DICHOTOMY THEOREMS FOR RANDOM MATRICES AND CLOSED IDEALS OF OPERATORS ON $\big(\bigoplus_{n=1}^\infty \ell_1^n\big)_{c_0}$

N. J. LAUSTSEN, E. ODELL, TH. SCHLUMPRECHT AND A. ZSÁK

Abstract. We prove two dichotomy theorems about sequences of operators into L_1 given by random matrices. In the second theorem we assume that the entries of each random matrix form a sequence of independent, symmetric random variables. Then the corresponding sequence of operators either uniformly factor the identity operators on ℓ_1^k ($k \in \mathbb{N}$) or uniformly approximately factor through c_0. The first theorem has a slightly weaker conclusion still related to factorization properties but makes no assumption on the random matrices. Indeed, it applies to operators defined on an arbitrary sequence of Banach spaces. These results provide information on the closed ideal structure of the Banach algebra of all operators on the space $\big(\bigoplus_{n=1}^\infty \ell_1^n\big)_{c_0}$.

Introduction

In this paper we study closed ideals of operators on the space $\big(\bigoplus_{n=1}^\infty \ell_1^n\big)_{c_0}$ with the ultimate goal of classifying all of them. When studying operators on this space one is quickly reduced to considering sequences of operators $T^{(m)}\colon \ell_\infty^m(\ell_1^m) \to \ell_1^m$ ($m \in \mathbb{N}$), where $\ell_\infty^m(\ell_1^m)$ is the ℓ_∞-sum of m copies of ℓ_1^m. Often it will be more convenient to use a different normalization and view $T^{(m)}$ as an operator into $L_1 = L_1[0,1]$. We shall denote by $e_{i,j} = e_{i,j}^{(m)}$ the unit vector basis of $\ell_\infty^m(\ell_1^m)$, where the norm of $\sum_{i,j} a_{i,j} e_{i,j}$ is given by $\max_i \sum_j |a_{i,j}|$. We then let $T_{i,j}^{(m)} = T^{(m)}(e_{i,j})$, so $T^{(m)}$ can be identified with the $m \times m$ matrix $\big(T_{i,j}^{(m)}\big)$ with entries in L_1. Our main results concern such random matrices. The first one is general with no extra assumptions on the random variables $T_{i,j}^{(m)}$.

Theorem A. *Let $T^{(m)}\colon \ell_\infty^m(\ell_1^m) \to L_1$ ($m \in \mathbb{N}$) be a uniformly bounded sequence of operators. Then*

(i) either the identity operators $\mathrm{Id}_{\ell_1^k}\colon \ell_1^k \to \ell_1^k$ ($k \in \mathbb{N}$) uniformly factor through the $T^{(m)}$,

(ii) or the operators $T^{(m)}$ have uniform approximate lattice bounds, i.e.,

$$\forall \varepsilon > 0 \quad \exists C > 0 \quad \forall m \in \mathbb{N} \quad \exists g_m \in L_1 \quad \text{such that} \quad \|g_m\|_{L_1} \le C \quad \text{and}$$
$$T^{(m)}\big(B_{\ell_\infty^m(\ell_1^m)}\big) \subset \big\{ f \in L_1 : |f| \le g_m \big\} + \varepsilon B_{L_1} \ .$$

Here and throughout the paper we denote by B_X the closed unit ball of a Banach space X. It turns out that this result does not depend on the domain spaces of the $T^{(m)}$ which can be replaced by an arbitrary sequence of Banach spaces (*c.f.* Theorem 2.1). One of the consequences of this theorem is that the Banach algebra $\mathcal{B}(X)$ of all bounded operators on $X = \big(\bigoplus_{n=1}^\infty \ell_1^n\big)_{c_0}$ has a unique maximal ideal. We thus obtain the following picture of the lattice of closed ideals of $\mathcal{B}(X)$. Here $\mathcal{K}$ is the ideal of compact operators while $\mathcal{G}_{c_0}$ denotes the

2000 *Mathematics Subject Classification.* 47L10 (primary), 46B09, 46B42, 47L20, 46B45 (secondary).

The authors gratefully acknowledge the financial support from the EPSRC, grant EP/F023537/1, and the NSF, grants DMS 0700126 and DMS 0856148. Commutative diagrams were drawn by Paul Taylor's package.

ideal of operators factoring through c_0. For an operator ideal $\mathcal{J}$ we let $\overline{\mathcal{J}}$ be the norm closure of $\mathcal{J}$ and we denote by $\mathcal{J}^{(\mathrm{sur})}$ the surjective hull of $\mathcal{J}$ (defined in Section 3).

Theorem B. *Let $X = \big(\bigoplus_{n=1}^{\infty} \ell_1^n\big)_{c_0}$. We have the following closed ideals in $\mathcal{B}(X)$:*

$$\{0\} \subsetneq \mathcal{K}(X) \subsetneq \overline{\mathcal{G}}_{c_0}(X) \subseteq \overline{\mathcal{G}}_{c_0}^{(\mathrm{sur})}(X) \subsetneq \mathcal{B}(X)\ .$$

Moreover, if there is another closed ideal $\mathcal{J}$ of $\mathcal{B}(X)$, then it must lie between $\overline{\mathcal{G}}_{c_0}(X)$ and its surjective hull. In particular, $\overline{\mathcal{G}}_{c_0}^{(\mathrm{sur})}(X)$ is the unique maximal ideal of $\mathcal{B}(X)$.

We do not know whether the inclusion $\overline{\mathcal{G}}_{c_0}(X) \subseteq \overline{\mathcal{G}}_{c_0}^{(\mathrm{sur})}(X)$ is proper. If it is in fact an equality, then $\mathcal{K}(X)$ and $\overline{\mathcal{G}}_{c_0}(X)$ are the only non-trivial (*i.e.,* non-zero), proper closed ideals of $\mathcal{B}(X)$ and we have a full description of the lattice of closed ideals of $\mathcal{B}(X)$. Otherwise $\overline{\mathcal{G}}_{c_0}^{(\mathrm{sur})}(X)$ may be the only non-trivial, proper closed ideal of $\mathcal{B}(X)$ besides $\mathcal{K}(X)$ and $\overline{\mathcal{G}}_{c_0}(X)$ or there may also be other new closed ideals strictly between $\overline{\mathcal{G}}_{c_0}(X)$ and $\overline{\mathcal{G}}_{c_0}^{(\mathrm{sur})}(X)$. Classifying the closed ideals of $\mathcal{B}(X)$, one is lead to the following problem.

Problem. *Let $T^{(m)}\colon \ell_\infty^m(\ell_1^m) \to L_1$ ($m \in \mathbb{N}$) be a uniformly bounded sequence of operators. Is it true that*

(i) either the identity operators $\mathrm{Id}_{\ell_1^k}$ ($k \in \mathbb{N}$) uniformly factor through the $T^{(m)}$,

(ii) or the $T^{(m)}$ uniformly approximately factor through ℓ_∞^k ($k \in \mathbb{N}$)?

Our final result gives a positive answer to this problem in the case when the entries of the matrix associated to $T^{(m)}$ are independent, symmetric random variables.

Theorem C. *For each $m \in \mathbb{N}$ let $T^{(m)}\colon \ell_\infty^m(\ell_1^m) \to L_1$ be an operator such that the entries of the corresponding random matrix $\big(T_{i,j}^{(m)}\big)$ form a sequence of independent, symmetric random variables with*

$$\big\|T^{(m)}\big\| = \max\left\{ \mathbb{E}\Big| \sum_{i=1}^{m} T_{i,j_i}^{(m)} \Big| : j_1, \dots, j_m \in \{1, \dots, m\} \right\} \le 1\ .$$

Then

(i) either the identity operators $\mathrm{Id}_{\ell_1^k}$ ($k \in \mathbb{N}$) uniformly factor through the $T^{(m)}$,

(ii) or the $T^{(m)}$ uniformly approximately factor through ℓ_∞^k ($k \in \mathbb{N}$).

The problem of classifying the closed ideals of operators on a Banach space goes back to Calkin who in 1941 proved that the compact operators are the only non-trivial, proper closed ideal in $\mathcal{B}(\ell_2)$ [1]. The same result was later proved for all ℓ_p spaces (p finite) and for c_0 by Gohberg, Markus, and Feldman in 1960 [5]. Remarkably, very little is known about the closed ideals of $\mathcal{B}(\ell_p \oplus \ell_q)$, and it is not even known if there are infinitely many of them. For the most recent results on the spaces $\ell_p \oplus \ell_q$ the reader is invited to consult [15].

In the late 1960's Gramsch [6] and Luft [12] independently extended Calkin's theorem in a different direction by classifying all the closed ideals of $\mathcal{B}(H)$ for each Hilbert space H (not necessarily separable). In particular, they showed that these ideals are well-ordered by inclusion.

It was not until fairly recently that new examples were added to the list of Banach spaces for which all of the closed ideals of operators can be determined. In 2004 Laustsen, Loy, and Read [9] proved that for the Banach space $E = \big(\bigoplus_{n=1}^{\infty} \ell_2^n\big)_{c_0}$ there are exactly four closed ideals of $\mathcal{B}(E)$, namely $\{0\}$, the compact operators $\mathcal{K}(E)$, the closure $\overline{\mathcal{G}}_{c_0}(E)$ of the set of operators factoring through c_0, and $\mathcal{B}(E)$ itself. A similar result was subsequently obtained by Laustsen, Schlumprecht and Zsák for the dual space $F = \big(\bigoplus_{n=1}^{\infty} \ell_2^n\big)_{\ell_1}$ [10]. In 2006 Daws [2] extended Gramsch and Luft's result to the Gohberg–Markus–Feldman case by classifying the closed ideals of $\mathcal{B}(\ell_p(I))$ (for p finite) and $\mathcal{B}(\mathrm{c}_0(I))$ where I is an index set of arbitrary cardinality. Again, these ideals are well-ordered by inclusion. Recently Argyros and Haydon constructed a space that solves the famous compact-plus-scalar problem: every operator on their space is a compact perturbation of a scalar multiple of the identity operator. This remarkable space has many interesting properties. In particular, as this space also has a basis, the compact operators are the only non-trivial, proper closed ideal of the algebra of all operators.

Our paper is organized as follows. In Section 1 we sketch the proofs of the more straightforward parts of Theorem B. We also reduce the ideal classification problem to the problem stated above (preceding the statement of Theorem C), and we introduce the notions of uniform factorization and uniform approximate factorization. In Section 2 we define the notions of uniform lattice bounds and uniform approximate lattice bounds, and we prove Theorem A. In Section 3 we complete the proof of Theorem B. The general dichotomy theorem, Theorem A, gives rise to a very natural conjecture that would solve the ideal classification problem completely. In Section 4 we present a counterexample to this conjecture. Section 5 contains a proof of Theorem C.

We use standard Banach space terminology throughout. For convenience we shall work with real scalars. All our results extend without difficulty to the complex case. The sign $|\cdot|$ will be used for absolute value (of a number or a function) as well as for the size of a finite set. Finally, we denote by $\mathbf{1}_A$ the indicator function of a set A, and use the probabilistic notation $\mathbb{P}$ for Lebesgue measure on $[0,1]$.

1. Preliminary results

Throughout this paper we fix X to be the Banach space $\big(\bigoplus_{n=1}^{\infty} \ell_1^n\big)_{\mathrm{c}_0}$. In this section we first prove those parts of Theorem B that follow easily from standard basis arguments. We then reduce the problem of finding the closed ideal structure of $\mathcal{B}(X)$ to a question about sequences of operators defined on finite ℓ_∞-direct sums of ℓ_1-spaces with values in L_1 (this reduction will also follow easily from standard basis arguments). We shall also be introducing definitions and notations to be used throughout the paper.

We shall only give sketch proofs. The results in this section extend without difficulty to more general unconditional sums of finite-dimensional spaces. For detailed proofs in the general case, we refer the reader to [9].

Proposition 1.1. *We have the following closed ideals in $\mathcal{B}(X)$:*

$$\{0\} \subsetneq \mathcal{K}(X) \subsetneq \overline{\mathcal{G}}_{\mathrm{c}_0}(X) \subsetneq \mathcal{B}(X)\ .$$

Moreover, if T is a non-compact operator on X, then the closed ideal generated by T contains $\overline{\mathcal{G}}_{\mathrm{c}_0}(X)$. It follows that any closed ideal of $\mathcal{B}(X)$ not in the above list must lie strictly between $\overline{\mathcal{G}}_{\mathrm{c}_0}(X)$ and $\mathcal{B}(X)$.

Proof. Since X has a basis, the compact operators are the smallest non-trivial closed ideal of $\mathcal{B}(X)$, and the inclusion $\mathcal{K}(X) \subset \overline{\mathcal{G}}_{\mathrm{c}_0}(X)$ follows. (Note, however, that not every compact operator on X factors through c_0.) This inclusion is strict, since c_0 is complemented in X and a projection onto a copy of c_0 is a non-compact operator in $\overline{\mathcal{G}}_{\mathrm{c}_0}(X)$.

We next show that $\overline{\mathcal{G}}_{\mathrm{c}_0}(X) \neq \mathcal{B}(X)$. Recall that if an idempotent element of a Banach algebra belongs to the closure of an ideal I, then in fact it belongs to I. Thus, if $\overline{\mathcal{G}}_{\mathrm{c}_0}(X) = \mathcal{B}(X)$, then the identity operator on X factors through c_0, *i.e.,* X is complemented in c_0, and thus isomorphic to it. It is well known, however, that X is not isomorphic to c_0 (*e.g.,* because ℓ_1 has cotype 2).

Finally, let T be a non-compact operator on X. To complete the proof it is enough to show that the identity on c_0 factors through T. Let (x_n) be a bounded sequence in X such that (Tx_n) has no convergent subsequence. After passing to a subsequence we can assume that both (x_n) and (Tx_n) converge coordinatewise (with respect to the obvious basis of X). We then extract a further subsequence for which the difference sequence $(Tx_n - Tx_{n+1})$ is bounded away from zero. This way we obtain a sequence (y_n) in X such that both (y_n) and (Ty_n) converge to zero coordinatewise and (Ty_n) is bounded away from zero. We can then pass to a further subsequence such that (y_n) and (Ty_n) are basic sequences equivalent to the unit vector basis of c_0 and such that their closed linear spans are complemented in X. It is now straightforward that Id_{c_0} factors through T. $\square$

For $n \in \mathbb{N}$ we let $J_n\colon \ell_1^n \to X$ be the canonical embedding given by $J_n x = (y_i)$ where $y_n = x$ and $y_i = 0$ for $i \neq n$. For each $m \in \mathbb{N}$ the map $Q_m\colon X \to \ell_1^m$ denotes the canonical quotient map defined by $Q_m(y) = y_m$ for $y = (y_i) \in X$. We introduce projections $P_n = J_n Q_n \in \mathcal{B}(X)$ for $n \in \mathbb{N}$, and $P_A(x) = \sum_{n \in A} P_n x$ for $A \subset \mathbb{N}$ and $x \in X$.

For an operator $T\colon X \to X$ we let $T_{m,n} = Q_m T J_n\colon \ell_1^n \to \ell_1^m$. We can identify T with the infinite matrix $\big(T_{m,n}\big)$: if $Tx = y$, then $y_m = \sum_n T_{m,n} x_n$. We say that T is *locally finite* if the sets $\{j \in \mathbb{N} : T_{m,j} = 0\}$ and $\{i \in \mathbb{N} : T_{i,n}\}$ are finite for all $m, n \in \mathbb{N}$, *i.e.,* if T has finitely supported rows and columns.

Lemma 1.2. *For any $T \in \mathcal{B}(X)$ and $\varepsilon > 0$ there is a compact operator $K \in \mathcal{B}(X)$ such that $\|K\| < \varepsilon$ and $T + K$ is locally finite.*

Proof. Fix a sequence (ε_i) in $(0,1)$ with $\sum_i \varepsilon_i < \varepsilon$. Let $n \in \mathbb{N}$. For each $x \in \ell_1^n$ there exists $N(n,x) \in \mathbb{N}$ such that $\|(I - P_{\{1,\dots,N\}})TJ_n x\| < \varepsilon_n/2$ for all $N \geq N(n,x)$. By compactness of $B_{\ell_1^n}$, there exists $N_n \in \mathbb{N}$ such that $\|(I - P_{\{1,\dots,N_n\}})TJ_n\| < \varepsilon_n$. Then the operator $K = \sum_n (I - P_{\{1,\dots,N_n\}})TJ_n$ is compact, $\|K\| < \varepsilon$ and $T - K$ has finite columns.

Next fix $m \in \mathbb{N}$. Since the unit vector basis of c_0 is shrinking, for each $f \in \ell_\infty^m$ there exists $M(m,f) \in \mathbb{N}$ such that $\|fQ_m T(I - P_{\{1,\dots,M\}})\| < \varepsilon_m/2$ for all $M \geq M(m,f)$. By compactness of $B_{\ell_\infty^m}$, there exists $M_m \in \mathbb{N}$ such that $\|fQ_m T(I - P_{\{1,\dots,M_m\}})\| < \varepsilon_m \|f\|$ for all $f \in \ell_\infty^m$ and hence, by Hahn–Banach, $\|Q_m T(I - P_{\{1,\dots,M_m\}})\| \leq \varepsilon_m$. As before, we now obtain a compact operator K such that $\|K\| < \varepsilon$ and $T - K$ has finite rows. $\square$

Definition. Given families $\big(U_i\colon E_i \to F_i\big)_{i \in I}$ and $\big(V_j\colon G_j \to H_j\big)_{j \in J}$ of operators between Banach spaces, we say *the U_i uniformly factor through the V_j* (or that *the V_j uniformly factor the U_i*) if

$$\exists\, C > 0 \quad \forall\, i \in I \quad \exists\, j_i \in J\,,\ A_i\colon E_i \to G_{j_i}\,,\ B_i\colon H_{j_i} \to F_i$$
$$\text{such that} \quad U_i = B_i V_{j_i} A_i \quad \text{and} \quad \|A_i\| \cdot \|B_i\| \leq C\ .$$

We say *the U_i uniformly approximately factor through the V_j* (or that *the V_j uniformly approximately factor the U_i*) if

$$\forall\, \varepsilon > 0 \quad \exists\, C > 0 \quad \forall\, i \in I \quad \exists\, j_i \in J\,,\ A_i\colon E_i \to G_{j_i}\,,\ B_i\colon H_{j_i} \to F_i$$
$$\text{such that} \quad \|U_i - B_i V_{j_i} A_i\| < \varepsilon \quad \text{and} \quad \|A_i\| \cdot \|B_i\| \leq C\ .$$

If $G_j = H_j$ and V_j is the identity operator Id_{G_j} on G_j for all $j \in J$, then we will also use the term *factoring through the G_j* instead of factoring through the Id_{G_j}, *etc.*

For a family $\big(U_i\colon E_i \to F_i\big)_{i \in I}$ of operators with $\sup_{i \in I} \|U_i\| < \infty$ we write $\mathrm{diag}(U_i)_{i \in I}$ for the diagonal operator $\big(\bigoplus_{i \in I} E_i\big)_{c_0} \to \big(\bigoplus_{i \in I} F_i\big)_{c_0}$ given by $(x_i)_{i \in I} \mapsto (U_i x_i)_{i \in I}$.

Now let $T \in \mathcal{B}(X)$ be a locally finite operator. For $m \in \mathbb{N}$ we let R_m be the support of the m^{th} row of T: this is the finite set $R_m = \{j \in \mathbb{N} : T_{m,j} \neq 0\}$. We set $X_m = \big(\bigoplus_{j \in R_m} \ell_1^j\big)_{\ell_\infty}$ and let $J^{(m)} \colon X_m \to X$ and $Q^{(m)} \colon X \to X_m$ be the canonical embedding and quotient maps given by $J^{(m)}\big((x_j)_{j\in R_m}\big) = \sum_{j\in R_m} J_j(x_j)$ and $Q^{(m)}(x) = \big(Q_j(x)\big)_{j\in R_m}$, respectively. We define $T^{(m)} \colon X_m \to \ell_1^m$ to be the m^{th} row of T ignoring the zero entries, *i.e.,* $T^{(m)}$ maps $x = (x_j)_{j\in R_m}$ to $Q_m T J^{(m)}(x) = \sum_{j\in R_m} T_{m,j} x_j$.

One final piece of notation before we relate factorization properties of T to those of the sequence $\big(T^{(m)}\big)$: for subsets A and B of $\mathbb{N}$ we write $A < B$ if $a < b$ for all $a \in A$ and $b \in B$.

Proposition 1.3. *Let $T \in \mathcal{B}(X)$ be a locally finite operator.*

(i) If the $T^{(m)}$ uniformly factor the identity operators $\mathrm{Id}_{\ell_1^k}$ ($k \in \mathbb{N}$), then T factors the identity operator on X.

(ii) T approximately factors through c_0 if and only if the $T^{(m)}$ uniformly approximately factor through ℓ_∞^n ($n \in \mathbb{N}$).

Proof. (i) By the assumption, there exist $C > 0$, positive integers $m_1 < m_2 < \dots$ and operators $A_k \colon \ell_1^k \to X_{m_k}$ and $B_k \colon \ell_1^{m_k} \to \ell_1^k$ such that $\mathrm{Id}_{\ell_1^k} = B_k T^{(m_k)} A_k$ and $\|A_k\| \cdot \|B_k\| \leq C$ for every $k \in \mathbb{N}$. We may assume, after passing to a subsequence if necessary, that $R_{m_1} < R_{m_2} < \dots$, so in particular the m_j^{th} and m_k^{th} rows of T have disjoint support whenever $j \neq k$. Observe that the identity operator $\mathrm{Id}_X = \mathrm{diag}\big(\mathrm{Id}_{\ell_1^k}\big)$ factors through the diagonal operator

$$\tilde{T} = \mathrm{diag}(T^{(m_k)}) \colon \Big(\bigoplus_k X_{m_k}\Big)_{\mathrm{c}_0} \longrightarrow \Big(\bigoplus_k \ell_1^{m_k}\Big)_{\mathrm{c}_0} .$$

Indeed, we have $\mathrm{Id}_X = B\tilde{T}A$, where $A = \mathrm{diag}(A_k)$ and $B = \mathrm{diag}(B_k)$. It is therefore sufficient to show that $\tilde{T}$ factors through T. Define $\tilde{A} \colon \big(\bigoplus_k X_{m_k}\big)_{\mathrm{c}_0} \to X$ by $(x_k) \mapsto \sum_k J^{(m_k)}(x_k)$ and $\tilde{B} \colon X \to \big(\bigoplus_k \ell_1^{m_k}\big)_{\mathrm{c}_0}$ by $x \mapsto \big(Q_{m_k}(x)\big)_{k=1}^\infty$. That $\tilde{A}$ is well-defined follows from the assumption $R_{m_1} < R_{m_2} < \dots$. Note that we have $\tilde{T} = \tilde{B} T \tilde{A}$, as required.

(ii) Assume the $T^{(m)}$ uniformly approximately factor through ℓ_∞^n ($n \in \mathbb{N}$). Then $\tilde{T} = \mathrm{diag}(T^{(m)})$ approximately factors through $\big(\bigoplus_k \ell_\infty^{n_k}\big)_{\mathrm{c}_0}$ for some $n_1 < n_2 < \dots$. This latter space is isomorphic to c_0, so it is enough to observe that T factors through $\tilde{T}$. Indeed, $T = \tilde{T}Q$, where $Qx = \big(Q^{(m)}(x)\big)$ for $x \in X$.

The converse implication is clear since each $T^{(m)}$ factors through T, and c_0 is a $\mathcal{L}_\infty$-space. □

2. The general dichotomy theorem

In this section we begin our study of factorization properties of sequences of operators $T_m \colon X_m \to L_1$ ($m \in \mathbb{N}$) where (X_m) is a sequence of *arbitrary* Banach spaces. We will prove a dichotomy theorem in this general setting. In the next section we shall apply this to an operator T on our space $X = \big(\bigoplus_{n=1}^\infty \ell_1^n\big)_{\mathrm{c}_0}$: the T_m will be the rows $T^{(m)}$ of T (as defined before Proposition 1.3). Before stating our main theorem we need a definition.

Definition. Let $T_i \colon X_i \to L_1$ ($i \in I$) be a family of operators. We say the T_i *have uniform lattice bounds* if

$$\exists\, C > 0 \quad \forall\, i \in I \quad \exists\, g_i \in L_1 \quad \text{with} \quad \|g_i\|_{L_1} \leq C \quad \text{and} \quad T_i\big(B_{X_i}\big) \subset \{f \in L_1 : |f| \leq g_i\}$$

(*i.e.,* $|T_i x| \leq g_i$ for all $x \in B_{X_i}$). The family $(g_i)_{i\in I}$ is a *uniform lattice bound for the T_i.*

We say the T_i *have uniform approximate lattice bounds* if

$$\forall\, \varepsilon > 0\ \exists\, C > 0\ \forall\, i \in I\ \exists\, g_i \in L_1^+ \text{ with } \|g_i\|_{L_1} \leq C \text{ and } T_i\big(B_{X_i}\big) \subset \{f \in L_1 : |f| \leq g_i\} + \varepsilon B_{L_1}$$

(*i.e.*, $\big\|\big(|T_i x| - g_i\big)^+\big\|_{L_1} \leq \varepsilon$ for all $x \in B_{X_i}$). The family $(g_i)_{i\in I}$ is a *uniform approximate lattice bound for the T_i corresponding to ε.*

We now come to one of the main results in this paper, which yields, as a special case, Theorem A stated in the Introduction.

Theorem 2.1. *Let $T_m\colon X_m \to L_1$ ($m \in \mathbb{N}$) be a uniformly bounded sequence of operators. Then the following dichotomy holds:*

(i) either the identity operators $\mathrm{Id}_{\ell_1^k}$ ($k \in \mathbb{N}$) uniformly factor through the T_m,

(ii) or the T_m have uniform approximate lattice bounds.

Remark. We observe that this is a genuine dichotomy. Indeed, assume that both alternatives hold. By (i) there exists $C > 0$ such that for all $k \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $T_m\big(B_{X_m}\big)$ contains a sequence $f_1, \dots, f_k$ which is C-equivalent to the unit vector basis of ℓ_1^k for some constant C independent of k. By a theorem of Dor [4, Theorem B] there exist $\delta > 0$ (depending only on C) and disjoint sets $E_1, \dots, E_k$ such that $\|f_j\restriction_{E_j}\| \geq \delta$ for all j. By (ii) there exists a uniform approximate lattice bound (g_m) for the T_m corresponding to $\varepsilon = \delta/2$. Then

$$\begin{aligned}\|g_m\|_{L_1} &\geq \sum_{j=1}^k \|g_m\restriction_{E_j}\|_{L_1} \geq \sum_{j=1}^k \|(|f_j| \wedge g_m)\restriction_{E_j}\|_{L_1}\\ &\geq \sum_{j=1}^k \big(\|f_j\restriction_{E_j}\|_{L_1} - \|(|f_j| - g_m)^+\restriction_{E_j}\|_{L_1}\big) \geq k\delta/2 \ .\end{aligned}$$

Thus $\sup_m \|g_m\|_{L_1} = \infty$ — a contradiction.

Before embarking on the proof of Theorem 2.1, we make a simple observation, which places uniform lattice bounds in the context of factorization.

Proposition 2.2. *Let $T_m\colon X_m \to L_1$ ($m \in \mathbb{N}$) be a uniformly bounded sequence of operators.*

(i) If the T_m have uniform lattice bounds then they uniformly factor through L_∞. In particular, if $\dim X_m < \infty$ for all m, then the T_m uniformly factor through ℓ_∞^n ($n \in \mathbb{N}$).

(ii) Suppose that for each $m \in \mathbb{N}$ we have $X_m = \ell_1^{N_m}$ for some $N_m \in \mathbb{N}$. If the T_m have uniform approximate lattice bounds, then they uniformly approximately factor through ℓ_∞^n ($n \in \mathbb{N}$).

Proof. (i) Let (g_m) be a bounded sequence in L_1 such that $|T_m x| \leq g_m$ for all $x \in B_{X_m}$ and for all $m \in \mathbb{N}$. Without loss of generality for each $m \in \mathbb{N}$ we have $g_m > 0$ everywhere. We can then define maps $A_m\colon X_m \to L_\infty$ by $A_m x = \frac{T_m x}{g_m}$ and $B_m\colon L_\infty \to L_1$ by $B_m f = g_m \cdot f$. This gives the required factorization $T_m = B_m A_m$ with $\sup\|A_m\| \cdot \|B_m\| = \sup\|g_m\|_{L_1} < \infty$. The second assertion follows immediately by virtue of the fact that L_∞ is a $\mathcal{L}_\infty$-space.
(ii) Let $\varepsilon > 0$ and let (g_m) be a corresponding uniform approximate lattice bound for the T_m. For $m \in \mathbb{N}$ define a linear operator $S_m\colon \ell_1^{N_m} \to L_1$ by setting $S_m e_i = (T_m e_i \wedge g_m) \vee (-g_m)$ ($i = 1, \dots, N_m$), where $(e_i)_{i=1}^{N_m}$ denotes the unit vector basis of $\ell_1^{N_m}$. Then

$$\|T_m - S_m\| = \max_{1\leq i\leq N_m} \|(T_m - S_m)(e_i)\|_{L_1} \leq \varepsilon \ .$$

Since (g_m) is a uniform lattice bound for the S_m, it follows from (i) that the S_m uniformly factor through ℓ_∞^n ($n \in \mathbb{N}$). □

We now begin the proof of Theorem 2.1. We will need two ingredients. The first of these is a sort of converse to the aformentioned result of Dor [4, Theorem B]. This converse result for an infinite sequence (f_i), from which the quantitative statement below follows easily, was proved by H. Rosenthal [14] using a combinatorial argument. Here we sketch a particularly elegant probabilistic proof from [8] which has the advantage of giving a linear bound (with respect to k) on the constant $n(\delta, k)$ in the statement of the theorem.

Theorem 2.3. *For each $\delta > 0$ and $k \in \mathbb{N}$ there exists $n = n(\delta, k) \in \mathbb{N}$ such that if $f_1, \dots, f_n$ are functions in B_{L_1} for which there are disjoint sets $E_1, \dots, E_n$ with $\|f_i\restriction_{E_i}\|_{L_1} \geq \delta$ for all i, then there is a subsequence $(f_{j_i})_{i=1}^k$ such that*

$$\Big\| \sum_{i=1}^k a_i f_{j_i} \Big\|_{L_1} \geq \frac{\delta}{2} \qquad \text{whenever } \sum_{i=1}^k |a_i| = 1 \ .$$

In particular, $(f_{j_i})_{i=1}^k$ is $\frac{2}{\delta}$-equivalent to the unit vector basis of ℓ_1^k.

Proof. Fix $\delta \in (0, 1]$ and $k \in \mathbb{N}$. Let $n = \lfloor \frac{10}{\delta} \rfloor \cdot k$, and let $A = (\alpha_{i,j})$ be the $n \times n$ matrix with $\alpha_{i,j} = \|f_i\restriction_{E_j}\|_{L_1}$ when $i \neq j$ and zeros on the diagonal. Note that the row sums of A satisfy $\sum_{j=1}^n \alpha_{i,j} \leq \|f_i\|_{L_1} \leq 1$. We will show the existence of a $k \times k$ submatrix $(\alpha_{i,j})_{i,j \in F}$ whose row sums are at most $\frac{\delta}{2}$. An easy direct computation then shows that the subsequence $(f_i)_{i \in F}$ has the required property.

Pick a subset E of $\{1, \dots, n\}$ of size $2k$ uniformly at random. Then

$$\mathbb{E} \sum_{i,j \in E} \alpha_{i,j} = \mathbb{E} \sum_{i,j=1}^n \alpha_{i,j} \mathbf{1}_{\{i,j \in E\}} = \sum_{i,j=1}^n \alpha_{i,j} \binom{n-2}{2k-2} \binom{n}{2k}^{-1} \leq \frac{(2k)^2}{n-1} \ .$$

It follows that for some subset E the row sums of the submatrix $(\alpha_{i,j})_{i,j \in E}$ are at most $\frac{2k}{n-1}$ on average. Hence, by Markov's inequality, at least half of the rows sum to at most twice this average. *I.e.*, for some $F \subset E$ with $|F| = k$, the row sums of $(\alpha_{i,j})_{i,j \in F}$ are at most $\frac{\delta}{2}$. □

The second ingredient is a theorem of Dor which shows, in particular, that a subspace of L_1 whose Banach–Mazur distance to ℓ_1^k is not too large is well complemented.

Theorem 2.4 ((Dor [4, Theorem A]))**.** *Let μ and ν be measures and $T\colon L_1(\nu) \to L_1(\mu)$ an isomorphic embedding with $\|T\| \cdot \|T^{-1}\| = \lambda < \sqrt{2}$. Then there is a projection P of $L_1(\mu)$ onto the range of T with*

$$\|P\| \leq \left(2\lambda^{-2} - 1\right)^{-1} \ .$$

In the proof of Theorem 2.1 we shall use an argument that will also be needed in Section 5, so we state and prove it separately.

Proposition 2.5. *Let $T_m\colon X_m \to L_1$ ($m \in \mathbb{N}$) be operators with $\|T_m\| \leq 1$ for all $m \in \mathbb{N}$. Assume that there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$, functions $f_1, \dots, f_n \in T_m(B_{X_m})$ and pairwise disjoint sets $E_1, \dots, E_n$ such that $\|f_i\restriction_{E_i}\|_{L_1} \geq \delta$ for all i. Then the identity operators $\mathrm{Id}_{\ell_1^k}$ uniformly factor through the T_m.*

Proof. By Theorem 2.3 we can deduce the following from the assumption:

(1) $\forall\, k \in \mathbb{N} \quad \exists\, m \in \mathbb{N} \quad \exists\, y_1, \dots, y_k \in B_{X_m}$ such that

$$\Big\| \sum_{i=1}^k a_i T_m y_i \Big\|_{L_1} \geq \frac{\delta}{2} \qquad \text{whenever } \sum_{i=1}^k |a_i| = 1 \ .$$

Thus, in particular, $T_m\big(B_{X_m}\big)$ contains a sequence $\frac{2}{\delta}$-equivalent to the unit vector basis of ℓ_1^k. We next use a well known argument of James (see *e.g.,* [13, Proposition 2]) to improve the equivalence constant $\frac{2}{\delta}$. Fix $1<\lambda<\sqrt{2}$. Choose $r\in\mathbb{N}$ such that $\big(\frac{2}{\delta}\big)^{1/r}<\lambda$, and then set $K=k^r$. By (1) there exist $m\in\mathbb{N}$ and $y_1,\dots,y_K\in B_{X_m}$ such that

$$\Big\|\sum_{i=1}^K a_iT_my_i\Big\|_{L_1}\geq\frac{\delta}{2}\qquad\text{whenever }\sum_{i=1}^K|a_i|=1\ . \tag{2}$$

Now James's argument shows that there is a block basis $z_j=\sum_{i=p_{j-1}+1}^{p_j}a_iy_i$, where $0=p_0<p_1<\dots<p_k=K$ and $\sum_{i=p_{j-1}+1}^{p_j}|a_i|=1$ for all j, such that $\big(T_mz_j\big)_{j=1}^k$ is $\big(\frac{2}{\delta}\big)^{1/r}$-equivalent to the unit vector basis of ℓ_1^k. Thus there exist constants $0<\alpha\leq\beta$ with $\frac{\beta}{\alpha}<\lambda$ such that

$$\alpha\leq\Big\|\sum_{j=1}^k b_jT_mz_j\Big\|_{L_1}\leq\beta\qquad\text{whenever }\sum_{j=1}^k|b_j|=1\ . \tag{3}$$

Note that by (2) we have $\beta\geq\|T_mz_j\|_{L_1}\geq\frac{\delta}{2}$. Now define $A_m\colon\ell_1^k\to X_m$ by $e_j\mapsto z_j$. We then have $\big\|T_mA_m\big\|\cdot\big\|(T_mA_m)^{-1}\big\|<\lambda$, so we can apply Theorem 2.4: there is a projection P of L_1 onto the range of T_mA_m with $\|P\|\leq\big(2\lambda^{-2}-1\big)^{-1}$. Let $B_m\colon L_1\to\ell_1^k$ be the composition of P with the map $\operatorname{span}\{T_mz_j:\,j=1,\dots,k\}\to\ell_1^k$ defined by $T_mz_j\mapsto e_j$. Using (3) and the above estimates involving α and β, we obtain

$$\|A_m\|\leq1\ ,\quad\|B_m\|\leq\|P\|\cdot\frac{1}{\alpha}\leq\|P\|\cdot\lambda\cdot\frac{2}{\delta}\leq\frac{2\lambda}{\delta}\cdot\big(2\lambda^{-2}-1\big)^{-1}\ ,$$

and $\mathrm{Id}_{\ell_1^k}=B_mT_mA_m$. Thus the T_m uniformly factor the identity operators $\mathrm{Id}_{\ell_1^k}$ ($k\in\mathbb{N}$), as required. □

Proof of Theorem 2.1. Without loss of generality we have $\|T_m\|\leq1$ for all m. We assume that (ii) fails: there exists an $\varepsilon>0$ such that for all $C>0$ there exists $m\in\mathbb{N}$ such that

$$\forall\, g\in L_1^+\text{ with }\|g\|_{L_1}\leq C\ \exists\, x\in B_{X_m}\text{ such that }\big\|\big(|T_mx|-g\big)^+\big\|_{L_1}>\varepsilon\ . \tag{4}$$

From this we deduce that the assumption of Proposition 2.5 is satisfied with $\delta=\varepsilon/2$.

Fix $n\in\mathbb{N}$ and set $N=\big\lfloor\frac{4n^2}{\varepsilon}\big\rfloor$. Putting $C=N-1$, we find $m\in\mathbb{N}$ such that (4) holds. From now on we let $T=T_m$. Successive applications of (4) yield $x_1,\dots,x_N\in B_{X_m}$ such that

$$\Big\|\Big(|Tx_i|-\bigvee_{1\leq j<i}|Tx_j|\Big)^+\Big\|_{L_1}>\varepsilon\qquad\text{for }i=1,\dots,N\ .$$

(Note that $\big\|\bigvee_{1\leq j<i}|Tx_j|\big\|_{L_1}\leq N-1=C$ for all $i\leq N$.) For each $i=1,\dots,N$ set

$$\begin{aligned}D_i&=\Big\{\omega\in[0,1]:\,|Tx_i|(\omega)>\bigvee_{1\leq j<i}|Tx_j|(\omega)\Big\}\ ,\text{ and}\\ \tilde D_i&=\Big\{(\omega,t)\in[0,1]\times\mathbb{R}:\,\omega\in D_i,\ |Tx_i|(\omega)>t>\bigvee_{1\leq j<i}|Tx_j|(\omega)\Big\}\ .\end{aligned}$$

(Thus $\tilde D_i$ is the region between the graphs of $|Tx_i|$ and $\bigvee_{1\leq j<i}|Tx_j|$ where the former is greater.) For each $1<i_0\leq N$, the regions $(D_{i_0}\times\mathbb{R})\cap\tilde D_i$, $i=1,\dots,i_0-1$, are pairwise disjoint and lie beneath the graph of $|Tx_{i_0}|$. It follows that

$$\sum_{i=1}^{i_0-1}\Big\|\Big(|Tx_i|-\bigvee_{1\leq j<i}|Tx_j|\Big)^+\cdot\mathbf{1}_{D_{i_0}}\Big\|_{L_1}\leq\|Tx_{i_0}\|_{L_1}\leq1\ ,$$

and hence

$$\Big|\Big\{i < i_0 : \Big\|\Big(|Tx_i| - \bigvee_{1\le j<i} |Tx_j|\Big)^+ \cdot \mathbf{1}_{D_{i_0}}\Big\|_{L_1} \geq \frac{\varepsilon}{2n}\Big\}\Big| \leq \frac{2n}{\varepsilon} \ .$$

By the choice of N, we can therefore find $N = i_1 > i_2 > \cdots > i_n \geq 1$ such that

$$\Big\|\Big(|Tx_{i_s}| - \bigvee_{1\le j<i_s} |Tx_j|\Big)^+ \cdot \mathbf{1}_{D_{i_r}}\Big\|_{L_1} < \frac{\varepsilon}{2n} \qquad \text{for } 1 \leq r < s \leq n \ .$$

Now set $f_s = Tx_{i_s}$ and $E_s = D_{i_s} \setminus \bigcup_{r<s} D_{i_r}$ for $s = 1, \dots, n$. Then $f_1, \dots, f_n \in T(B_{X_m})$, the sets $E_1, \dots, E_n$ are pairwise disjoint, and $\|f_i\restriction_{E_i}\|_{L_1} \geq \frac{\varepsilon}{2}$ for all $i = 1, \dots, n$. This completes the proof of the theorem. □

3. The existence of a unique maximal ideal

Let $\mathcal{J}$ be an operator ideal. We say $\mathcal{J}$ is *injective* if, given any operator $T\colon E \to F$ between Banach spaces and an (isomorphic) embedding $J\colon F \to G$, we have $JT \in \mathcal{J}(E, G)$ implies $T \in \mathcal{J}(E, F)$. The *injective hull* of $\mathcal{J}$ is defined to be

$$\mathcal{J}^{(\mathrm{inj})}(E, F) = \big\{T \in \mathcal{B}(E, F) : \exists \text{ embedding } J\colon F \to G \text{ such that } JT \in \mathcal{J}(E, G)\big\} \ .$$

It is easy to see that $\mathcal{J}^{(\mathrm{inj})}$ is an injective operator ideal and it is the smallest injective ideal containing $\mathcal{J}$.

The dual concept is that of a surjective ideal. We say $\mathcal{J}$ is *surjective* if, given any operator $T\colon E \to F$ and a quotient map (*i.e.,* an onto bounded linear map) $Q\colon D \to E$, we have $TQ \in \mathcal{J}(D, F)$ implies $T \in \mathcal{J}(E, F)$. The *surjective hull* of $\mathcal{J}$ is

$$\mathcal{J}^{(\mathrm{sur})}(E, F) = \big\{T \in \mathcal{B}(E, F) : \exists \text{ quotient map } Q\colon D \to E \text{ such that } TQ \in \mathcal{J}(D, F)\big\} \ .$$

One can again verify that $\mathcal{J}^{(\mathrm{sur})}$ is a surjective operator ideal and it is the smallest such ideal containing $\mathcal{J}$.

In this section we investigate what happens if we apply these two ways of obtaining a new ideal from a given one in the algebra $\mathcal{B}(X)$. Recall that throughout $X = \big(\bigoplus_{n=1}^\infty \ell_1^n\big)_{\mathrm{c}_0}$. Since $\mathcal{K}$ is an injective and surjective operator ideal, we only need to consider $\overline{\mathcal{G}}_{\mathrm{c}_0}(X)$. Taking the injective hull, we obtain nothing new.

Theorem 3.1. $\overline{\mathcal{G}}_{\mathrm{c}_0}^{(\mathrm{inj})}(X) = \mathcal{B}(X)$.

Proof. Since X is the c_0-sum of finite-dimensional spaces, we have an embedding $J\colon X \to \mathrm{c}_0$ and $JI_X \in \mathcal{G}_{\mathrm{c}_0}(X, \mathrm{c}_0)$. □

The surjective hull, however, does give new information about the ideal structure of $\mathcal{B}(X)$. This is the main result of this section.

Theorem 3.2. *$\overline{\mathcal{G}}_{\mathrm{c}_0}^{(\mathrm{sur})}(X)$ is the unique maximal ideal of $\mathcal{B}(X)$.*

Proof. We first show that $\overline{\mathcal{G}}_{\mathrm{c}_0}^{(\mathrm{sur})}(X)$ is a proper ideal. Assume, for a contradiction, that this ideal contains Id_X, *i.e.,* that some quotient map $Z \to X$ approximately factors through c_0. Without loss of generality we can assume that Z is separable. By considering a quotient map $\ell_1 \to Z$, we may also assume that $Z = \ell_1$, so there is an embedding $X^* = \big(\bigoplus_{n=1}^\infty \ell_\infty^n\big)_{\ell_1} \to \ell_\infty$ which approximately factors through ℓ_1. It follows easily that ℓ_1 contains ℓ_∞^n $(n \in \mathbb{N})$ uniformly. This is impossible, *e.g.,* because ℓ_1 has cotype 2.

Now fix $T \in \mathcal{B}(X)$. We are going to show that if Id_X does not factor through T, then T belongs to $\overline{\mathcal{G}}_{\mathrm{c}_0}^{(\mathrm{sur})}(X)$. This will prove that every proper ideal is contained in $\overline{\mathcal{G}}_{\mathrm{c}_0}^{(\mathrm{sur})}(X)$, and our proof is then complete.

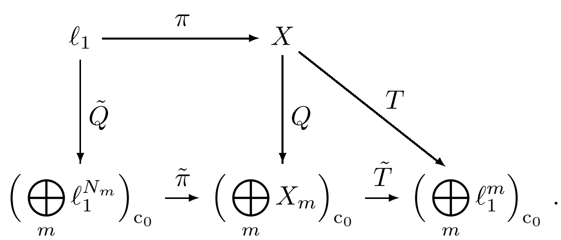
Without loss of generality we can assume that T is locally finite (Lemma 1.2). We are going to use the notation introduced before Proposition 1.3: $R_m = \{j \in \mathbb{N} : T_{m,j} \neq 0\}$ is the m^{th} row support of T, $X_m = \big(\bigoplus_{j\in R_m} \ell_1^j\big)_{\ell_\infty}$, and $T^{(m)}\colon X_m \to \ell_1^m$ is the m^{th} row of T.

Fix quotient maps $\pi\colon \ell_1 \to X$ and $\pi_m\colon \ell_1^{N_m} \to X_m$ with

$$\frac{1}{2}B_{X_m} \subset \pi_m\big(B_{\ell_1^{N_m}}\big) \subset B_{X_m} \qquad (m \in \mathbb{N}).$$

Note that $\tilde{\pi} = \operatorname{diag}(\pi_m)\colon \Big(\bigoplus_m \ell_1^{N_m}\Big)_{c_0} \to \big(\bigoplus_m X_m\big)_{c_0}$ is also a quotient map.

Recall from the proof of Proposition 1.3(ii) that T factors through $\tilde{T} = \operatorname{diag}(T^{(m)})$ via the map $Q\colon X \to \big(\bigoplus_m X_m\big)_{c_0}$ given by $Qx = \big(Q^{(m)}(x)\big)_{m=1}^\infty$ for $x \in X$. By the lifting property of ℓ_1 there is a map $\tilde{Q}\colon \ell_1 \to \Big(\bigoplus_m \ell_1^{N_m}\Big)_{c_0}$ with $\|\tilde{Q}\| \leq 2$ such that $Q\pi = \tilde{\pi}\tilde{Q}$. We thus have the following commuting diagram:



We claim that $T\pi$ approximately factors through c_0. Since T does not factor Id_X, the $T^{(m)}$ do not factor $\mathrm{Id}_{\ell_1^k}$ ($k \in \mathbb{N}$) uniformly (Proposition 1.3(i)). By Theorem 2.1, the $T^{(m)}$, and hence the $T^{(m)}\pi_m$, have uniform approximate lattice bounds. It follows by Proposition 2.2(ii) that the $T^{(m)}\pi_m$ uniformly approximately factor through ℓ_∞^n ($n \in \mathbb{N}$). This implies that $\tilde{T}\tilde{\pi}$ approximately factors through c_0, and hence so does $T\pi$. □

Remark. Of course, we have $\overline{\mathcal{G}}_{c_0}(X) \subset \overline{\mathcal{G}}_{c_0}^{(\mathrm{sur})}(X)$, but we do not know whether this inclusion is strict *i.e.,* whether there exist closed ideals of $\mathcal{B}(X)$ other than those listed in Proposition 1.1.

4. Perturbing operators with uniform approximate lattice bounds

In Proposition 2.2(ii), can we replace $\ell_1^{N_m}$ by more general spaces X_m? *I.e.,* given operators $T_m\colon X_m \to L_1$ ($m \in \mathbb{N}$) with uniform approximate lattice bounds, do the T_m uniformly approximately factor through ℓ_∞^n ($n \in \mathbb{N}$)? Proposition 2.2(i) gives an affirmative answer to this question *provided* there exist arbitrarily small perturbations of the T_m with uniform lattice bounds. This leads to the following question.

Question. *Let $T_m\colon X_m \to L_1$ ($m \in \mathbb{N}$) be a uniformly bounded sequence of operators. Assume that the T_m have uniform approximate lattice bounds. Does there exist, for all $\varepsilon > 0$, a sequence $S_m\colon X_m \to L_1$ ($m \in \mathbb{N}$) of operators with $\|T_m - S_m\| < \varepsilon$ for all m such that the S_m have uniform lattice bounds?*

One cannot hope for a positive answer for a general sequence (X_m) of Banach spaces: *e.g.,* the diagonal operators $A_m\colon \ell_2^m \to \ell_1^m$ below give a simple counterexample (*c.f.* Proposition 4.2). However, the proof of Proposition 2.2(ii) shows that we do have a positive answer in the case when each X_m is an ℓ_1-space. We would hope to generalize this to the case when each X_m is a finite ℓ_∞-direct sum of finite-dimensional ℓ_1-spaces. A positive answer in that case together with Theorem 2.1 and Proposition 2.2(ii) would provide a positive answer to the problem raised in the Introduction (stated before Theorem C). In turn, this would imply (by Proposition 1.3) that the list in Proposition 1.1 is a complete list of the closed ideals of $\mathcal{B}(X)$ for our space $X = \big(\bigoplus_{n=1}^{\infty} \ell_1^n\big)_{c_0}$.

In this section we present an example that shows that the above question has a negative answer even in the case when each X_m is a finite-dimensional ℓ_∞-space. Here it will be convenient to use a different normalization: the range spaces will be ℓ_1^m ($m \in \mathbb{N}$) instead of L_1.

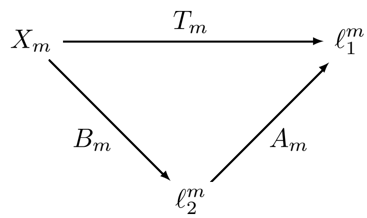
For $m \in \mathbb{N}$ set $N_m = 2^m$ and $X_m = \ell_\infty^{N_m}$. Let $r_i^m \in X_m^*$ be the i^{th} Rademacher function, $i = 1, \dots, m$, normalized with respect to the ℓ_∞-norm, *i.e.,* the coordinates of each r_i^m are ± 1. Let e_i^m, $i = 1, \dots, m$, denote the standard basis of $\mathbb{R}^m$. Define $T_m \colon X_m \to \ell_1^m$ by defining its adjoint

$$T_m^* \colon \ell_\infty^m \longrightarrow X_m^* \ , \qquad e_i^m \longmapsto \frac{1}{\sqrt{m}} \frac{1}{N_m} r_i^m \ , \qquad i = 1, \dots, m \ .$$

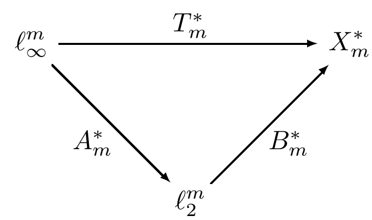
Thus we have

$$\langle T_m x, e_i^m \rangle = \tfrac{1}{\sqrt{m} N_m} \langle x, r_i^m \rangle \ , \qquad x \in \ell_\infty^{N_m},\ i = 1, \dots, m \ .$$

Note that $\|T_m\| \leq 1$ for all m. We now show that the T_m have uniform approximate lattice bounds. We have factorizations



obtained from its dual



where $A_m^*(e_i^m) = \frac{1}{\sqrt{m}} e_i^m$ and $B_m^*(e_i^m) = \frac{1}{N_m} r_i^m$ for $i = 1, \dots, m$. Note that $\|A_m\| = 1$ (consider extreme points of $B_{\ell_\infty^m}$) and $\|B_m\| = 1$ for all $m \in \mathbb{N}$. Thus, in particular, it is sufficient to show that the A_m have uniform approximate lattice bounds.

Proposition 4.1. *Given $\varepsilon > 0$, let $C = \frac{1}{\varepsilon}$. Then for each $m \in \mathbb{N}$ and $x = \sum_{i=1}^m x_i e_i^m \in B_{\ell_2^m}$ we have $\big\| A_m x \restriction_L \big\|_{\ell_1^m} \leq \varepsilon$, where $L = L(m, x) = \{i : |\langle A_m x, e_i^m \rangle| > C/m\}$.*

Proof. For $x \in B_{\ell_2^m}$ we have $L = L(m, x) = \big\{i : |x_i| > C/\sqrt{m}\big\}$. Since $|L| \frac{C^2}{m} \leq \|x\|_{\ell_2^m}^2$, by Cauchy–Schwarz we get

$$\big\| A_m x \restriction_L \big\|_{\ell_1^m} = \sum_{i \in L} \frac{|x_i|}{\sqrt{m}} \leq \sqrt{\frac{|L|}{m}} \cdot \|x\|_{\ell_2^m} \leq \frac{1}{C} = \varepsilon \ . \qquad \square$$

This shows that for any $\varepsilon > 0$ and for $C = \frac{1}{\varepsilon}$ we have

$$T_m(B_{X_m}) \subset \big\{ y = \textstyle\sum y_i e_i^m \in \ell_1^m : |y_i| \leq \frac{C}{m} \text{ for } i = 1, \dots, m \big\} + \varepsilon B_{\ell_1^m} \ .$$

Thus the T_m have uniform approximate lattice bounds. The difficult part is to show that the T_m cannot be perturbed to get uniform lattice bounds. We first show this for the A_m. Although we do not need this, the proof is much simpler than for the T_m and contains some of the ideas used later.

Proposition 4.2. *Let $\varepsilon \in (0,1)$. Assume that for all $m \in \mathbb{N}$ there exist $S_m \colon \ell_2^m \to \ell_1^m$ and $g_m \in \ell_1^m$ such that*

$$\|S_m - A_m\| < \varepsilon \tag{5}$$

$$|S_m x| \le g_m \quad \text{for all } x \in B_{\ell_2^m} \ . \tag{6}$$

Then $\sup \|g_m\|_{\ell_1^m} = \infty$.

Proof. Fix $m \in \mathbb{N}$. We will show that $\|g_m\|_{\ell_1^m} \ge \frac{(1-\varepsilon)\sqrt{m}}{3}$. For the rest of the proof we drop the subscript m; π will denote a permutation of $\{1,\dots,m\}$ as well as the corresponding linear map on $\mathbb{R}^m$ given by $e_i \mapsto e_{\pi(i)}$. Note that $A = \pi^{-1} A \pi$ for all π. Let

$$\overline{S} = \frac{1}{m!} \sum_{\pi} \pi^{-1} S \pi \qquad \text{and} \qquad C = \|g\|_{\ell_1^m} = \sum_{i=1}^m g(i) \ .$$

Then $\|\overline{S} - A\| < \varepsilon$ and

$$\big|\langle \overline{S} x, e_i\rangle\big| \le \frac{1}{m!} \sum_{\pi} \big|\langle S\pi(x), \pi(e_i)\rangle\big| \le \frac{1}{m!} \sum_{\pi} g(\pi(i)) = \frac{C}{m} \ .$$

Thus, without of loss of generality, g is the constant function $\frac{C}{m}$ and $S = \pi^{-1} S \pi$ for all π. It follows that for some $a, b \in \mathbb{R}$ we have $\langle Se_i, e_i\rangle = \frac{a}{m}$ for all i, and $\langle Se_i, e_j\rangle = \frac{b}{m(m-1)}$ for all $i \ne j$.

Now by (6) we have $|a| \le C$ and $|b| \le C(m-1)$. We next apply (5) to $x = \frac{1}{\sqrt{m}} \sum (-1)^i e_i$ to obtain

$$\varepsilon > \|Ax - Sx\|_{\ell_1^m} \ge \|Ax\|_{\ell_1^m} - \|Sx\|_{\ell_1^m} \ge 1 - \tfrac{1}{\sqrt{m}} \Big(\tfrac{|a|}{m} + \tfrac{2|b|}{m(m-1)} \Big) \cdot m \ge 1 - \tfrac{3C}{\sqrt{m}} \ ,$$

from which our claim follows. ☐

Remark. The motivation behind the proof of Proposition 4.2 is as follows. In contrast to A, S cannot be large on the diagonal because it has a lattice bound. On the other hand, being close to A, S has norm close to 1, so the off-diagonal entries of S must make a significant contribution to the norm of S. Next, since A is symmetric, we could "symmetrize" S, and hence assume that S is constant off the diagonal. Applying S to a "flat" vector whose coefficients alternate in sign, we produce a small vector due to cancellations. On the other hand, when we apply the diagonal operator A to the same vector, no cancellations occur making the outcome large. This contradicts that A and S are close in norm. The idea behind the proof of Theorem 4.3 below is exactly the same.

We now turn to the proof that the T_m cannot be perturbed to have uniform lattice bounds. By Khintchine's inequality in L_1 (see, for example [7]), with $K = \sqrt{2}$ we have

$$\frac{1}{K} \Big(\sum_{i=1}^m a_i^2 \Big)^{1/2} \le \Big\| \sum_{i=1}^m a_i \tfrac{1}{N_m} r_i^m \Big\|_{\ell_1^{N_m}} \le \Big(\sum_{i=1}^m a_i^2 \Big)^{1/2} \qquad \text{for all } (a_i)_{i=1}^m \in \mathbb{R}^m \ . \tag{7}$$

Theorem 4.3. *Let $0 < \varepsilon < \frac{1}{4K}$. Assume that for all $m \in \mathbb{N}$ there exist $g_m \in \ell_1^m$ and $S_m \colon \ell_\infty^{N_m} \to \ell_1^m$ such that*

$$\|S_m - T_m\| < \varepsilon \ , \tag{8}$$

$$|S_m x| \le g_m \quad \text{for all } x \in B_{\ell_\infty^{N_m}} \ . \tag{9}$$

Then $\sup_m \|g_m\|_{\ell_1^m} = \infty$.

Proof. We shall argue by contradiction. Assume that for some $0<\varepsilon<\frac{1}{4K}$ there is a $C>0$ such that for all $m\in\mathbb{N}$ there exist $g_m\in\ell_1^m$ and $S_m\colon \ell_\infty^{N_m}\to\ell_1^m$ such that (8) and (9) hold, and moreover $\|g_m\|_{\ell_1^m}\le C$ for all $m\in\mathbb{N}$.

We will obtain a contradiction in a number of steps. From now on we fix a large m (to be specified at the end of the proof), and drop m in the various subscripts and superscripts. We denote by N the power set of $\{1,\dots,m\}$ and write the standard basis of $\mathbb{R}^N$ as $e_\alpha,\ \alpha\in N$. The Rademacher functions can then be expressed as

$$r_i=\sum_{\alpha,\, i\in\alpha}e_\alpha-\sum_{\alpha,\, i\notin\alpha}e_\alpha \qquad i=1,\dots,m\ .$$

The letter π will always denote a permutation of $\{1,\dots,m\}$ as well as the following induced maps:

$$\ell_1^m\xrightarrow{\ \pi\ }\ell_1^m\ ,\qquad e_i\longmapsto e_{\pi(i)}$$

$$N\xrightarrow{\ \pi\ }N\ ,\qquad \alpha\longmapsto\{\pi(i)\,:\,i\in\alpha\}$$

$$\ell_\infty^N\xrightarrow{\ \pi\ }\ell_\infty^N\ ,\qquad e_\alpha\longmapsto e_{\pi(\alpha)}\ .$$

Note that the first and third interpretations of π are isometries. The letter R will also stand for a number of different maps:

$$\ell_1^m\xrightarrow{\ R\ }\ell_1^m\ ,\qquad e_i\longmapsto -e_i$$

$$N\xrightarrow{\ R\ }N\ ,\qquad \alpha\longmapsto\neg\alpha=\{1,\dots,m\}\setminus\alpha$$

$$\ell_\infty^N\xrightarrow{\ R\ }\ell_\infty^N\ ,\qquad e_\alpha\longmapsto e_{R(\alpha)}\ .$$

Here again R is an isometry in the first and third definitions. Note also that the last map satisfies $R\big(r_i\big)=-r_i$, and that R and π commute in each their interpretations.

Having fixed our notation, we next show that S can be assumed to have various symmetries. We begin with the observation that T is symmetric in the sense that it equals the composite $\pi^{-1}T\pi$:

$$\ell_\infty^N\xrightarrow{\ \pi\ }\ell_\infty^N\xrightarrow{\ T\ }\ell_1^m\xrightarrow{\ \pi^{-1}\ }\ell_1^m\ .$$

Similarly, $T=RTR$. Set $\overline{S}=\frac{1}{m!}\sum_\pi\pi^{-1}S\pi$ and $C=\|g\|_{\ell_1^m}=\sum_{i=1}^m g(i)$. Then $\|\overline{S}-T\|<\varepsilon$, and, by (9), for all $x\in B_{\ell_\infty^N}$ and for $i=1,\dots,m$ we have

$$\big|\langle\overline{S}x,e_i\rangle\big|\le\frac{1}{m!}\sum_\pi\big|\langle S\pi(x),\pi(e_i)\rangle\big|\le\frac{1}{m!}\sum_\pi g(\pi(i))=\frac{C}{m}\ .$$

Thus, without of loss of generality, we may assume that g is the constant function $\frac{C}{m}$ and that $S=\pi^{-1}S\pi$ for all π.

Next we set $\overline{S}=\frac12(S+RSR)$. Then $\|\overline{S}-T\|<\varepsilon$, $\big|\langle\overline{S}x,e_i\rangle\big|\le\frac{C}{m}$ for all $x\in B_{\ell_\infty^N}$ and for $i=1,\dots,m$. We can thus also assume that $S=RSR$.

The above two symmetrization procedures have the following implications for the matrix of S: there exist $a_k\in\mathbb{R},\ k=1,\dots,m$, such that

$$S_{i,\alpha}=\langle Se_\alpha,e_i\rangle=\begin{cases}a_{|\alpha|}&\text{if } i\in\alpha\\ -a_{|\neg\alpha|}&\text{if } i\notin\alpha\end{cases}\qquad\qquad \alpha\in N,\ i=1,\dots,m\ .$$

To complete the proof of Theorem 4.3 we require a number of lemmas.

Lemma 4.4. $2\sum_{k=1}^{m}|a_k|\binom{m-1}{k-1} \le \frac{C}{m}$.

Proof. For $x \in \ell_\infty^N$ and $i = 1, \dots, m$ we have

$$\langle Sx, e_i\rangle = \sum_\alpha x_\alpha S_{i,\alpha} = \sum_{k=1}^m a_k \sum_{|\alpha|=k,\ i\in\alpha} (x_\alpha - x_{\neg\alpha}) \ . \tag{10}$$

Fix an arbitrary $i \in \{1, \dots, m\}$, set

$$x_\alpha = \begin{cases} \operatorname{sign}(a_k) & \text{if } |\alpha| = k,\ i \in \alpha \\ -\operatorname{sign}(a_k) & \text{if } |\alpha| = m-k,\ i \notin \alpha \ , \end{cases}$$

and use (9) to obtain

$$\big|\langle Sx, e_i\rangle\big| = 2\sum_{k=1}^m |a_k|\binom{m-1}{k-1} \le \frac{C}{m} \ ,$$

as required. □

Lemma 4.5. *Fix $k_0 \in \mathbb{N}$. Let $\varepsilon_i \in \{-1, +1\}$, $i = 1, \dots, m$. For $\alpha \in N$ set*

$$x_\alpha = \operatorname{sign}\Big(\sum_{i\in\alpha}\varepsilon_i - \sum_{i\notin\alpha}\varepsilon_i\Big)$$

whenever $k_0 \le |\alpha| \le m - k_0$ (we let $\operatorname{sign}(0) = 0$), otherwise set $x_\alpha = 0$. Then there exists $m(k_0) \in \mathbb{N}$ such that $\|Tx\|_{\ell_1^m} \ge \frac{1}{4K}$ provided $m \ge m(k_0)$.

Proof. Recall that $T^*\colon \ell_\infty^m \to \ell_1^N$ is given by $T^*(e_i) = \frac{1}{\sqrt{mN}} r_i$, $i = 1, \dots, m$. For $y = \sum_{i=1}^m \varepsilon_i e_i$ Khintchine's inequality (7) yields

$$\|T^*y\|_{\ell_1^N} = \Big\|\sum_{i=1}^m \tfrac{\varepsilon_i}{\sqrt{mN}} r_i\Big\|_{\ell_1^N} \ge \frac{1}{K}\Big\|\sum_{i=1}^m \tfrac{\varepsilon_i}{\sqrt{m}} e_i\Big\|_{\ell_2^m} = \frac{1}{K} \ .$$

It follows that setting $z = \operatorname{sign}\big(T^*y\big)$, we have

$$\|Tz\|_{\ell_1^m} \ge \langle Tz, y\rangle = \langle z, T^*y\rangle = \|T^*y\|_{\ell_1^N} \ge \frac{1}{K} \ .$$

Now for any $\alpha \in N$ we have

$$\langle T^*y, e_\alpha\rangle = \tfrac{1}{\sqrt{mN}}\sum_{i=1}^m \varepsilon_i\langle r_i, e_\alpha\rangle = \tfrac{1}{\sqrt{mN}}\Big(\sum_{i\in\alpha}\varepsilon_i - \sum_{i\notin\alpha}\varepsilon_i\Big) \ ,$$

and hence

$$z_\alpha = \operatorname{sign}\Big(\sum_{i\in\alpha}\varepsilon_i - \sum_{i\notin\alpha}\varepsilon_i\Big) \ .$$

Note that $x_\alpha = z_\alpha$ whenever $k_0 \le |\alpha| \le m - k_0$.

Observe that if we add an element to the set $\alpha \in N$, then the expression $\sum_{i\in\alpha}\varepsilon_i - \sum_{i\notin\alpha}\varepsilon_i$ changes by at most 2 in absolute value. It follows that

$$\sum_{|\alpha|=k+1}\big|\langle T^*y, e_\alpha\rangle\big| \ge \sum_{|\alpha|=k}\big|\langle T^*y, e_\alpha\rangle\big| - \binom{m}{k}\frac{2}{\sqrt{mN}}$$

whenever $0 \le k < \frac{m}{2}$. (Indeed, there exists an injection from sets of size k to sets of size $k+1$ mapping each α to some set $\beta \supset \alpha$. This can be seen using Hall's marriage theorem.) Iterating k_0 times, we get

$$\begin{aligned}\sum_{|\alpha|=k+k_0} \big|\langle T^*y, e_\alpha\rangle\big| &\ge \sum_{|\alpha|=k} \big|\langle T^*y, e_\alpha\rangle\big| - \sum_{j=0}^{k_0-1} \binom{m}{k+j} \frac{2}{\sqrt{m}N} \\ &\ge \sum_{|\alpha|=k} \big|\langle T^*y, e_\alpha\rangle\big| - \frac{2}{\sqrt{m}}\end{aligned}$$

whenever $0 \le k < \frac{m}{2} - k_0$. Summing over k, we obtain

$$\sum_{k=k_0}^{2k_0-1} \sum_{|\alpha|=k} \big|\langle T^*y, e_\alpha\rangle\big| \ge \sum_{k=0}^{k_0-1} \sum_{|\alpha|=k} \big|\langle T^*y, e_\alpha\rangle\big| - \frac{2k_0}{\sqrt{m}}$$

provided $k_0 < \frac{m}{4}$. Similarly (or using $\langle T^*y, e_{\neg\alpha}\rangle = -\langle T^*y, e_\alpha\rangle$), we obtain

$$\sum_{k=k_0}^{2k_0-1} \sum_{|\alpha|=m-k} \big|\langle T^*y, e_\alpha\rangle\big| \ge \sum_{k=0}^{k_0-1} \sum_{|\alpha|=m-k} \big|\langle T^*y, e_\alpha\rangle\big| - \frac{2k_0}{\sqrt{m}} \ .$$

Putting these together, we finally get

$$\begin{aligned}\|Tx\|_{\ell_1^m} \ge \langle Tx, y\rangle &= \sum_{k_0 \le |\alpha| \le m-k_0} \big|\langle e_\alpha, T^*y\rangle\big| \\ &\ge \frac{1}{3} \sum_{\alpha} \big|\langle e_\alpha, T^*y\rangle\big| - \frac{4k_0}{3\sqrt{m}} > \frac{1}{4K}\end{aligned}$$

provided m is sufficiently large. ☐

The quantity $d(m,k)$ in Lemmas 4.6 and 4.7 is defined for an even integer m as follows:

$$d(m,k) = \begin{cases} \binom{\frac{m}{2}-1}{\frac{k-1}{2}}^2 & \text{if } k \text{ is odd} \\ \binom{\frac{m}{2}-1}{\frac{k}{2}-1}\binom{\frac{m}{2}-1}{\frac{k}{2}} & \text{if } k \text{ is even.} \end{cases}$$

Lemma 4.6. *Fix $k_0 \in \mathbb{N}$, let $m \in \mathbb{N}$ be even, and set $\varepsilon_i = (-1)^i$ for $i = 1, \dots, m$. Define $x = (x_\alpha) \in \ell_\infty^N$ as in Lemma 4.5. Then for $k_0 \le k \le m - k_0$ and for $j = 1, \dots, m$ we have*

$$\sum_{|\alpha|=k,\ j\in\alpha} x_\alpha = (-1)^j \cdot d(m,k) \ .$$

Proof. It is sufficient to consider $j = m$. Let E be the set of all even numbers in $\{1, \dots, m\}$. Note that for the given choice of signs $\varepsilon_1, \dots, \varepsilon_m$ we have

$$x_\alpha = \operatorname{sign}\Big(\sum_{i\in\alpha} \varepsilon_i - \sum_{i\notin\alpha} \varepsilon_i\Big) = \operatorname{sign}\Big(\sum_{i\in\alpha} \varepsilon_i\Big) \ .$$

Given $\alpha \in N$ with $|\alpha| = k$ and $m \in \alpha$, let

$$\beta = \{i+1 : i \le m-2,\ i \in \alpha \setminus E\} \cup \{i-1 : i \le m-2,\ i \in \alpha \cap E\} \cup \big(\alpha \cap \{m-1, m\}\big) \ .$$

Then $|\beta| = k$, $m \in \beta$ and $x_\alpha + x_\beta = 0$ unless $m-1 \notin \alpha$ and either (k is odd and) $|\alpha \cap E| = \frac{k+1}{2}$, or ($k$ is even and) $|\alpha \cap E| = \frac{k}{2}$ or $\frac{k}{2}+1$. The result follows. ☐

Lemma 4.7. *Let $m \in \mathbb{N}$ be even. Then*

$$d(m,k) \leq 2\binom{m-1}{k-1}\binom{k}{\lfloor \frac{k}{2} \rfloor}\frac{1}{2^k} \qquad \text{for each } k = 1,\dots,m\ .$$

Proof. Assume k is even. Then

$$d(m,k)\binom{m-1}{k-1}^{-1} =$$

$$\frac{1}{2^{k-1}} \cdot \frac{\big[(m-2)(m-4)\dots(m-k+2)\big]\cdot\big[(m-2)(m-4)\dots(m-k)\big]}{(m-1)(m-2)\dots(m-k+1)} \cdot \frac{(k-1)!}{(\frac{k}{2}-1)!\cdot(\frac{k}{2})!}$$

$$= \frac{m-2}{m-1}\cdot\frac{m-4}{m-3}\cdots\frac{m-k}{m-k+1}\cdot\binom{k}{k/2}\cdot\frac{1}{2^k} \leq \binom{k}{k/2}\cdot\frac{1}{2^k}\ .$$

An almost identical computation works for odd k except we get an extra factor of 2 in that case. ☐

Lemma 4.8. *There is a universal constant U such that*

$$\binom{k}{\lfloor \frac{k}{2} \rfloor} \leq U\frac{2^k}{\sqrt{k}} \qquad \text{for all } k \in \mathbb{N}\ .$$

Proof. For $\frac{k}{2} - \sqrt{k} \leq j < \frac{k}{2}$ we have

$$\binom{k}{j+1} = \binom{k}{j}\cdot\frac{k-j}{j+1} \leq \binom{k}{j}\cdot\frac{\frac{k}{2}+\sqrt{k}}{\frac{k}{2}-\sqrt{k}} \leq \binom{k}{j}\cdot\Big(1+\frac{6}{\sqrt{k}}\Big)$$

provided k is sufficiently large. It follows that for $\frac{k}{2} - \sqrt{k} \leq j < \frac{k}{2}$ we have

$$\binom{k}{\lfloor \frac{k}{2} \rfloor} \leq \Big(1+\frac{6}{\sqrt{k}}\Big)^{\sqrt{k}}\cdot\binom{k}{j} \leq \mathrm{e}^6\cdot\binom{k}{j}$$

for sufficiently large k. Hence for a universal constant U and for all $k \in \mathbb{N}$ we have

$$\sqrt{k}\cdot\binom{k}{\lfloor \frac{k}{2} \rfloor} \leq U\cdot 2^k\ ,$$

as required. ☐

Proof of Theorem 4.3 continued. We finally have all the ingredients to obtain the required contradiction. Choose $k_0, m \in \mathbb{N}$ with $\frac{2UC}{\sqrt{k_0}} < \frac{1}{4K} - \varepsilon$, $m \geq m(k_0)$ and m even. Recall that C and ε were fixed at the very beginning of the proof, K is the Khintchine constant, U is the universal constant obtained in Lemma 4.8 above, and $m(k_0)$ is given by Lemma 4.5.

Let $x = (x_\alpha) \in \ell_\infty^N$ be as in Lemma 4.5 with $\varepsilon_i = (-1)^i$. Note that $x_{\neg\alpha} = -x_\alpha$ for all $\alpha \in N$. We have

$$\begin{aligned}
\|Sx\|_{\ell_1^m} &= \sum_{i=1}^m \big|\langle Sx, e_i\rangle\big| \\
&\leq \sum_{i=1}^m \sum_{k=1}^m 2|a_k| \cdot \Big| \sum_{|\alpha|=k,\ i\in\alpha} x_\alpha \Big| && \text{by (10)} \\
&\leq m \sum_{k_0 \leq k \leq m-k_0} 2|a_k| d(m,k) && \text{by Lemma 4.6} \\
&\leq m \sum_{k_0 \leq k \leq m-k_0} 4|a_k| \binom{m-1}{k-1} U \frac{1}{\sqrt{k}} && \text{by Lemmas 4.7 and 4.8} \\
&\leq \frac{2UC}{\sqrt{k_0}} < \frac{1}{4K} - \varepsilon && \text{by Lemma 4.4.}
\end{aligned}$$

Finally, by Lemma 4.5 we have $\|Tx\|_{\ell_1^m} \geq \frac{1}{4K}$, and this contradicts (8). □

5. Searching for new ideals

Proposition 1.3 tells us that a possible new closed ideal in $\mathcal{B}(X)$ (if there is one) is generated by an operator defined by a sequence $T^{(m)}\colon \ell_\infty^m(\ell_1^m) \to L_1$ ($m \in \mathbb{N}$) which neither factors the identity operators $\mathrm{Id}_{\ell_1^k}$ ($k \in \mathbb{N}$) uniformly, nor does it factor through ℓ_∞^k ($k \in \mathbb{N}$) approximately uniformly. The main result of this section, stated as Theorem C in the Introduction, shows that there is no such sequence when for each $m \in \mathbb{N}$, the entries of the random matrix $\big(T_{i,j}^{(m)}\big)$ are independent, symmetric random variables.

We begin with a characterization of sequences of operators which uniformly factor through ℓ_∞^k ($k \in \mathbb{N}$) in terms of the 2-summing norm. The 2-summing norm is defined for an operator $U\colon E \to F$ between Banach spaces as

$$\pi_2(U) = \sup\Big\{ \big(\textstyle\sum_{s=1}^k \|Uz^{(s)}\|^2\big)^{1/2} :\ k \in \mathbb{N},\ z^{(1)}, \dots, z^{(k)} \in E\ , \quad \textstyle\sum_{s=1}^k |\langle z^{(s)}, z^*\rangle|^2 \leq 1 \quad \forall\, z^* \in B_{E^*} \Big\}\ .$$

We denote by Ω_k the probability space $\big(\{1, \dots, k\}, \mu_k\big)$, where μ_k is the uniform probability measure given by $\mu_k\big(\{i\}\big) = \frac{1}{k}$ for $i = 1, \dots, k$.

Theorem 5.1. *Let $T^{(m)}\colon \ell_\infty^m(\ell_1^m) \to L_1$ ($m \in \mathbb{N}$) be a uniformly bounded sequence of operators. Then the following are equivalent.*

(i) The $T^{(m)}$ uniformly factor through ℓ_∞^k ($k \in \mathbb{N}$).

(ii) $\sup_m \pi_2\big(T^{(m)}\big) < \infty$.

(iii) The $T^{(m)}$ uniformly factor through the formal identity maps

$$\iota_k\colon \ell_\infty^k \to L_2(\Omega_k)\ , \quad \textstyle\sum_{i=1}^k x_i e_i \mapsto \sum_{i=1}^k x_i \mathbf{1}_{\{i\}} \qquad (k \in \mathbb{N}).$$

Proof. (i)⇒(ii) follows from the fact that $\pi_2(\cdot)$ is an ideal norm and from the following consequence of Grothendieck's theorem (*c.f.* [3, Theorem 3.5]).

Theorem 5.2. *Let Φ be a compact, Hausdorff space and μ an arbitrary measure on some measurable space. Then for $1 \leq p \leq 2$ any operator $U\colon C(\Phi) \to L_p(\mu)$ is 2-summing with $\pi_2(U) \leq K_G\|U\|$, where K_G is the Grothendieck constant.*

(ii)⇒(iii) is a consequence of the following special case of Pietsch's Factorization Theorem (*c.f.* [3, Corollary 2.16]).

Theorem 5.3. *Let E and F be Banach spaces, and let Φ be a w^*-compact subset of B_{E^*} which is 1-norming for E. Let $\kappa\colon E\to C(\Phi)$ denote the canonical embedding: $\kappa(x)(x^*)=x^*(x),\ x\in E,\ x^*\in\Phi$.*

Then an operator $u\colon E\to F$ is 2-summing if and only if there is a probability measure μ on the Borel σ-algebra of Φ and an operator $\tilde{u}\colon L_2(\mu)\to F$ such that $u=\tilde{u}\circ\iota\circ\kappa$, where $\iota\colon C(\Phi)\to L_2(\mu)$ is the formal identity. Moreover, $\tilde{u}$ can be chosen with $\|\tilde{u}\|=\pi_2(u)$.

(iii)⇒(i) is of course obvious. □

Recall that for each $m\in\mathbb{N}$ we denote by $e_{i,j}=e_{i,j}^{(m)}$ the unit vector basis of $\ell_\infty^m(\ell_1^m)$ such that the norm of $\sum_{i,j}a_{i,j}e_{i,j}$ is given by $\max_i\sum_j|a_{i,j}|$. We identify an operator $U\colon\ell_\infty^m(\ell_1^m)\to L_1$ with the $m\times m$ matrix $(U_{i,j})$ in L_1, where $U_{i,j}=U(e_{i,j})$. We now estimate $\pi_2(U)$ in the case the matrix entries $U_{i,j}$ form a symmetric sequence of random variables. Here and elsewhere we will make use of the *square function inequality:* if $f_1,\dots,f_n\in L_1$ form a symmetric sequence of random variables, then

$$\frac{1}{K}\Big\|\Big(\sum_{i=1}^n|f_i|^2\Big)^{1/2}\Big\|_{L_1}\leq\Big\|\sum_{i=1}^n f_i\Big\|_{L_1}\leq\Big\|\Big(\sum_{i=1}^n|f_i|^2\Big)^{1/2}\Big\|_{L_1}\ . \tag{11}$$

This is a well known consequence of Khintchine's inequality (7).

Lemma 5.4. *Let $m\in\mathbb{N}$, and let $U\colon\ell_\infty^m(\ell_1^m)\to L_1$ be an operator such that the matrix entries $U_{i,j}$ form a symmetric sequence of random variables. Then*

$$\pi_2(U)\leq\Big(\sum_{i=1}^m\max_{1\leq j\leq m}\|U_{i,j}\|_{L_2}^2\Big)^{1/2}\ .$$

Proof. By definition

$$\pi_2^2(U)=\sup_{(z^{(s)})_{s=1}^k}\frac{\sum_{s=1}^k\|Uz^{(s)}\|_{L_1}^2}{\sup_{z^*\in B_{\ell_1^m(\ell_\infty^m)}}\sum_{s=1}^k|\langle z^{(s)},z^*\rangle|^2} \tag{12}$$

where the supremum is over all $k\in\mathbb{N}$ and $z^{(1)},\dots,z^{(k)}\in\ell_\infty^m(\ell_1^m)$. We will estimate the denominator and numerator of the above expression separately. We will denote by ρ an arbitrary element $(\rho_j)_{j=1}^m$ of $\{\pm1\}^m$. We begin with the denominator:

$$\sup_{z^*\in B_{\ell_1^m(\ell_\infty^m)}}\sum_{s=1}^k|\langle z^{(s)},z^*\rangle|^2=\max_{1\leq i\leq m}\max_\rho\sum_{s=1}^k\Big|\sum_{j=1}^m\rho_j z_{i,j}^{(s)}\Big|^2\geq\max_{1\leq i\leq m}\sum_{s=1}^k\sum_{j=1}^m|z_{i,j}^{(s)}|^2\ .$$

The equality follows since the sup is attained at an extreme point of $B_{\ell_1^m(\ell_\infty^m)}$. We then replace $\max_\rho$ by Ave_ρ, interchange Ave_ρ and $\sum_{s=1}^k$, and compute the variance of a linear combination of independent Bernoulli random variables. This yields the inequality. Now the numerator:

$$\begin{aligned}\sum_{s=1}^{k}\|Uz^{(s)}\|_{L_1}^2 &\le \sum_{s=1}^{k}\Big\|\Big(\sum_{i,j=1}^{m}\big|z_{i,j}^{(s)}U_{i,j}\big|^2\Big)^{1/2}\Big\|_{L_1}^2 \le \sum_{s=1}^{k}\Big\|\sum_{i,j=1}^{m}\big|z_{i,j}^{(s)}U_{i,j}\big|^2\Big\|_{L_1}\\ &= \sum_{s=1}^{k}\sum_{i,j=1}^{m}|z_{i,j}^{(s)}|^2\|U_{i,j}\|_{L_2}^2 = \sum_{i=1}^{m}\left(\sum_{s=1}^{k}\sum_{j=1}^{m}|z_{i,j}^{(s)}|^2\|U_{i,j}\|_{L_2}^2\right)\\ &\le \sum_{i=1}^{m}\max_{1\le j\le m}\|U_{i,j}\|_{L_2}^2\cdot\left(\sum_{s=1}^{k}\sum_{j=1}^{m}|z_{i,j}^{(s)}|^2\right).\end{aligned}$$

Here the first inequality is the square function inequality (11), the second inequality follows from Jensen's inequality whereas the rest is straightforward. Substitution of our estimates into (12) yields the result. □

Proof of Theorem C. For $m\in\mathbb{N}$ we let $\mathcal{F}_m$ be the set of functions $\{1,\dots,m\}\to\{1,\dots,m\}$. Functions $j,j'\in\mathcal{F}_m$ are said to be *disjoint* if $j_i\ne j_i'$ for all $i=1,\dots,m$. Since $\|T^{(m)}\|$ is attained at an extreme point of $B_{\ell_\infty^m(\ell_1^m)}$, we have

$$\|T^{(m)}\| = \sup\Big\{\mathbb{E}\Big|\sum_{i=1}^{m}\rho_i T_{i,j_i}^{(m)}\Big| : j\in\mathcal{F}_m,\ \rho\in\{\pm1\}^m\Big\}.$$

By the symmetry of the $T_{i,j}^{(m)}$, we in fact have

$$\|T^{(m)}\| = \sup\Big\{\mathbb{E}\Big|\sum_{i=1}^{m}T_{i,j_i}^{(m)}\Big| : j\in\mathcal{F}_m\Big\}.$$

We consider two cases motivated by the notion of uniform approximate lattice bounds. The second case is the negation of the first.

(i') $\exists\,\varepsilon>0\ \forall\,C>0\ \forall\,n\in\mathbb{N}\ \exists\,m\in\mathbb{N}$ and pairwise disjoint functions $j^{(s)}\in\mathcal{F}_m$ $(s=1,\dots,n)$ such that

$$\Big\|\sum_{i=1}^{m}T_{i,j_i^{(s)}}^{(m)}\cdot\mathbf{1}_{\big\{\big|T_{i,j_i^{(s)}}^{(m)}\big|>C\big\}}\Big\|_{L_1}\ge\varepsilon \qquad \text{for } s=1,\dots,n. \tag{13}$$

(ii') $\forall\,\varepsilon>0\quad\exists\,C>0\quad\exists\,n\in\mathbb{N}\quad\forall\,m\ge n$ there exist pairwise disjoint functions $j^{(s)}\in\mathcal{F}_m$ $(s=1,\dots,n)$ such that

$$\Big\|\sum_{i=1}^{m}T_{i,j_i}^{(m)}\cdot\mathbf{1}_{\{|T_{i,j_i}^{(m)}|>C\}}\Big\|_{L_1}<\varepsilon \tag{14}$$

for each $j\in\mathcal{F}_m$ that is disjoint from all the $j^{(s)}$.

We will deduce alternatives (i) and (ii) of Theorem C from the above cases (i') and (ii'), respectively. We begin with case (i'). Fix $n\in\mathbb{N}$ and choose $C>0$ such that $\big(1-\frac{2}{C}\big)^n\ge\frac{1}{2}$. Now case (i') gives $m\in\mathbb{N}$ and pairwise disjoint functions $j^{(s)}\in\mathcal{F}_m$ $(s=1,\dots,n)$ such that (13) holds. To avoid cumbersome notation, we assume, after permuting entries in each row if necessary, that $j_i^{(s)}=s$ for all $i=1,\dots,m$ and $s=1,\dots,n$. We also drop the superscript m from $T^{(m)}$ for the rest of this case.

Fix $s \in \{1, \dots, n\}$. We apply the square function inequality (11) twice and monotonicity of $\|\cdot\|_{L_1}$ to (13), to obtain

$$\Big\| \sum_{i=1}^m T_{i,s} \cdot \mathbf{1}_{\{\max_{i'}|T_{i',s}|>C\}} \Big\|_{L_1} \geq \frac{1}{K} \Big\| \Big(\sum_{i=1}^m T_{i,s}^2 \cdot \mathbf{1}_{\{\max_{i'}|T_{i',s}|>C\}} \Big)^{1/2} \Big\|_{L_1}$$
$$\geq \frac{1}{K} \Big\| \Big(\sum_{i=1}^m T_{i,s}^2 \cdot \mathbf{1}_{\{|T_{i,s}|>C\}} \Big)^{1/2} \Big\|_{L_1}$$
$$\geq \frac{1}{K} \Big\| \sum_{i=1}^m T_{i,s} \cdot \mathbf{1}_{\{|T_{i,s}|>C\}} \Big\|_{L_1} \geq \frac{\varepsilon}{K} \ .$$

Now set $f_s = \sum_{i=1}^m T_{i,s}$, $E'_s = \big\{ \max_i |T_{i,s}| > C \big\}$ and $E_s = E'_s \cap \bigcap_{r \neq s} (E'_r)^{\complement}$. We have $\|f_s\|_{L_1} = \mathbb{E}\big| \sum_{i=1}^m T_{i,s} \big| \leq 1$ and $\|f_s \restriction_{E'_s}\|_{L_1} \geq \frac{\varepsilon}{K}$. By an inequality of Lévy (*c.f.* [11, Proposition 2.3]) and Markov's inequality we have

$$\mathbb{P}(E'_s) = \mathbb{P}\big(\max_i |T_{i,s}| > C \big) \leq 2 \cdot \mathbb{P}\big(\big| \textstyle\sum_{i=1}^m T_{i,s} \big| > C \big) \leq \dfrac{2}{C} \ .$$

Since the $T_{i,j}$ are independent, it follows that

$$\|f_s \restriction_{E_s}\|_{L_1} = \mathbb{E}\big| f_s \mathbf{1}_{E'_s} \cdot \mathbf{1}_{\bigcap_{r\neq s}(E'_r)^{\complement}} \big| = \mathbb{E}\big| f_s \mathbf{1}_{E'_s} \big| \cdot \mathbb{P}\Big(\bigcap_{r \neq s} (E'_r)^{\complement} \Big) \geq \frac{\varepsilon}{K} \cdot \Big(1 - \frac{2}{C} \Big)^{n-1} \geq \frac{\varepsilon}{2K} \ .$$

Thus we have proved that for all $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$, $f_1, \dots, f_n \in T^{(m)}\big(B_{\ell_\infty^m(\ell_1^m)} \big)$ and disjoint sets $E_1, \dots, E_n$ with $\|f_s \restriction_{E_s}\| \geq \frac{\varepsilon}{2K}$ for $s = 1, \dots, n$. By Proposition 2.5 the identity maps $\mathrm{Id}_{\ell_1^k}$ $(k \in \mathbb{N})$ uniformly factor through the $T^{(m)}$.

We now turn to case (ii'). Fix $\varepsilon > 0$ and choose the corresponding $C > 0$ and $n \in \mathbb{N}$. We will show that for every $m \in \mathbb{N}$ there exists $S^{(m)} \colon \ell_\infty^m(\ell_1^m) \to L_1$ such that $\|T^{(m)} - S^{(m)}\| < \varepsilon$, and moreover $\sup_m \pi_2(S^{(m)}) < \infty$. We can then complete the proof by applying Theorem 5.1 to deduce that the $S^{(m)}$ uniformly factor through ℓ_∞^k $(k \in \mathbb{N})$. Since ε was arbitrary, it follows that the $T^{(m)}$ uniformly approximately factor through ℓ_∞^k $(k \in \mathbb{N})$.

Fix $m \in \mathbb{N}$. If $m < n$, then we can take $S^{(m)} = T^{(m)}$. So assume $m \geq n$, put $T = T^{(m)}$, $\mathcal{F} = \mathcal{F}_m$, and let $j^{(s)} \in \mathcal{F}$ $(s = 1, \dots, n)$ be pairwise disjoint functions such that (14) holds for each $j \in \mathcal{F}$ that is disjoint from all the $j^{(s)}$. We may again assume for convenience of notation that $j^{(s)}$ is the constant function with value s for each $s = 1, \dots, n$. We now define

$$S = S^{(1)} + S^{(2)} \colon \ell_\infty^m(\ell_1^m) \to L_1$$

by letting, for each $i = 1, \dots, m$,

$$S_{i,j}^{(1)} = \begin{cases} T_{i,j} & \text{if } 1 \leq j \leq n \\ 0 & \text{if } n < j \leq m \end{cases}$$

$$S_{i,j}^{(2)} = \begin{cases} 0 & \text{if } 1 \leq j \leq n \\ T_{i,j} \cdot \mathbf{1}_{\big\{|T_{i,j}| \leq C\big\}} & \text{if } n < j \leq m \ . \end{cases}$$

We first check that $\|T - S\| < \varepsilon$. Here the suprema are taken over all $j \in \mathcal{F}$ and $\rho \in \{\pm 1\}^m$.

$$\|T - S\| = \sup_{j,\rho} \mathbb{E}\Big| \sum_{i=1}^m \rho_i (T - S)_{i,j_i} \Big|$$
$$= \sup_{j,\rho} \mathbb{E}\Big| \sum_{i \colon n < j_i} \rho_i T_{i,j_i} \cdot \mathbf{1}_{\big\{|T_{i,j_i}| > C\big\}} \Big| < \varepsilon \ .$$

The first line comes from looking at the extreme points of $B_{\ell_\infty^m(\ell_1^m)}$. The second line follows from the definition of S and (14) as well as the use of convexity and the symmetry of the $T_{i,j}$.

We next estimate $\pi_2(S)$ from above. First, $S^{(1)}$ clearly factors through $\ell_\infty^m(\ell_1^n)$ with constant 1. Since $\ell_\infty^m(\ell_1^n)$ is n-isomorphic to ℓ_∞^{mn}, it follows by Theorem 5.2 that $\pi_2(S^{(1)}) \leq K_G \cdot n$. Second, we can estimate $\pi_2(S^{(2)})$ as follows. First, by Lemma 5.4 we have

$$\pi_2^2(S^{(2)}) \leq \sum_{i=1}^m \max_{1\leq j\leq m} \big\|S_{i,j}^{(2)}\big\|_{L_2}^2 = \max_{j\in\mathcal{F}} \sum_{i=1}^m \big\|S_{i,j_i}^{(2)}\big\|_{L_2}^2 = \max_{j\in\mathcal{F}} \Big\| \sum_{i=1}^m S_{i,j_i}^{(2)} \Big\|_{L_2}^2 ,$$

where the last equality is the variance of a sum of independent, mean zero random variables. To continue, we need the following consequence of the Hoffman-Jørgensen inequality (*c.f.* [11, Proposition 6.10]). Here the notation $a \overset{\kappa}{\sim} b$ means that $a \leq \kappa \cdot b$ and $b \leq \kappa \cdot a$.

Theorem 5.5. *Given $0 < p, q < \infty$, there is a constant $K_{p,q}$ such that if $\mathcal{X}_1, \ldots, \mathcal{X}_N$ are independent, symmetric random variables in L_p then*

$$\Big\| \sum_{i=1}^N \mathcal{X}_i \Big\|_{L_p} \overset{K_{p,q}}{\sim} \Big\| \max_{1\leq i\leq N} |\mathcal{X}_i| \Big\|_{L_p} + \Big\| \sum_{i=1}^N \mathcal{X}_i \cdot \mathbf{1}_{\{|\mathcal{X}_i|\leq\delta_0\}} \Big\|_{L_q}$$

where $\delta_0 = \inf\Big\{ t > 0 : \sum_{i=1}^N \mathbb{P}\big(|\mathcal{X}_i| > t\big) \leq \frac{1}{8\cdot 3^p} \Big\}$.

We apply this theorem to the sequence $\big(S_{i,j_i}^{(2)}\big)_{i=1}^m$ (where $j \in \mathcal{F}$) with $p = 2,\ q = 1$ to obtain

$$\begin{aligned} K_{2,1}^{-1} \cdot \Big\| \sum_{i=1}^m S_{i,j_i}^{(2)} \Big\|_{L_2} &\leq \Big\| \max_{1\leq i\leq m} |S_{i,j_i}^{(2)}| \Big\|_{L_2} + \Big\| \sum_{i=1}^m S_{i,j_i}^{(2)} \cdot \mathbf{1}_{|S_{i,j_i}^{(2)}|\leq\delta_0} \Big\|_{L_1} \\ &\leq C + K \cdot \Big\| \sum_{i=1}^m T_{i,j_i} \Big\|_{L_1} \leq C + K \ . \end{aligned}$$

The second inequality follows by applying the square function inequality twice and monotonicity of expectation. Substituting this into the previous inequality, we obtain $\pi_2(S^{(2)}) \leq K_{2,1} \cdot (C + K)$.

We have thus shown that $\pi_2(S) \leq \pi_2(S^{(1)}) + \pi_2(S^{(2)}) \leq K_G \cdot n + K_{2,1} \cdot (C + K)$. This upper bound is independent of m, and so the proof is complete. □

References

[1] J. W. Calkin. Two-sided ideals and congruences in the ring of bounded operators in Hilbert space. *Ann. of Math. (2)*, 42:839–873, 1941.

[2] Matthew Daws. Closed ideals in the Banach algebra of operators on classical non-separable spaces. *Math. Proc. Cambridge Philos. Soc.*, 140(2):317–332, 2006.

[3] Joe Diestel, Hans Jarchow, and Andrew Tonge. *Absolutely summing operators*, volume 43 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[4] Leonard E. Dor. On projections in L_1. *Ann. of Math. (2)*, 102(3):463–474, 1975.

[5] I. C. Gohberg, A. S. Markus, and I. A. Fel′dman. Normally solvable operators and ideals associated with them. *Bul. Akad. Štiince RSS Moldoven.*, 1960(10 (76)):51–70, 1960.

[6] Bernhard Gramsch. Eine Idealstruktur Banachscher Operatoralgebren. *J. Reine Angew. Math.*, 225:97–115, 1967.

[7] G. J. O. Jameson. *Summing and nuclear norms in Banach space theory*, volume 8 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1987.

[8] W. B. Johnson and G. Schechtman. On subspaces of L_1 with maximal distances to Euclidean space. In *Proceedings of research workshop on Banach space theory (Iowa City, Iowa, 1981)*, pages 83–96, Iowa City, IA, 1982. Univ. Iowa.

[9] Niels Jakob Laustsen, Richard J. Loy, and Charles J. Read. The lattice of closed ideals in the Banach algebra of operators on certain Banach spaces. *J. Funct. Anal.*, 214(1):106–131, 2004.

[10] Niels Jakob Laustsen, Thomas Schlumprecht, and András Zsák. The lattice of closed ideals in the Banach algebra of operators on a certain dual Banach space. *J. Operator Theory*, 56(2):391–402, 2006.
[11] Michel Ledoux and Michel Talagrand. *Probability in Banach spaces*, volume 23 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991. Isoperimetry and processes.
[12] Erhard Luft. The two-sided closed ideals of the algebra of bounded linear operators of a Hilbert space. *Czechoslovak Math. J.*, 18 (93):595–605, 1968.
[13] Edward Odell and Th. Schlumprecht. Distortion and asymptotic structure. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1333–1360. North-Holland, Amsterdam, 2003.
[14] Haskell P. Rosenthal. On relatively disjoint families of measures, with some applications to Banach space theory. *Studia Math.*, 37:13–36, 1970.
[15] B. Sari, Th. Schlumprecht, N. Tomczak-Jaegermann, and V. G. Troitsky. On norm closed ideals in $L(l_p, l_q)$. *Studia Math.*, 179(3):239–262, 2007.

N. J. Laustsen and A. Zsák
Department of Mathematics and Statistics
Lancaster University
Lancaster
LA1 4YF, United Kingdom
email: n.laustsen@lancaster.ac.uk
a.zsak@dpmms.cam.ac.uk

E. Odell
Department of Mathematics,
The University of Texas,
1 University Station C1200,
Austin, TX 78712, USA
email: odell@math.utexas.edu

A. Zsák also at:
Peterhouse
Cambridge
CB2 1RD, United Kingdom

Th. Schlumprecht
Department of Mathematics,
Texas A&M University,
College Station, TX 78712, USA
email: schlump@math.tamu.edu