On Avoider-Enforcer games

József Balogh* Ryan Martin†

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Abstract

In the Avoider-Enforcer game on the complete graph K_n, the players (Avoider and Enforcer) each take an edge in turn. Given a graph property $\mathcal{P}$, Enforcer wins the game if Avoider's graph has the property $\mathcal{P}$. An important parameter is $\tau_E(\mathcal{P})$, the smallest integer t such that Enforcer can win the game against any opponent in t rounds.

In this paper, let $\mathcal{F}$ be an arbitrary family of graphs and $\mathcal{P}$ be the property that a member of $\mathcal{F}$ is a subgraph or is an induced subgraph. We determine the asymptotic value of $\tau_E(\mathcal{P})$ when $\mathcal{F}$ contains no bipartite graph and establish that $\tau_E(\mathcal{P}) = o(n^2)$ if $\mathcal{F}$ contains a bipartite graph.

The proof uses the game of JumbleG and the Szemerédi Regularity Lemma.

1 Introduction

An **unbiased positional game** is one in which two players alternately select a vertex from a hypergraph. In the most common formulation of the game, one player is Maker and the other Breaker. Maker attempts to occupy all vertices in some hyperedge and Breaker attempts to occupy at least one vertex in every hyperedge. In the graph context, players select edges of a complete graph K_n or possibly some other graph, such as a random graph [9]. Here Maker attempts to create a graph with a given monotone property and Breaker attempts to prevent Maker from achieving this.

Hefetz, Krivelevich and Szabó [7] recently investigated the so-called **Avoider-Enforcer game**. The players are, unsurprisingly, Avoider and Enforcer. In the context of graph games, Avoider attempts to prevent having her edges induce a graph with a property $\mathcal{P}$ for as many rounds as possible. Enforcer selects his edges in such a way as to force **Avoider's graph** to have property $\mathcal{P}$ as early as possible.

At the end of r rounds, Avoider and Enforcer each have chosen exactly r edges. Let $\tau_E(\mathcal{P})$ be the smallest integer t such that Enforcer can win the $\mathcal{P}$-property game in t rounds. Let $\mathcal{C}_n^k$ denote

*Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL, 61801, `jobal@math.uiuc.edu`.
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†Department of Mathematics, Iowa State University, Ames, IA 50011, `rymartin@iastate.edu`. This author's research supported in part by NSA grant H98230-05-1-0257.

the property that an n-vertex graph is k-colorable and let $\mathcal{NC}_n^k$ denote the property that an n-vertex graph is not k-colorable.

In [5] (see also [6] for similar questions on Maker-Breaker games), Hefetz, Krivelevich, Stojaković and Szabó establish that

$$\frac{n^2}{8} + \frac{n-2}{12} \leq \tau_E(\mathcal{NC}_n^2) \leq \frac{n^2}{8} + \frac{n}{2} + 1,$$

but for $k \geq 3$ show only that

$$(1-o(1))\frac{(k-1)n^2}{4k} \leq \tau_E(\mathcal{NC}_n^k) < \frac{1}{2}\binom{n}{2}.$$

They state that it "seems reasonable" that $\tau_E(\mathcal{NC}_n^k) \leq (1+o(1))\frac{(k-1)n^2}{4k}$.

In this paper, we prove that this is the case, as a consequence of a stronger result.

1.1 Main Results

Let $t(n,k)$ denote the Turán number, which is the maximum number of edges in a graph on n vertices with no K_{k+1}. In particular, Turán's theorem gives $\left(\frac{k-1}{k}\right)\frac{n^2}{2} - \frac{k}{8} \leq t(n,k) \leq \left(\frac{k-1}{k}\right)\frac{n^2}{2}$.

Definition 1 *For a family of graphs, $\mathcal{F}$, denote $\mathcal{P}_n(\mathcal{F})$ to be the property that a graph on n vertices has a copy of some member of $\mathcal{F}$ as a subgraph and let $\mathcal{P}_n^{\mathrm{ind}}(\mathcal{F})$ be the property that a graph on n vertices has a copy of some member of $\mathcal{F}$ as an* ***induced*** *subgraph.*

Theorem 2 *Let $\mathcal{F}$ be a family of graphs such that $k = \min\{\chi(F) : F \in \mathcal{F}\} \geq 3$. In the Avoider-Enforcer game for properties $\mathcal{P}_n(\mathcal{F})$ and $\mathcal{P}_n^{\mathrm{ind}}(\mathcal{F})$, for n large enough, we have*

$$\left(\frac{k-2}{k-1}\right)\frac{n^2}{4} - O(k) = \left\lfloor \frac{1}{2}t(n,k-1)\right\rfloor \leq \tau_E(\mathcal{P}_n(\mathcal{F})) \leq \tau_E(\mathcal{P}_n^{\mathrm{ind}}(\mathcal{F})) \leq \left(\frac{k-2}{k-1}\right)\frac{n^2}{4} + o(n^2).$$

In addition, the last two inequalities hold if $k = 2$.

The last inequality in Theorem 2 is our main result. Enforcer's strategy is simple, he tries to achieve to maintain that the board "looks like a random graph" after each round. We simply show that in this case Avoider's graph when it has $\frac{k-2}{k-1}\left(\frac{n^2}{4}\right) + o(n^2)$ edges, will satisfy property $\mathcal{P}_n^{\mathrm{ind}}(\mathcal{F})$. The answer to the question of Hefetz, Krivelevich, Stojaković and Szabó is a corollary of Theorem 2:

Theorem 3 *Let $k \geq 2$ be an integer and let $\mathcal{NC}_n^k$ be the property that a graph is not k-colorable. In the Avoider-Enforcer game,*

$$\left(\frac{k-1}{k}\right)\frac{n^2}{4} - O(k) = \left\lfloor \frac{1}{2}t(n,k)\right\rfloor \leq \tau_E(\mathcal{NC}_n^k) \leq \left(\frac{k-1}{k}\right)\frac{n^2}{4} + o(n^2).$$

In addition, the last two inequalities hold if $k = 1$.

Let $\mathcal{F} = \{K_{k+1}\}$. The upper bound for Theorem 3 is an immediate consequence of Theorem 2. The lower bound comes from the same trivial strategy that establishes the lower bound in Theorem 2.

This result is weaker than that in [5] for $k = 2$, but gives the correct asymptotic approximation for all $k \geq 2$.

This paper is organized as follows: In Section 2, we describe the game of JumbleG [4], along with important consequences thereof. In Section 3, we state the version of the Szemerédi Regularity Lemma that is useful for our purposes. Section 4 contains the proofs of the theorems. Section 5 contains some concluding remarks.

2 The game of JumbleG

The game of JumbleG is a traditional Maker-Breaker game. The goal of Maker is to create a pseudo-random graph. Frieze, Krivelevich, Pikhurko and Szabó [4] use two different conditions for pseudo-randomness. We will use their first definition, including the use of "ϵ-regular" to describe a graph, which is not to be confused with ϵ-regular pairs, as defined by Szemerédi's Regularity Lemma.

Definition 4 *If G is a graph and S, T are disjoint vertex sets in $V(G)$, then denote by $e_G(S,T)$ to be the number of edges which have one endpoint in S and the other in T.*

*A pair of disjoint vertex sets (S,T) is ϵ-**unbiased** if*

$$\left| \frac{e_G(S,T)}{|S||T|} - \frac{1}{2} \right| \leq \epsilon.$$

*A graph G on n vertices, with minimum degree $\delta(G)$, is ϵ-**regular** if both:*

P1 *$\delta(G) \geq (1/2 - \epsilon)n$.*

P2 *Any pair S, T of disjoint subsets of $V(G)$ with $|S|, |T| > \epsilon n$ is ϵ-unbiased.*

Formally, the game of JumbleG(ϵ) is won by Maker if Maker can ensure that his graph is an ϵ-regular graph.

Theorem 5 ([4]) *Maker has a winning strategy in JumbleG(ϵ), provided $\epsilon \geq 2(\log n/n)^{1/3}$ and n is sufficiently large.*

In our proofs, Enforcer will simply play as Maker in the game of JumbleG(ϵ). Note that the game stops before all the edges are claimed. So Theorem 5 is used so that if the game were continued further, then from the resulting graph, Enforcer could win the game. As a result, between any pair of disjoint sets, both of them large enough, Avoider cannot occupy too many of the edges. We formalize this as follows.

Definition 6 *After round r in the game, let $e_B(S,T;r)$ denote the number of edges that Breaker (Avoider) occupies in the pair (S,T) and $e_M(S,T;r)$ denote the number of edges that Maker (Enforcer) occupies.*

Lemma 7 *Let $S,T \subset V(G)$ be disjoint sets and $|S|,|T| \geq \epsilon n$. If Maker plays a strategy to win JumbleG(ϵ), then for every r*

$$e_B(S,T;r) - e_M(S,T;r) \leq 2\epsilon|S||T| + 1.$$

Proof. Fix any integer $r > 0$. Suppose $e_B(S,T;r) - e_M(S,T;r) > 2\epsilon|S||T| + 1$. From this point on, we let Breaker to occupy edges only between S and T whenever possible. When the game concludes, the number of edges that Breaker occupies between S and T is at least

$$\begin{aligned}
& e_B(S,T;r) + \left\lfloor \frac{1}{2}\left(|S||T| - e_B(S,T;r) - e_M(S,T;r)\right) \right\rfloor \\
\geq \quad & e_B(S,T;r) + \frac{1}{2}\left(|S||T| - e_B(S,T;r) - e_M(S,T;r)\right) - \frac{1}{2} \\
> \quad & \frac{1}{2}|S||T| + \frac{1}{2}\left(2\epsilon|S||T| + 1\right) - \frac{1}{2} \\
\geq \quad & \left(\frac{1}{2} + \epsilon\right)|S||T|.
\end{aligned}$$

This contradicts the assumption that Maker played a winning strategy for JumbleG(ϵ). ■

3 The Regularity Lemma

Definition 8 *Let A and B be disjoint vertex sets. The number of edges with one endpoint in A and the other in B is denoted by $e(A,B)$. The **density** of (A,B) is denoted by*

$$d(A,B) = \frac{e(A,B)}{|A||B|}.$$

*A pair (A,B) is α-**regular** if, for every $X \subseteq A$ and $Y \subseteq B$ with $|X| > \alpha|A|$ and $|Y| > \alpha|B|$,*

$$|d(A,B) - d(X,Y)| < \alpha.$$

*A partition $V_1,\ldots,V_\ell$ is an **equipartition** of V if for every i,j we have $||V_i| - |V_j|| \leq 1$.*

We use a form of the Szemerédi Regularity Lemma, introduced by Alon, Fischer, Krivelevich and M. Szegedy [1]. Lemma 9 is a simplified version.

Lemma 9 (Regularity Lemma [1]) *For every integer m and constant $\mathcal{E} > 0$, there is an $S = S(m, \mathcal{E})$ which satisfies the following: For any graph G on $n \geq S$ vertices, there exists an equipartition $\mathcal{A} = \{V_i : 1 \leq i \leq \ell\}$ of $V(G)$ and an induced subgraph U of G, with an equipartition $\mathcal{B} = \{U_i : 1 \leq i \leq \ell\}$ of the vertices of U, that satisfy:*

- *$m \leq \ell \leq S$.*
- *$U_i \subseteq V_i$ and $|U_i| = L \geq \lceil n/S \rceil$, for all $i \geq 1$.*
- *In the equipartition $\mathcal{B}$, **all** pairs are $\mathcal{E}$-regular.*
- *All but at most $\mathcal{E}\binom{\ell}{2}$ of the pairs $1 \leq i < j \leq \ell$ are such that $|d(V_i, V_j) - d(U_i, U_j)| < \mathcal{E}$.*

Since we want to establish that a large enough density in G will enable us to apply Turán's Theorem, we need to bound $e(U)$ in terms of $e(G)$.

Lemma 10 *Let m be an integer, and $\mathcal{E}$ be a constant such that $m \geq \mathcal{E}^{-1}$ and let $S = S(m, \mathcal{E})$ be the integer provided by Lemma 9. Let $n \geq 2S/\mathcal{E}$ and G be a graph on n vertices. Let U an the induced subgraph and an equipartition of $V(U)$ be $\{U_i\}_{i=1}^{\ell}$ with $|U_1| = \ldots = |U_\ell| = L$, provided by Lemma 9. Let $\tilde{U}$ be the graph formed by deleting all edges which have both endpoints in the same set U_i for $i = 1, \ldots, \ell$. In this case,*

$$e(U) \geq e(\tilde{U}) \geq e(G)\frac{\ell^2 L^2}{n^2} - 3\mathcal{E}\ell^2 L^2.$$

Proof. First, we bound $e(G)$, using the conditions in Lemma 9.

$$\begin{aligned}
e(G) &\leq \sum_i \binom{|V_i|}{2} + \sum_{1 \leq i < j \leq \ell} e(V_i, V_j) \\
&\leq \ell\frac{\lceil n/\ell \rceil^2}{2} + \mathcal{E}\binom{\ell}{2}\left\lceil \frac{n}{\ell} \right\rceil^2 + \left\lceil \frac{n}{\ell} \right\rceil^2 \sum_{1 \leq i < j \leq \ell} [d(U_i, U_j) + \mathcal{E}] \\
&\leq \frac{\ell}{2}\left\lceil \frac{n}{\ell} \right\rceil^2 + 2\mathcal{E}\binom{\ell}{2}\left\lceil \frac{n}{\ell} \right\rceil^2 + \left\lceil \frac{n}{\ell} \right\rceil^2 \sum_{1 \leq i < j \leq \ell} d(U_i, U_j) \\
&\leq \frac{\ell}{2}\left\lceil \frac{n}{\ell} \right\rceil^2 + \mathcal{E}\ell^2\left\lceil \frac{n}{\ell} \right\rceil^2 + \frac{\lceil n/\ell \rceil^2}{L^2} e(\tilde{U}). \qquad (1)
\end{aligned}$$

Recall that $L = |U_1| = \ldots = |U_\ell|$. The calculations below, use the fact that $\ell^{-1} \leq m^{-1} \leq \mathcal{E}$ and $\ell/n \leq S/n \leq \mathcal{E}/2$. By rearranging the terms in inequality (1), we get a lower bound for $e(\tilde{U})$.

$$\begin{aligned}
e(\tilde{U}) &\geq e(G)\frac{L^2}{\lceil n/\ell \rceil^2} - \frac{\ell L^2}{2} - \mathcal{E}\ell^2 L^2 \\
&\geq e(G)\frac{\ell^2 L^2}{n^2} - e(G)\left(\frac{\ell^2 L^2}{n^2} - \frac{L^2}{\lceil n/\ell \rceil^2}\right) - \frac{1}{2\ell}\ell^2 L^2 - \mathcal{E}\ell^2 L^2 \\
&\geq e(G)\frac{\ell^2 L^2}{n^2} - \frac{n^2}{2}\left(\frac{\ell^2 L^2}{n^2} - \frac{\ell^2 L^2}{(n+\ell)^2}\right) - \frac{\mathcal{E}}{2}\ell^2 L^2 - \mathcal{E}\ell^2 L^2 \\
&= e(G)\frac{\ell^2 L^2}{n^2} - \left(1 - \frac{n^2}{(n+\ell)^2}\right)\frac{\ell^2 L^2}{2} - \frac{\mathcal{E}}{2}\ell^2 L^2 - \mathcal{E}\ell^2 L^2 \\
&\geq e(G)\frac{\ell^2 L^2}{n^2} - 3\mathcal{E}\ell^2 L^2.
\end{aligned}$$

Trivially, $e(U) \geq e(\tilde{U})$ and this concludes the proof. ∎

Lemma 11 establishes that a regular f-tuple will induce any graph on f vertices, given necessary density conditions. We take the version from Alon and Shapira [2]. (It has appeared previously in other forms, see [8] and [3].)

Lemma 11 *For every real η, $0 < \eta < 1$, and integer $f \geq 1$, there exists a $\gamma = \gamma(\eta, f)$ with the following property: Suppose that H is a graph on f vertices $v_1, \ldots, v_f$, and that $U_1, \ldots, U_f$ is an f-tuple of disjoint nonempty vertex sets of a graph G such that for every $1 \leq i < j \leq f$, the pair (U_i, U_j) is γ-regular. Moreover, whenever $(v_i, v_j) \in E(H)$ we have $d(U_i, U_j) \geq \eta$ and whenever $(v_i, v_j) \notin E(H)$ we have $d(U_i, U_j) \leq 1 - \eta$. Then, some f-tuple $u_1 \in U_1, \ldots, u_f \in U_f$ spans an* ***induced*** *copy of H, where each u_i plays the role of v_i.*

The Slicing Lemma (Fact 1.5 in [8]) is a basic fact of regular pairs, common in proofs involving the Regularity Lemma.

Lemma 12 (Slicing Lemma) *Let (U_i, U_j) be an α-regular pair with density d and $|U_i| = |U_j| = L_0$. If $X \subseteq U_i$ and $Y \subseteq U_j$ with $|X| \geq L_i$ and $|Y| \geq L_j$, then (X, Y) is α'-regular, where*

$$\alpha' = \max\left\{2\alpha, \frac{L_0}{L_i}\,\alpha, \frac{L_0}{L_j}\,\alpha\right\},$$

with density in $(d - \alpha, d + \alpha)$.

4 Proof of Theorem 2

As in many proofs involving the Regularity Lemma, there is a sequence of constants. Fix an $F \in \mathcal{F}$ with chromatic number k and order f. With $a \ll b$ meaning that a is small enough relative to b, the constants are

$$\epsilon \ll \mathcal{E}_0 \ll \mathcal{E}_1 \ll \eta \ll \delta \ll f^{-1}.$$

We will determine the precise relationships later.

Let $k \geq 3$. In order to establish the lower bound for $\tau_E(\mathcal{P}_n(\mathcal{F}))$, Avoider equipartitions the n vertices into $k-1$ clusters, and chooses edges only between different clusters. By this strategy, Avoider's graph will always be $(k-1)$-colorable, thus, have no member of $\mathcal{F}$ as a subgraph. Avoider can make at least $\lfloor (1/2)t(n,k-1) \rfloor$ moves. Careful calculations establish that

$$t(n,k) = \left(\frac{k-1}{k}\right)\frac{n^2}{2} - \frac{k}{2}\left(\left\lceil \frac{n}{k} \right\rceil - \frac{n}{k}\right)\left(\frac{n}{k} - \left\lfloor \frac{n}{k} \right\rfloor\right).$$

Having established the first inequality of Theorem 2 for $k \geq 3$, we assume for the rest of the proof that $k \geq 2$. Since the existence of an induced copy of some $F \in \mathcal{F}$ implies the existence of a copy of F as a subgraph, it is trivial that $\tau_E(\mathcal{P}_n(\mathcal{F})) \leq \tau_E(\mathcal{P}_n^{\mathrm{ind}}(\mathcal{F}))$.

Finally, we shall prove the upper bound. Among all $F \in \mathcal{F}$ choose an F with $\chi(F) = k$ and let $f = |V(F)|$. Enforcer will play the game JumbleG(ϵ), and we shall prove that Avoider's graph, after $\left(\frac{k-2}{4(k-1)} + o(1)\right) n^2$ rounds will contain F as an induced subgraph. Assume, for the sake of contradiction, that for (a small) $\delta > 0$, Avoider managed to build a graph G of order n (where n is sufficiently large) and $\left(\frac{k-2}{4(k-1)} + 2\delta\right) n^2$ edges. Apply the Regularity Lemma (Lemma 9) to Avoider's graph with parameters $\mathcal{E}_0$ and $m := \max\{k, \lceil \mathcal{E}_0^{-1} \rceil\}$, (where $\mathcal{E}_0$ will be determined later) which yields the subsets $U_1, \ldots, U_{\ell_0}$ and a constant $S_0 = S_0(m, \mathcal{E}_0)$ so that $|U_1| = \ldots = |U_{\ell_0}| = L_0 \geq n/S_0$. Construct an auxiliary graph H on the vertex set $\{1, \ldots, \ell_0\}$ where $i \sim j$ if and only if $d(U_i, U_j) \geq \delta + \epsilon$. Let $\tilde{U}$ be the graph induced by $U_1 \cup \ldots \cup U_{\ell_0}$, with the edges inside of each cluster deleted. We use $e(\tilde{U})$ in order to compute $e(H)$. Observe that Lemma 7 gives that $d(U_i, U_j) \leq (1/2 + \epsilon)$ as long as $L_0 \geq n/S_0 > \epsilon n$.

$$\begin{aligned} e(\tilde{U}) &\leq e(H)\left(\frac{1}{2} + \epsilon\right) L_0^2 + \left[\binom{\ell_0}{2} - e(H)\right](\delta + \epsilon) L_0^2 \\ &\leq e(H)\left(\frac{1}{2} - \delta\right) L_0^2 + \frac{\delta + \epsilon}{2} \ell_0^2 L_0^2. \end{aligned}$$

We use Lemma 10 to bound $e(\tilde{U})$ by $e(G)$.

$$e(H) \geq \frac{e(\tilde{U}) - \frac{\delta+\epsilon}{2} \ell_0^2 L_0^2}{\left(\frac{1}{2} - \delta\right) L_0^2} \geq \frac{e(G)\,\frac{\ell_0^2}{n^2} - 3\mathcal{E}_0 \ell_0^2 - \frac{\delta+\epsilon}{2} \ell_0^2}{1/2 - \delta}. \qquad (2)$$

If

$$\delta \geq 2\mathcal{E}_0 + \frac{\epsilon}{3}, \qquad \text{which is equivalent to} \qquad 2\delta \geq \frac{\delta}{2} + 3\mathcal{E}_0 + \frac{\epsilon}{2}, \qquad (3)$$

then

$$e(G) \geq \left(\frac{k-2}{4(k-1)} + 2\delta\right) n^2 > \left(\frac{k-2}{4(k-1)} + 3\mathcal{E}_0 + \frac{\delta+\epsilon}{2}\right) n^2.$$

Plugging this into (2), we obtain that $e(H) > \left(\frac{k-2}{k-1}\right)\frac{\ell_0^2}{2}$. Therefore, H contains a K_k by Turán's Theorem. Without loss of generality, this copy of K_k is spanned by $(U_1, \ldots, U_k)$. Each pair (U_i, U_j) is $\mathcal{E}_0$-regular with density in the interval $(\delta + \epsilon, 1/2 + \epsilon)$.

Note: At this stage of the proof, we could use, say, the Blow-up lemma (see [8]) to show that if n is large enough, a not necessarily induced copy of F occurs as a subgraph in $(U_1, \ldots, U_k)$, hence in G. This provides an upper bound for $\tau_E(\mathcal{P}_n(\mathcal{F}))$. However, we want the stronger result that produces an upper bound for $\tau_E(\mathcal{P}_n^{\mathrm{ind}}(\mathcal{F}))$. To prove such a result, we will apply Szemerédi's Regularity Lemma inside the clusters that were formed by its first application. An alternative way to finish the proof would have been to use the Erdős-Stone Theorem instead of Turán's Theorem.

So, we apply the Regularity Lemma (Lemma 9) to the portion of Avoider's graph in U_i with parameters $\mathcal{E}_1$ and $m = f$, for each $i = 1, \ldots, k$ (where $\mathcal{E}_1$ will be determined later), which yields the constants ℓ_i, the subclasses $U_{i,1}, \ldots, U_{i,\ell_i}$ and a constant $S_1 = S_1(m, \mathcal{E}_1)$ so that $|U_{i,1}| = \ldots = |U_{i,\ell_i}| = L_i \geq L_0/S_1 \geq n/(S_0 S_1)$. If

$$\epsilon \leq S_0^{-1} S_1^{-1} \tag{4}$$

then each pair $(U_{i,x}, U_{j,y})$ has density at most $1/2 + \epsilon$, by Lemma 7.

Each pair $(U_{i,x}, U_{i,y})$ is $\mathcal{E}_1$-regular. Now consider a pair $(U_{i,x}, U_{j,y})$ where $i \neq j$. The pair (U_i, U_j) is $\mathcal{E}_0$-regular with density at least $\delta + \epsilon$. Using the Slicing Lemma (Lemma 12), the pair $(U_{i,x}, U_{j,y})$ is $\mathcal{E}_0 L_0 \cdot \max\{L_i^{-1}, L_j^{-1}\}$-regular. Since $L_i, L_j \geq L_0/S_1$, the pair $(U_{i,x}, U_{j,y})$ is $\mathcal{E}_0 S_1$-regular with density at most $1/2 + \epsilon$ and at least $\delta/2$, as long as

$$\mathcal{E}_0 \leq \delta/2 + \epsilon. \tag{5}$$

Finally, we apply Lemma 11 with $\eta = \delta/2$ to F and the tuple $\{U_{i,x}\}_{1 \leq i \leq k, 1 \leq x \leq \ell_i}$. This implies the existence of the constant $\gamma = \gamma(f, \delta/2)$. As F is k-colorable graph of order f, and $f \leq \ell_i$ for every i, all pair of classes are $\mathcal{E}_0 S_1$-regular. In order to apply Lemma 11, it is sufficient to have

$$\mathcal{E}_0 S_1 \leq \gamma\left(f, \delta/2\right). \tag{6}$$

This way F can be embedded into G, Avoider's graph.

The order of choosing the constants. First, a forbidden graph F with chromatic number k, and order f is fixed, and an arbitrary $\delta > 0$ is chosen, assuming that Avoider built a graph with $\left(\frac{k-2}{4(k-1)} + 2\delta\right) n^2$ edges, where $n > n(f, \delta)$. Note that if δ is too large, i.e., $\left(\frac{k-2}{4(k-1)} + 2\delta\right) n^2 > \binom{n}{2}/2$, then our theorem is true by default. Then from Lemma 11, with $\eta = \delta/2$ we obtain the existence of a γ, which is less than m^{-1}. The Regularity Lemma (Lemma 9) is applied to the classes U_i with $\mathcal{E}_1 = \gamma$ and $m = f$. This also implies the existence of an $S_1 = S_1(\mathcal{E}_1, m)$. We need that the pairs $(U_{i,x}, U_{j,y})$ are γ-regular, hence $\mathcal{E}_0 S_1 \leq \gamma$ needed, so a positive constant $\mathcal{E}_0 < \gamma / S_1$ is chosen. The Regularity Lemma (Lemma 9) is applied to G with $\mathcal{E}_0$ and $m = f$. This also implies the existence of an $S_0 = S_0(\mathcal{E}_0, m)$. For JumbleG($\epsilon$), we use $\epsilon = S_0^{-1} S_1^{-1}$. In order to apply all these Lemmas, we need to assume that n is sufficiently large. This specific choice of constants satisfies the inequalities (3), (4), (5) and (6).

To summarize, we proved that for every $\delta > 0$, if the Avoider-Enforcer induced-$\mathcal{F}$-avoidance game is played for $\left(\frac{k-2}{4(k-1)} + 2\delta\right) n^2$ rounds, then the Avoider's graph will contain an induced F, and the Enforcer will win. Thus, $\tau_E(\mathcal{P}_n^{\text{ind}}(\mathcal{F})) \leq \left(\frac{k-2}{k-1}\right) \frac{n^2}{4} + o(n^2)$. ■

5 Concluding remarks

The game $\mathcal{NC}_n^1$ is trivial, but the $\mathcal{P}_n(\mathcal{F})$ question is not for $k = 2$. That is, the case in which $\mathcal{F}$ contains a bipartite graph. Theorem 2 gives only that, if $k = 2$ and $\{F\}$ contains a bipartite graph, then

$$\tau_E(\mathcal{P}_n(\mathcal{F})) \leq \tau_E(\mathcal{P}_n^{\text{ind}}(\mathcal{F})) \leq o(n^2),$$

which is not very helpful. And certainly in the non-induced case, we have trivially

$$\tau_E(\mathcal{P}_n(\mathcal{F})) \leq t(n, F),$$

where $t(n, F)$ denotes the maximum number of edges in a graph with n vertices with no F as a subgraph.

The authors believe the approach needed to prove Theorem 2 may solve some other problems in the area of positional games, as our proof is the first instance of Szemerédi's Regularity Lemma being used in the context of positional game theory.

Finally, we remark that one can prove the non-induced case of $k = 3$ without the use of the Szemerédi Regularity Lemma.

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