisfy the normalization conditions (3.11).

Now we are ready to begin showing that $\{\alpha_s\}_{s=1}^\infty, \{\beta_s\}_{s=0}^\infty$ are given by the recursive formulas (3.12)-(3.14). The first key step is proving the following:

$$(A_0 - \lambda_0 I) \beta_s = - \sum_{k=1}^s \left(A_{\frac{k}{m}} - \alpha_k I \right) \beta_{s-k}, \text{ for } s \geq 1, \quad (4.7)$$

$$\beta_0 = u_1, \beta_s = \Lambda (A_0 - \lambda_0 I) \beta_s, \text{ for } s \geq 1, \quad (4.8)$$

where we define $A_{\frac{k}{m}} := 0$, if $\frac{k}{m} \notin \mathbb{N}$.

The first equality holds since in a neighborhood of the origin

$$0 = (A(\varepsilon) - \lambda_0(\varepsilon) I) x_0(\varepsilon) = \sum_{s=0}^{\infty} \left(\sum_{k=0}^s \left(A_{\frac{k}{m}} - \alpha_k I \right) \beta_{s-k} \right) \varepsilon^{\frac{s}{m}}.$$

The second equality will be proven once we show $\beta_0 = u_1$ and $\beta_s \in S := \text{span}\{Ue_i | 2 \leq i \leq n\}$, for $s \geq 1$, where U is the matrix from (3.1). This will prove (4.8) because $\Lambda(A_0 - \lambda_0 I)$ acts as the identity on S by Proposition A.1.i. But these follow from the facts that $S = \{x \in \mathbb{C}^{n \times 1} | (v_1, x) = 0\}$ and the normalization conditions (3.11) imply that $(v_1, \beta_0) = 1$ and $(v_1, \beta_s) = 0$, for $s \geq 1$.

The next key step in this proof is the following lemma:

LEMMA 4.1. *For all $s \geq 0$ the following identity holds*

$$(A_0 - \lambda_0 I) \beta_s = \begin{cases} \sum_{i=0}^s p_{s,i} u_i, & \text{for } 0 \leq s \leq m-1 \\ \sum_{i=0}^m p_{s,i} u_i - \sum_{j=0}^{s-m} \sum_{k=0}^j \sum_{l=1}^{\lfloor \frac{s-j}{m} \rfloor} p_{j,k} \Lambda^k A_l \beta_{s-j-lm}, & \text{for } s \geq m \end{cases} \quad (4.9)$$

where we define $u_0 := 0$.

Proof. The proof is by induction on s . The statement is true for $s = 0$ since $p_{0,0} u_0 = 0 = (A_0 - \lambda_0 I) \beta_0$. Now suppose it was true for all r with $0 \leq r \leq s$ for some nonnegative integer s . We will show the statement is true for $s+1$ as well.

Suppose $s+1 \leq m-1$ then $(A_0 - \lambda_0 I) \beta_r = \sum_{i=0}^r p_{r,i} u_i$ for $0 \leq r \leq s$ and we must show that $(A_0 - \lambda_0 I) \beta_{s+1} = \sum_{i=0}^{s+1} p_{s+1,i} u_i$. Well, for $1 \leq r \leq s$,

$$\beta_r \stackrel{(4.8)}{=} \Lambda (A_0 - \lambda_0 I) \beta_r = \sum_{i=0}^r p_{r,i} \Lambda u_i \stackrel{(A.1)}{=} \sum_{i=1}^r p_{r,i} u_{i+1}. \quad (4.10)$$

Hence the statement is true if $s+1 \leq m-1$ since

$$\begin{aligned} (A_0 - \lambda_0 I) \beta_{s+1} &\stackrel{(4.7)}{=} - \sum_{k=1}^{s+1} \left(A_{\frac{k}{m}} - \alpha_k I \right) \beta_{s+1-k} \stackrel{(4.10)}{=} \sum_{k=1}^{s+1} \sum_{i=0}^{s+1-k} \alpha_k p_{s+1-k,i} u_{i+1} \\ &\stackrel{(C.1)}{=} \sum_{i=0}^s \left(\sum_{k=1}^{s+1-i} \alpha_k p_{s+1-k,i} \right) u_{i+1} \stackrel{(B.3)}{=} \sum_{i=0}^{s+1} p_{s+1,i} u_i. \end{aligned}$$

Now suppose that $s + 1 \geq m$. The proof is similar to what we just proved. By the induction hypothesis (4.9) is true for $1 \leq r \leq s$ and $\beta_r \stackrel{(4.8)}{=} \Lambda(A_0 - \lambda_0 I) \beta_r$ thus

$$\beta_r \stackrel{(A.1)}{=} \begin{cases} \sum_{i=0}^r p_{r,i} u_{i+1}, & \text{for } 0 \leq r \leq m-1 \\ \sum_{i=0}^{m-1} p_{r,i} u_{i+1} - \sum_{j=0}^{r-m} \sum_{k=0}^j \sum_{l=1}^{\lfloor \frac{r-j}{m} \rfloor} p_{j,k} \Lambda^{k+1} A_l \beta_{r-j-lm}, & \text{for } r \geq m. \end{cases} \quad (4.11)$$

Hence we have

$$\begin{aligned} (A_0 - \lambda_0 I) \beta_{s+1} &\stackrel{(4.7)}{=} - \sum_{k=1}^{s+1} \left(A_{\frac{k}{m}} - \alpha_k I \right) \beta_{s+1-k} \\ &\stackrel{(4.11)}{=} - \sum_{l=1}^{\lfloor \frac{s+1}{m} \rfloor} A_l \beta_{s+1-lm} + \sum_{k=1}^{s+1-m} \sum_{i=0}^{m-1} \alpha_k p_{s+1-k,i} u_{i+1} \\ &\quad - \sum_{k=1}^{s+1-m} \sum_{j=0}^{s+1-k-m} \sum_{i=0}^j \sum_{l=1}^{\lfloor \frac{s+1-k-j}{m} \rfloor} \alpha_k p_{j,i} \Lambda^{i+1} A_l \beta_{s+1-k-j-lm} \\ &\quad + \sum_{k>s+1-m}^{s+1} \sum_{i=0}^{s+1-k} \alpha_k p_{s+1-k,i} u_{i+1} \\ &\stackrel{(C.1)}{=} - \sum_{l=1}^{\lfloor \frac{s+1}{m} \rfloor} A_l \beta_{s+1-lm} + \sum_{i=0}^{m-1} \left(\sum_{k=1}^{s+1-i} \alpha_k p_{s+1-k,i} \right) u_{i+1} \\ &\quad - \sum_{k=1}^{s+1-m} \sum_{j=0}^{s+1-k-m} \sum_{i=0}^j \sum_{l=1}^{\lfloor \frac{s+1-k-j}{m} \rfloor} \alpha_k p_{j,i} \Lambda^{i+1} A_l \beta_{s+1-k-j-lm} \\ &\stackrel{(B.3)}{=} - \sum_{l=1}^{\lfloor \frac{s+1}{m} \rfloor} A_l \beta_{s+1-lm} + \sum_{i=0}^m p_{s+1,i} u_i \\ &\quad - \sum_{k=1}^{s+1-m} \sum_{j=0}^{s+1-k-m} \sum_{i=0}^j \sum_{l=1}^{\lfloor \frac{s+1-k-j}{m} \rfloor} \alpha_k p_{j,i} \Lambda^{i+1} A_l \beta_{s+1-k-j-lm}. \end{aligned}$$

Now let $a_{k,j,i} := \sum_{l=1}^{\lfloor \frac{s+1-k-j}{m} \rfloor} \alpha_k p_{j,i} \Lambda^{i+1} A_l \beta_{s+1-k-j-lm}$. Then using the sum identity

$$\begin{aligned} \sum_{k=1}^{s+1-m} \sum_{j=0}^{s+1-k-m} \sum_{i=0}^j a_{k,j,i} &\stackrel{(C.1)}{=} \sum_{j=0}^{s-m} \sum_{k=1}^{s+1-j-m} \sum_{i=0}^j a_{k,j,i} = \sum_{j=0}^{s-m} \sum_{i=0}^j \sum_{k=1}^{s+1-j-m} a_{k,j,i} \\ &\stackrel{(C.3)}{=} \sum_{i=0}^{s-m} \sum_{j=i}^{s-m} \sum_{k=1}^{s+1-j-m} a_{k,j,i} \stackrel{(C.4)}{=} \sum_{i=0}^{s-m} \sum_{q=i+1}^{s+1-m} \sum_{k=1}^{q-i} a_{k,q-k,i} \end{aligned}$$

we can concluded that

$$\begin{aligned} (A_0 - \lambda_0 I) \beta_{s+1} &= - \sum_{l=1}^{\lfloor \frac{s+1}{m} \rfloor} A_l \beta_{s+1-lm} + \sum_{i=0}^m p_{s+1,i} u_i \\ &\quad - \sum_{i=0}^{s-m} \sum_{q=i+1}^{s+1-m} \sum_{k=1}^{q-i} \sum_{l=1}^{\lfloor \frac{s+1-q}{m} \rfloor} \alpha_k p_{q-k,i} \Lambda^{i+1} A_l \beta_{s+1-q-lm} \\ &= - \sum_{l=1}^{\lfloor \frac{s+1}{m} \rfloor} A_l \beta_{s+1-lm} + \sum_{i=0}^m p_{s+1,i} u_i \end{aligned}$$

$$\begin{aligned}
 & - \sum_{i=0}^{s-m} \sum_{q=i+1}^{s+1-m} \left[\sum_{l=1}^{\lfloor \frac{s+1-q}{m} \rfloor} \right] \left(\sum_{k=1}^{q-i} \alpha_k p_{q-k,i} \right) \Lambda^{i+1} A_l \beta_{s+1-q-lm} \\
 \stackrel{\text{(B.3)}}{=} & - \sum_{l=1}^{\lfloor \frac{s+1}{m} \rfloor} A_l \beta_{s+1-lm} + \sum_{i=0}^m p_{s+1,i} u_i \\
 & - \sum_{i=0}^{s-m} \sum_{q=i+1}^{s+1-m} \left[\sum_{l=1}^{\lfloor \frac{s+1-q}{m} \rfloor} \right] p_{q,i+1} \Lambda^{i+1} A_l \beta_{s+1-q-lm} \\
 \stackrel{\text{(C.2)}}{=} & - \sum_{l=1}^{\lfloor \frac{s+1}{m} \rfloor} A_l \beta_{s+1-lm} + \sum_{i=0}^m p_{s+1,i} u_i \\
 & - \sum_{q=1}^{s+1-m} \sum_{i=0}^{q-1} \left[\sum_{l=1}^{\lfloor \frac{s+1-q}{m} \rfloor} \right] p_{q,i+1} \Lambda^{i+1} A_l \beta_{s+1-q-lm} \\
 \stackrel{\text{(3.5)}}{=} & \sum_{i=0}^m p_{s+1,i} u_i - \sum_{j=0}^{s+1-m} \sum_{k=0}^j \left[\sum_{l=1}^{\lfloor \frac{s+1-j}{m} \rfloor} \right] p_{j,k} \Lambda^k A_l \beta_{s+1-j-lm}.
 \end{aligned}$$

But this is the statement we needed to prove for $s+1 \geq m$. Therefore by induction the statement (4.9) is true for all $s \geq 0$ and the lemma is proved. \square

The lemma above is the key to prove the recursive formulas for α_s and β_s as given by (3.12)-(3.14). First we prove that β_s is given by (3.14). For $s=0$ we have already shown $\beta_0 = u_1 = p_{0,0} u_1$. So suppose $s \geq 1$. Then by (4.8) and (4.9) we find that

$$\beta_s \stackrel{\text{(A.1)}}{=} \begin{cases} \sum_{i=0}^s p_{s,i} u_{i+1}, & \text{if } 0 \leq s \leq m-1 \\ \sum_{i=0}^{m-1} p_{s,i} u_{i+1} - \sum_{j=0}^{s-m} \sum_{k=0}^j \left[\sum_{l=1}^{\lfloor \frac{s-j}{m} \rfloor} \right] p_{j,k} \Lambda^{k+1} A_l \beta_{s-j-lm}, & \text{if } s \geq m. \end{cases}$$

This proves that β_s is given by (3.14).

Next we will prove that α_s is given by (3.12) and (3.13). We start with $s=1$ and prove α_1 is given by (3.12). First, $(A_0 - \lambda_0 I)^* v_m = 0$ and $(v_m, u_i) = \delta_{m,i}$ hence

$$0 = (v_m, (A_0 - \lambda_0 I) \beta_m) \stackrel{\text{(4.9)}}{=} \left(v_m, \sum_{i=0}^m p_{m,i} u_i - A_1 u_1 \right) \stackrel{\text{(B.2)}}{=} \alpha_1^m - (v_m, A_1 u_1)$$

so that $\alpha_1^m = (v_m, A_1 u_1)$. This and identity (4.6) imply that formula (3.12) is true.

Finally, suppose that $s \geq 2$. Then $(A_0 - \lambda_0 I)^* v_m = 0$ and $(v_m, u_i) = \delta_{m,i}$ implies

$$\begin{aligned}
 0 & = (v_m, (A_0 - \lambda_0 I) \beta_{m+s-1}) \\
 \stackrel{\text{(4.9)}}{=} & \left(v_m, \sum_{i=0}^m p_{m+s-1,i} u_i - \sum_{j=0}^{s-1} \sum_{k=0}^j \left[\sum_{l=1}^{\lfloor \frac{m+s-1-j}{m} \rfloor} \right] p_{j,k} \Lambda^k A_l \beta_{m+s-1-j-lm} \right) \\
 \stackrel{\text{(C.3)}}{=} & p_{m+s-1,m} - \sum_{k=0}^{s-1} \sum_{j=k}^{s-1} p_{j,k} \left((\Lambda^*)^k v_m, \sum_{l=1}^{\lfloor \frac{m+s-1-j}{m} \rfloor} A_l \beta_{m+s-1-j-lm} \right) \\
 \stackrel{\text{(B.1)}}{=} & r_{s-1} + m \alpha_1^{m-1} \alpha_s - \sum_{i=0}^{s-1} \sum_{j=i}^{s-1} p_{j,i} \left((\Lambda^*)^i v_m, \sum_{k=1}^{\lfloor \frac{m+s-1-j}{m} \rfloor} A_k \beta_{m+s-1-j-km} \right)
 \end{aligned}$$

Therefore with this equality, the fact $\alpha_1 \neq 0$, and Proposition A.1.iii, we can solve for α_s and we will find that it is given by (3.13). This completes the proof. \square

4.3. Proof of Corollaries 3.2 and 3.3. Both corollaries follow almost trivially now. To prove Corollary 3.2, we just examine the recursive formulas (3.12)-(3.14) in Theorem 3.1 to see that α_k, β_k requires only $A_0, \dots, A_{\lfloor \frac{m+k-1}{m} \rfloor}$. To prove Corollary 3.3, we use Proposition B.1 to show that

$$p_{0,0} = 1, p_{1,0} = p_{2,0} = 0, p_{1,1} = \alpha_1, p_{2,1} = \alpha_2, p_{2,2} = \alpha_1^2$$

and then from this and (3.12)-(3.14) we get the desired result for $\alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2$ in terms of A_0, A_1, A_2 . The last part to prove is the formula for α_2 in terms of $f(\varepsilon, \lambda)$ and its partial derivatives. But the formula follows from the series representation of $f(\varepsilon, \lambda)$ in (4.1) and $\lambda_0(\varepsilon)$ in (3.8) since, for ε in a neighborhood of the origin,

$$\begin{aligned} 0 &= f(\varepsilon, \lambda_0(\varepsilon)) \\ &= (a_{10} + a_{0m}p_{m,m})\varepsilon \\ &\quad + \begin{cases} (a_{01}p_{2,1} + a_{02}p_{2,2} + a_{11}p_{1,1} + a_{20}p_{00})\varepsilon^2, & \text{for } m = 1 \\ (a_{0m}p_{m+1,m} + a_{0m+1}p_{m+1,m+1} + a_{11}p_{1,1})\varepsilon^{\frac{m+1}{m}}, & \text{for } m > 1 \end{cases} \\ &\quad + O\left(\varepsilon^{\frac{m+2}{m}}\right) \end{aligned}$$

which together with Proposition B.1 implies the formula for α_2 . \square

Appendix A.

The fundamental properties of the matrix Λ defined in (3.4) which are needed in this paper are given in the following proposition:

PROPOSITION A.1.

- (i) We have $\Lambda(A_0 - \lambda_0 I)Ue_1 = 0$, $\Lambda(A_0 - \lambda_0 I)Ue_i = Ue_i$, for $2 \leq i \leq n$,
- (ii) For $1 \leq i \leq m-1$ we have

$$\Lambda u_m = 0, \Lambda u_i = u_{i+1} \tag{A.1}$$

- (iii) $\Lambda^* v_1 = 0$, and $\Lambda^* v_i = v_{i-1}$, for $2 \leq i \leq m$.

Proof. i. Using the fact $J_m(0)^* J_m(0) = \text{diag}[0, I_{m-1}]$, (3.1), and (3.4) we find by block multiplication that $U^{-1}\Lambda(A_0 - \lambda_0 I)U = \text{diag}[0, I_{n-1}]$. This implies the result.

- ii. & iii. The results follow from the definition of u_i, v_i in (3.2), (3.3) and the fact

$$(U^{-1}\Lambda U)^* = U^* \Lambda^* (U^{-1})^* = \left[\begin{array}{c|c} J_m(0) & \\ \hline & [(W_0 - \lambda_0 I_{n-m})^{-1}]^* \end{array} \right]. \quad \square$$

Appendix B.

This appendix contains two propositions. The first proposition gives fundamental identities that help to characterize the polynomials $\{p_{j,i}\}_{j=i}^\infty$ and $\{r_l\}_{l \in \mathbb{N}}$ in (3.5) and (3.6). The second proposition gives explicit recursive formulas to calculate these polynomials.

We may assume $\sum_{j=1}^\infty \alpha_j z^j$ is a convergent Taylor series and $\alpha_1 \neq 0$.

PROPOSITION B.1. *The polynomials $\{p_{j,i}\}_{j=i}^\infty$ and $\{r_l\}_{l \in \mathbb{N}}$ have the following properties:*

- (i) $\sum_{j=i}^\infty p_{j,i} z^j = \left(\sum_{j=1}^\infty \alpha_j z^j \right)^i$, for $j \geq i \geq 0$.
- (ii) For $l \geq 1$ we have

$$r_l = p_{m+l,m} - m\alpha_1^{m-1}\alpha_{l+1}. \tag{B.1}$$

- (iii) $p_{j,1} = \alpha_j$, for $j \geq 1$.
 (iv) For $j \geq 0$ we have

$$p_{j,j} = \alpha_1^j. \quad (\text{B.2})$$

- (v) $p_{j+1,j} = j\alpha_1^{j-1}\alpha_2$, for $j > 0$.
 (vi) For $j \geq i > 0$ we have

$$\sum_{q=1}^{j-i+1} \alpha_q p_{j-q,i-1} = p_{j,i}. \quad (\text{B.3})$$

Proof. i. For $i \geq 0$, $\left(\sum_{j=1}^{\infty} \alpha_j z^j\right)^i = \sum_{s_1=1}^{\infty} \cdots \sum_{s_i=1}^{\infty} \left(\prod_{\varrho=1}^i \alpha_{s_{\varrho}}\right) z^{s_1+\cdots+s_i} = \sum_{j=i}^{\infty} p_{j,i} z^j$.

ii. Let $l \geq 1$. Then by definition of $p_{m+l,m}$ we have

$$\begin{aligned} p_{m+l,m} &= \sum_{\substack{s_1+\cdots+s_m=m+l \\ 1 \leq s_{\varrho} \leq l+1 \\ \exists \varrho \in \{1, \dots, m\} \text{ such that } s_{\varrho}=l+1}} \prod_{\varrho=1}^m \alpha_{s_{\varrho}} + \sum_{\substack{s_1+\cdots+s_m=m+l \\ 1 \leq s_{\varrho} \leq l+1 \\ \nexists \varrho \in \{1, \dots, m\} \text{ such that } s_{\varrho}=l+1}} \prod_{\varrho=1}^m \alpha_{s_{\varrho}} \\ &= m\alpha_1^{m-1}\alpha_{l+1} + r_l. \end{aligned}$$

iii. For $j \geq 1$ we have $p_{j,1} = \sum_{\substack{s_1=j \\ 1 \leq s_{\varrho} \leq j}} \prod_{\varrho=1}^1 \alpha_{s_{\varrho}} = \alpha_j$.

iv. For $j \geq 0$, $p_{0,0} = 1$ and $p_{j,j} = \sum_{\substack{s_1+\cdots+s_j=j \\ 1 \leq s_{\varrho} \leq 1}} \prod_{\varrho=1}^j \alpha_{s_{\varrho}} = \prod_{\varrho=1}^j \alpha_1 = \alpha_1^j$.

v. For $j > 0$, $p_{j+1,j} = \sum_{\substack{s_1+\cdots+s_j=j+1 \\ 1 \leq s_{\varrho} \leq 2}} \prod_{\varrho=1}^j \alpha_{s_{\varrho}} = \sum_{\varrho=1}^j \alpha_1^{j-1} \alpha_2 = j\alpha_1^{j-1}\alpha_2$.

vi. It follows by

$$\sum_{j=i}^{\infty} p_{j,i} z^j \stackrel{(i)}{=} \sum_{j=i-1}^{\infty} p_{j,i-1} z^j \sum_{j=1}^{\infty} p_{j,1} z^j = \sum_{j=i}^{\infty} \left(\sum_{q=1}^{j-i+1} p_{j-q,i-1} p_{q,1} \right) z^j. \quad \square$$

This next proposition gives explicit recursive formulas to calculate the polynomials $\{p_{j,i}\}_{j=i}^{\infty}$ and $\{r_l\}_{l \in \mathbb{N}}$.

PROPOSITION B.2. *For each $i \geq 0$, the sequence of polynomials, $\{p_{j,i}\}_{j=i}^{\infty}$, is given by the recursive formula*

$$p_{i,i} = \alpha_1^i, p_{j,i} = \frac{1}{(j-i)\alpha_1} \sum_{k=i}^{j-1} [(j+1-k)i-k] \alpha_{j+1-k} p_{k,i}, \text{ for } j > i. \quad (\text{B.4})$$

Furthermore, the polynomials $\{r_l\}_{l \in \mathbb{N}}$ are given by the recursive formula:

$$\begin{aligned} r_1 &= 0, r_l = \frac{1}{l\alpha_1} \sum_{j=1}^{l-1} [(l+1-j)m - (m+j)] \alpha_{l+1-j} r_j + \\ &\quad \frac{m}{l} \alpha_1^{m-2} \sum_{j=1}^{l-1} [(l+1-j)m - (m+j)] \alpha_{l+1-j} \alpha_{j+1}, \text{ for } l > 1. \end{aligned} \quad (\text{B.5})$$

Proof. We begin by showing (B.4) is true. For $i = 0$, (B.4) follows from the definition of the $p_{j,0}$. If $i > 0$ then by (B.2) and [27, (1.1) & (3.2)] it follows that (B.4) is true. Lets now prove (B.5). From (B.1) and (B.4), we have

$$\begin{aligned}
r_l &= p_{m+l,m} - m\alpha_1^{m-1}\alpha_{l+1} \\
&= \frac{1}{l\alpha_1} \sum_{k=m}^{m+l-1} [(m+l+1-k)m-k]\alpha_{m+l+1-k}p_{k,m} - m\alpha_1^{m-1}\alpha_{l+1} \\
&\stackrel{(B.2)}{=} \frac{1}{l\alpha_1} \sum_{k=m+1}^{m+l-1} [(m+l+1-k)m-k]\alpha_{m+l+1-k}p_{k,m} \\
&= \frac{1}{l\alpha_1} \sum_{j=1}^{l-1} [(l+1-j)m-(m+j)]\alpha_{l+1-j}p_{m+j,m} \\
&\stackrel{(B.1)}{=} \frac{1}{l\alpha_1} \sum_{j=1}^{l-1} [(l+1-j)m-(m+j)]\alpha_{l+1-j} (r_j + m\alpha_1^{m-1}\alpha_{j+1}) \\
&= \frac{1}{l\alpha_1} \sum_{j=1}^{l-1} [(l+1-j)m-(m+j)]\alpha_{l+1-j}r_j + \\
&\quad \frac{m}{l}\alpha_1^{m-2} \sum_{j=1}^{l-1} [(l+1-j)m-(m+j)]\alpha_{l+1-j}\alpha_{j+1},
\end{aligned}$$

for $l > 1$. This completes the proof. \square

Appendix C. These double sum identities are used in the proof of Theorem 3.1

$$\sum_{x=c}^d \sum_{y=0}^{d-x} a_{x,y} = \sum_{y=0}^{d-c} \sum_{x=c}^{d-y} a_{x,y}, \quad (C.1)$$

$$\sum_{x=0}^{d-1} \sum_{y=x+1}^d a_{x,y} = \sum_{y=1}^d \sum_{x=0}^{y-1} a_{x,y}, \quad (C.2)$$

$$\sum_{x=0}^d \sum_{y=0}^x a_{x,y} = \sum_{y=0}^d \sum_{x=y}^d a_{x,y}, \quad (C.3)$$

$$\sum_{y=c}^{d-1} \sum_{x=1}^{d-y} a_{x,y} = \sum_{q=c+1}^d \sum_{x=1}^{q-c} a_{x,q-x}. \quad (C.4)$$

Acknowledgments. I would like to thank Prof. Alexander Figotin for bringing the subject of this paper to my attention and for all the helpful suggestions and encouragement given in the various stages of writing this paper. I am also indebted to the anonymous referees for their valuable comments on my original manuscript.

REFERENCES

- [1] H. BAUMGÄRTEL, *Analytic perturbation theory for matrices and operators*, vol. 15 of Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, 1985.
- [2] TOSIO KATO, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

- [3] V. B. LIDSKII, *Perturbation theory of non-conjugate operators*, USSR Computational Mathematics and Mathematical Physics, 6 (1966), pp. 73–85.
- [4] JULIO MORO, JAMES V. BURKE, AND MICHAEL L. OVERTON, *On the Lidskii-Vishik-Lyusternik perturbation theory for eigenvalues of matrices with arbitrary Jordan structure*, SIAM J. Matrix Anal. Appl., 18 (1997), pp. 793–817.
- [5] M. M. VAĪNBERG AND V. A. TRENIGIN, *Theory of branching of solutions of non-linear equations*, Noordhoff International Publishing, Leyden, 1974. Translated from the Russian by Israel Program for Scientific Translations.
- [6] M. I. VIŠIK AND L. A. LJUSTERNIK, *Solution of some perturbation problems in the case of matrices and self-adjoint or non-selfadjoint differential equations. I*, Russian Math. Surveys, 15 (1960), pp. 1–73.
- [7] ALEX FIGOTIN AND ILYA VITEBSKIY, *Slow light in photonic crystals*, Waves in Random and Complex Media, 16 (2006), pp. 293–382.
- [8] G. MUMCU, K. SERTEL, J. L. VOLAKIS, I. VITEBSKIY, AND A. FIGOTIN, *RF propagation in finite thickness unidirectional magnetic photonic crystals*, Antennas and Propagation, IEEE Transactions on, 53 (2006), pp. 4026–4034.
- [9] S. YARGA, K. SERTEL, AND J. L. VOLAKIS, *Degenerate Band Edge Crystals for Directive Antennas*, Antennas and Propagation, IEEE Transactions on, 56 (2008), pp. 119–126.
- [10] PETER LANCASTER AND MIRON TISMENETSKY, *The theory of matrices*, Computer Science and Applied Mathematics, Academic Press Inc., Orlando, FL, second ed., 1985.
- [11] ALAN L. ANDREW, K.-W. ERIC CHU, AND PETER LANCASTER, *Derivatives of eigenvalues and eigenvectors of matrix functions*, SIAM J. Matrix Anal. Appl., 14 (1993), pp. 903–926.
- [12] ALAN L. ANDREW AND ROGER C. E. TAN, *Iterative computation of derivatives of repeated eigenvalues and the corresponding eigenvectors*, Numer. Linear Algebra Appl., 7 (2000), pp. 151–167.
- [13] KING-WAH ERIC CHU, *On multiple eigenvalues of matrices depending on several parameters*, SIAM J. Numer. Anal., 27 (1990), pp. 1368–1385.
- [14] R. HRYNIV AND P. LANCASTER, *On the perturbation of analytic matrix functions*, Integral Equations Operator Theory, 34 (1999), pp. 325–338.
- [15] C.-P. JEANNEROD AND E. PFLÜGEL, *A reduction algorithm for matrices depending on a parameter*, in Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation (Vancouver, BC), New York, 1999, ACM, pp. 121–128 (electronic).
- [16] P. LANCASTER, *On eigenvalues of matrices dependent on a parameter*, Numer. Math., 6 (1964), pp. 377–387.
- [17] P. LANCASTER, A. S. MARKUS, AND F. ZHOU, *Perturbation theory for analytic matrix functions: the semisimple case*, SIAM J. Matrix Anal. Appl., 25 (2003), pp. 606–626 (electronic).
- [18] H. LANGER AND B. NAJMAN, *Remarks on the perturbation of analytic matrix functions. II*, Integral Equations Operator Theory, 12 (1989), pp. 392–407.
- [19] YANYUAN MA AND ALAN EDELMAN, *Nongeneric eigenvalue perturbations of Jordan blocks*, Linear Algebra Appl., 273 (1998), pp. 45–63.
- [20] CARL D. MEYER AND G. W. STEWART, *Derivatives and perturbations of eigenvectors*, SIAM J. Numer. Anal., 25 (1988), pp. 679–691.
- [21] JULIO MORO AND FROILÁN M. DOPICO, *First order eigenvalue perturbation theory and the Newton diagram*, in Applied mathematics and scientific computing (Dubrovnik, 2001), Kluwer/Plenum, New York, 2003, pp. 143–175.
- [22] A. P. SEYRANIAN AND A. A. MAILYBAEV, *Multiparameter stability theory with mechanical applications*, vol. 13 of Series on Stability, Vibration and Control of Systems. Series A: Textbooks, Monographs and Treatises, World Scientific Publishing Co. Inc., River Edge, NJ, 2003.
- [23] JI GUANG SUN, *Eigenvalues and eigenvectors of a matrix dependent on several parameters*, J. Comput. Math., 3 (1985), pp. 351–364.
- [24] ———, *Multiple eigenvalue sensitivity analysis*, Linear Algebra Appl., 137/138 (1990), pp. 183–211.
- [25] STEVEN G. KRANTZ, *Function theory of several complex variables*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, second ed., 1992.
- [26] ILSE C. F. IPSEN AND RIZWANA REHMAN, *Perturbation bounds for determinants and characteristic polynomials*, SIAM J. Matrix Anal. Appl., 30 (2008), pp. 762–776.
- [27] H. W. GOULD, *Coefficient identities for powers of Taylor and Dirichlet series*, Amer. Math. Monthly, 81 (1974), pp. 3–14.