On q-ary codes with two distances d and $d+1$

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Abstract

The q-ary block codes with two distances d and $d+1$ are considered. Several constructions of such codes are given, as in the linear case all codes can be obtained by a simple modification of linear equidistant codes. Upper bounds for the maximum cardinality of such codes is derived. Tables of lower and upper bounds for small q and n are presented.

1 Introduction

Let $Q = \{0, 1, ..., q-1\}$. Any subset $C \subseteq Q^n$ is a code denoted by $(n, N, d)_q$ of length n, cardinality $N = |C|$ and the minimum (Hamming) distance d. For linear codes we use notation $[n, k, d]_q$ (i.e., $N = q^k$). An $(n, N, d)_q$ code C is equidistant if for any two distinct codewords x and y we have $d(x, y) = d$, where $d(x, y)$ is the (Hamming) distance between x and y. A code C is constant weight and denoted $(n, N, w, d)_q$ if every codeword is of weight w.

We consider codes with only two distances d and $d+1$. As we will observe, such codes are sometimes connected to equidistant codes. We are not aware, however, of any investigations of codes with two consecutive distances.

Denote by $(n, N, \{d, d+1\})_q$ an $(n, N, d)_q$ code $C \subset Q^n$ with the following property: for any two distinct codewords x and y from C we have $d(x, y) \in \{d, d+1\}$. We are interested in constructions, classification results and upper bounds on the maximal possible size of $(n, N, \{d, d+1\})_q$ codes. We show that the linear q-ary codes with two distances d and $d+1$ are completely known and can be obtained by simple modification of linear equidistant codes [1, 2]. The preliminary results of this paper were announced partly in [1].

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2 Preliminary results

We recall the following classical Johnson bound for the size $N_q(n,d,w)$ of a q-ary constant weight $(n,N,w,d)_q$-code [3]:

$$N_q(n,d,w) \le \frac{(q-1)dn}{qw^2-(q-1)(2w-d)n} \tag{1}$$

if $qw^2 > (q-1)(2w-d)n$.

Definition 1. *A balanced incomplete block (BIB) design $B(v,k,\lambda)$ is an incidence structure (X,B), where $X=\{x_1,\dots,x_v\}$ is a set of elements and B is a collection of k-sets of elements (called* blocks*) such that every two distinct elements of X are contained in exactly $\lambda \ge 0$ blocks of B (here $1 \le k \le v-1$).*

Two other parameters of a $B(v,k,\lambda)$-design are $b=|B|$ (the number of blocks) and r (the number of blocks containing one fixed element):

$$r=\lambda\frac{v-1}{k-1},\quad b=\lambda\frac{v(v-1)}{k(k-1)},\quad \text{if } \lambda>0;$$

($\lambda=0$ corresponds to the case $k=1$ and hence $b=rv$).

In terms of the binary incidence matrix a $B(v,k,\lambda)$-design is a binary $(v\times b)$ matrix A with columns of weight k such that any two distinct rows contain exactly λ common nonzero positions.

We need the following result from [4, 5].

Theorem 1. *Any m-nearly resolvable $NRB_m(v,k,\lambda)$-design induces a q-ary equidistant constant weight $(n,N,w,d)_q$ code C with the additional property and with parameters $q=(v-m+k)/k$, $N=v$,*

$$n=\frac{\lambda v(v-1)}{(k-1)(v-m)},\quad w=\frac{\lambda(k-1)}{v-1},\quad d=\frac{\lambda(v+m-k)}{k-1},$$

meeting the Johnson bound (1).

Recall the following wide class of q-ary equidistant codes constructed in [6].

Theorem 2. *Let p be a prime and let s,ℓ,h be any positive integers. Then there exists an equidistant $(n,N,d)_q$ code with parameters*

$$q=p^{sh},\ n=\frac{p^{s(h+\ell)}-1}{p^s-1},\ N=p^{s(h+\ell)},\ d=p^{s\ell}\cdot\frac{p^{sh}-1}{p^s-1}.$$

Definition 2. *Let G be an abelian group of order q written additively. A square matrix D of order $q\mu$ is called a difference matrix and denoted $D(q,\mu)$, if the component-wise difference of any two different rows of D contains any element of G exactly μ times.*

Clearly a matrix $D(q,\mu)$ induces an equidistant $(q\mu-1,q\mu,\mu(q-1))_q$ code [6].

3 Constructions

3.1 Combinatorial constructions

Denote by $W_q(n)$ a ball of radius 1 with center at the zero vector, i.e. $W_q(n) = \{x \in Q^n : \mathrm{wt}(x) \le 1\}$.

Construction 1a. The ball $W_q(n)$ is an $(n, (q-1)n+1, \{1,2\})_q$ code.

Construction 1b. Parity checking (modulo 2) of Construction 1a implies an $(n+1, (q-1)n+1, \{2,3\})_q$ code, which we denote by $W_q^*(n+1)$. For any codeword $(0\cdots 0\ a\ 0\cdots 0)$ from $W_q(n)$ we form the codeword $(0\cdots 0\ a\ 0\cdots 0 \mid a)$ from $W_q^*(n+1)$.

Construction 2. An equidistant $(n, N, d)_q$ code C produces two $(n', N, \{d', d'+1\})_q$ codes, namely, $(n-1, N, \{d-1, d\})_q$ code C_1 obtained by deleting (any) position from C, and $(n+1, N, \{d, d+1\})_q$ code C_2 obtained by adding one position to C.

Combining Constructions 1a and 1b with Construction 2 we obtain the following two constructions.

Construction 3a. An equidistant $(n_1, N_1, d)_{q_1}$ code and $W_{q_2}(n_2) = (n_2, N_2, \{1,2\})$ give an $(n, N, \{d+1, d+2\})_q$ code with parameters

$$q = \max\{q_1, q_2\}, \quad n = n_1 + n_2, \quad N = \min\{N_1, N_2\}.$$

Construction 3b. An equidistant $(n_1, N_1, d)_{q_1}$ code and $W_{q_2}^*(n_2) = (n_2, N_2, \{2,3\})$ give an $(n, N, \{d+2, d+3\})_q$ code with parameters

$$q = \max\{q_1, q_2\}, \quad n = n_1 + n_2, \quad N = \min\{N_1, N_2\}.$$

Construction 4. If there exist r mutually orthogonal Latin squares of order q, then there exists a family of $(s+2, q^2, \{s+1, s+2\})_q$ codes C_s, where $s = 1, \ldots, r$.

Combining Constructions 2 and 4, we obtain:

Construction 5. For any prime power q there exists a family of $(n, q^2, \{d, d+1\})_q$ codes with parameters

$$n = s(q+1) + r, \quad d = sq + r - 1, \quad s \ge 1, \quad r = 1, \ldots, q+1.$$

Construction 6. If there exists a difference matrix $D(q, \mu)$, then there exist $(n, N, \{d, d+1\})_q$ codes with parameters:

$$\begin{aligned} n &= q\mu - 2, \quad & N &= q\mu, \quad & d &= (q-1)\mu - 1, \\ n &= q\mu, \quad & N &= q\mu, \quad & d &= (q-1)\mu. \end{aligned}$$

The well known equidistant $(4, 9, 3)_3$ code C_1 and a $(4, 9, \{1,2\})_3$ code C_2 (Construction 1a) give by Construction 3 an $(8, 9, \{4,5\})$ code C, which is not good. Using the $(5, 9, \{2,3\})_3$ code C_3 (Construction 1b) gives by Construction 3 a $(9, 9, \{5,6\})$ code.

The equidistant $(13, 27, 9)_3$ (Theorem 2) implies by Construction 2 a $(14, 27, \{9, 10\})_3$ code, which is better than the random $(14, 18, \{9, 10\})_3$ code and also a $(12, 27, \{8, 9\})_3$ code which meets the upper bound (the best found random code has cardinality 18).

The difference matrix $D(4, 3)$ (see [7]) without the trivial column is an optimal equidistant $(11, 12, 8)_3$ code. The difference matrix $D(3, 4)$ (see [7]) without the trivial column is an equidistant $(11, 12, 9)_4$ code.

The well known equidistant $(5, 16, 4)_4$ code C_1 and a $(5, 16, \{1, 2\})_4$ code C_2 (Construction 1) give by Construction 3 a $(10, 16, \{5, 6\})_4$ code (not good – there is a random $(10, 20, \{5, 6\})_4$ code). Twofold repetition of $(5, 16, 4)_4$ code C_1 gives an optimal $(10, 16, 8)_4$ code.

The equidistant $(6, 9, 5)_4$ code [5] implies by twofold repetition a $(12, 9, \{10, 11\})_4$ code (better than the random code). The equidistant $(21, 64, 16)_4$ code [6] implies $(22, 64, \{16, 17\})_4$ and $(20, 64, \{15, 16\})_4$ codes by Construction 2. The equidistant $(9, 10, 8)_5$ code [4] implies $(8, 10, \{7, 8\})_5$ and $(10, 10, \{8, 9\})_5$ codes by Construction 2. By Construction 5 we obtain the following family of $(n, N, \{d, d + 1\})_5$ codes:

$$n = 9 + s, \quad N = 10, \quad d = 8 + s - 1, \quad s = 0, 1, \ldots, 6.$$

In particular, for $s = 0$ we obtain an optimal $(9, 10, 8)_5$ code and for $s \geq 2$ all resulting codes are new. By Construction 1 this equidistant $(9, 10, 8)_5$ code implies the $(11, 9, \{9, 10\})_5$ code.

The equidistant $(6, 25, 5)_5$ code implies the family of $(6+s, 25, \{5+s-1, 5+s\})_5$ codes where $s = 0, 1, \ldots, 6$, which give better (or new codes) for $s \geq 1$.

The well known resolvable design $(15, 35, 7, 3, 1)$ is equivalent to the optimal equidistant $(7, 15, 6)_5$ code. Now using Construction 5 we obtain from this code the following codes: $n = 7 + s, \quad N = 15, \quad d = 6 + s - 1, \quad s = 1, \ldots, 6.$

The affine design $(16, 20, 5, 4, 1)$ implies [4] the equidistant constant weight $(16, 16, 15, 14)_6$ code which implies in turn the $(16, 17, \{14, 15\})_6$ code (by adding the zero codeword).

3.2 Random codes

We use a computer program for generation of random codes by a simple heuristic algorithm. We start with a seed (at least the zero vector), then generate the search space and choose consecutively random vectors until the resulting code is good (i.e. has only distances d and $d + 1$). It is possible to take for a seed the best code constructed earlier. Many iterations can be implemented but usually the best codes (found this way) are obtained quickly. The cardinalities of such random codes are shown in Section 6 together with those of the codes obtained from constructions from this section.

4 Linear $(n, N, \{d, d+1\})_q$ codes

In this section we obtain the classification results in the case of linear codes with distances d and $d+1$. As we already mentioned the linear codes with two distances are completely known. The next theorem was proved for the binary case in [1]. The q-ary case also was conjectured in [1]. Here we give a simple proof of our conjecture for the case $q \geq 2$ and $k \geq 2$ based on purely coding theoretic arguments. Simultaneously the corresponding result for $k \geq 3$ was proved in [2], based on geometrical arguments.

Let C be a q-ary (linear) $[n, 3, q^2]_q$ equidistant code of length $n = q^2+q+1$, the distance q^2 and cardinality q^3.

Lemma 1. *Suppose that C is the code above presented as a $(n \times q^3)$-matrix over $\mathbb{F}_q$ (which we denote by $[C]$). Then, $[C]$ cannot be written as a concatenation of two matrices, i.e. $[C] = [C_1|C_2]$, where $[C_1]$ is a $(x \times q^3)$-matrix (where $x < (n-1)/2$) which represents a linear $(x, q^3, \{d, d+1\})_q$ code C_1.*

Proof. If $\mathbf{x} \in C$ then clearly $\alpha\mathbf{x} \in C$ for all $\alpha \in \mathbb{F}_q^*$. Thus, all non zero codewords of C can be split into q^2+q+1 classes. So, by $\mathbb{P}C$ we denote the code of classes of such elements. It is given by a matrix P of n by n, where $n = q^2+q+1$.

Suppose that the matrix P is a concatenation of two matrices P_1 and P_2, i.e. $P = [P_1|P_2]$, where P_1 is a x by q^2+q+1 matrix that corresponds to the equivalence classes of a $[x, q^3, \{d, d+1\}]$ linear code C_1. Consequently every word of C_1 has $x-d$ or $x-d-1$ positions with zero entries. To simplify further computations, let $\ell = x-(d+1)$ (since we will consider the number of zero entries of any word instead of its weight).

Since P corresponds to an equidistant code with code distance q^2, clearly P_2 corresponds to a (linear) $[n-x, 3, q^2-d-1]_q$ code C_2 with two distances q^2-d-1 and q^2-d. Therefore, without loss of generality we can assume that $x \leq n/2$. Since $n = q^2+q+1$, we assume that $x \leq q(q+1)/2$ and $\ell \leq (q+1)/2$ (indeed, every column of P contains $q+1$ zeroes).

Since any column in P has $q+1$ zeroes, the matrix P contains $n(q+1)$ zero entries. Suppose that the matrix P_1 has exactly η words of weight $d+1$ (i.e. ℓ zeroes) and the remaining $n-\eta$ words of weight d (i.e. $\ell+1$ zeroes). Thus, we can write

$$\ell\eta + (\ell+1)(n-\eta) = (q+1)x.$$

Solving for η, we obtain

$$\eta = (l+1)n - (q+1)x. \tag{2}$$

Since C_1 is a linear of dimension 3, for any pair of coordinate positions there exists exactly one row in P_1 with zeroes at these positions. There are x coordinate positions and $x(x-1)$ pairs of positions. On the other hand, there are η rows with ℓ zeroes (every row provides $\ell(\ell-1)/2$ pairs of coordinates) and $n-\eta$ rows with $\ell+1$ (every row provides $\ell(\ell+1)/2$ pairs of coordinates). Thus, we obtain the following equality:

$$\frac{x(x-1)}{2} = \frac{\ell(\ell-1)}{2}\eta + \frac{\ell(\ell+1)}{2}(n-\eta). \tag{3}$$

Our goal is to show that the equality (3) can not be valid for any x in the interval $[2, q(q+1)/2]$. Using (2), the expression (3) becomes

$$\begin{aligned} x(x-1) &= \ell(\ell-1)\eta + \ell(\ell+1)n - \ell(\ell+1)\eta \\ &= \ell(\ell+1)n - 2\ell\eta \\ &= \ell(\ell+1)n - 2\ell[(\ell+1)n - (q+1)x] \\ &= 2\ell(q+1)x - \ell(\ell+1)n. \end{aligned}$$

Thus, we arrive at the following quadratic equation for x:

$$x^2 - (2\ell(q+1)+1)x + \ell(\ell+1)n = 0. \tag{4}$$

We will show that the discriminant of this equation is negative. Thus, we have to verify that

$$(2\ell(q+1)+1)^2 < 4\ell(\ell+1)n.$$

Recalling that $n = q^2+q+1$, this is equivalent to

$$\begin{aligned} 4\ell^2(q^2+2q+1) + 4\ell(q+1) + 1 &< 4\ell(\ell+1)(q^2+q+1) \\ &= 4\ell^2(q^2+q+1) + 4\ell(q^2+q+1). \end{aligned}$$

Once simplified, it becomes

$$4\ell^2 q + 1 < 4\ell q^2.$$

Since $\ell \leq (q+1)/2$, the last inequality is obviously true for $\ell \geq 1$. Thuw, we obtained that there is no submatrix P_1, and consequently, the linear $[q^2+q+1, 3, q^2]_q$ code C cannot be presented as a concatenation of two linear codes C_1 and C_2 of type $(n, N, \{d, d+1\})_q$. $\square$

Theorem 3. *Let C be a q-ary linear $[n,k,d]_q$ code with two distances d and $d+1$ and $k \geq 2$. Then C is obtained by Construction 2 from the previous section, i.e. by deleting or adding an arbitrary vector column in the parity check matrix of a linear q-ary equidistant code with the following exception for the case $k = 2$ and $q \geq 3$, when C can be obtained by Construction 2 or by Construction 5.*

Proof. First consider the case $k = 2$. For this case we can have a $[n, 2, d]_q$ code C with two distances d and $d+1$ obtained also by Construction 5. Let C_1 be an equidistant $[n_1, 2, d_1]_q$ code with parameters $n_1 = s(q+1)$, $d_1 = s\,q$ and C_2 be a $[n_2, 2, d_2]_q$ code with parameters $n_2 = r$, $d_2 = r-1$. The generator matrix G of C is of the form $G = [G_1 \,|\, G_2]$, where G_1 and G_2 are the generator matrices of the codes C_1 and C_2, which (up to equivalence) look as follows: the matrix $G_1 = [G_0|\cdots|G_0]$ is the s-time repetition of G_0,

$$G_0 = \begin{bmatrix} a_0\; a_1\; a_1\; a_1 & \cdots & a_1 \\ a_1\; a_0\; a_1\; a_2 & \cdots & a_{q-1} \end{bmatrix}$$

where we denote $\mathbb{F}_q = \{a_0 = 0, a_1 = 1, a_2, \ldots, a_{q-1}\}$ and the matrix G_2 is of the form

$$G_2 = \left[\begin{array}{cccccc} a_0 & a_1 & a_1 & a_1 & \cdots & a_1 \\ a_1 & a_0 & a_1 & a_2 & \cdots & a_{r-2} \end{array} \right].$$

All these facts are commonly known and do not need any proofs. The only thing we have to say is that the all elements of the second row of G_2 starting from the second position should be different and this condition is necessary and sufficient in order for G_2 to be a generator matrix of the code C_2.

Now we claim that any $[n, 2, d]_q$ code with two distances should be of the form described above. It is clear for the case $n \leq q$. For larger n assume that the code C_1 of length $q + 1$ is not equdistant $[q + 1, 2, q]_q$ code, i.e. it has minimal distance $d = q - 1$. Since its averige distance is known (and it equals q), we conclude that this code has three distances, namely, $q - 1$, q and $q + 1$. Denoting by α_w the number of codewords of weight w, and taking into account that $\alpha_{q-1} = \alpha_{q+1}$, we obtain that

$$\alpha_{q-1} = \alpha_{q+1} = q - 1, \;\; \alpha_q = (q - 1)^2. \tag{5}$$

As we know the $[r, 2, r - 1]_q$ code C_2 has weights $r - 1$ and r. Denoting β_w the number of codewords of weight w, we deduce that

$$\beta_{r-1} = (q - 1)r, \;\; \beta_r = (q - 1)(q + 1 - r). \tag{6}$$

So, the concatenation of these two codes C_1 and C_2 would be a code C with at least three distances $d, d + 1$ and $d + 2$ where $d \leq q + r - 1$, i.e. we obtain a contradiction. So, C_1 of length $(q + 1)s$ should be an equidistant code. Therefore, any $[n, 2, d]_q$ code C with two distances $d, d + 1$ is obtained by one of two constructions, namely, Constructions 2 or 5.

Now to finish the proof we have only to show that any $[n, 3, d]_q$-code with two distances d and $d + 1$ can be obtained only by Construction 2. In contrary, assume that C_1 is a $[n_1, 3, d_1]_q$-code with two distances d_1 and $d_1 + 1$ of length n_1 in the interval $2 \leq n_1 \leq q^2 + q - 1$. It means that there is a $[n_2, 3, d_2]_q$-code C_2 (complementary to C_1) with two distances d_2 and $d_2 + 1$ of length $n_2 = q^2 + q + 1 - n_1$. Hence, there exists a q-ary $[n, 3, q^2]_q$ equidistant code C of length $n = q^2 + q + 1$, which can be written as a concatenation of two codes C_1 and C_2. But by Lemma 1 it is impossible, that finishes the proof. $\square$

5 Upper bounds

We are interested in upper bounds for the quantity

$$A_q(n; \{d, d + 1\}) = \max\{|C| : C \text{ is an } (n, |C|, \{d, d + 1\}) \text{ code}\},$$

the maximal possible cardinality of a code in Q^n with two distances d and $d + 1$.

5.1 Linear programming bounds

For fixed n and q, the (normalized) Krawtchouk polynomials are defined by

$$Q_i^{(n,q)}(t) := \frac{1}{r_i} K_i^{(n,q)}(d),\ d = \frac{n(1-t)}{2},\ r_i = (q-1)^i \binom{n}{i},$$

where

$$K_i^{(n,q)}(d) = \sum_{j=0}^{i} (-1)^j (q-1)^{i-j} \binom{d}{j} \binom{n-d}{i-j}$$

are the (usual) Krawtchouk polynomials. If $f(t) \in \mathbb{R}[t]$ is of degree $m \geq 0$, then it can be uniquely expanded as

$$f(t) = \sum_{i=0}^{m} f_i Q_i^{(n,q)}(t).$$

The next theorem is adapted for estimation of $A_q(n; \{d, d+1\})$ from the general Delsarte linear programming bound . Proofs of such bounds are usually considered folklore.

Theorem 4. *Let $n \geq q \geq 2$ and $f(t)$ be a real polynomial of degree $m \leq n$ such that:*

(A1) *$f(t) \leq 0$ for $t \in \{1 - 2d/n, 1 - 2(d+1)/n\}$;*

(A2) *the coefficients in the Krawtchouk expansion $f(t) = \sum_{i=0}^{m} f_i Q_i^{(n,q)}(t)$ satisfy $f_i \geq 0$ for every i.*

Then $A_q(n; \{d, d+1\}) \leq f(1)/f_0$.

Most of the upper bounds in the table below are obtained by Theorem 4 with the simplex method. We describe now other cases where analytic forms of good bounds are possible.

The first degree polynomial $f(t) = t - 1 + 2d/n$ gives the Plotkin bound which is attained for many large d. Optimization over the second degree polynomials gives the following result.

Theorem 5. *If $d \geq (n-1)(q-1)/q$, then*

$$A_q(n; \{d, d+1\}) \leq \frac{q^2 d(d+1)}{n^2(q-1)^2 - n(q-1)(2dq + q - 1) + dq^2(d+1)}. \tag{7}$$

Proof. Consider the second degree polynomial

$$f(t) = \left(t - 1 + \frac{2d}{n}\right)\left(t - 1 + \frac{2d+2}{n}\right) = f_0 + f_1 Q_1^{(n,q)}(t) + f_2 Q_2^{(n,q)}(t),$$

where $f_0 = \frac{4(n^2(q-1)^2 - n(q-1)(2dq+q-1) + dq^2(d+1))}{n^2q^2}$, $f_1 = \frac{8(q-1)(dq-(q-1)(n-1))}{nq^2}$, and $f_2 = \frac{4(q-1)^2(n-1)}{nq^2}$. The condition (A1) is obviously satisfied.

The condition $f_0 > 0$ is equivalent to a quadratic inequality with respect to dq, giving that $n \geq q$ implies it. The condition $f_1 \geq 0$ is equivalent to $dq \geq (n-1)(q-1)$ and $f_2 > 0$ is obvious. Thus $f(t)$ satisfies (A1) and (A2) provided $d \geq (n-1)(q-1)/q$.

Now the calculation of $f(1)/f_0$ gives the bound (7). □

The bound (7) is attained in some cases. It gives $A_q(n;\{d,d+1\}) \leq q^2$ for $d = n-1$ which is attained for $(q,n) = (3,3), (3,4), (4,5)$, and $(5,6)$. Further, we have $A_2(7;\{4,5\}) = A_2(7;\{3,4\}) = 8$, $A_2(10;\{5,6\}) = 12$, $A_3(12,\{8,9\}) = A_3(13,\{9,10\}) = 27$ by (7). The cases of attaining (7) are marked by $d2$ in the tables below.

Furthermore, the bound (7) is attained by some code C and $d > (n-1)(q-1)/q$ (i.e. $f_1 > 0$), then C is an orthogonal array of strength 2. In particular, the cardinality of C is divisible by q^2. This argument implies improvements of (7) by one giving the exact values $A_2(12,\{5,6\}) = A_2(12,\{6,7\}) = A_2(13,\{6,7\}) = 13$ and the bounds $13 \leq A_3(6,\{4,5\}) \leq 14$. These cases are marked by n in the tables. One more interesting case is $A_3(7,\{4,5\}) = 15$ where (7) is attaned in the case $d = (n-1)(q-1)/q$.

Further bounds can be obtained by some ad-hoc polynomials. For example, the polynomial

$$f(t) = 1 + (q-1)nQ^{(n,q)}_{(n(q-1)+1)/q}(t)$$

gives $A_q(n,\{1,2\}) = (q-1)n+1$ (see Construction 1a) whenever q divides $(q-1)n+1$. Similarly, the polynomial

$$f(t) = 1 + \frac{n+2}{2}Q^{(n,2)}_{n/2}(t) + \frac{n}{2}Q^{(n,2)}_{1+n/2}(t),$$

where n is even, gives $A_2(n,\{1,2\}) \leq f(1)/f_0 = n+2$. This bound cannot be attained since it implies impossible distance distributions for the corresponding codes. Both polynomials prove that $A_2(n,\{1,2\}) = n+1$ (attained by Construction 1a). Such cases are marked with a in the tables.

Further careful examination of the conditions for attaining the linear programming bounds could probably lead to other improvements in the tables.

5.2 Bounds via spherical codes

Codes from Q^n are naturally mapped to the sphere $\mathbb{S}^{(q-1)n-1}$. We first map bijectively the alphabet symbols $0, 1, \ldots, q-1$ to the vertices of the regular simplex in $q-1$ dimensions and then map the codewords of a q-ary code $C \subset Q^n$ coordinate-wise to $\mathbb{R}^{(q-1)n}$. It is easy to see that all vectors have the same norm and after a normalization we obtain a spherical code on $\mathbb{S}^{(q-1)n-1}$. This spherical code has cardinality $|C|$ and maximal inner product $1 - 2dq/(q-1)n$ (equivalently, squared minimum distance $2dq/(q-1)n$). Clearly, q-ary codes with distances d and $d+1$ are mapped to spherical 2-distance codes with squared distances $2dq/(q-1)n$ and $2(d+1)q/(q-1)n$. This relation implies the following upper bound for $A_q(n,\{d,d+1\})$.

Theorem 6. *If* $d > (\sqrt{2(q-1)n} - 1)/2$, *then*

$$A_q(n, \{d, d+1\}) \leq 2(q-1)n + 1.$$

Proof. Larman, Rogers, and Seidel [11] proved that if the cardinality of a two-distance set in $\mathbb{R}^n$ with distances a and b, $a < b$, is greater than $2n+3$, then the ratio a^2/b^2 equals $(k-1)/k$, where k is a positive integer satisfying $2 \leq k \leq (\sqrt{2n}+1)/2$. The restriction $2n+3$ was moved to $2n+1$ in [12].

In our situation $a^2/b^2 = d/(d+1) = (k-1)/k$ holds, whence we conclude that $d = k-1$ has to belong to the interval $[1, (\sqrt{2(q-1)n} - 1)/2]$. In other words, there exist no q-ary codes with distances d and $d+1$ and cardinality greater than $2(q-1)n+1$, whenever $d > (\sqrt{2(q-1)n} - 1)/2$; i.e., we have $A_q(n, \{d, d+1\}) \leq 2(q-1)n + 1$. □

The bound from Theorem 6 is usually better than the simplex method for large enough n and middle range d. The first time where this happens is $(n, d) = (13, 4)$ for $q = 2$, $(9, 3$ for $q = 3$, $(8, 3)$ for $q = 4$, and $(7, 4)$ for $q = 5$.

6 Tables

The tables below are for $q = 2, 3, 4, 5$. Horizontally we give d, vertically n. The lower bounds show the better of the computer generated random codes and the constructions from Section 3. All our random codes are available upon request.

The upper bounds are taken from the best of the linear programming bound obtained by the simplex method (unmarked), ad-hoc approaches as in Section 5.1 (marked with $d2$, n and a, respectively), the corresponding best known upper bound on $A_q(n, d)$ [8] (marked with $*$), and the bound from Theorem 6 (marked with $t6$).

$q=2$																	
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
7	8	7-10	8^{d2}	8^{d2}	2*	2*											
8	9^{a}	8-12	8-10	8-10	4*	2*	2*										
9	10	9-14	8-16	8-10	6*	4*	2*	2*									
10	11^{a}	10-16	8-16	10-16	12^{d2}	6^{d2}	2-3	2*	2*								
11	12	11-18	8-19	10-20	12^{d2}	12^{d2}	4*	2*	2*	2*							
12	13^{a}	12-20	8-25	10-21	13^{n}	13^{n}	4*	4*	2*	2*	2*						
13	14	13-22	8-26	10-27	13-19	13^{n}	8^{d2}	4*	2*	2*	2*	2*					
14	15^{a}	14-24	$8\text{-}29^{t6}$	$10\text{-}29^{t6}$	14-27	14-19	16^{d2}	8*	4*	2-3	2*	2*	2*				
15	16	15-26	$8\text{-}31^{t6}$	$11\text{-}31^{t6}$	14-29	14-30	16	16*	4*	4*	2*	2*	2*	2*			
16	17^{a}	16-28	$8\text{-}33^{t6}$	$11\text{-}33^{t6}$	$14\text{-}33^{t6}$	$15\text{-}33^{t6}$	16-18	16-18	6*	4*	2*	2*	2*	2*	2*		
17	18	17-30	$9\text{-}35^{t6}$	$12\text{-}35^{t6}$	$14\text{-}35^{t6}$	$15\text{-}35^{t6}$	17-22	16-18	10*	6*	4*	2*	2*	2*	2*	2*	
18	19^{a}	18-32	$9\text{-}37^{t6}$	$12\text{-}37^{t6}$	$14\text{-}37^{t6}$	$15\text{-}37^{t6}$	17-35	18-22	20*	10	4*	2-4	2*	2*	2*	2*	2*

$q=3$													
	1	2	3	4	5	6	7	8	9	10	11	12	13
3	9	9											
4	9^a	9	9										
5	11-13	9-17	11-13	6									
6	13-15	11-18	11-16	13-14^n	4								
7	15^a	13-27	11-27	15^{d2}	10	3							
8	17-19	15-31	11-30	15-31	18-19	9	3						
9	19-21	17-33	11-37^{t6}	15-36	18-25	18-21	6	3					
10	21^a	19-45	11-41^{t6}	15-41^{t6}	18-41^{t6}	18-21	13-14	3					
11	23-25	21-45	11-45^{t6}	15-45^{t6}	18-45	18-45	18-25	12*	4*	3*			
12	25-27	23-51	12-49^{t6}	15-49^{t6}	18-49^{t6}	18-49^{t6}	18-30	27^{d2}	9*	4*	3*		
13	27^a	25-63	13-53^{t6}	15-53^{t6}	18-53^{t6}	18-53^{t6}	18-53^{t6}	18-27	27^{d2}	6*	3*	3*	
14	29-31	27-63	14-57^{t6}	15-57^{t6}	18-57^{t6}	18-57^{t6}	18-57^{t6}	18-45	27-31	12-13	6*	3*	3*

$q=4$											
	1	2	3	4	5	6	7	8	9	10	11
5	16^a	16-25	16	16*							
6	19-22	16-37	16-37	18-22	9*						
7	22-26	19-41	16-43	18-41	21-26	8*					
8	25-28	22-50	16-49^{t6}	18-49^{t6}	21-32	19-28	5*				
9	28^a	25-67	16-86	18-55^{t6}	21-55^{t6}	19-28	15-20*	5*			
10	31-34	28-72	16-90	18-61^{t6}	20-61^{t6}	19-61^{t6}	21-34	16*	5*		
11	34-38	31-78	16-134	18-67^{t6}	21-67^{t6}	19-67^{t6}	20-56	22-38	12*	4*	
12	37-40	34-97	18-152	18-73^{t6}	21-73^{t6}	19-73^{t6}	20-73^{t6}	22-43	21-40	9*	4*

$q=5$									
n/d	1	2	3	4	5	6	7	8	9
5	25	25-30	25-30	19-25*					
6	25^a	25-51	25-51	19-25	15-25*				
7	29-34	25-66	25-81	19-57^{t6}	25-34	12-15*			
8	33-40	29-75	25-88	19-65^{t6}	22-65^{t6}	26-40	10*		
9	37-43	33-83	25-130	21-73^{t6}	22-73^{t6}	26-65	25-43	8-10*	
10	41-45	37-114	25-177	21-81^{t6}	22-81^{t6}	26-81^{t6}	25-49	25-45	7*

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