# The algebra of integro-differential operators on a polynomial algebra

V. V. Bavula

#### Abstract

We prove that the algebra  $\mathbb{I}_n := K\langle x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \int_1, \ldots, \int_n \rangle$  of integro-differential operators on a polynomial algebra is a prime, central, catenary, self-dual, non-Noetherian algebra of classical Krull dimension n and of Gelfand-Kirillov dimension 2n. Its weak homological dimension is n, and  $n \leq \operatorname{gldim}(\mathbb{I}_n) \leq 2n$ . All the ideals of  $\mathbb{I}_n$  are found explicitly, there are only finitely many of them  $(\leq 2^{2^n})$ , they commute  $(\mathfrak{ab} = \mathfrak{ba})$  and are idempotent ideals  $(\mathfrak{a}^2 = \mathfrak{a})$ . The number of ideals of  $\mathbb{I}_n$  is equal to the *Dedekind number*  $\mathfrak{d}_n$ . An analogue of Hilbert's Syzygy Theorem is proved for  $\mathbb{I}_n$ . The group of units of the algebra  $\mathbb{I}_n$  is described (it is a huge group). A canonical form is found for each integro-differential operators (by proving that the algebra  $\mathbb{I}_n$  is a generalized Weyl algebra). All the mentioned results hold for the Jacobian algebra  $\mathbb{A}_n$  (but GK ( $\mathbb{A}_n$ ) = 3n, note that  $\mathbb{I}_n \subset \mathbb{A}_n$ ). It is proved that the algebra  $\mathbb{I}_n$  are ideal equivalent.

Key Words: the algebra of integro-differential operators on a polynomial algebra, catenary algebra, the classical Krull dimension, the global dimension, the weak homological dimension, the Gelfand-Kirillov dimension, the Weyl algebras, the Jacobian algebras, the prime spectrum.

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#### 1 Introduction

Throughout, ring means an associative ring with 1; module means a left module;  $\mathbb{N} := \{0, 1, \ldots\}$  is the set of natural numbers; K is a field of characteristic zero and  $K^*$  is its group of units;  $P_n := K[x_1, \ldots, x_n]$  is a polynomial algebra over K;  $\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n}$  are the partial derivatives (K-linear derivations) of  $P_n$ ;  $\operatorname{End}_K(P_n)$  is the algebra of all K-linear maps from  $P_n$  to  $P_n$ ; the subalgebra  $A_n := K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$  of  $\operatorname{End}_K(P_n)$  is called the *n*'th Weyl algebra.

Definition, [11]: The Jacobian algebra  $\mathbb{A}_n$  is the subalgebra of  $\operatorname{End}_K(P_n)$  generated by the Weyl algebra  $A_n$  and the elements  $H_1^{-1}, \ldots, H_n^{-1} \in \operatorname{End}_K(P_n)$  where

$$H_1 := \partial_1 x_1, \dots, H_n := \partial_n x_n.$$

Clearly,  $\mathbb{A}_n = \bigotimes_{i=1}^n \mathbb{A}_1(i) \simeq \mathbb{A}_1^{\otimes n}$  where  $\mathbb{A}_1(i) := K\langle x_i, \partial_i, H_i^{-1} \rangle \simeq \mathbb{A}_1$ . The algebra  $\mathbb{A}_n$  contains all the integrations  $\int_i : P_n \to P_n, p \mapsto \int p \, dx_i$ , since

$$\int_i = x_i H_i^{-1} : x^{\alpha} \mapsto (\alpha_i + 1)^{-1} x_i x^{\alpha}$$

In particular, the algebra  $\mathbb{A}_n$  contains the algebra  $\mathbb{I}_n := K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, \int_1, \ldots, \int_n \rangle$ of polynomial integro-differential operators. Note that  $\mathbb{I}_n = \bigotimes_{i=1}^n \mathbb{I}_1(i) \simeq \mathbb{I}_1^{\otimes n}$  where  $\mathbb{I}_1(i) := K\langle x_i, \partial_i, \int_i \rangle$ .

The paper proceeds as follows. In Section 2, two sets of defining relations are given for the algebra  $\mathbb{I}_n$  (Proposition 2.2); a canonical form is found for each element of  $\mathbb{I}_n$  by showing that the algebra  $\mathbb{I}_n$  is a generalized Weyl algebra (Proposition 2.2.(2)); the Gelfand-Kirillov dimension of the algebra  $\mathbb{I}_n$  is 2n (Theorem 2.3).

In Section 3, a new equivalence relation, the *ideal equivalence*, on the class of algebras is introduced: two algebras A and B are *ideal equivalent* if there exists a bijection f from the set  $\mathcal{J}(A)$  of all the ideals of the algebra A to the set  $\mathcal{J}(B)$  of all the ideals of the algebra B such that, for all  $\mathfrak{a}, \mathfrak{b} \in \mathcal{J}(A)$ ,

$$f(\mathfrak{a} + \mathfrak{b}) = f(\mathfrak{a}) + f(\mathfrak{b}), \ f(\mathfrak{a} \cap \mathfrak{b}) = f(\mathfrak{a}) \cap f(\mathfrak{b}), \ f(\mathfrak{a}\mathfrak{b}) = f(\mathfrak{a})f(\mathfrak{b}).$$

The algebras  $\mathbb{I}_n$  and  $\mathbb{A}_n$  are ideal equivalent (Theorem 3.1). As a result, we have for free many results for the ideals of  $\mathbb{I}_n$  using similar known results for the ideals of  $\mathbb{A}_n$  of [11]. Name just a few:

- $\mathbb{I}_n$  is a prime, catenary algebra of classical Krull dimension n, and there is a unique maximal ideal  $\mathfrak{a}_n$  of the algebra  $\mathbb{I}_n$ .
- $\mathfrak{ab} = \mathfrak{ba}$  and  $\mathfrak{a}^2 = \mathfrak{a}$  for all  $\mathfrak{a}, \mathfrak{b} \in \mathcal{J}(\mathbb{I}_n)$ .
- The lattice  $\mathcal{J}(\mathbb{I}_n)$  is distributive.
- Classifications of all the ideals and the prime ideals of the algebra  $\mathbb{I}_n$  are given.
- The set  $\mathcal{J}(\mathbb{I}_n)$  is finite. Moreover,  $|\mathcal{J}(\mathbb{I}_n)| = \mathfrak{d}_n$  where  $\mathfrak{d}_n$  is the Dedekind number, and  $2 n + \sum_{i=1}^n 2^{\binom{n}{i}} \leq \mathfrak{d}_n \leq 2^{2^n}$ .
- $P_n$  is the only (up to isomorphism) faithful simple  $\mathbb{I}_n$ -module.

The fact that certain rings of differential operators are catenary was proved by Brown, Goodearl and Lenagan in [24].

In Section 4, it is proved that the factor algebra  $\mathbb{I}_n/\mathfrak{a}$  is Noetherian iff the ideal  $\mathfrak{a}$  is maximal (Proposition 4.1); and  $\mathrm{GK}(\mathbb{I}_n/\mathfrak{a}) = 2n$  for all the ideals  $\mathfrak{a}$  of  $\mathbb{I}_n$  distinct from  $\mathbb{I}_n$  (Lemma 4.2).

In Section 5, for the algebra  $\mathbb{I}_n$  an involution \* is introduced such that  $\partial_i^* = \int_i$ ,  $\int_i^* = \partial_i$ , and  $H_i^* = H_i$ , see (18). This means that the algebra  $\mathbb{I}_n$  is 'symmetrical' with respect to derivations and integrations.  $\mathfrak{a}^* = \mathfrak{a}$  for all ideals  $\mathfrak{a}$  of the algebra  $\mathbb{I}_n$  (Lemma 5.1.(1)). Each ideal of the algebra  $\mathbb{I}_n$  is an essential left and right submodule of  $\mathbb{I}_n$  (Lemma 5.2.(2)). The group  $\mathbb{I}_n^*$  of units of the algebra  $\mathbb{I}_n$  is described:

$$\mathbb{I}_n^* = K^* \times (1 + \mathfrak{a}_n)^* \supseteq K^* \times \underbrace{\operatorname{GL}_{\infty}(K) \ltimes \cdots \ltimes \operatorname{GL}_{\infty}(K)}_{2^n - 1 \text{ times}}$$

and its centre is  $K^*$  (Theorem 5.6). For n = 1, the group  $\mathbb{I}_n^*$  is found explicitly,  $\mathbb{I}_1^* \simeq K^* \times \operatorname{GL}_{\infty}(K)$  (Corollary 5.7). The centre of the algebra  $\mathbb{I}_n$  is K (Lemma 5.4.(2)). It is proved that, for a K-algebra A, the algebra  $A \otimes \mathbb{I}_n$  is prime iff the algebra A is prime (Corollary 5.3).

In Section 6, we prove that the weak (w.dim) dimension of the algebra  $\mathbb{I}_n$  is n (Theorem 6.2). Moreover, wdim $(\mathbb{I}_n/\mathfrak{p}) = n$  for all the prime ideals  $\mathfrak{p} \in \text{Spec}(\mathbb{I}_n)$  (Corollary 6.4). Recall that for each Noetherian ring its weak dimension coincides with its global dimension (in general, this is wrong for non-Noetherian rings). In 1972, Roos proved that the global dimension of the Weyl algebra  $A_n$  is n, [55]. This result was generalized by Chase [28] to the ring of differential operators on a smooth affine variety. Goodearl obtained formulae for the global dimension of certain rings of differential operators [36], [37] (see also Levasseur [48], and van den Bergh [61]). Holland and Stafford found the global dimension of the ring of differential operators on a rational projective curve [35] (see also Smith and Stafford [59]).

Many classical algebras are tensor product of algebras (eg,  $P_n = P_1^{\otimes n}$ ,  $A_n = A_1^{\otimes n}$ ,  $\mathbb{A}_n = \mathbb{A}_1^{\otimes n}$ ,  $\mathbb{I}_n = \mathbb{I}_1^{\otimes n}$ , etc). In general, it is difficult to find the dimension  $d(A \otimes B)$  of the tensor product of two algebras (even to answer the question of when it is finite). In [33], it was pointed out by Eilenberg, Rosenberg and Zelinsky that 'the questions concerning the dimension of the tensor product of two algebras have turned out to be surprisingly difficult.' An answer is known if one of the algebras is a polynomial algebra:

#### **Hilbert's Syzygy Theorem**: $d(P_n \otimes B) = d(P_n) + d(B) = n + d(B)$ ,

where d = wdim, gldim. In [8], [9], an analogue of Hilbert's Syzygy Theorem was established for certain generalized Weyl algebras A (eg,  $A = A_n$ , the Weyl algebra):

$$l.gldim(A \otimes B) = l.gldim(A) + l.gldim(B)$$

for all left Noetherian finitely generated algebras B (K is an algebraically closed uncountable field of characteristic zero). In this paper, a similar result is proved for the algebra  $\mathbb{I}_n$  and for all its prime factor algebras but for the weak dimension (Theorem 6.5). It is shown that  $n \leq \text{gldim}(\mathbb{I}_n) \leq 2n$ (Proposition 6.7).

In Section 7, we prove that the weak dimension of the Jacobian algebra  $\mathbb{A}_n$  is n (Theorem 7.2), and wdim $(\mathbb{A}_n/\mathfrak{p}) = n$  for all the prime ideals  $\mathfrak{p} \in \text{Spec}(\mathbb{A}_n)$  (Corollary 7.3). An analogue of Hilbert's Syzygy Theorem is proved for the Jacobian algebra  $\mathbb{A}_n$  and its prime factor algebras (Theorem 7.4). It is shown that  $n \leq \text{gldim}(\mathbb{A}_n) \leq 2n$  (Proposition 7.5).

The algebra  $\mathbb{I}_1 = A_1 \langle f \rangle$  is an example of the *Rota-Baxter* algebra. The latter appeared in the work of Baxter [21] and further explored by Rota [56, 57], Cartier [25], and Atkinson [4], and more recently by many others: Aguiar, Moreira [1]; Cassidy, Guo, Keigher, Sit, Ebrahimi-Fard [26], [32]; Connes, Kreimer, Marcoli [29], [30], name just a few. From the angle of the Rota-Baxter algebras the algebra  $\mathbb{I}_1$  was studied by Regensburger, Rosenkranz and Middeke [54].

### 2 Defining relations for the algebra $\mathbb{I}_n$

In this section defining relations are found for the algebra  $\mathbb{I}_n$  and it is proved that the algebra  $\mathbb{I}_n$  is a generalized Weyl algebra (Proposition 2.2) of Gelfand-Kirillov dimension 2n (Theorem 2.3) which is neither left nor right Noetherian (Lemma 2.4).

**Generalized Weyl Algebras.** Let *D* be a ring,  $\sigma = (\sigma_1, ..., \sigma_n)$  be an *n*-tuple of commuting ring endomorphisms of *D*, and  $a = (a_1, ..., a_n)$  be an *n*-tuple of elements of *D*. The generalized Weyl algebra  $A = D(\sigma, a)$  (briefly, GWA) of degree *n* is a ring generated by *D* and 2*n* elements  $x_1, ..., x_n, y_1, ..., y_n$  subject to the defining relations [6], [7]:

$$\begin{aligned} y_i x_i &= a_i, & x_i y_i = \sigma_i(a_i), \\ x_i d &= \sigma_i(d) x_i, & dy_i = y_i \sigma_i(d), \quad d \in D, \\ [x_i, x_j] &= [y_i, y_j] = [x_i, y_j] = 0, \quad i \neq j, \end{aligned}$$

where [x, y] = xy - yx. We say that a and  $\sigma$  are the sets of defining elements and endomorphisms of A respectively. For a vector  $k = (k_1, ..., k_n) \in \mathbb{Z}^n$ , let  $v_k = v_{k_1}(1) \cdots v_{k_n}(n)$  where, for  $1 \leq i \leq n$  and  $m \geq 0$ :  $v_m(i) = x_i^m$ ,  $v_{-m}(i) = y_i^m$ ,  $v_0(i) = 1$ . It follows from the definition that  $A = \bigoplus_{k \in \mathbb{Z}^n} A_k$  is a  $\mathbb{Z}^n$ -graded algebra  $(A_k A_e \subseteq A_{k+e}, \text{ for all } k, e \in \mathbb{Z}^n)$ , where  $A_k = v_{k,-}Dv_{k,+}$ ;  $v_{k,+} := \prod_{k_i>0} v_{k_i}(i)$  and  $v_{k,-} = \prod_{k_i<0} v_{k_i}(i)$ . The tensor product (over the ground field)  $A \otimes A'$ of generalized Weyl algebras of degree n and n' respectively is a GWA of degree n + n':

$$A \otimes A' = D \otimes D'((\tau, \tau'), (a, a')).$$

Let  $\mathcal{P}_n$  be a polynomial algebra  $K[H_1, \ldots, H_n]$  in *n* indeterminates and let  $\sigma = (\sigma_1, \ldots, \sigma_n)$  be the *n*-tuple of commuting automorphisms of  $\mathcal{P}_n$  such that  $\sigma_i(H_i) = H_i - 1$  and  $\sigma_i(H_j) = H_j$ , for  $i \neq j$ . The algebra homomorphism

$$A_n \to \mathcal{P}_n((\sigma_1, ..., \sigma_n), (H_1, \dots, H_n)), \quad x_i \mapsto x_i, \quad \partial_i \mapsto y_i, \quad i = 1, \dots, n,$$
(1)

is an isomorphism. We identify the Weyl algebra  $A_n$  with the GWA above via this isomorphism. Note that  $H_i = \partial_i x_i = x_i \partial_i + 1$ .

It is an experimental fact that many small quantum algebras/groups are GWAs. More about GWAs and their generalizations the interested reader can find in [2, 3, 12, 15, 16, 17, 18, 19, 20, 22, 27, 34, 40, 42, 43, 44, 45, 50, 51, 52, 53, 58, 60].

Suppose that A is a K-algebra that admits two elements x and y with yx = 1. The element  $xy \in A$  is an idempotent,  $(xy)^2 = xy$ , and so the set xyAxy is a K-algebra where xy is its identity element. Consider the linear maps  $\sigma = \sigma_{x,y}, \tau = \tau_{x,y} : A \to A$  which are defined as follows

$$\sigma(a) = xay, \ \tau(a) = yax. \tag{2}$$

Then  $\tau \sigma = \mathrm{id}_A$  and  $\sigma \tau(a) = xy \cdot a \cdot xy$ , and so the map  $\sigma$  is an algebra monomorphism with  $\sigma(1) = xy$  and

$$A = \sigma(A) \bigoplus \ker(\tau). \tag{3}$$

In more details,  $\sigma(A) \cap \ker(\tau) = 0$  since  $\tau\sigma = \operatorname{id}_A$ . Since  $(\sigma\tau)^2 = \sigma\tau$ , we have the equality  $A = \operatorname{im}(\sigma\tau) \bigoplus \operatorname{im}(1 - \sigma\tau)$ . Clearly,  $\operatorname{im}(\sigma\tau) \subseteq \operatorname{im}(\sigma)$  and  $\operatorname{im}(1 - \sigma\tau) \subseteq \ker(\tau)$  as  $\tau\sigma = \operatorname{id}_A$ . Then  $A = \operatorname{im}(\sigma) + \ker(\tau)$ , i.e. (3) holds. In general, the map  $\tau$  is not an algebra endomorphism and its kernel is not an ideal of the algebra A. Suppose that the algebra A contains a subalgebra D such that  $\sigma(D) \subseteq D$  (and so  $xy = \sigma(1) \in D$ ), and that the algebra A is generated by D, x, and y. Since yx = 1, we have  $x^i D_D \simeq D$  and  $_D Dy^i \simeq D$ . It follows from the relations:

$$yx = 1, xy = \sigma(1), xd = \sigma(d)x, dy = y\sigma(d), d \in D,$$

that  $A = \sum_{i \ge 1} y^i D + \sum_{i \ge 0} Dx^i$ . Suppose, in addition, that the sum is a *direct* one. Then the algebra A is the GWA  $D(\sigma, 1)$ .

**Lemma 2.1** [14] Keep the assumptions as above, i.e.  $A = D\langle x, y \rangle = \bigoplus_{i \ge 1} y^i D \bigoplus_{i \ge 0} Dx^i$  and  $\sigma(D) \subseteq D$ . Then  $A = D(\sigma, 1)$ . If, in addition,  $\tau(D) \subseteq D$  and the element xy is central in D. Then  $Dx^i = x^i D$  and  $Dy^i = y^i D$  for all  $i \ge 1$ .

Definition, [13]. The algebra  $\mathbb{S}_n$  of one-sided inverses of  $P_n$  is an algebra generated over a field K by 2n elements  $x_1, \ldots, x_n, y_n, \ldots, y_n$  that satisfy the defining relations:

$$y_1 x_1 = \dots = y_n x_n = 1$$
,  $[x_i, y_j] = [x_i, x_j] = [y_i, y_j] = 0$  for all  $i \neq j$ ,

where [a, b] := ab - ba is the algebra commutator of elements a and b.

By the very definition, the algebra  $\mathbb{S}_n \simeq \mathbb{S}_1^{\otimes n}$  is obtained from the polynomial algebra  $P_n$  by adding commuting, left (but not two-sided) inverses of its canonical generators. The algebra  $\mathbb{S}_1$ is a well-known primitive algebra [39], p. 35, Example 2. Over the field  $\mathbb{C}$  of complex numbers, the completion of the algebra  $\mathbb{S}_1$  is the *Toeplitz algebra* which is the  $\mathbb{C}^*$ -algebra generated by a unilateral shift on the Hilbert space  $l^2(\mathbb{N})$  (note that  $y_1 = x_1^*$ ). The Toeplitz algebra is the universal  $\mathbb{C}^*$ -algebra generated by a proper isometry.

The Jacobian algebra  $\mathbb{A}_n$  contains the algebra  $\mathbb{S}_n$  where

$$y_1 := H_1^{-1}\partial_1, \dots, y_n := H_n^{-1}\partial_n.$$

Moreover, the algebra  $\mathbb{A}_n$  is the subalgebra of  $\operatorname{End}_K(P_n)$  generated by the algebra  $\mathbb{S}_n$  and the 2n invertible elements  $H_1^{\pm 1}, \ldots, H_n^{\pm 1}$  of  $\operatorname{End}_K(P_n)$ .

The algebras  $\mathbb{S}_n$  and  $\mathbb{A}_n$  are much more better understood than the algebra  $\mathbb{I}_n$ . We will see that the three classes of algebras have much in common. In particular, they are GWAs. Moreover, we will deduce many results for the algebra  $\mathbb{I}_n$  from known results for the algebras  $\mathbb{S}_n$  and  $\mathbb{A}_n$  in [11], [13], [14].

The algebra  $\mathbb{S}_n$  is a GWA. Clearly,  $\mathbb{S}_n = \mathbb{S}_1(1) \otimes \cdots \otimes \mathbb{S}_1(n) \simeq \mathbb{S}_1^{\otimes n}$  where  $\mathbb{S}_1(i) := K\langle x_i, y_i | y_i x_i = 1 \rangle \simeq \mathbb{S}_1$  and  $\mathbb{S}_n = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} K x^{\alpha} y^{\beta}$  where  $x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ ,  $\alpha = (\alpha_1, \ldots, \alpha_n)$ ,  $y^{\beta} := y_1^{\beta_1} \cdots y_n^{\beta_n}$ ,  $\beta = (\beta_1, \ldots, \beta_n)$ . In particular, the algebra  $\mathbb{S}_n$  contains two polynomial subalgebras  $P_n$  and  $Q_n := K[y_1, \ldots, y_n]$  and is equal, as a vector space, to their tensor product  $P_n \otimes Q_n$ . Note that also the Weyl algebra  $A_n$  is a tensor product (as a vector space)  $P_n \otimes K[\partial_1, \ldots, \partial_n]$  of its two polynomial subalgebras.

When n = 1, we usually drop the subscript '1' if this does not lead to confusion (we do the same also for the algebras  $A_1$ ,  $A_1$  and  $\mathbb{I}_1$ ). So,  $\mathbb{S}_1 = K\langle x, y | yx = 1 \rangle = \bigoplus_{i,j \ge 0} Kx^i y^j$ . For each natural number  $d \ge 1$ , let  $M_d(K) := \bigoplus_{i,j=0}^{d-1} KE_{ij}$  be the algebra of d-dimensional matrices where  $\{E_{ij}\}$  are the matrix units, and

$$M_{\infty}(K) := \varinjlim M_d(K) = \bigoplus_{i,j \in \mathbb{N}} K E_{ij}$$

be the algebra (without 1) of infinite dimensional matrices. The algebra  $M_{\infty}(K) = \bigoplus_{k \in \mathbb{Z}} M_{\infty}(K)_k$ is  $\mathbb{Z}$ -graded where  $M_{\infty}(K)_k := \bigoplus_{i-j=k} KE_{ij} (M_{\infty}(K)_k M_{\infty}(K)_l \subseteq M_{\infty}(K)_{k+l}$  for all  $k, l \in \mathbb{Z}$ ). The algebra  $\mathbb{S}_1$  contains the ideal  $F := \bigoplus_{i,j \in \mathbb{N}} KE_{ij}$ , where

$$E_{ij} := x^i y^j - x^{i+1} y^{j+1}, \ i, j \ge 0.$$
(4)

Note that  $E_{ij} = x^i E_{00} y^j$  and  $E_{00} = 1 - xy$ . For all natural numbers i, j, k, and  $l, E_{ij} E_{kl} = \delta_{jk} E_{il}$ where  $\delta_{jk}$  is the Kronecker delta function. The ideal F is an algebra (without 1) isomorphic to the algebra  $M_{\infty}(K)$  via  $E_{ij} \mapsto E_{ij}$ . In particular, the algebra  $F = \bigoplus_{k \in \mathbb{Z}} F_{1,k}$  is  $\mathbb{Z}$ -graded where  $F_{1,k} := \bigoplus_{i-j=k} K E_{ij} (F_{1,k}F_{1,l} \subseteq F_{1,k+l} \text{ for all } k, l \in \mathbb{Z})$ . For all  $i, j \ge 0$ ,

$$xE_{ij} = E_{i+1,j}, \ yE_{ij} = E_{i-1,j}, \ E_{ij}x = E_{i,j-1}, \ E_{ij}y = E_{i,j+1},$$
 (5)

where  $E_{-1,j} := 0$  and  $E_{i,-1} := 0$ .

$$xE_{i,j} = E_{i+1,j+1}x, \ E_{ij}y = yE_{i+1,j+1}.$$
 (6)

$$\mathbb{S}_1 = K \oplus xK[x] \oplus yK[y] \oplus F,\tag{7}$$

the direct sum of vector spaces. Then

$$\mathbb{S}_1/F \simeq K[x, x^{-1}] =: L_1, \ x \mapsto x, \ y \mapsto x^{-1},$$
(8)

since yx = 1,  $xy = 1 - E_{00}$  and  $E_{00} \in F$ .

The algebra  $\mathbb{S}_n = \bigotimes_{i=1}^n \mathbb{S}_1(i)$  contains the ideal

$$F_n := F^{\otimes n} = \bigoplus_{\alpha,\beta \in \mathbb{N}^n} K E_{\alpha\beta}, \text{ where } E_{\alpha\beta} := \prod_{i=1}^n E_{\alpha_i\beta_i}(i), \ E_{\alpha_i\beta_i}(i) := x_i^{\alpha_i} y_i^{\beta_i} - x_i^{\alpha_i+1} y_i^{\beta_i+1}.$$

Note that  $E_{\alpha\beta}E_{\gamma\rho} = \delta_{\beta\gamma}E_{\alpha\rho}$  for all elements  $\alpha, \beta, \gamma, \rho \in \mathbb{N}^n$  where  $\delta_{\beta\gamma}$  is the Kronecker delta function.

Using Lemma 2.1, we can show that the algebra  $\mathbb{S}_1(i)$  is the GWA  $\mathbb{F}_{1,0}(i)(\sigma_i, 1)$  where  $\mathbb{F}_{1,0}(i) := K \bigoplus \bigoplus_{k \ge 0} KE_{kk}(i)$  and  $\sigma_i(a) = x_i a y_i$  (moreover,  $\sigma_i(1) = 1 - E_{00}(i)$  and  $\sigma_i(E_{kk}(i)) = E_{k+1,k+1}(i)$ ) Therefore,  $\mathbb{S}_n = \bigotimes_{i=1}^n \mathbb{F}_{1,0}(i)(\sigma_i, 1) = \mathbb{F}_{n,0}((\sigma_1, \ldots, \sigma_n), (1, \ldots, 1))$  is a GWA (Lemma 3.3, [14]) where  $\mathbb{F}_{n,0} := \bigotimes_{i=1}^{n} \mathbb{F}_{1,0}(i)$ . The algebra  $\mathbb{F}_{n,0}$  is a commutative, non-finitely generated, non-Noetherian algebra, it contains the direct sum  $\bigoplus_{\alpha \in \mathbb{N}^n} KE_{\alpha\alpha}$  of ideals, hence  $\mathbb{F}_{n,0}$  is not a prime algebra. The algebra  $\mathbb{S}_n = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathbb{S}_{n,\alpha}$  is a  $\mathbb{Z}^n$ -graded algebra where  $\mathbb{S}_{n,\alpha} = \mathbb{F}_{n,0}v_\alpha = v_\alpha \mathbb{F}_{n,0}$  for

all  $\alpha \in \mathbb{Z}^n$  where  $v_{\alpha} := \prod_{i=1}^n v_{\alpha_i}(i)$  and  $v_j(i) := \begin{cases} x_i^j & \text{if } j \ge 0, \\ y_i^{-j} & \text{if } j < 0. \end{cases}$ 

The map  $\tau_i : \mathbb{S}_n \to \mathbb{S}_n$ ,  $a \mapsto y_i a x_i$ , is not an algebra endomorphism but its restriction to the subalgebra  $\mathbb{F}_{n,0}$  of  $\mathbb{S}_n$  is a K-algebra epimorphism,  $\tau_i(\mathbb{F}_{n,0}) = \mathbb{F}_{n,0}$ , with  $\ker(\tau_i|_{\mathbb{F}_{n,0}}) = KE_{00}(i) \bigotimes \bigotimes_{j \neq i} \mathbb{F}_{1,0}(j)$ . For all  $j \in \mathbb{N}$  and  $d \in \mathbb{F}_{n,0}$ ,  $dx_i^j = x_i^j \tau_i^j(d)$  and  $y_i^j d = \tau_i^j(d) y_i^j$ .

The algebra  $\mathcal{I}_n := K\langle \partial_1, \ldots, \partial_n, \int_1, \ldots, \int_n \rangle$  of integro-differential operators with constant coefficients is canonically isomorphic to the algebra  $\mathbb{S}_n$ :

$$\mathbb{S}_n \to \mathcal{I}_n, \ x_i \mapsto \int_i, \ y_i \mapsto \partial_i, \ i = 1, \dots, n.$$
 (9)

For n = 1 this is obvious since the map above is a well-defined epimorphism (since  $\partial \int = 1$ ) which must be an isomorphism as the algebra  $\mathcal{I}_1$  is non-commutative but any proper factor algebra of  $\mathbb{S}_1$  is commutative [13]. Then the general case follows since  $\mathbb{S}_n \simeq \mathbb{S}_1^{\otimes n}$  and  $\mathcal{I}_n \simeq \mathcal{I}_1^{\otimes n}$ .

The Jacobian algebra  $\mathbb{A}_n$  is a GWA. The Jacobian algebra  $\mathbb{A}_n = \mathbb{D}_n((\sigma_1, \ldots, \sigma_n), (1, \ldots, 1))$ is a GWA (Lemma 3.4, [14]) where  $\mathbb{D}_n := \bigotimes_{i=1}^n \mathbb{D}_1(i)$ ,

$$\mathbb{D}_{1}(i) = \mathcal{L}_{1}^{-}(i) \bigoplus \mathcal{L}_{1}^{+}(i) \bigoplus F_{1,0}(i), \quad F_{1,0}(i) := \bigoplus_{s \ge 0} KE_{ss}(i),$$

$$\mathcal{L}_{1}^{-}(i) := \bigoplus_{s,t \ge 1} K \frac{1}{(H_{i} - s)_{1}^{t}}, \quad (H_{i} - s)_{1} := H_{i} - s + E_{s-1,s-1}(i),$$

$$\mathcal{L}_{1}^{+}(i) := K[H_{i}^{\pm 1}, (H_{i} + 1)^{-1}, (H_{i} + 2)^{-1}, \ldots],$$

and  $\sigma_i(a) = x_i a y_i$ . In more detail, for all natural numbers  $s \ge 0$  and  $t \ge 1$ ,

$$\begin{aligned} \sigma_i(E_{ss}(i)) &= E_{s+1,s+1}(i), \quad \sigma_i((H_i - t)_1) = (H_i - t - 1)_1 \sigma_i(1), \\ \sigma_i(H_i) &= H_i - 1 = (H_i - 1) \sigma_i(1) = (H_i - 1)_1 \sigma_i(1). \end{aligned}$$

The algebra  $\mathbb{D}_n$  is a commutative, non-finitely generated, non-Noetherian algebra, it contains the direct sum  $\bigoplus_{\alpha \in \mathbb{N}^n} KE_{\alpha\alpha}$  of ideals (and so  $\mathbb{D}_n$  is not a prime algebra). Note that  $H_i E_{ss}(i) = E_{ss}(i)H_i = (s+1)E_{ss}(i)$ . Clearly,  $\mathbb{A}_n = \bigotimes_{i=1}^n \mathbb{A}_1(i) = \bigotimes_{i=1}^n \mathbb{D}_1(i)(\sigma_i, 1)$ . The algebra  $\mathbb{A}_n = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathbb{A}_{n,\alpha}$  is a  $\mathbb{Z}^n$ -graded algebra where  $\mathbb{A}_{n,\alpha} = \mathbb{D}_n v_\alpha = v_\alpha \mathbb{D}_n$ ,  $v_\alpha := \prod_{i=1}^n v_{\alpha_i}(i)$  and

$$v_j(i) := \begin{cases} x_i^j & \text{if } j \ge 0, \\ y_i^{-j} & \text{if } j < 0. \end{cases}$$
(10)

The map  $\tau_i : \mathbb{A}_n \to \mathbb{A}_n$ ,  $a \mapsto y_i a x_i$ , is not an algebra endomorphism but its restriction to the subalgebra  $\mathbb{D}_n$  of  $\mathbb{A}_n$  is a K-algebra epimorphism,  $\tau_i(\mathbb{D}_n) = \mathbb{D}_n$ ,  $\ker(\tau_i|_{\mathbb{D}_n}) = KE_{00}(i) \bigotimes \bigotimes_{j \neq i} \mathbb{D}_1(j)$ . In more detail, for  $a, b \in \mathbb{D}_1$ ,

$$\tau(a)\tau(b) = ya(1 - E_{00})bx = \tau(ab) - yaE_{00}bx = \tau(ab),$$

since  $aE_{00}b \in KE_{00}$  and  $yE_{00} = 0$ . For all  $j \in \mathbb{N}$  and  $d \in \mathbb{D}_n$ ,  $dx_i^j = x_i^j \tau_i^j(d)$  and  $y_i^j d = \tau_i^j(d)y_i^j$ . Indeed, when n = 1,  $x\tau(d) = (1 - E_{00})dx = dx - E_{00}dx = dx$  since  $E_{00}d \in KE_{00}$  and  $E_{00}x = 0$ .

**The algebra**  $\mathbb{I}_n$  **is a GWA**. Since  $x_i = \int_i H_i$ , the algebra  $\mathbb{I}_n$  is generated by the elements  $\{\partial_i, H_i, \int_i | i = 1, ..., n\}$ , and  $\mathbb{I}_n = \bigoplus_{i=1}^n \mathbb{I}_1(i)$  where  $\mathbb{I}_1(i) := K \langle \partial_i, H_i, \int_i \rangle = K \langle \partial_i, x_i, \int_i \rangle \simeq \mathbb{I}_1$ . By (9), when n = 1 the following elements of the algebra  $\mathbb{I}_1 = K \langle \partial, H, f \rangle$ ,

$$e_{ij} := \int^{i} \partial^{j} - \int^{i+1} \partial^{j+1}, \ i, j \in \mathbb{N},$$
(11)

satisfy the relations:  $e_{ij}e_{kl} = \delta_{jk}e_{il}$ . Note that  $e_{ij} = \int^i e_{00}\partial^j$ . The matrices of the linear maps  $e_{ij} \in \operatorname{End}_K(K[x])$  with respect to the basis  $\{x^{[s]} := \frac{x^s}{s!}\}_{s \in \mathbb{N}}$  of the polynomial algebra K[x] are the elementary matrices, i.e.

$$e_{ij} * x^{[s]} = \begin{cases} x^{[i]} & \text{if } j = s, \\ 0 & \text{if } j \neq s. \end{cases}$$
$$e_{ij} = \frac{j!}{i!} E_{ij}, \tag{12}$$

It follows that

$$Ke_{ij} = KE_{ij}, \text{ and } F = \bigoplus_{i,j \ge 0} Ke_{ij} \simeq M_{\infty}(K). \text{ Moreover, } F_n = \bigoplus_{\alpha,\beta \in \mathbb{N}^n} Ke_{\alpha\beta} \text{ where } e_{\alpha\beta} := \prod_{i=1}^n e_{\alpha_i\beta_i}(i) \text{ and } e_{\alpha_i\beta_i}(i) := \int_i^{\alpha_i} \partial_i^{\beta_i} - \int_i^{\alpha_i+1} \partial_i^{\beta_i+1}.$$

The next proposition gives a finite set of defining relations for the algebra  $\mathbb{I}_n$  and shows that the algebra  $\mathbb{I}_n$  is a GWA (and so we have another set of defining relations for the algebra  $\mathbb{I}_n$ ).

**Proposition 2.2** 1. The algebra  $\mathbb{I}_n$  is generated by the elements  $\{\partial_i, \int_i, H_i | i = 1, ..., n\}$  that satisfy the defining relations:

$$\forall i : \partial_i \int_i = 1, \quad [H_i, \int_i] = \int_i, \quad [H_i, \partial_i] = -\partial_i, \quad H_i(1 - \int_i \partial_i) = (1 - \int_i \partial_i)H_i = 1 - \int_i \partial_i \partial_i H_i = 1 - \int_i \partial_i H_i = 1 - \int$$

2. The algebra  $\mathbb{I}_n = \bigotimes_{i=1}^n D_1(i)(\sigma_i, 1) = D_n((\sigma_1, \dots, \sigma_n), (1, \dots, 1))$  is a GWA  $(\int_i \leftrightarrow x_i, \partial_i \leftrightarrow y_i, H_i \leftrightarrow H_i)$  where  $D_n := \bigotimes_{i=1}^n D_1(i), D_1(i) := K[H_i] \bigoplus \bigoplus_{j \ge 0} Ke_{jj}(i), H_i e_{jj}(i) = e_{jj}(i)H_i = (j+1)e_{jj}(i)$ , and the K-algebra endomorphisms  $\sigma_i$  of  $D_n$  are given by the rule  $\sigma_i(a) := \int_i a\partial_i \ (\sigma_i(H_i) = H_i - 1, \ \sigma_i e_{jj}(i)) = e_{j+1,j+1}(i))$ . Moreover, the algebra  $\mathbb{I}_n = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathbb{I}_{n,\alpha}$  is  $\mathbb{Z}^n$ -graded where  $\mathbb{I}_{n,\alpha} = D_n v_\alpha = v_\alpha D_n$  for all  $\alpha \in \mathbb{Z}^n$  where  $v_\alpha := \prod_{i=1}^n v_{\alpha_i}(i)$ 

and 
$$v_j(i) := \begin{cases} \int_i^j & \text{if } j > 0, \\ 1 & \text{if } j = 0. \\ \partial_i^{-j} & \text{if } j < 0. \end{cases}$$

3. (The canonical basis for the algebra  $\mathbb{I}_n$ )  $\mathbb{I}_n = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathbb{I}_{n,\alpha}$  and, for all  $\alpha \in \mathbb{Z}^n$ ,  $\mathbb{I}_{n,\alpha} = v_{\alpha,+}D_n v_{\alpha,-} \simeq D_n$  ( $v_{\alpha,+}dv_{\alpha,-} \leftrightarrow d$ ) where  $v_{\alpha,+} := \prod_{\alpha_i>0} v_{\alpha_i}(i)$  and  $v_{\alpha,-} := \prod_{\alpha_i<0} v_{\alpha_i}(i)$ . So, each element  $a \in \mathbb{I}_n$  is a unique finite sum  $a = \sum_{\alpha \in \mathbb{Z}^n} v_{\alpha,+}a_\alpha v_{\alpha,-}$  for unique elements  $a_\alpha \in D_n$ .

Proof. It suffices to prove the statements for n = 1 since  $\mathbb{I}_n = \bigotimes_{i=1}^n \mathbb{I}_1(i)$ . So, let n = 1 and  $\mathbb{I}'_1$  be an algebra generated by symbols  $\partial$ ,  $\int$ , and H that satisfy the defining relations of statement 1. The algebra  $\mathbb{I}_1$  is generated by the elements  $\partial$ ,  $\int$ , and H; and they satisfy the defining relations of statement 1 as we can easily verify. Therefore, there is the natural algebra epimorphism  $\mathbb{I}'_1 \to \mathbb{I}_1$  given by the rule:  $\partial \mapsto \partial$ ,  $\int \mapsto \int$ ,  $H \mapsto H$ . It follows from the relations of statement 1 and from the equalities  $e_{ij} = \begin{cases} \int_{i=j}^{i-j} e_{jj} & \text{if } i \geq j, \\ e_{ii}\partial^{j-i} & \text{if } i < j, \end{cases}$  that

$$\mathbb{I}'_{1} = \sum_{i \ge 1} D'_{1} \partial^{i} + D'_{1} + \sum_{i \ge 1} \int^{i} D'_{1} = \sum_{i \ge 1} \partial^{i} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} = \sum_{i \ge 1} D'_{1} \partial^{i} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} = \sum_{i \ge 1} D'_{1} \partial^{i} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} = \sum_{i \ge 1} D'_{1} \partial^{i} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} = \sum_{i \ge 1} D'_{1} \partial^{i} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} = \sum_{i \ge 1} D'_{1} \partial^{i} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} \int^{i} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} + \sum_{i \ge 1} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{1} + D'_{1} + \sum_{i \ge 1} D'_{i} + \sum_{i \ge 1} D'_{i} + \sum_{i \ge 1} D'_{i} + \sum_{i$$

where  $D'_1 := K\langle H \rangle + \sum_{i \geq 0} Ke_{ii}$ . Since  $\partial \int = 1$ , the left  $D'_1$ -modules  $D'_1$  and  $D'_1 \partial^i$  are isomorphic, and the right  $D'_1$ -modules  $\int^i D'_1$  and  $D'_1$  are isomorphic. Using the  $\mathbb{Z}$ -grading of the Jacobian algebra  $\mathbb{A}_1$  and the fact that  $\mathbb{I}_1 \subseteq \mathbb{A}_1$ , we have

$$\mathbb{I}_1 = \bigoplus_{i \ge 1} D_1 \partial^i \bigoplus D_1 \bigoplus \bigoplus_{i \ge 1} \int^i D_1 = \bigoplus_{i \ge 1} \partial^i D_1 \bigoplus D_1 \bigoplus \bigoplus_{i \ge 1} D_1 \int^i$$

where  $D_1 = K[H] \bigoplus \bigoplus_{i \ge 0} Ke_{ii} = K[H] \bigoplus \bigoplus_{i \ge 0} KE_{ii}$  since  $Ke_{ii} = KE_{ii}$ . Note that the left  $D_1$ -modules  $D_1$  and  $D_1\partial^i$  are isomorphic and the right  $D_1$ -modules  $D_1$  and  $\int^i D_1$  are isomorphic since  $\partial \int = 1$ . This implies that the sum for  $\mathbb{I}'_1$  above is a direct one. Therefore,  $\mathbb{I}'_1 \simeq \mathbb{I}_1$  and the relations in statement 1 are defining relations for the algebra  $\mathbb{I}_1$  and  $D'_1 = D_1$ . The condition of Lemma 2.1 hold, and so  $\mathbb{I}_1 = D_1(\sigma, 1)$  with  $D_1 \int^i = \int^i D_1$  and  $D_1\partial^i = \partial^i D_1$  for all  $i \ge 1$ . The proof of statements 1 and 2 of the proposition is complete.

Statement 3 follows from statement 2 and the fact that, for all  $\alpha \in \mathbb{Z}^n$ , the linear map  $\mathbb{I}_{n,\alpha} \to D_n$ ,  $b \mapsto u_{\alpha,-}bu_{\alpha,+}$ , is a bijection since  $u_{\alpha,-}v_{\alpha,+} = 1$  and  $v_{\alpha,-}u_{\alpha,+} = 1$  where  $u_{\alpha,-} := \prod_{\alpha_i > 0} v_{-\alpha_i}(i)$  and  $u_{\alpha,+} := \prod_{\alpha_i < 0} v_{-\alpha_i}(i)$ .  $\Box$ 

Definition. For each element  $a \in \mathbb{I}_n$ , the unique sum for a in statement 3 of Proposition 2.2 is called the *canonical form* of a.

The map  $\tau_i : \mathbb{I}_n \to \mathbb{I}_n, a \mapsto \partial_i a \int_i$ , is not an algebra endomorphism but its restriction to the subalgebra  $D_n$  of  $\mathbb{I}_n$  is a K-algebra epimorphism,  $\tau_i(D_n) = D_n$  with  $\ker(\tau_i|_{D_n}) = Ke_{00}(i) \bigoplus \bigoplus_{j \neq i} D_1(j)$ . In more detail, for n = 1 and  $a, b \in D_1$ ,

$$\tau(a)\tau(b) = \partial a(1 - e_{00})b\int = \tau(ab) - \partial ae_{00}b\int = \tau(ab)$$

since  $ae_{00}b \in Ke_{00}$  and  $\partial e_{00} = 0$ . For all  $j \in \mathbb{N}$  and  $d \in D_n$ ,  $d\int_i^j = \int_i^j \tau_i^j(d)$  and  $\partial_i^j d = \tau_i^j(d)\partial_i^j$ . Indeed, for n = 1,  $\int \tau(d) = (1 - e_{00})d\int = d\int -e_{00}d\int = d\int$  since  $e_{00}d \in Ke_{00}$  and  $e_{00}\int = 0$ . Note that

$$\tau_i(H_j) = H_j + \delta_{ij} \text{ and } \tau_i(e_{st}(j)) = \begin{cases} e_{s-1,t-1}(i) & \text{if } i = j, \\ e_{st}(j) & \text{if } i \neq j. \end{cases}$$
(13)

It follows that  $\bigcap_{i=1}^{n} \ker(\tau_i|_{D_n}) = K \prod_{i=1}^{n} e_{00}(i) = K e_{00}$  and  $\bigcap_{i=1}^{n} \ker(\tau_i|_{D_n} - 1) = K$ .

For the definition and properties of the Gelfand-Kirillov dimension GK the reader is referred to [47] and [49].

**Theorem 2.3** The Gelfand-Kirillov dimension  $GK(\mathbb{I}_n)$  of the algebra  $\mathbb{I}_n$  is 2n.

Proof. Since  $A_n \subseteq \mathbb{I}_n$ , we have the inequality  $2n = \operatorname{GK}(A_n) \leq \operatorname{GK}(\mathbb{I}_n)$ . To prove the reverse inequality let us consider the standard filtration  $\{\mathbb{I}_{n,i}\}_{i\in\mathbb{N}}$  of the algebra  $\mathbb{I}_n$  with respect to the set of generators  $\{\partial_i, H_i, \int_i | i = 1, \ldots, n\}$  of the algebra  $\mathbb{I}_n$ . By Proposition 2.3,  $\mathbb{I}_{n,i} \subseteq \mathbb{I}'_{n,i} := \bigoplus_{|\alpha| \leq i} v_{\alpha} D_{n,i}$  where  $D_{n,i} := \bigotimes_{j=1}^n D_{1,i}(j)$  and  $D_{1,i}(j) := \bigoplus_{s=0}^i KH_j^s \bigoplus_{t=0}^i Ke_{tt}(j)$ . Then  $\dim(\mathbb{I}_{n,i}) \leq \dim(\mathbb{I}'_{n,i}) \leq (2i+1)^n (2i+2)^n$ , and so  $\operatorname{GK}(\mathbb{I}_n) \leq 2n$ , as required.  $\Box$ 

**Lemma 2.4** The algebra  $\mathbb{I}_n$  is neither left nor right Noetherian. Moreover, it contains infinite direct sums of nonzero left (resp. right) ideals.

*Proof.* Since  $\mathbb{I}_n \simeq \mathbb{I}_1^{\otimes n}$ , it suffices to prove the lemma for n = 1. The ideal  $F = \bigoplus_{i,j\geq 0} KE_{ij}$  of the algebra  $\mathbb{I}_1$  is the infinite direct sum  $\bigoplus_{j\geq 0} (\bigoplus_{i\geq 0} KE_{ij})$  (resp.  $\bigoplus_{i\geq 0} (\bigoplus_{j\geq 0} KE_{ij})$ ) of nonzero left (resp. right) ideals, and the statements follow.  $\Box$ 

#### 3 Ideals of the algebra $\mathbb{I}_n$

In this section, we prove that the restriction map (Theorem 3.1) from the set of ideals  $\mathcal{J}(\mathbb{A}_n)$  of the algebra  $\mathbb{A}_n$  to the set of ideals  $\mathcal{J}(\mathbb{I}_n)$  of the algebra  $\mathbb{I}_n$  is a bijection that respects the three operations on ideals: sum, intersection and product. As a consequence, we obtain many results for the ideals of the algebra  $\mathbb{I}_n$  using similar results for the ideals of the algebra  $\mathbb{A}_n$  in [11], see Corollary 3.3 and Corollary 3.4: a classification of all the ideals of  $\mathbb{I}_n$  (there are only finitely many of them) and a classification of prime ideals of  $\mathbb{I}_n$ , etc. Definition. Let A and B be algebras, and let  $\mathcal{J}(A)$  and  $\mathcal{J}(B)$  be their lattices of ideals. We say that a bijection  $f : \mathcal{J}(A) \to \mathcal{J}(B)$  is an *isomorphism* if  $f(\mathfrak{a} * \mathfrak{b}) = f(\mathfrak{a}) * f(\mathfrak{b})$  for  $* \in \{+, \cdot, \cap\}$ , and in this case we say that the algebras A and B are *ideal equivalent*. The ideal equivalence is an equivalence relation on the class of algebras.

The next theorem shows that the algebras  $\mathbb{A}_n$  and  $\mathbb{I}_n$  are ideal equivalent.

**Theorem 3.1** The restriction map  $\mathcal{J}(\mathbb{A}_n) \to \mathcal{J}(\mathbb{I}_n)$ ,  $\mathfrak{a} \mapsto \mathfrak{a}^r := \mathfrak{a} \cap \mathbb{I}_n$ , is an isomorphism (i.e.  $(\mathfrak{a}_1 * \mathfrak{a}_2)^r = \mathfrak{a}_1^r * \mathfrak{a}_2^r$  for  $* \in \{+, \cdot, \cap\}$ ) and its inverse is the extension map  $\mathfrak{b} \mapsto \mathfrak{b}^e := \mathbb{A}_n \mathfrak{b} \mathbb{A}_n$ .

*Proof.* The theorem follows from Theorem 3.2.  $\Box$ 

Recall that  $\mathbb{F}_{n,0} \subset \mathbb{I}_n \subset \mathbb{A}_n \subset \operatorname{End}_K(P_n)$ . The subset of  $\mathcal{J}(\mathbb{F}_{n,0})$ ,  $\mathcal{J}(\mathbb{F}_{n,0})_{\sigma,\tau} := \{ \mathfrak{b} \in \mathcal{J}(\mathbb{F}_{n,0}) \mid \sigma_i(\mathfrak{b}) \subseteq \mathfrak{b}, \tau_i(\mathfrak{b}) \subseteq \mathfrak{b}$  for all  $i = 1, \ldots, n \}$ , is closed under addition, multiplication and intersection of ideals where  $\sigma_i(a) = \int_i a\partial_i$  and  $\tau_i(a) = \partial_i a \int_i$  (recall that the maps  $\sigma_i, \tau_i : \mathbb{F}_{n,0} \to \mathbb{F}_{n,0}$  are K-algebra homomorphisms;  $\tau_i(1) = 1$  but  $\sigma_i(1) = 1 - e_{00}(i)$ ).

- **Theorem 3.2** 1. The restriction map  $\mathcal{J}(\mathbb{I}_n) \to \mathcal{J}(\mathbb{F}_{n,0})_{\sigma,\tau}$ ,  $\mathfrak{a} \mapsto \mathfrak{a}^r := \mathfrak{a} \cap \mathbb{F}_{n,0}$ , is an isomorphism (i.e.  $(\mathfrak{a}_1 * \mathfrak{a}_2)^r = \mathfrak{a}_1^r * \mathfrak{a}_2^r$  for  $* \in \{+, \cdot, \cap\}$ ) and its inverse is the extension map  $\mathfrak{b} \mapsto \mathfrak{b}^e := \mathbb{I}_n \mathfrak{b} \mathbb{I}_n$ .
  - 2. The restriction map  $\mathcal{J}(\mathbb{A}_n) \to \mathcal{J}(\mathbb{F}_{n,0})_{\sigma,\tau}$ ,  $\mathfrak{a} \mapsto \mathfrak{a}^r := \mathfrak{a} \cap \mathbb{F}_{n,0}$ , is an isomorphism (i.e.  $(\mathfrak{a}_1 * \mathfrak{a}_2)^r = \mathfrak{a}_1^r * \mathfrak{a}_2^r$  for  $* \in \{+, \cdot, \cap\}$ ) and its inverse is the extension map  $\mathfrak{b} \mapsto \mathfrak{b}^e := \mathbb{A}_n \mathfrak{b} \mathbb{A}_n$ .

The proof of Theorem 3.2 is given at the end of this section. Now, we obtain some consequences of Theorem 3.1.

The next corollary shows that the ideal theory of  $\mathbb{I}_n$  is 'very arithmetic.' Let  $\mathcal{B}_n$  be the set of all functions  $f : \{1, 2, \ldots, n\} \to \{0, 1\}$ . For each function  $f \in \mathcal{B}_n$ ,  $I_f := I_{f(1)} \otimes \cdots \otimes I_{f(n)}$  is the ideal of  $\mathbb{I}_n$  where  $I_0 := F$  and  $I_1 := \mathbb{I}_1$ . Let  $\mathcal{C}_n$  be the set of all subsets of  $\mathcal{B}_n$  all distinct elements of which are incomparable (two distinct elements f and g of  $\mathcal{B}_n$  are *incomparable* if neither  $f(i) \leq g(i)$ nor  $f(i) \geq g(i)$  for all i). For each  $C \in \mathcal{C}_n$ , let  $I_C := \sum_{f \in C} I_f$ , the ideal of  $\mathbb{I}_n$ . The number  $\mathfrak{d}_n$  of elements in the set  $\mathcal{C}_n$  is called the *Dedekind number*. It appeared in the paper of Dedekind [31]. An asymptotic of the Dedekind numbers was found by Korshunov [46].

**Corollary 3.3** 1. The algebra  $\mathbb{I}_n$  is a prime algebra.

- 2. The set of height one prime ideals of the algebra  $\mathbb{I}_n$  is  $\{\mathfrak{p}_1 := F \otimes \mathbb{I}_{n-1}, \mathfrak{p}_1 := \mathbb{I}_1 \otimes F \otimes \mathbb{I}_{n-2}, \dots, \mathfrak{p}_n := \mathbb{I}_{n-1} \otimes F\}.$
- 3. Each ideal of the algebra  $\mathbb{I}_n$  is an idempotent ideal  $(\mathfrak{a}^2 = \mathfrak{a})$ .
- 4. The ideals of the algebra  $\mathbb{I}_n$  commute  $(\mathfrak{ab} = \mathfrak{ba})$ .
- 5. The lattice  $\mathcal{J}(\mathbb{I}_n)$  of ideals of the algebra  $\mathbb{I}_n$  is distributive.
- 6. The classical Krull dimension  $\operatorname{cl.Kdim}(\mathbb{I}_n)$  of the algebra  $\mathbb{I}_n$  is n.
- 7.  $\mathfrak{ab} = \mathfrak{a} \cap \mathfrak{b}$  for all ideals  $\mathfrak{a}$  and  $\mathfrak{b}$  of the algebra  $\mathbb{I}_n$ .
- 8. The ideal  $\mathfrak{a}_n := \mathfrak{p}_1 + \cdots + \mathfrak{p}_n$  is the largest (hence, the only maximal) ideal of  $\mathbb{I}_n$  distinct from  $\mathbb{I}_n$ , and  $F_n = F^{\otimes n} = \bigcap_{i=1}^n \mathfrak{p}_i$  is the smallest nonzero ideal of  $\mathbb{I}_n$ .
- 9. (A classification of ideals of  $\mathbb{I}_n$ ) The map  $\mathcal{C}_n \to \mathcal{J}(\mathbb{I}_n)$ ,  $C \mapsto I_C := \sum_{f \in C} I_f$  is a bijection where  $I_{\emptyset} := 0$ . The number of ideals of  $\mathbb{I}_n$  is the Dedekind number  $\mathfrak{d}_n$ . Moreover,  $2 n + \sum_{i=1}^n 2^{\binom{n}{i}} \leq \mathfrak{d}_n \leq 2^{2^n}$ . For n = 1, F is the unique proper ideal of the algebra  $\mathbb{I}_1$ .
- 10. (A classification of prime ideals of  $\mathbb{I}_n$ ) Let  $\operatorname{Sub}_n$  be the set of all subsets of  $\{1, \ldots, n\}$ . The map  $\operatorname{Sub}_n \to \operatorname{Spec}(\mathbb{I}_n)$ ,  $I \mapsto \mathfrak{p}_I := \sum_{i \in I} \mathfrak{p}_i$ ,  $\emptyset \mapsto 0$ , is a bijection, i.e. any nonzero prime ideal of  $\mathbb{I}_n$  is a unique sum of primes of height 1;  $|\operatorname{Spec}(\mathbb{I}_n)| = 2^n$ ; the height of  $\mathfrak{p}_I$  is |I|; and

11.  $\mathfrak{p}_I \subset \mathfrak{p}_J$  iff  $I \subset J$ .

*Proof.* By Theorem 3.1, the statements hold for the algebra  $\mathbb{I}_n$  since the same statements hold for the algebra  $\mathbb{A}_n$ , and below references are given for their proofs in [11].

1. Corollary 2.7.(5).

2. Corollary 3.5.

- 3. Theorem 3.1.(2).
- 4. Corollary 3.10.(3).
- 5. Theorem 3.11.
- 6. Corollary 3.7.
- 7. Corollary 3.10.(3).
- 8. Corollary 2.7.(4,7).
- 9. Theorem 3.1.
- 10. Corollary 3.5.
- 11. Corollary 3.6.  $\Box$

For each ideal  $\mathfrak{a}$  of  $\mathbb{I}_n$ ,  $\operatorname{Min}(\mathfrak{a})$  denotes the set of minimal primes over  $\mathfrak{a}$ . Two distinct prime ideals  $\mathfrak{p}$  and  $\mathfrak{q}$  are called *incomparable* if neither  $\mathfrak{p} \subseteq \mathfrak{q}$  nor  $\mathfrak{p} \supseteq \mathfrak{q}$ . The algebras  $\mathbb{I}_n$  have beautiful ideal theory as the following unique factorization properties demonstrate.

- **Corollary 3.4** 1. Each ideal  $\mathfrak{a}$  of  $\mathbb{I}_n$  such that  $\mathfrak{a} \neq \mathbb{I}_n$  is a unique product of incomparable primes, i.e. if  $\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_s = \mathfrak{r}_1 \cdots \mathfrak{r}_t$  are two such products then s = t and  $\mathfrak{q}_1 = \mathfrak{r}_{\sigma(1)}, \ldots, \mathfrak{q}_s = \mathfrak{r}_{\sigma(s)}$  for a permutation  $\sigma$  of  $\{1, \ldots, n\}$ .
  - 2. Each ideal  $\mathfrak{a}$  of  $\mathbb{I}_n$  such that  $\mathfrak{a} \neq \mathbb{I}_n$  is a unique intersection of incomparable primes, i.e. if  $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s = \mathfrak{r}_1 \cap \cdots \cap \mathfrak{r}_t$  are two such intersections then s = t and  $\mathfrak{q}_1 = \mathfrak{r}_{\sigma(1)}, \ldots, \mathfrak{q}_s = \mathfrak{r}_{\sigma(s)}$  for a permutation  $\sigma$  of  $\{1, \ldots, n\}$ .
  - 3. For each ideal  $\mathfrak{a}$  of  $\mathbb{I}_n$  such that  $\mathfrak{a} \neq \mathbb{I}_n$ , the sets of incomparable primes in statements 1 and 2 are the same, and so  $\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_s = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ .
  - 4. The ideals  $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$  in statement 3 are the minimal primes of  $\mathfrak{a}$ , and so  $\mathfrak{a} = \prod_{\mathfrak{p} \in \operatorname{Min}(\mathfrak{a})} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \operatorname{Min}(\mathfrak{a})} \mathfrak{p}$ .

*Proof.* The same statements are true for the algebra  $\mathbb{A}_n$  (Theorem 3.8, [11]). Now, the corollary follows from Theorem 3.1.  $\Box$ 

The next corollary gives all decompositions of an ideal as a product or intersection of ideals.

**Corollary 3.5** Let  $\mathfrak{a}$  be an ideal of  $\mathbb{I}_n$ , and  $\mathcal{M}$  be the minimal elements with respect to inclusion of the set of minimal primes of a set of ideals  $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$  of  $\mathbb{I}_n$ . Then

- 1.  $\mathfrak{a} = \mathfrak{a}_1 \cdots \mathfrak{a}_k$  iff  $\operatorname{Min}(\mathfrak{a}) = \mathcal{M}$ .
- 2.  $\mathfrak{a} = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_k$  iff  $Min(\mathfrak{a}) = \mathcal{M}$ .

*Proof.* The same statements are true for the algebra  $\mathbb{A}_n$  (Theorem 3.12, [11]), and the corollary follows from Theorem 3.1.  $\Box$ 

This is a rare example of a noncommutative algebra of classical Krull dimension > 1 where one has a complete picture of decompositions of ideals. Recall that a ring R of finite classical Krull dimension is called *catenary* if, for each pair of prime ideals  $\mathfrak{p}$  and  $\mathfrak{q}$  with  $\mathfrak{p} \subseteq \mathfrak{q}$ , all maximal chains of prime ideals,  $\mathfrak{p}_0 = \mathfrak{p} \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_l = \mathfrak{q}$ , have the same length l.

**Corollary 3.6** The algebra  $\mathbb{I}_n$  is catenary.

*Proof.* This follows from Corollary 3.3.(10, 11).

**Corollary 3.7** The same statements (with obvious modifications) of Corollaries 3.3 and 3.4 hold for the ideals  $\mathcal{J}(\mathbb{F}_{n,0})_{\sigma,\tau}$  of the algebra  $\mathbb{F}_{n,0}$  rather than  $\mathbb{I}_n$  (we leave it to the reader to formulate them).

**Proposition 3.8** The polynomial algebra  $P_n$  is the only (up to isomorphism) faithful simple  $\mathbb{I}_n$ module.

Proof. The  $\mathbb{I}_n$ -module  $P_n$  is faithful (as  $\mathbb{I}_n \subset \operatorname{End}_K(P_n)$ ) and simple since the  $A_n$ -module  $P_n$  is simple and  $A_n \subset \mathbb{I}_n$ . Let M be a faithful simple  $\mathbb{I}_n$ -module. Then  $F_nM \neq 0$ , i.e.  $e_{0\beta}m \neq 0$  for some elements  $\beta \in \mathbb{N}^n$  and  $m \in M$ . The  $\mathbb{I}_n$ -module  $P_n \simeq \mathbb{I}_n / \sum_{i=1}^n \mathbb{I}_n \partial_i$  is simple. Therefore, the  $\mathbb{I}_n$ -module epimorphism  $P_n \to M = \mathbb{I}_n e_{0\beta}m = \sum_{\alpha \in \mathbb{N}^n} K e_{\alpha\beta}m, 1 \mapsto e_{0\beta}m$ , is an isomorphism. The proof of the proposition is complete.  $\Box$ 

For a ring R and its element  $r \in R$ ,  $ad(r) : s \mapsto [r, s] := rs - sr$  is the *inner derivation* of the ring R associated with the element r.

The proof of Theorem 3.2. We split the proof of Theorem 3.2 into several statements (which are interesting on their own) to make the proof clearer.

The algebra  $D_n$  is the zero component of the  $\mathbb{Z}^n$ -graded algebra  $\mathbb{I}_n = D_n((\sigma_1, \ldots, \sigma_n), (1, \ldots, 1))$ , hence  $\sigma_i(D_n) \subseteq D_n$  and  $\tau_i(D_n) \subseteq D_n$  for all i where  $\sigma_i(a) = \int_i a\partial_i$  and  $\tau_i(a) = \partial_i a \int_i$ . Let  $\mathcal{J}(D_n)_{\sigma,\tau} := \{ \mathfrak{b} \in \mathcal{J}(D_n) | \sigma_i(\mathfrak{b}) \subseteq \mathfrak{b}, \tau_i(\mathfrak{b}) \subseteq \mathfrak{b}$  for all  $i = 1, \ldots, n \}$ . Similarly, the algebra  $\mathbb{D}_n$  is the zero component of the  $\mathbb{Z}^n$ -graded algebra  $\mathbb{A}_n = \mathbb{D}_n((\sigma_1, \ldots, \sigma_n), (1, \ldots, 1))$ , hence  $\sigma_i(\mathbb{D}_n) \subseteq \mathbb{D}_n$ and  $\tau_i(\mathbb{D}_n) \subseteq \mathbb{D}_n$  for all i where  $\sigma_i(a) = x_i a y_i$  and  $\tau_i(a) = y_i a x_i$ . Let  $\mathcal{J}(\mathbb{D}_n)_{\sigma,\tau} := \{\mathfrak{b} \in \mathcal{J}(\mathbb{D}_n) | \sigma_i(\mathfrak{b}) \subseteq \mathfrak{b}, \tau_i(\mathfrak{b}) \subseteq \mathfrak{b}$  for all  $i = 1, \ldots, n \}$ .

- **Theorem 3.9** 1. For each ideal  $\mathfrak{a}$  of the algebra  $\mathbb{I}_n$ ,  $\mathfrak{a} = \bigoplus_{\alpha \in \mathbb{Z}^n} v_{\alpha,+} \mathfrak{a}^r v_{\alpha,-}$  where  $\mathfrak{a}^r := \mathfrak{a} \cap D_n \in \mathcal{J}(D_n)_{\sigma,\tau}$  and, for each ideal  $\mathfrak{b} \in \mathcal{J}(D_n)_{\sigma,\tau}$ ,  $\mathfrak{b}^e := \mathbb{I}_n \mathfrak{b} \mathbb{I}_n = \bigoplus_{\alpha \in \mathbb{Z}^n} v_{\alpha,+} \mathfrak{b} v_{\alpha,-}$  where  $v_{\alpha,+} := \prod_{\alpha_i > 0} v_{\alpha_i}(i)$ ,  $v_{\alpha,-} := \prod_{\alpha_i < 0} v_{\alpha_i}(i)$ , and  $v_j(i)$  is defined in Proposition 2.2.(2).
  - 2. For each ideal  $\mathfrak{a}$  of the algebra  $\mathbb{A}_n$ ,  $\mathfrak{a} = \bigoplus_{\alpha \in \mathbb{Z}^n} v_{\alpha,+} \mathfrak{a}^r v_{\alpha,-}$  where  $\mathfrak{a}^r := \mathfrak{a} \cap \mathbb{D}_n \in \mathcal{J}(\mathbb{D}_n)_{\sigma,\tau}$ and, for each ideal  $\mathfrak{b} \in \mathcal{J}(\mathbb{D}_n)_{\sigma,\tau}$ ,  $\mathfrak{b}^e := \mathbb{A}_n \mathfrak{b} \mathbb{A}_n = \bigoplus_{\alpha \in \mathbb{Z}^n} v_{\alpha,+} \mathfrak{b} v_{\alpha,-}$  where  $v_{\alpha,\pm}$  are as above but the elements  $v_i(i)$  are defined in (10).

Proof. 1. Let  $\mathfrak{a}$  be an ideal of the algebra  $\mathbb{I}_n$ . The algebra  $\mathbb{I}_n = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathbb{I}_{n,\alpha}$  is a  $\mathbb{Z}^n$ -graded algebra with  $\mathbb{I}_{n,\alpha} := \bigcap_{i=1}^n \ker(\operatorname{ad}(H_i) - \alpha_i)$  for all  $\alpha \in \mathbb{Z}^n$ . Then  $\mathfrak{a}$  is a homogeneous ideal, that is  $\mathfrak{a} = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathfrak{a}_\alpha$  where  $\mathfrak{a}_\alpha := \mathfrak{a} \cap \mathbb{I}_{n,\alpha}$ . The ideal  $\mathfrak{a}_0 := \mathfrak{a} \cap D_n = \mathfrak{a}^r$  of the algebra  $D_n$  belongs to the set  $\mathcal{J}(D_n)_{\sigma,\tau}$  since  $\sigma_i(\mathfrak{a}_0) = \int_i \mathfrak{a}_0 \partial_i \subseteq \mathfrak{a}_0$  and  $\tau_i(\mathfrak{a}_0) = \partial_i \mathfrak{a}_0 \int_i \subseteq \mathfrak{a}_0$  for all  $i = 1, \ldots, n$ . By Proposition 2.2.(2),  $\mathfrak{a}_\alpha = v_{\alpha, +} \mathfrak{b}_\alpha v_{\alpha, -}$  for some ideal  $\mathfrak{b}_\alpha$  of the algebra  $D_n$ :  $\mathfrak{a}_\alpha = D_n \mathfrak{a}_\alpha D_n = D_n v_{\alpha, +} \mathfrak{b}_\alpha v_{\alpha, -} D_n = v_{\alpha, +} \tau_{\alpha, +} (D_n) \mathfrak{b}_\alpha \tau_{\alpha, -} (D_n) v_{\alpha, -} = v_{\alpha, +} D_n \mathfrak{b}_\alpha D_n v_{\alpha, -}$  since  $\tau_{\alpha, \pm}(D_n) = D_n$  where  $\tau_{\alpha, +} := \prod_{\alpha_i > 0} \tau_i$  and  $\tau_{\alpha, -} := \prod_{\alpha_i < 0} \tau_i$ . Let

$$u_{\alpha,-} := \prod_{\alpha_i > 0} v_{-\alpha_i}(i), \ \ u_{\alpha,+} := \prod_{\alpha_i < 0} v_{-\alpha_i}(i).$$
(14)

The ideal  $\mathfrak{b}_{\alpha}$  is unique since  $v_{\alpha,+}\mathfrak{b}_{\alpha}v_{\alpha,-} = v_{\alpha,+}\mathfrak{b}'_{\alpha}v_{\alpha,-}$  implies

$$\mathfrak{b}_{\alpha} = 1 \cdot \mathfrak{b}_{\alpha} \cdot 1 = u_{\alpha,-} v_{\alpha,+} \mathfrak{b}_{\alpha} v_{\alpha,-} u_{\alpha,+} = u_{\alpha,-} v_{\alpha,+} \mathfrak{b}'_{\alpha} v_{\alpha,-} u_{\alpha,+} = 1 \cdot \mathfrak{b}'_{\alpha} \cdot 1 = \mathfrak{b}'_{\alpha}.$$

Moreover,  $\mathfrak{b}_{\alpha} = \mathfrak{a}_{0}$  for all  $\alpha \in \mathbb{Z}^{n}$  since  $\mathfrak{a}_{0} \supseteq u_{\alpha,-}\mathfrak{a}_{n,\alpha}u_{\alpha,+} = u_{\alpha,-}v_{\alpha,+}\mathfrak{b}_{\alpha}v_{\alpha,-}u_{\alpha,+} = 1 \cdot \mathfrak{b}_{\alpha} \cdot 1 = \mathfrak{b}_{\alpha}$ . On the other hand,  $\mathfrak{a}_{n,\alpha} \supseteq v_{\alpha,+}\mathfrak{a}_{0}v_{\alpha,-}$ , and so  $\mathfrak{a}_{0} \subseteq \mathfrak{b}_{\alpha}$ .

Let  $\mathfrak{b} \in \mathcal{J}(D_n)_{\sigma,\tau}$ . Then  $\mathfrak{b}^{er} = \mathfrak{b}$  since

$$\mathfrak{b} \subseteq \mathfrak{b}^{er} = (\mathbb{I}_n \mathfrak{b} \mathbb{I}_n)^r = \sum_{\alpha \in \mathbb{Z}^n} \mathbb{I}_{n,\alpha} \mathfrak{b} \mathbb{I}_{n,-\alpha}$$
$$= \sum_{\alpha \in \mathbb{Z}^n} v_\alpha D_n \mathfrak{b} D_n v_{-\alpha} \quad (\text{Proposition 2.2.(2)})$$
$$= \sum_{\alpha \in \mathbb{Z}^n} \sigma_{\alpha,+} \tau_{\alpha,-}(\mathfrak{b}) v_\alpha v_{-\alpha} \subseteq \mathfrak{b} D_n = \mathfrak{b},$$

where  $\sigma_{\alpha,+} := \prod_{\alpha_i > 0} \sigma_i$ . Therefore,  $\mathfrak{b}^e = \bigoplus_{\alpha \in \mathbb{Z}^n} v_{\alpha,+} \mathfrak{b} v_{\alpha,-}$ , by the first part of statement 1.

2. Repeat the proof of statement 1 replacing  $(\mathbb{I}_n, D_n)$  by  $(\mathbb{A}_n, \mathbb{D}_n)$  and making obvious modifications.  $\Box$ 

For each function  $f \in \mathcal{B}_n$ , let

 $\begin{array}{rcl} \mathfrak{f}_{f} &:=& \mathfrak{f}_{f(1)} \otimes \cdots \otimes \mathfrak{f}_{f(n)} \mbox{ where } \mathfrak{f}_{0} := F_{1,0}, \ \mathfrak{f}_{1} := \mathbb{F}_{1,0}; \\ \mathfrak{d}_{f} &:=& \mathfrak{d}_{f(1)} \otimes \cdots \otimes \mathfrak{d}_{f(n)} \mbox{ where } \mathfrak{d}_{0} := F_{1,0}, \ \mathfrak{d}_{1} := D_{1}; \\ \mathfrak{d}'_{f} &:=& \mathfrak{d}'_{f(1)} \otimes \cdots \otimes \mathfrak{d}'_{f(n)} \mbox{ where } \mathfrak{d}'_{0} := F_{1,0}, \ \mathfrak{d}'_{1} := \mathbb{D}_{1}. \end{array}$ 

Note that  $\mathfrak{f}_f \in \mathcal{J}(\mathbb{F}_{n,0})_{\sigma,\tau}, \mathfrak{d}_f \in \mathcal{J}(D_n)_{\sigma,\tau}, \text{ and } \mathfrak{d}'_f \in \mathcal{J}(\mathbb{D}_n)_{\sigma,\tau}.$ 

**Lemma 3.10** 1. The map  $\mathcal{C}_n \to \mathcal{J}(\mathbb{F}_{n,0})_{\sigma,\tau}, C \mapsto \mathfrak{f}_C := \sum_{f \in C} \mathfrak{f}_f$ , is a bijection where  $\mathfrak{f}_{\emptyset} := 0$ .

- 2. The map  $\mathcal{C}_n \to \mathcal{J}(D_n)_{\sigma,\tau}, C \mapsto \mathfrak{d}_C := \sum_{f \in C} \mathfrak{d}_f$ , is a bijection where  $\mathfrak{d}_{\emptyset} := 0$ .
- 3. The map  $\mathcal{C}_n \to \mathcal{J}(\mathbb{D}_n)_{\sigma,\tau}, C \mapsto \mathfrak{d}'_C := \sum_{f \in C} \mathfrak{d}'_f$ , is a bijection where  $\mathfrak{d}'_{\emptyset} := 0$ .

Proof. 1. It follows from  $\mathbb{F}_{n,0} = \bigotimes_{i=1}^{n} (K + \sum_{j \in \mathbb{N}} Ke_{jj}(i)), \sigma_i(e_{jj}(i)) = e_{j+1,j+1}(i), \tau_i(e_{jj}(i)) = e_{j-1,j-1}(i), e_{kk}(i)e_{jj}(i) = \delta_{jk}e_{jj}(i)$  that any ideal  $\mathfrak{b} \in \mathcal{J}(\mathbb{F}_{n,0})_{\sigma,\tau}$  is a sum  $\sum_{f \in C'} \mathfrak{f}_f$ . Then  $\mathfrak{b} = \mathfrak{f}_C$  for a unique element  $C \in \mathcal{C}_n$  (C is the set of all the maximal elements of C', it does not depend on C'), and so the map  $C \mapsto \mathfrak{f}_C$  is a bijection.

2. Similarly, it follows from  $D_n = \bigotimes_{i=1}^n (K[H_i] + \sum_{j \in \mathbb{N}} Ke_{jj}(i)), \tau_i(H_i) = H_i + 1$  (hence  $K[H_i] = \bigcup_{s \ge 1} \ker(\tau_i - 1)^s$ ) and the actions of the endomorphisms  $\sigma_i, \tau_i$  on the matrix units  $e_{jj}(i)$  that any ideal  $\mathfrak{b} \in \mathcal{J}(D_n)_{\sigma,\tau}$  is a sum  $\sum_{f \in C'} \mathfrak{d}_f$ . Then  $\mathfrak{b} = \mathfrak{d}_C$  for a unique element  $C \in \mathcal{C}_n$  (C is the set of all the maximal elements of C', it does not depend on C'), and so the map  $C \mapsto \mathfrak{d}_C$  is a bijection.

3. Statement 3 follows from statement 2 since the commutative algebra  $\mathbb{D}_n$  is a localization of the commutative algebra  $D_n$  at the monoid generated by the set  $\{(H_i-j)_1 \mid i=1,\ldots,n; 0 \neq j \in \mathbb{N}\}$  of nonzero divisors and  $\mathfrak{d}'_C = \mathbb{D}_n \mathfrak{d}_C$  for all  $C \in \mathcal{C}_n$ .  $\Box$ 

- **Corollary 3.11** 1. The restriction map  $\mathcal{J}(D_n)_{\sigma,\tau} \to \mathcal{J}(\mathbb{F}_{n,0})_{\sigma,\tau}$ ,  $\mathfrak{b} \mapsto \mathfrak{b}^r := \mathfrak{b} \cap \mathbb{F}_{n,0}$ , is an isomorphism (i.e.  $(\mathfrak{b}_1 * \mathfrak{b}_2)^r = \mathfrak{b}_1^r * \mathfrak{b}_2^r$  for  $* \in \{+, \cdot, \cap\}$ ) and its inverse is the extension map  $\mathfrak{c} \mapsto \mathfrak{c}^e := D_n \mathfrak{c}$ .
  - 2. The restriction map  $\mathcal{J}(\mathbb{D}_n)_{\sigma,\tau} \to \mathcal{J}(\mathbb{F}_{n,0})_{\sigma,\tau}$ ,  $\mathfrak{b} \mapsto \mathfrak{b}^r := \mathfrak{b} \cap \mathbb{F}_{n,0}$ , is an isomorphism (i.e.  $(\mathfrak{b}_1 * \mathfrak{b}_2)^r = \mathfrak{b}_1^r * \mathfrak{b}_2^r$  for  $* \in \{+, \cdot, \cap\}$ ) and its inverse is the extension map  $\mathfrak{c} \mapsto \mathfrak{c}^e := \mathbb{D}_n \mathfrak{c}$ .

*Proof.* 1. Statement 1 follows from Corollary 3.10.(1,2) and the fact that  $\mathfrak{d}_C^r = \mathfrak{f}_C$  for all  $C \in \mathcal{C}_n$ .

2. Statement 2 follows from Corollary 3.10.(1,3) and the fact that  $(\mathfrak{d}'_C)^r = \mathfrak{f}_C$  for all  $C \in \mathcal{C}_n$ .

Proof of Theorem 3.2. 1. By Theorem 3.9.(1), the restriction map  $\mathcal{J}(\mathbb{I}_n) \to \mathcal{J}(D_n)_{\sigma,\tau}$  is an isomorphism and its inverse map is the extension map. By Corollary 3.11.(1), the restriction map  $\mathcal{J}(D_n)_{\sigma,\tau} \to \mathcal{J}(\mathbb{F}_n)_{\sigma,\tau}$  is an isomorphism and its inverse map is the extension map. Now, statement 1 is obvious.

2. Similarly, by Theorem 3.9.(2), the restriction map  $\mathcal{J}(\mathbb{A}_n) \to \mathcal{J}(\mathbb{D}_n)_{\sigma,\tau}$  is an isomorphism and its inverse map is the extension map. By Corollary 3.11.(2), the restriction map  $\mathcal{J}(\mathbb{D}_n)_{\sigma,\tau} \to \mathcal{J}(\mathbb{F}_{n,0})_{\sigma,\tau}$  is an isomorphism and its inverse map is the extension map. Now, statement 2 is obvious.  $\Box$ 

**Theorem 3.12** Let  $Id(\mathcal{I}_n)$  be the set of all the idempotent ideals of the algebra  $\mathcal{I}_n$ . Then

- 1. the restriction map  $\mathcal{I}(\mathbb{I}_n) \to \mathrm{Id}(\mathcal{I}_n)$ ,  $\mathfrak{a} \mapsto \mathfrak{a}^e := \mathfrak{a} \cap \mathcal{I}_n$  is a bijection such that  $(\mathfrak{a}_1 * \mathfrak{a}_2)^r = \mathfrak{a}_1^r * \mathfrak{a}_2^r$ for  $* \in \{+, \cdot, \cap\}$ , and its inverse is the extension map  $\mathfrak{b} \mapsto \mathfrak{b}^e := \mathbb{I}_n \mathfrak{b} \mathbb{I}_n$ .
- 2. The restriction map  $\mathrm{Id}(\mathcal{I}_n) \to \mathcal{J}(\mathbb{F}_{n,0})_{\sigma,\tau}, \ \mathfrak{b} \mapsto \mathfrak{b}^r := \mathfrak{b} \cap \mathbb{F}_{n,0}, \text{ is a bijection such that} (\mathfrak{b}_1 * \mathfrak{b}_2)^r = \mathfrak{b}_1^r * \mathfrak{b}_2^r \text{ for } * \in \{+, \cdot, \cap\}, \text{ and its inverse is the extension map } \mathfrak{c} \mapsto \mathfrak{c}^e := \mathcal{I}_n \mathfrak{c} \mathcal{I}_n.$

*Proof.* 1. Statement 1 follows from Theorem 3.2.(1) and statement 2.

2. Statement 2 follows at once from a classification of the idempotent ideals of the algebra  $\mathbb{S}_n \simeq \mathcal{I}_n$  (Theorem 7.2, [13]).  $\Box$ 

#### 4 The Noetherian factor algebra of the algebra $\mathbb{I}_n$

The aim of this section is to show that the factor algebra  $\mathbb{I}_n/\mathfrak{a}_n$  of the algebra  $\mathbb{I}_n$  at its maximal ideal  $\mathfrak{a}_n = \mathfrak{p}_1 + \cdots + \mathfrak{p}_n$  is the only Noetherian factor algebra of the algebra  $\mathbb{I}_n$  (Proposition 4.1).

The factor algebra  $\mathbb{I}_n/\mathfrak{a}_n$ . Recall that the Weyl algebra  $A_n$  is the generalized Weyl algebra  $\mathcal{P}_n((\sigma_1, ..., \sigma_n), (H_1, ..., H_n))$ . Denote by  $S_n$  the multiplicative submonoid of  $\mathcal{P}_n$  generated by the elements  $H_i + j$  where i = 1, ..., n and  $j \in \mathbb{Z}$ . It follows from the above presentation of the Weyl algebra  $A_n$  as a GWA that  $S_n$  is an Ore set in  $A_n$ , and, using the  $\mathbb{Z}^n$ -grading of  $A_n$ , that the (two-sided) localization  $\mathcal{A}_n := S_n^{-1} A_n$  of the Weyl algebra  $A_n$  at  $S_n$  is the skew Laurent polynomial ring

$$\mathcal{A}_{n} = S_{n}^{-1} \mathcal{P}_{n}[x_{1}^{\pm 1}, \dots, x_{n}^{\pm 1}; \sigma_{1}, \dots, \sigma_{n}]$$
(15)

with coefficients in the algebra

 $\mathcal{L}_n := S_n^{-1} \mathcal{P}_n = K[H_1^{\pm 1}, (H_1 \pm 1)^{-1}, (H_1 \pm 2)^{-1}, \dots, H_n^{\pm 1}, (H_n \pm 1)^{-1}, (H_n \pm 2)^{-1}, \dots],$ 

which is the localization of  $\mathcal{P}_n$  at  $S_n$ . We identify the Weyl algebra  $A_n$  with its image in the algebra  $\mathcal{A}_n$  via the monomorphism,

$$A_n \to \mathcal{A}_n, \ x_i \mapsto x_i, \ \partial_i \mapsto H_i x_i^{-1}, \ i = 1, \dots, n.$$

Let  $k_n$  be the *n*'th Weyl skew field, that is the full ring of quotients of the *n*'th Weyl algebra  $A_n$  (it exists by Goldie's Theorem since  $A_n$  is a Noetherian domain). Then the algebra  $\mathcal{A}_n$  is a K-subalgebra of  $k_n$  generated by the elements  $x_i$ ,  $x_i^{-1}$ ,  $H_i$  and  $H_i^{-1}$ ,  $i = 1, \ldots, n$  since, for all  $j \in \mathbb{N}$ ,

$$(H_i \mp j)^{-1} = x_i^{\pm j} H_i^{-1} x_i^{\mp j}, \quad i = 1, \dots, n.$$
(16)

Clearly,  $\mathcal{A}_n \simeq \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_1$  (*n* times).

Recall that the algebra  $\mathbb{I}_n$  is a subalgebra of  $\mathbb{A}_n$  and the extension  $\mathfrak{a}_n^e$  of the maximal ideal  $\mathfrak{a}_n$  of the algebra  $\mathbb{I}_n$  is the maximal ideal of the algebra  $\mathbb{A}_n$ . By (22) of [11], there is the algebra isomorphism (where  $\overline{a} := a + \mathfrak{a}_n^e$ ):

$$\mathbb{A}_n/\mathfrak{a}_n^e \to \mathcal{A}_n, \ \overline{x_i} \mapsto x_i, \ \overline{\partial}_i \mapsto H_i x_i^{-1}, \ \overline{H_i^{\pm 1}} \mapsto H_i^{\pm 1}, \ i = 1, \dots, n.$$

Since  $\mathfrak{a}_n^{er} = \mathfrak{a}_n$  (Theorem 3.1), the algebra  $B_n := \mathbb{I}_n/\mathfrak{a}_n$  is a subalgebra of the algebra  $\mathbb{A}_n/\mathfrak{a}_n^e$ , and so there is the algebra monomorphism (where  $\overline{a} := a + \mathfrak{a}_n^e$ ):

$$B_n \to \mathcal{A}_n, \ \overline{x_i} \mapsto x_i, \ \overline{\partial}_i \mapsto H_i x_i^{-1}, \ \overline{\int_i} \mapsto x_i H_i^{-1}, \ \overline{H_i} \mapsto H_i, \ i = 1, \dots, n.$$

It follows that there is the algebra isomorphism:

$$B_n \to \bigotimes_{i=1}^n K[H_i][\partial_i, \partial_i^{-1}; \tau_i] = \mathcal{P}_n[\partial_1^{\pm 1}, \dots, \partial_n^{\pm 1}; \tau_1, \dots, \tau_n],$$

the RHS is the skew Laurent polynomial algebra with coefficients in the polynomial algebra  $\mathcal{P}_n = K[H_1, \ldots, H_n]$  where  $\tau_i(H_j) = H_j + \delta_{ij}$ . It is a standard fact that

$$B_n = (A_n)_{\partial_1, \dots, \partial_n} \tag{17}$$

where  $(A_n)_{\partial_1,\ldots,\partial_n}$  is the localization of the Weyl algebra  $A_n$  at the Ore subset of  $A_n$  which is the submonoid of  $A_n$  generated by the elements  $\partial_1,\ldots,\partial_n$ . Note that  $(A_n)_{\partial_1,\ldots,\partial_n} \simeq (A_n)_{x_1,\ldots,x_n}$ . It is well-known that the algebra  $B_n$  is a simple, Noetherian, finitely generated algebra of Gelfand-Kirillov dimension 2n and 1.gldim $(B_n) = r$ .gldim $(B_n) = n$ .

**Proposition 4.1** Let  $\mathfrak{a}$  be an ideal of the algebra  $\mathbb{I}_n$  such that  $\mathfrak{a} \neq \mathbb{I}_n$ . The following statements are equivalent.

- 1. The factor algebra  $\mathbb{I}_n/\mathfrak{a}$  is a left Noetherian algebra.
- 2. The factor algebra  $\mathbb{I}_n/\mathfrak{a}$  is a right Noetherian algebra.
- 3. The factor algebra  $\mathbb{I}_n/\mathfrak{a}$  is a Noetherian algebra.
- 4.  $\mathfrak{a} = \mathfrak{a}_n$ .

Proof. Note that the algebra  $B_n = \mathbb{I}_n/\mathfrak{a}_n$  is a Noetherian algebra as a two-sided localization of the Noetherian algebra  $A_n$ . Suppose that  $\mathfrak{a} \neq \mathfrak{a}_n$ . Fix  $\mathfrak{p} \in \operatorname{Min}(\mathfrak{a})$ . Then  $\mathfrak{p} = \mathfrak{p}_I := \sum_{i \in I} \mathfrak{p}_i$  for a non-empty subset I of the set  $\{1, \ldots, n\}$  with m := |I| < n (Corollary 3.3.(10) and Corollary 3.4). The factor algebra  $\mathbb{I}_n/\mathfrak{p} \simeq B_m \otimes \mathbb{I}_{n-m}$  is neither left nor right Noetherian since the algebra  $\mathbb{I}_{n/\mathfrak{a}}$  is neither left nor right Noetherian. Now, the proposition is obvious.  $\Box$ 

**Lemma 4.2** Let  $\mathfrak{a}$  be an ideal of the algebra  $\mathbb{I}_n$  distinct from  $\mathbb{I}_n$ . Then  $\mathrm{GK}(\mathbb{I}_n/\mathfrak{a}) = 2n$ .

*Proof.* It is well-known that  $\operatorname{GK}(B_n) = 2n$ . Now,  $2n = \operatorname{GK}(\mathbb{I}_n) \ge \operatorname{GK}(\mathbb{I}_n/\mathfrak{a}) \ge \operatorname{GK}(\mathbb{I}_n/\mathfrak{a}_n) = \operatorname{GK}(B_n) = 2n$ . Therefore,  $\operatorname{GK}(\mathbb{I}_n/\mathfrak{a}) = 2n$ .  $\Box$ 

#### 5 The group of units of the algebra $\mathbb{I}_n$ and its centre

In this section, the group  $\mathbb{I}_n^*$  of units of the algebra  $\mathbb{I}_n$  is described (Theorem 5.6.(1)) and its centre is found (Theorem 5.6.(2)). It is proved that the algebra  $\mathbb{I}_n$  is central (Lemma 5.4.(2)) and self-dual.

The involution \* on the algebra  $\mathbb{I}_n$ . Using the defining relations in Proposition 2.2.(1), we see that the algebra  $\mathbb{I}_n$  admits the involution:

$$*: \mathbb{I}_n \to \mathbb{I}_n, \ \partial_i \mapsto \int_i, \ \int_i \mapsto \partial_i, \ H_i \mapsto H_i, \ i = 1, \dots, n,$$
(18)

i.e. it is a K-algebra anti-isomorphism  $((ab)^* = b^*a^*)$  such that  $* \circ * = \mathrm{id}_{\mathbb{I}_n}$ . Therefore, the algebra  $\mathbb{I}_n$  is self-dual, i.e. is isomorphic to its opposite algebra  $\mathbb{I}_n^{op}$ . As a result, the left and the right properties of the algebra  $\mathbb{I}_n$  are the same. For all elements  $\alpha, \beta \in \mathbb{N}^n$ ,

$$e_{\alpha\beta}^* = e_{\beta\alpha}.\tag{19}$$

An element  $a \in \mathbb{I}_n$  is called *hermitian* if  $a^* = a$ .

**Lemma 5.1** 1.  $\mathfrak{a}^* = \mathfrak{a}$  for all ideals  $\mathfrak{a}$  of the algebra  $\mathbb{I}_n$ .

2.  $(\mathbb{I}_{n,\alpha})^* = \mathbb{I}_{n,-\alpha}$  for all  $\alpha \in \mathbb{Z}^n$ .

3. The set  $\operatorname{Fix}_{\mathbb{I}_n}(*) = \{a \in \mathbb{I}_n \mid a^* = a\}$  of all the hermitian elements of the algebra  $\mathbb{I}_n$  is the commutative subalgebra  $D_n$  of the algebra  $\mathbb{I}_n$ .

*Proof.* 1. By (19),  $\mathfrak{p}_i^* = \mathfrak{p}_i$  for all i = 1, ..., n (see Corollary 3.3.(2)). By Corollary 3.3.(4,9),  $\mathfrak{a}^* = \mathfrak{a}$ .

2. Note that  $D_n^* = D_n$  and  $v_{\alpha}^* = v_{-\alpha}$ . By Proposition 2.2.(2),  $(\mathbb{I}_{n,\alpha})^* = (v_{\alpha}D_n)^* = D_n v_{-\alpha} = \mathbb{I}_{n,-\alpha}$ .

3. By statement 2,  $\operatorname{Fix}_{\mathbb{I}_n}(*) \subseteq D_n$ . The opposite inclusion is obvious. Therefore,  $\operatorname{Fix}_{\mathbb{I}_n}(*) = D_n$ .  $\Box$ 

The involution \* of the algebra  $\mathbb{I}_n$  respects the maximal ideal  $\mathfrak{a}_n$   $(\mathfrak{a}_n^* = \mathfrak{a}_n)$ . Therefore, the factor algebra  $B_n = \mathbb{I}_n/\mathfrak{a}_n$  inherits the involution \*:  $\partial_i^* = \partial_i^{-1}$ ,  $x_i^* = x_i + \partial_i^{-1}$ ,  $H_i^* = H_i$  for  $i = 1, \ldots, n$  (since  $\partial_i^* = \overline{\int_i} = \partial_i^{-1}$  and  $x_i^* = (\partial_i H_i)^* = H_i \partial_i^{-1} = \partial_i x_i \partial_i^{-1} = x_i + \partial_i^{-1}$ ).

The involution \* of the algebra  $\mathbb{I}_n$  can be extended to an involution of the algebra  $\mathbb{A}_n$  by setting

$$x_i^* = H_i \partial_i, \ \ \partial_i^* = \int_i, \ \ (H_i^{\pm 1})^* = H_i^{\pm 1}, \ \ i = 1, \dots, n.$$

This can be checked using the defining relations coming from the presentation of the algebra  $\mathbb{A}_n$  as a GWA. Note that  $y_i^* = (H_i^{-1}\partial_i)^* = \int_i H_i^{-1} = x_i H_i^{-2}$ ,  $A_n^* \not\subseteq A_n$ ,  $\mathbb{S}_n^* \not\subseteq \mathbb{S}_n$ , but  $\mathcal{I}_n^* = \mathcal{I}_n$  where  $\mathcal{I}_n$  is the algebra of integro-differential operators with constant coefficients.

For a subset S of a ring R, the sets  $l.ann_R(S) := \{r \in R | rS = 0\}$  and  $r.ann_R(S) := \{r \in R | Sr = 0\}$  are called the *left* and the *right annihilators* of the set S in R. Using the fact that the algebra  $\mathbb{I}_n$  is a GWA and its  $\mathbb{Z}^n$ -grading, we see that

$$\operatorname{l.ann}_{\mathbb{I}_n}(\int_i) = \bigoplus_{k \in \mathbb{N}} Ke_{k0}(i) \bigotimes \bigotimes_{i \neq j} \mathbb{I}_1(j), \quad \operatorname{r.ann}_{\mathbb{I}_n}(\int_i) = 0.$$
(20)

$$\operatorname{r.ann}_{\mathbb{I}_n}(\partial_i) = \bigoplus_{k \in \mathbb{N}} Ke_{0k}(i) \bigotimes \bigotimes_{i \neq j} \mathbb{I}_1(j), \quad \operatorname{l.ann}_{\mathbb{I}_n}(\partial_i) = 0.$$
(21)

Recall that a submodule of a module that intersects non-trivially each nonzero submodule of the module is called an *essential* submodule.

**Lemma 5.2** 1. For all nonzero ideals  $\mathfrak{a}$  of the algebra  $\mathbb{I}_n$ ,  $\operatorname{l.ann}_{\mathbb{I}_n}(\mathfrak{a}) = \operatorname{r.ann}_{\mathbb{I}_n}(\mathfrak{a}) = 0$ .

2. Each nonzero ideal of the algebra  $\mathbb{I}_n$  is an essential left and right submodule of  $\mathbb{I}_n$ .

*Proof.* The algebra  $\mathbb{I}_n$  is self-dual, so it suffices to prove only, say, the left versions of the statements.

1. Suppose that  $\mathfrak{b} := \text{l.ann}_{\mathbb{I}_n}(\mathfrak{a}) \neq 0$ , we seek a contradiction. By Corollary 3.3.(8), the nonzero ideals  $\mathfrak{a}$  and  $\mathfrak{b}$  contain the ideal  $F_n$ . Then  $0 = \mathfrak{ba} \supseteq F_n^2 = F_n \neq 0$ , a contradiction. Therefore,  $\mathfrak{b} = 0$ .

2. Let I be a nonzero left ideal of the algebra  $\mathbb{I}_n$ . By statement  $1, 0 \neq F_n I \subseteq F_n \cap I$ . Therefore,  $F_n$  is an essential left submodule of the algebra  $\mathbb{I}_n$ . Then so are all the nonzero ideals of the algebra  $\mathbb{I}_n$  since  $F_n$  is the least nonzero ideal of the algebra  $\mathbb{I}_n$ .  $\Box$ 

**Corollary 5.3** Let A be a K-algebra. Then the algebra  $\mathbb{I}_n \otimes A$  is a prime algebra iff the algebra A is so.

*Proof.* It is obvious that if the algebra A is not prime  $(\mathfrak{ab} = 0 \text{ for some nonzero ideals } \mathfrak{a} \text{ and } \mathfrak{b}$  of A) then the algebra  $\mathbb{I}_n \otimes A$  is neither (since  $\mathbb{I}_n \otimes \mathfrak{a} \cdot \mathbb{I}_n \otimes \mathfrak{b} = 0$ ).

It suffices to show that if the algebra A is prime then so is the algebra  $\mathbb{I}_n \otimes A$ . Let  $\mathfrak{c}$  be a nonzero ideal of the algebra  $\mathbb{I}_n \otimes A$ . Then  $F_n \mathfrak{c} \neq 0$ , by Lemma 5.2.(1). Note that  $F_n \mathfrak{c} \subseteq \mathfrak{c}$ . Let  $u = E_{\alpha\beta} \otimes a + \cdots + E_{\sigma\rho} \otimes a'$  be a nonzero element of  $F_n \mathfrak{c}$  where  $E_{\alpha\beta}, \ldots, E_{\sigma\rho}$  are distinct matrix units;  $a, \ldots, a' \in A$ , and  $a \neq 0$ . Then  $0 \neq E_{\alpha\beta} \otimes a = E_{\alpha\alpha} u E_{\beta\beta} \in \mathfrak{a}$ , and so  $F_n \otimes AaA \subseteq \mathfrak{c}$ . Let  $\mathfrak{d}$ 

be a nonzero ideal of the algebra  $\mathbb{I}_n \otimes A$ . Then  $F_n \otimes AbA \subseteq \mathfrak{d}$  for some nonzero element  $b \in A$ . Then

$$\mathfrak{cd} \supseteq F_n \otimes AaA \cdot F_n \otimes AbA = F_n \otimes (AaA \cdot AbA) \neq 0$$

since  $F_n^2 = F_n$  and  $AaA \cdot AbA \neq 0$  (A is a prime algebra). Therefore,  $\mathbb{I}_n \otimes A$  is a prime algebra.  $\Box$ **The centre of the algebra**  $\mathbb{I}_n$ . For an algebra A and its subset S, the subalgebra of A,

The centre of the algebra  $\mathbb{I}_n$ . For an algebra A and its subset S, the subalgebra of A,  $\operatorname{Cen}_A(S) := \{a \in A \mid as = sa \text{ for all } s \in S\}$ , is called the *centralizer* of S in A. The next lemma shows that the algebra  $\mathbb{I}_n$  is a *central* algebra, i.e. its centre  $Z(\mathbb{I}_n)$  is K.

Lemma 5.4 1.  $\operatorname{Cen}_{\mathbb{I}_n}(\mathbb{F}_{n,0}) = \operatorname{Cen}_{\mathbb{I}_n}(D_n) = D_n.$ 

- 2. The centre of the algebra  $\mathbb{I}_n$  is K.
- 3.  $\operatorname{Cen}_{\mathbb{I}_n}(\mathcal{I}_n) = K.$

Proof. 1. Since  $\mathbb{F}_{n,0} \subset D_n$  and  $D_n$  is a commutative algebra, we have the inclusions  $D_n \subseteq \operatorname{Cen}_{\mathbb{I}_n}(D_n) \subseteq \operatorname{Cen}_{\mathbb{I}_n}(\mathbb{F}_{n,0})$ . It remains to show that the inclusion  $C := \operatorname{Cen}_{\mathbb{I}_n}(\mathbb{F}_{n,0}) \subseteq D_n$  holds. Recall that the algebra  $\mathbb{I}_n = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathbb{I}_{n,\alpha}$  is a  $\mathbb{Z}^n$ -graded algebra with  $\mathbb{F}_{n,0} \subset D_n = \mathbb{I}_{n,0}$ . Therefore, C is a homogeneous subalgebra of  $\mathbb{I}_n$ , i.e.  $C = \bigoplus_{\alpha \in \mathbb{Z}^n} C_\alpha$  where  $C_\alpha := C \cap \mathbb{I}_{n,\alpha}$ . We have to show that  $C_\alpha = 0$  for all  $\alpha \neq 0$ . Let  $c \in C_\alpha$  for some  $\alpha \neq 0$ . Then  $c = v_{\alpha,+} dv_{\alpha,-}$  for some element  $d \in D_n$  (the elements  $v_{\alpha,+}$  and  $v_{\alpha,-}$  are defined in Theorem 3.9.(1)). For all elements  $E_{\beta\beta} \in \mathbb{F}_{n,0}$  where  $\beta \in \mathbb{N}^n$ ,

$$cE_{\beta\beta} = v_{\alpha,+}d\tau_{\alpha,-}(E_{\beta\beta})v_{\alpha,-} = v_{\alpha,+}dE_{\beta-\alpha_-,\beta-\alpha_-}v_{\alpha,-},$$
  

$$E_{\beta\beta}c = v_{\alpha,+}\tau_{\alpha,+}(E_{\beta\beta})dv_{\alpha,-} = v_{\alpha,+}E_{\beta-\alpha_+,\beta-\alpha_+}dv_{\alpha,-},$$

where  $\tau_{\alpha,-} := \prod_{\alpha_i < 0} \tau_i, \ \tau_{\alpha,+} := \prod_{\alpha_i > 0} \tau_i, \ \alpha_- := -\sum_{\alpha_i < 0} \alpha_i e_i \text{ and } \alpha_+ := \sum_{\alpha_i > 0} \alpha_i e_i \ (E_{st} = 0)$ if either  $s \notin \mathbb{N}^n$  or  $t \notin \mathbb{N}^n$ ). Since  $cE_{\beta\beta} = E_{\beta\beta}c$  and the map  $a \mapsto v_{\alpha,+} av_{\alpha,-}$  is injective (its left inverse is the map  $a \mapsto u_{\alpha,-} au_{\alpha,+}$ , see (14)), we have the equality  $E_{\beta-\alpha_+,\beta-\alpha_+}d = E_{\beta-\alpha-,\beta-\alpha-}d$ for each  $\beta \in \mathbb{N}^n$ . Since  $\bigoplus_{\gamma \in \mathbb{N}^n} KE_{\gamma\gamma}$  is the direct sum of ideals of the algebra  $D_n$ , it follows that  $E_{\gamma\gamma}d = 0$  for all elements  $\gamma \in \mathbb{N}^n$ . Then it is not difficult to show that d = 0 (using the fact that each polynomial of  $K[H_1, \ldots, H_n]$  is uniquely determined by its values on the set  $\mathbb{N}^n$ ).

2. By statement 1, the centre Z of the algebra  $\mathbb{I}_n$  is a subalgebra of  $D_n$ . Let  $d \in Z$ . For all elements  $i = 1, \ldots, n, 0 = dx_i - x_i d = x_i(\tau_i(d) - d)$ . Since  $\mathbb{I}_n \subseteq \mathbb{A}_n$ , we see that  $0 = y_i x_i(\tau_i(d) - d) = \tau_i(d) - d$ , and so  $d \in \bigcap_{i=1}^n \ker_{D_n}(\tau_i - 1) = K$ . Therefore, Z = K.

3. By (12),  $\mathbb{F}_{n,0} \subseteq \mathcal{I}_n$ . This implies that  $C := \operatorname{Cen}_{\mathbb{I}_n}(\mathcal{I}_n) \subseteq \operatorname{Cen}_{\mathbb{I}_n}(\mathbb{F}_{n,0}) = D_n$ , by statement 1. Let  $d \in C$ . Then

$$0 = \partial_i \cdot 0 = \partial_i (d \int_i - \int_i d) = \partial_i \int_i (\tau_i(d) - d) = \tau_i(d) - d \text{ for all } i = 1, \dots, n,$$

where  $\tau_i(a) = \partial_i a \int_i$ . Hence  $d \in \bigcap_{i=1}^n \ker_{D_n}(\tau_i - 1) = K$ , and so C = K.  $\Box$ 

**Lemma 5.5** Let  $C = P_n$ ,  $K[\partial_1, \ldots, \partial_n]$ ,  $K[\int_1, \ldots, \int_n]$  or  $D_n$ . Then  $\operatorname{Cen}_{\mathbb{I}_n}(C) = C$  and C is a maximal commutative subalgebra of the algebra  $\mathbb{I}_n$ .

*Proof.* The first statement,  $\operatorname{Cen}_{\mathbb{I}_n}(C) = C$ , follows from the fact that the algebra  $\mathbb{I}_n$  is  $\mathbb{Z}^n$ -graded and the canonical generators of the algebra C are homogeneous elements of the algebra  $\mathbb{I}_n$  (we leave this as an exercise for the reader). Then C is a maximal commutative subalgebra of the algebra  $\mathbb{I}_n$  since  $\operatorname{Cen}_{\mathbb{I}_n}(C) = C$  and C is a commutative algebra.  $\Box$ 

The group  $\mathbb{I}_n^*$  of units of the algebra  $\mathbb{I}_n$  and its centre. The group  $\mathbb{A}_1^*$  of units of the algebra  $\mathbb{A}_1$  contains the following infinite discrete subgroup Theorem 4.2, [11]:

$$\mathcal{H} := \{ \prod_{i \ge 0} (H+i)^{n_i} \cdot \prod_{i \ge 1} (H-i)_1^{n_{-i}} \, | \, (n_i)_{i \in \mathbb{Z}} \in \mathbb{Z}^{(\mathbb{Z})} \} \simeq \mathbb{Z}^{(\mathbb{Z})}.$$
(22)

For each tensor multiple  $\mathbb{A}_1(i)$  of the algebra  $\mathbb{A}_n = \bigotimes_{i=1}^n \mathbb{A}_1(i)$ , let  $\mathcal{H}_1(i)$  be the corresponding group  $\mathcal{H}$ . Their (direct) product

$$\mathcal{H}_n := \mathcal{H}_1(1) \cdots \mathcal{H}_1(n) = \prod_{i=1}^n \mathcal{H}_1(i)$$
(23)

is a (discrete) subgroup of the group  $\mathbb{A}_n^*$  of units of the algebra  $\mathbb{A}_n$ , and  $\mathcal{H}_n \simeq \mathcal{H}^n \simeq (\mathbb{Z}^n)^{(\mathbb{Z})}$ . Note that  $\mathbb{A}_n^* = K^* \times (\mathcal{H}_n \ltimes (1 + \mathfrak{a}_n^e)^*)$  and  $Z(\mathbb{A}_n^*) = K^*$  (Theorem 4.4, [11]). A similar result holds for the group  $\mathbb{I}_n^*$  of the algebra  $\mathbb{I}_n$  (Theorem 5.6). Since  $\mathfrak{a}_n$  is an ideal of the algebra  $\mathbb{I}_n$ , the intersection  $(1 + \mathfrak{a}_n)^* := \mathbb{I}_n^* \cap (1 + \mathfrak{a}_n)$  is a subgroup of the group  $\mathbb{I}_n^*$  of units of the algebra  $\mathbb{I}_n$ .

**Theorem 5.6** 1. Let  $\mathbb{F}_n := \bigoplus_{i=1}^n (K + F(i))$ . Then

$$\mathbb{I}_n^* = K^* \times (1 + \mathfrak{a}_n)^* \text{ and } \mathbb{I}_n^* \supseteq (1 + \mathbb{F}_n \cap \mathfrak{a}_n)^* \simeq \underbrace{\operatorname{GL}_{\infty}(K) \ltimes \cdots \ltimes \operatorname{GL}_{\infty}(K)}_{2^n - 1 \text{ times}}.$$

2. The centre of the group  $\mathbb{I}_n^*$  is  $K^*$ .

Proof. 1. The commutative diagram of algebra homomorphisms



yields the commutative diagram of group homomorphisms



Since  $B_n^* = \bigcup_{\alpha \in \mathbb{Z}^n} K^* \partial^{\alpha}$  and  $\mathbb{A}_n^* = K^* \times (\mathcal{H}_n \ltimes (1 + \mathfrak{a}_n^e)^*)$ , we see that

$$K^* \times (1 + \mathfrak{a}_n)^* \subseteq \mathbb{I}_n^* \subseteq \mathbb{I}_n \cap \mathbb{A}_n^* = K^* \times (\mathbb{I}_n \cap (1 + \mathfrak{a}_n^e)^*) = K^* \times (1 + \mathfrak{a}_n^{er})^* = K^* \times (1 + \mathfrak{a}_n)^*$$

since  $\mathfrak{a}_n^{er} = \mathfrak{a}_n$  (Theorem 3.1). Therefore,  $\mathbb{I}_n^* = K^* \times (1 + \mathfrak{a}_n)^*$ . Since  $\mathbb{F}_n \subset \mathbb{I}_n \subset \mathbb{A}_n$ , it is obvious that

$$\mathbb{I}_n^* \supseteq (1 + \mathbb{F}_n \cap \mathfrak{a}_n)^* = (1 + \mathbb{F}_n \cap \mathfrak{a}_n^e)^* \simeq \underbrace{\operatorname{GL}_\infty(K) \ltimes \cdots \ltimes \operatorname{GL}_\infty(K)}_{2^n - 1 \text{ times}}.$$

The isomorphism is established in Corollary 7.3, [14].

2. Let S be the set of elements of the type  $1 + \prod_{i \in I} e_{s_i s_i}(i)$  where  $\emptyset \neq I \subseteq \{1, \ldots, n\}$ . Then  $S \subseteq \mathbb{I}_n^*$  and  $\operatorname{Cen}_{\mathbb{I}_n}(S) = \operatorname{Cen}_{\mathbb{I}_n}(\mathbb{F}_{n,0}) = D_n$ , by Lemma 5.4.(1). Therefore,  $\operatorname{Cen}_{\mathbb{I}_n^*}(S) = \operatorname{Cen}_{\mathbb{I}_n}(S) \cap \mathbb{I}_n^* = D_n \cap \mathbb{I}_n^* = D_n^* = \mathbb{F}_{n,0}^*$ . We see that  $\operatorname{Cen}_{\mathbb{F}_{n,0}}(S) = K$ . Therefore, the centre of the group  $\mathbb{I}_n^*$  is  $K^*$ .  $\Box$ 

The group of units  $(1+F)^*$  and  $\mathbb{I}_1^*$ . Recall that the algebra (without 1)  $F = \bigoplus_{i,j \in \mathbb{N}} Ke_{ij}$  is the union  $M_{\infty}(K) := \bigcup_{d \ge 1} M_d(K) = \varinjlim M_d(K)$  of the matrix algebras  $M_d(K) := \bigoplus_{1 \le i,j \le d-1} Ke_{ij}$ . i.e.  $F = M_{\infty}(K)$ . For each  $d \ge 1$ , consider the (usual) determinant  $\det_d = \det : 1 + M_d(K) \to K$ ,  $u \mapsto \det(u)$ . These determinants determine the (global) determinant,

$$\det: 1 + M_{\infty}(K) = 1 + F \to K, \ u \mapsto \det(u), \tag{24}$$

where  $\det(u)$  is the common value of all the determinants  $\det_d(u), d \gg 1$ . The (global) determinant has usual properties of the determinant. In particular, for all  $u, v \in 1 + M_{\infty}(K)$ ,  $\det(uv) =$   $\det(u) \cdot \det(v)$ . It follows from this equality and the Cramer's formula for the inverse of a matrix that the group  $\operatorname{GL}_{\infty}(K) := (1 + M_{\infty}(K))^*$  of units of the monoid  $1 + M_{\infty}(K)$  is equal to

$$GL_{\infty}(K) = \{ u \in 1 + M_{\infty}(K) \, | \, \det(u) \neq 0 \}.$$
(25)

Therefore,

$$(1+F)^* = \{ u \in 1+F \,|\, \det(u) \neq 0 \} = \operatorname{GL}_{\infty}(K).$$
(26)

**Corollary 5.7**  $\mathbb{I}_1^* = K^* \times (1+F)^* = K^* \times \operatorname{GL}_{\infty}(K)$ , *i.e.*  $\mathbb{I}_1^* = \{\lambda(1+f) | \det(1+f) \neq 0, \lambda \in K^*, f \in F\}$ . The elements  $\lambda \in K^*$ ,  $1 + \mu e_{ij}$  where  $\mu \in K$  and  $i \neq j$ , and  $1 + \gamma e_{00}$  where  $\gamma \in K \setminus \{-1\}$  are generators for the group  $\mathbb{I}_n^*$ .

## ${\bf 6} \quad {\bf The weak and the global dimensions of the algebra } \mathbb{I}_n$

In this section, we prove that the weak dimension of the algebra  $\mathbb{I}_n$  and of all its prime factor algebras is n (Theorem 6.2). An analogue of Hilbert's Syzygy Theorem is established for the algebra  $\mathbb{I}_n$  and for all its prime factor algebras (Theorem 6.5).

The weak dimension of the algebra  $\mathbb{I}_n$ . Let S be a non-empty multiplicatively closed subset of a ring R, and let  $\operatorname{ass}(S) := \{r \in R \mid sr = 0 \text{ for some } s \in S\}$ . Then a *left quotient ring* of R with respect to S is a ring Q together with a homomorphism  $\varphi : R \to Q$  such that

- (i) for all  $s \in S$ ,  $\varphi(s)$  is a unit in Q;
- (ii) for all  $q \in Q$ ,  $q = \varphi(s)^{-1}\varphi(r)$  for some  $r \in R$  and  $s \in S$ , and
- (iii)  $\ker(\varphi) = \operatorname{ass}(S)$ .

If there exists a left quotient ring Q of R with respect to S then it is unique up to isomorphism, and it is denoted  $S^{-1}R$ . It is also said that the ring Q is the *left localization* of the ring R at S.

Example 1. Let  $S := S_{\partial} := \{\partial^i, i \ge 0\}$  and  $R = \mathbb{I}_1$ . Then  $\operatorname{ass}(S) = F$ ,  $\mathbb{I}_1/\operatorname{ass}(S) = B_1$  and the conditions (i)-(iii) hold where  $Q = B_1$ . This means that the ring  $B_1 = \mathbb{I}_1/F$  is the left quotient ring of  $\mathbb{I}_1$  at S, i.e.  $B_1 \simeq S_{\partial}^{-1} \mathbb{I}_1$ .

ring of  $\mathbb{I}_1$  at S, i.e.  $B_1 \simeq S_{\partial}^{-1}\mathbb{I}_1$ . Example 2. Let  $S := S_{\partial_1,...,\partial_n} := \{\partial^{\alpha}, \alpha \in \mathbb{N}^n\}$  and  $R = \mathbb{I}_n$ . Then  $\operatorname{ass}(S_{\partial_1,...,\partial_n}) = \mathfrak{a}_n$ ,  $\mathbb{I}_n/\mathfrak{a}_n = B_n$ , and

$$S_{\partial_1,\dots,\partial_n}^{-1} \mathbb{I}_n \simeq B_n, \tag{27}$$

i.e.  $B_n$  is the left quotient ring of  $\mathbb{I}_n$  at  $S_{\partial_1,\ldots,\partial_n}$ . Note that the right localization  $\mathbb{I}_n S_{\partial_1,\ldots,\partial_n}^{-1}$  of  $\mathbb{I}_n$  at  $S_{\partial_1,\ldots,\partial_n}$  does not exist. Otherwise, we would have  $S_{\partial_1,\ldots,\partial_n}^{-1}\mathbb{I}_n \simeq \mathbb{I}_n S_{\partial_1,\ldots,\partial_n}^{-1}$  but all the elements  $\partial^{\alpha}$  are left regular, and we would have a monomorphism  $\mathbb{I}_n \to S_{\partial_1,\ldots,\partial_n}^{-1}\mathbb{I}_n \simeq B_n$ , which would be impossible since the elements  $\partial_i$  of the algebra  $\mathbb{I}_n$  are not regular. By applying the involution \* to (27), we see that

$$\mathbb{I}_n S_{\int_1,\dots,\int_n}^{-1} \simeq B_n,\tag{28}$$

i.e. the algebra  $B_n$  is the right localization of  $\mathbb{I}_n$  at the multiplicatively closed set  $S_{\int_1, \dots, \int_n} := \{\int_{\alpha}^{\alpha} | \alpha \in \mathbb{N}^n\}.$ 

Given a ring R and modules  $_RM$  and  $N_R$ , we denote by  $pd(_RM)$  and  $pd(N_R)$  their projective dimensions. Let us recall a result which will be used repeatedly in proofs later.

It is obvious that  $P_n \simeq A_n / \sum_{i=1}^n A_n \partial_i$ . A similar result is true for the  $\mathbb{I}_n$ -module  $P_n$  (Proposition 6.1.(2)). Note that  $\mathrm{pd}_{A_n}(P_n) = n$  but  $\mathrm{pd}_{\mathbb{I}_n}(P_n) = 0$  (Proposition 6.1.(3)).

**Proposition 6.1** *1.*  $\mathbb{I}_1 = \mathbb{I}_1 \partial \bigoplus \mathbb{I}_1 e_{00}$  and  $\mathbb{I}_1 = \int \mathbb{I}_1 \bigoplus e_{00} \mathbb{I}_1$ .

- 2.  $\mathbb{I}_n P_n \simeq \mathbb{I}_n / \sum_{i=1}^n \mathbb{I}_n \partial_i$ .
- 3. The  $\mathbb{I}_n$ -module  $P_n$  is projective.
- 4.  $F_n = F^{\otimes n}$  is a left and right projective  $\mathbb{I}_n$ -module.
- 5. The projective dimension of the left and right  $\mathbb{I}_n$ -module  $\mathbb{I}_n/F_n$  is 1.

6. For each element  $\alpha \in \mathbb{N}^n$ , the  $\mathbb{I}_n$ -module  $\mathbb{I}_n/\mathbb{I}_n\partial^{\alpha}$  is projective, moreover,  $\mathbb{I}_n/\mathbb{I}_n\partial^{\alpha} \simeq \bigoplus_{i=1}^n (K[x_i] \bigotimes \bigotimes_{j \neq i} \mathbb{I}_1(i))^{\alpha_j}$ .

*Proof.* 1. Using the equality  $\int \partial = 1 - e_{00}$ , we see that  $\mathbb{I}_1 = \mathbb{I}_1 \partial + \mathbb{I}_1 e_{00}$ . Since  $\partial e_{00} = 0$  and  $e_{00}^2 = e_{00}$ , we have  $\mathbb{I}_1 \partial \cap \mathbb{I}_1 e_{00} = (\mathbb{I}_1 \partial \cap \mathbb{I}_1 e_{00}) e_{00} \subseteq \mathbb{I}_1 \partial e_{00} = 0$ . Therefore,  $\mathbb{I}_1 = \mathbb{I}_1 \partial \bigoplus \mathbb{I}_1 e_{00}$ . Then applying the involution \* to this equality we obtain the equality  $\mathbb{I}_1 = \int \mathbb{I}_1 \bigoplus e_{00} \mathbb{I}_1$ .

2. Since  $\mathbb{I}_1 P_1 \simeq \mathbb{I}_1 e_{00} = \bigoplus_{i \in \mathbb{N}} K e_{i0}, 1 \mapsto e_{00}$ , we have  $\mathbb{I}_1 P_1 \simeq \mathbb{I}_1 / \mathbb{I}_1 \partial$ , by statement 1. Therefore,  $\mathbb{I}_n P_n \simeq \bigotimes_{i=1}^n P_1(i) \simeq \bigotimes_{i=1}^n \mathbb{I}_1(i) / \mathbb{I}_1(i) \partial_i \simeq \mathbb{I}_n / \sum_{i=1}^n \mathbb{I}_n \partial_i$ .

3. By statement 1,

$$\mathbb{I}_n = \bigotimes_{i=1}^n \mathbb{I}_1(i) = \bigotimes_{i=1}^n (\mathbb{I}_1(i)\partial_i \bigoplus \mathbb{I}_1(i)e_{00}(i)) = \mathbb{I}_n \prod_{i=1}^n e_{00}(i) \bigoplus (\sum_{i=1}^n \mathbb{I}_n\partial_i) \simeq P_n \bigoplus (\sum_{i=1}^n \mathbb{I}_n\partial_i).$$

Therefore,  $P_n$  is a projective  $\mathbb{I}_n$ -module.

4. Note that the left  $\mathbb{I}_1$ -module  $F = \bigoplus_{i \ge 0} \mathbb{I}_1 E_{ii} \simeq \bigoplus_{i \ge 0} P_1$  is projective by statement 2. Therefore,  $F_n = F^{\otimes n}$  is a projective left  $\mathbb{I}_n$ -module. Since the ideal  $F_n$  is stable under the involution  $*, F_n^* = F_n$ , the right  $\mathbb{I}_n$ -module  $F_n$  is projective.

5. The short exact sequence of left and right  $\mathbb{I}_n$ -modules  $0 \to F_n \to \mathbb{I}_n \to \mathbb{I}_n/F_n \to 0$  does not split since  $F_n$  is an essential left and right submodule of  $\mathbb{I}_n$  (Lemma 5.2.(2)). By statement 4, the projective dimension of the left and right  $\mathbb{I}_n$ -module  $\mathbb{I}_n/F_n$  is 1.

6. Let  $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i$  where  $e_1, \ldots, e_n$  is the canonical free  $\mathbb{Z}$ -basis for  $\mathbb{Z}^n$ . Let  $m = |\alpha|$ . Fix a chain of elements of  $\mathbb{Z}^n$ ,  $\beta_0 = 0, \beta_1, \ldots, \beta_m = \alpha$  such that, for all  $i, \beta_{i+1} = \beta_i + e_j$  for some index j = j(i). Then all the factors of the chain of left ideals

$$\mathbb{I}_n \partial^{\alpha} = \mathbb{I}_n \partial^{\beta_m} \subset \mathbb{I}_n \partial^{\beta_{m-1}} \subset \dots \subset \mathbb{I}_n \partial^{\beta_1} \subset \mathbb{I}_n$$

are projective  $\mathbb{I}_n$ -modules since  $\mathbb{I}_n \partial^{\beta_i} / \mathbb{I}_n \partial^{\beta_{i+1}} \simeq \mathbb{I}_n / \mathbb{I}_n \partial_j \simeq K[x_j] \otimes \mathbb{I}_{n-1}$  is the projective  $\mathbb{I}_n$ -module (statement 3). The first isomorphism is due to the fact that the element  $\partial_i$  is left regular, i.e.  $a\partial_i = b\partial_i$  implies a = b (by multiplying the equation on the right by  $\int_i$ ). Therefore, the  $\mathbb{I}_n$ -module  $\mathbb{I}_n / \mathbb{I}_n \partial^{\alpha}$  is projective. Moreover,  $\mathbb{I}_n / \mathbb{I}_n \partial^{\alpha} \simeq \bigoplus_{i=1}^n (K[x_i] \bigotimes \bigotimes_{j \neq i} \mathbb{I}_1(i))^{\alpha_j}$ .  $\Box$ 

**Theorem 6.2** Let  $\mathbb{I}_{n,m} := B_{n-m} \otimes \mathbb{I}_m$  where  $m = 0, 1, \ldots, n$  and  $\mathbb{I}_0 = B_0 := K$ . Then  $\operatorname{wdim}(\mathbb{I}_{n,m}) = n$  for all  $m = 0, 1, \ldots, n$ . In particular,  $\operatorname{wdim}(\mathbb{I}_n) = n$ .

*Proof.* The algebra  $B_n$  is Noetherian, hence  $n = \text{l.gldim}(S_{\partial_1,\ldots,\partial_n}^{-1}\mathbb{I}_{n,m}) = \text{wdim}(B_n) \leq \text{wdim}(\mathbb{I}_{n,m})$  (Corollary 7.4.3, [49]). To finish the proof of the theorem it suffices to show that the inequality wdim $(\mathbb{I}_{n,m}) \leq n$  holds for all numbers n and m. We use induction on n. The case n = 0 is trivial. So, let  $n \geq 1$  and we assume the inequality holds for all n' < n and all  $m = 0, 1, \ldots, n'$ . For n, we use the second induction on  $m = 0, 1, \ldots, n$ . When m = 0, the inequality holds since  $\mathbb{I}_{n,0} = B_n$  and wdim $(B_n) = n$ .

Suppose that m > 0 and wdim $(\mathbb{I}_{n,m'}) \leq n$  for all m' < m. We have to show that wdim $(\mathbb{I}_{n,m}) \leq n$  or, equivalently,  $\mathrm{fd}_{\mathbb{I}_{n,m}}(M) \leq n$  for all  $\mathbb{I}_{n,m}$ -modules M (fd denotes the flat dimension). Changing the order of the tensor multiples we can write  $\mathbb{I}_{n,m} = \mathbb{I}_1 \otimes \mathbb{I}_{n-1,m-1}$ . Then wdim $(\mathbb{I}_{n-1,m-1}) \leq n-1$ , by the inductive hypothesis. Recall that  $B_1 = S_{\partial}^{-1}\mathbb{I}_1 = \mathbb{I}_1/F$  and every  $\partial$ -torsion  $\mathbb{I}_1$ -module V is a direct sum of several (maybe an infinite number of) copies of the projective simple  $\mathbb{I}_1$ -module K[x] (Proposition 6.1.(6)), hence V is projective, hence V is flat. Note that  $S_{\partial}^{-1}\mathbb{I}_{n,m} \simeq \mathbb{I}_{n,m-1}$  and wdim $(\mathbb{I}_{n,m-1}) \leq n$ , by the inductive hypothesis. The  $\mathbb{I}_{n,m}$ -module tor $_{\partial}(M) := \{m \in M \mid \partial^i m = 0 \text{ for some } i\}$  is the  $\partial$ -torsion submodule of the  $\mathbb{I}_{n,m}$ -module M. There are two short exact sequences of  $\mathbb{I}_{n,m}$ -modules,

$$0 \to \operatorname{tor}_{\partial}(M) \to M \to \overline{M} \to 0, \tag{29}$$

$$0 \to \overline{M} \to S_{\partial}^{-1} M \to M' \to 0, \tag{30}$$

where the  $\mathbb{I}_{n,m}$ -modules  $\operatorname{tor}_{\partial}(M)$  and M' are  $\partial$ -torsion, and the  $\mathbb{I}_n$ -module  $S_{\partial}^{-1}M$  is  $\partial$ -torsion free. To prove that  $\operatorname{fd}_{\mathbb{I}_{n,m}}(M) \leq n$  it suffices to show that the flat dimensions of the  $\mathbb{I}_{n,m}$ modules  $\operatorname{tor}_{\partial}(M)$ ,  $S_{\partial}^{-1}M$  and M' are less or equal to n. Indeed, then by (30),  $\operatorname{fd}_{\mathbb{I}_{n,m}}(\overline{M}) \leq$   $\max\{\mathrm{fd}_{\mathbb{I}_{n,m}}(S_{\partial}^{-1}M), \mathrm{fd}_{\mathbb{I}_{n,m}}(M')\} \leq n; \text{ and by (29), } \mathrm{fd}_{\mathbb{I}_{n,m}}(M) \leq \max\{\mathrm{fd}_{\mathbb{I}_{n,m}}(\mathrm{tor}_{\partial}(M)), \mathrm{fd}_{\mathbb{I}_{n,m}}(\overline{M})\} \leq n.$ 

The  $\mathbb{I}_1$ -module  $\operatorname{tor}_{\partial}(M)$  (where  $\mathbb{I}_{n,m} = \mathbb{I}_1 \otimes \mathbb{I}_{n-1,m-1}$ ) is a direct sum of copies (may be infinitely many) of the projective simple  $\mathbb{I}_1$ -module K[x]. Note that  $\operatorname{End}_{\mathbb{I}_1}(K[x]) \simeq \ker_{K[x]}(\partial) = K$  since  $\mathbb{I}_1 K[x] \simeq \mathbb{I}_1/\mathbb{I}_1 \partial$ . Using this fact and Proposition 6.1.(6), for each finitely generated submodule Tof the  $\mathbb{I}_{n,m}$ -module  $\operatorname{tor}_{\partial}(M)$  there exists a family  $\{T_i\}_{i\in I}$  of its submodules  $T_i$  where  $(I, \leq)$  is a wellordered set such that if  $i, j \in I$  and  $i \leq j$  then  $T_i \subseteq T_j, T = \bigcup_{i \in I} T_i$  and  $T_i/\bigcup_{j < i} T_j \simeq K[x] \otimes \mathcal{T}_i$ for some  $\mathbb{I}_{n-1,m-1}$ -module  $\mathcal{T}_i$ . Note that

$$\operatorname{fd}_{\mathbb{I}_{n,m}}(K[x] \otimes \mathcal{T}_i) \le \operatorname{fd}_{\mathbb{I}_{n-1,m-1}}(\mathcal{T}_i) \le n-1,$$

since the  $\mathbb{I}_1$ -module K[x] is projective. Therefore,  $\mathrm{fd}_{\mathbb{I}_{n,m}}(T) \leq n-1$ . The module  $\mathrm{tor}_{\partial}(M) = \bigcup_{\theta \in \Theta} T_{\theta}$  is the union of its finitely generated submodules  $T_{\theta}$ , hence

$$\mathrm{fd}_{\mathbb{I}_{n,m}}(\mathrm{tor}_{\partial}(M)) = \mathrm{fd}_{\mathbb{I}_{n,m}}(\bigcup_{\theta \in \Theta} T_{\theta}) \le \sup\{\mathrm{fd}_{\mathbb{I}_{n,m}}(T_{\theta})\}_{\theta \in \Theta} = n - 1.$$

Similarly,  $\operatorname{fd}_{\mathbb{I}_{n,m}}(\overline{M}) \leq n-1$  since the  $\mathbb{I}_{n,m}$ -module  $\overline{M}$  is  $\partial$ -torsion.

It remains to show that  $\operatorname{fd}_{\mathbb{I}_{n,m}}(S_{\partial}^{-1}(M)) \leq n$ . By (27), the left  $\mathbb{I}_1$ -module  $B_1$  is flat, hence the left  $\mathbb{I}_{n,m}$ -module  $B_1 \otimes \mathbb{I}_{n-1,m-1}$  is flat. Then, by Proposition 7.2.2.(ii), [49],

 $\mathrm{fd}_{\mathbb{I}_{n,m}}(S_{\partial}^{-1}M) \leq \mathrm{fd}_{B_1 \otimes \mathbb{I}_{n-1,m-1}}(S_{\partial}^{-1}M) + \mathrm{fd}_{\mathbb{I}_{n,m}}(B_1 \otimes \mathbb{I}_{n-1,m-1}) \leq \mathrm{wdim}(\mathbb{I}_{n,m-1}) \leq n.$ 

The proof of the theorem is complete.  $\Box$ 

**Corollary 6.3** Let M be a  $\partial$ -torsion  $\mathbb{I}_{n,m}$ -module, i.e.  $S_{\partial}^{-1}M = 0$ , where  $S_{\partial}^{-1} : I_{n,m} = \mathbb{I}_1 \otimes \mathbb{I}_{n-1,m-1} \to B_1 \otimes \mathbb{I}_{n-1,m-1} = \mathbb{I}_{n,m-1}$  is the localization map and  $n, m \geq 1$ . Then there exists a family  $\{T_i\}_{i \in I}$  of  $\mathbb{I}_{n,m}$ -submodules of M such that  $M = \bigcup_{i \in I} T_i$ ,  $(I, \leq)$  is a well-ordered set such that if  $i, j \in I$  and  $i \leq j$  then  $T_i \subseteq T_j$ , and  $T_i / \bigcup_{j < i} T_j \simeq K[x] \otimes \mathcal{T}_i$  for some  $\mathbb{I}_{n-1,m-1}$ -module  $\mathcal{T}_i$ .

Proof. The  $\mathbb{I}_1$ -module M is a direct sum of (may be infinitely many) copies of the projective simple  $\mathbb{I}_1$ -module K[x]. Note that  $\operatorname{End}_{\mathbb{I}_1}(K[x]) \simeq \ker_{K[x]}(\partial) = K$  since  $\mathbb{I}_1 K[x] \simeq \mathbb{I}_1 / \mathbb{I}_1 \partial$ . Using this fact, for the  $\mathbb{I}_{n,m}$ -module M there exists a family  $\{T_i\}_{i\in I}$  of its submodules  $T_i$  where  $(I, \leq)$  is a wellordered set such that if  $i, j \in I$  and  $i \leq j$  then  $T_i \subseteq T_j$ ,  $M = \bigcup_{i \in I} T_i$  and  $T_i / \bigcup_{j < i} T_j \simeq K[x] \otimes \mathcal{T}_i$ for some  $\mathbb{I}_{n-1,m-1}$ -module  $\mathcal{T}_i$ .  $\Box$ 

**Corollary 6.4** Let A be a prime factor algebra of the algebra  $\mathbb{I}_n$ . Then wdim(A) = n.

*Proof.* By Corollary 3.3.(10), the algebra A is isomorphic to the algebra  $\mathbb{I}_{n,m}$  for some m. Now, the corollary follows from Theorem 6.2.  $\Box$ 

The next theorem is an analogue of Hilbert's Syzygy Theorem for the algebra  $\mathbb{I}_n$  and its prime factor algebras. The *flat dimension* of an A-module M is denoted by  $\mathrm{fd}_A(M)$ .

**Theorem 6.5** Let K be an algebraically closed uncountable field of characteristic zero. Let A be a prime factor algebra of  $\mathbb{I}_n$  (for example,  $A = \mathbb{I}_n$ ) and B be a Noetherian finitely generated algebra over K. Then wdim $(A \otimes B) =$ wdim(A) +wdim(B) = n +wdim(B).

*Proof.* Recall that  $A \simeq \mathbb{I}_{n,m}$  for some  $m \in \{0, 1, ..., n\}$  and wdim $(\mathbb{I}_{n,m}) = n$  (Theorem 6.2). Since

$$n + \operatorname{wdim}(B) = \operatorname{wdim}(\mathbb{I}_{n,m}) + \operatorname{wdim}(B) \leq \operatorname{wdim}(\mathbb{I}_{n,m} \otimes B),$$

it suffices to show that wdim( $\mathbb{I}_{n,m} \otimes B$ )  $\leq n + wdim(B)$  for all numbers n and m. We use induction on n. The case n = 0 is trivial since A = K. So, let  $n \geq 1$ , and we assume that the inequality holds for all n' < n and all  $m' = 0, 1, \ldots, n'$ . For the number  $n \geq 1$ , we use the second induction on  $m = 0, 1, \ldots, n$ . The case m = 0, i.e.  $\mathbb{I}_{n,0} = B_n$ , is known, Corollary 6.3, [8] (this can also be deduced from Proposition 9.1.12, [49]; see also [9]). So, let m > 0 and we assume that the inequality holds for all numbers m' < m. Let M be an  $\mathbb{I}_{n,m} \otimes B$ -module. We have to show that  $\mathrm{fd}_{\mathbb{I}_{n,m} \otimes B}(M) \leq n + \mathrm{wdim}(B)$ . We can treat M as an  $\mathbb{I}_{n,m}$ -module. Then we have the short exact sequences (29) and (30) which are, in fact, short exact sequence of  $\mathbb{I}_{n,m} \otimes B$ -modules. To prove that  $\mathrm{fd}_{\mathbb{I}_{n,m} \otimes B}(M) \leq n + \mathrm{wdim}(B)$  it suffices to show that the flat dimensions of the  $\mathbb{I}_{n,m} \otimes B$ -modules  $\mathrm{tor}_{\partial}(M)$ ,  $S_{\partial}^{-1}M$  and M' are less or equal to  $n + \mathrm{wdim}(B)$ , by the same reason as in the proof of Theorem 6.2. Repeating the same argument as at the end of the proof of Theorem 6.2, for each finitely generated submodule T of the  $\mathbb{I}_{n,m} \otimes B$ module  $\mathrm{tor}_{\partial}(M)$  (where  $\mathbb{I}_{n,m} = \mathbb{I}_1 \otimes \mathbb{I}_{n-1,m-1}$ ) there exists a family  $\{T_i\}_{i \in I}$  of its submodules  $T_i$ where  $(I, \leq)$  is a well-ordered set such that if  $i, j \in I$  and  $i \leq j$  then  $T_i \subseteq T_j, T = \bigcup_{i \in I} T_i$  and  $T_i/\bigcup_{j < i} T_j \simeq K[x] \otimes \mathcal{T}_i$  for some  $\mathbb{I}_{n-1,m-1} \otimes B$ -module  $\mathcal{T}_i$ . Note that  $\mathbb{I}_{n,m} \otimes B = \mathbb{I}_1 \otimes \mathbb{I}_{n-1,m-1} \otimes B$ and

$$\mathrm{fd}_{\mathbb{I}_{n,m}\otimes B}(K[x]\otimes\mathcal{T}_{i})\leq\mathrm{fd}_{\mathbb{I}_{n-1,m-1}\otimes B}(\mathcal{T}_{i})\leq\mathrm{wdim}(\mathbb{I}_{n-1,m-1}\otimes B)=n-1+\mathrm{wdim}(B),$$

since the  $\mathbb{I}_1$ -module K[x] is projective. Therefore,  $\mathrm{fd}_{\mathbb{I}_{n,m}\otimes B}(T_i) \leq n-1 + \mathrm{wdim}(B)$ . The  $\mathbb{I}_{n,m}\otimes B$ -module  $\mathrm{tor}_{\partial}(M) = \bigcup_{\theta \in \Theta} T_{\theta}$  is the union of its finitely generated submodules  $T_{\theta}$ , hence

$$\mathrm{fd}_{\mathbb{I}_{n,m}\otimes B}(\mathrm{tor}_{\partial}(M)) = \mathrm{fd}_{\mathbb{I}_{n,m}\otimes B}(\bigcup_{\theta\in\Theta}T_{\theta}) \leq \sup\{\mathrm{fd}_{\mathbb{I}_{n,m}\otimes B}(T_{\theta})\}_{\theta\in\Theta} = n-1 + \mathrm{wdim}(B).$$

Similarly,  $\operatorname{fd}_{\mathbb{I}_{n,m}\otimes B}(M') \leq n-1 + \operatorname{wdim}(B)$  since the  $\mathbb{I}_1$ -module M' is  $\partial$ -torsion.

It remains to show that  $\operatorname{fd}_{\mathbb{I}_{n,m}\otimes B}(S_{\partial}^{-1}M) \leq n + \operatorname{wdim}(B)$ . By (28), the left  $\mathbb{I}_1$ -module  $B_1$  is flat, hence the left  $\mathbb{I}_{n,m}\otimes B$ -module  $B_1\otimes \mathbb{I}_{n-1,m-1}\otimes B$  is flat. Then, by Proposition 7.2.2.(ii), [49],

$$\begin{aligned} \mathrm{fd}_{\mathbb{I}_{n,m}\otimes B}(S_{\partial}^{-1}M) &\leq & \mathrm{fd}_{B_{1}\otimes\mathbb{I}_{n-1,m-1}\otimes B}(S_{\partial}^{-1}M) + \mathrm{fd}_{\mathbb{I}_{n,m}\otimes B}(B_{1}\otimes\mathbb{I}_{n-1,m-1}\otimes B) \\ &\leq & \mathrm{wdim}(B_{1}\otimes\mathbb{I}_{n-1,m-1}\otimes B) \leq n-1 + \mathrm{wdim}(B_{1}\otimes B) \\ &= & n-1 + \mathrm{wdim}(B_{1}) + \mathrm{wdim}(B) = n-1 + 1 + \mathrm{wdim}(B) = n + \mathrm{wdim}(B), \end{aligned}$$

since the algebra B is Noetherian and finitely generated. The proof of the theorem is complete.  $\Box$ 

The global dimension of the algebra  $\mathbb{I}_n$ . For all Noetherian rings, wdim(M) = gldim(M), and this is not true for *non-Noetherian* ring, in general. For many Noetherian rings, including the Weyl algebras  $A_n$  and the algebras  $B_n$ , the known proofs of finding their global dimensions are, in fact, about their weak dimensions as localizations and faithfully flat extensions are used. This fact together with the fact that the algebras  $\mathbb{I}_n$  are not Noetherian are main difficulties in finding their global dimensions.

**Proposition 6.6** [5] Let M be a module over an algebra A, I a non-empty well-ordered set,  $\{M_i\}_{i\in I}$  be a family of submodules of M such that if  $i, j \in I$  and  $i \leq j$  then  $M_i \subseteq M_j$ . If  $M = \bigcup_{i\in I} M_i$  and  $pd_A(M_i/M_{\leq i}) \leq n$  for all  $i \in I$  where  $M_{\leq i} := \bigcup_{j \leq i} M_j$  then  $pd_A(M) \leq n$ .

Let  $V \subseteq U \subseteq W$  be modules. Then the factor module U/V is called a *sub-factor* of the module W. Each algebra  $\mathbb{I}_{n,m}$  is self-dual, so its left and right global dimensions coincide. Their common value is denoted by  $\operatorname{gldim}(\mathbb{I}_{n,m})$ .

**Proposition 6.7**  $n \leq \text{gldim}(\mathbb{I}_{n,m}) \leq n+m$  for all  $n \in \mathbb{N}$  and  $m = 0, 1, \ldots, n$ . In particular,  $n \leq \text{gldim}(\mathbb{I}_n) \leq 2n$ .

Proof.  $n = \text{gldim}(B_n) \leq \text{gldim}(\mathbb{I}_{n,m})$ , by (27). It remains to show that  $\text{gldim}(\mathbb{I}_{n,m}) \leq n + m$ . We use induction on n. The case n = 0 is trivial as  $\mathbb{I}_{0,0} = K$ . So, let  $n \geq 1$  and we assume that the inequality holds for all n' < n and all  $m = 0, 1, \ldots, n'$ . For n, we use the second induction on  $m = 0, 1, \ldots, n$ . When m = 0, the inequality holds since  $\mathbb{I}_{n,0} = B_n$  and  $\text{gldim}(B_n) = n$ .

Suppose that m > 0 and  $\operatorname{gldim}(\mathbb{I}_{n,m'}) \leq n + m'$  for all m' < m. We have to show that  $\operatorname{gldim}(\mathbb{I}_{n,m}) \leq n + m$ , or equivalently  $\operatorname{pd}_{\mathbb{I}_{n,m}}(M) \leq n + m$  for all  $\mathbb{I}_{n,m}$ -modules M. Changing

the order of the tensor multiples we can write  $\mathbb{I}_{n,m} = \mathbb{I}_1 \otimes \mathbb{I}_{n-1,m-1}$ . For the  $\mathbb{I}_{n,m}$ -module Mwe have the short exact sequence of  $\mathbb{I}_{n,m}$ -modules (29). By Corollary 6.3, for the  $\mathbb{I}_{n,m}$ -module  $\operatorname{tor}_{\partial}(M)$  there exists a family  $\{T_i\}_{i \in I}$  of its submodules  $T_i$  where  $(I, \leq)$  is a well-ordered set such that if  $i, j \in I$  and  $i \leq j$  then  $T_i \subseteq T_j$ ,  $\operatorname{tor}_{\partial}(M) = \bigcup_{i \in I} T_i$  and  $T_i / \bigcup_{j < i} T_j \simeq K[x] \otimes \mathcal{T}_i$  for some  $\mathbb{I}_{n-1,m-1}$ -module  $\mathcal{T}_i$ . Note that

$$\operatorname{pd}_{\mathbb{I}_{n,m}}(K[x] \otimes \mathcal{T}_i) \le \operatorname{pd}_{\mathbb{I}_1}(K[x]) + \operatorname{pd}_{\mathbb{I}_{n-1,m-1}}(\mathcal{T}_i) \le n + m - 2.$$

By Proposition 6.6,  $\operatorname{pd}_{\mathbb{I}_{n,m}}(\operatorname{tor}_{\partial}(M)) \leq n+m-2$ . Note that  $\operatorname{pd}_{\mathbb{I}_{n,m}}(\mathbb{I}_{n,m-1}) \leq 1$  (since  $0 \to F \to \mathbb{I}_1 \to B_1 \to 0$  is a projective resolution for the  $\mathbb{I}_1$ -module  $B_1$ ) and, by Proposition 7.2.2.(ii), [49],

$$\operatorname{pd}_{\mathbb{I}_{n,m}}(\overline{M}) \le \operatorname{pd}_{\mathbb{I}_{n,m-1}}(\overline{M}) + \operatorname{pd}_{\mathbb{I}_{n,m}}(\mathbb{I}_{n,m-1}) \le n+m-1+1 = n+m.$$

By (29),

 $\operatorname{pd}_{\mathbb{I}_{n,m}}(M) \le \max\{\operatorname{pd}_{\mathbb{I}_{n,m}}(\operatorname{tor}_{\partial}(M)), \operatorname{pd}_{\mathbb{I}_{n,m}}(\overline{M})\} \le n+m,$ 

as required. The proof of the theorem is complete.  $\Box$ 

Conjecture. gldim $(\mathbb{I}_n) = n$ .

## 7 The weak and the global dimensions of the Jacobian algebra $\mathbb{A}_n$

In this section, we prove that the weak dimension of the Jacobian algebra  $\mathbb{A}_n$  and of all its prime factor algebras is n (Theorem 7.2, Corollary 7.3). An analogue of Hilbert's Syzygy Theorem is established for the Jacobian algebras  $\mathbb{A}_n$  and for all its prime factor algebras (Theorem 7.4).

A K-algebra R has the endomorphism property over K if, for each simple R-module M, End<sub>R</sub>(M) is algebraic over K.

**Theorem 7.1** [10] Let K be a field of characteristic zero.

- 1. The algebra  $\mathcal{A}_n$  is a simple, affine, Noetherian domain.
- 2. The Gelfand-Kirillov dimension  $GK(\mathcal{A}_n) = 3n \ (\neq 2n = GK(\mathcal{A}_n)).$
- 3. The (left and right) global dimension  $gl.dim(\mathcal{A}_n) = n$ .
- 4. The (left and right) Krull dimension  $K.dim(\mathcal{A}_n) = n$ .
- 5. Let d = gl.dim or d = K.dim. Let R be a Noetherian K-algebra with  $d(R) < \infty$  such that R[t], the polynomial ring in a central indeterminate, has the endomorphism property over K. Then  $d(\mathcal{A}_1 \otimes R) = d(R) + 1$ . If, in addition, the field K is algebraically closed and uncountable, and the algebra R is affine, then  $d(\mathcal{A}_n \otimes R) = d(R) + n$ .

 $GK(A_1) = 3$  is due to A. Joseph [41], p. 336; see also [47], Example 4.11, p. 45.

The Jacobian algebra  $\mathbb{A}_n$  is a localization of the algebra  $\mathbb{I}_n$ . Using the presentations of the algebras  $\mathbb{I}_n$  and  $\mathbb{A}_n$  as GWAs, it is obvious that the algebra  $\mathbb{I}_n$  is the two-sided localization,

$$\mathbb{A}_n = S^{-1} \mathbb{I}_n = \mathbb{I}_n S^{-1},\tag{31}$$

of the algebra  $\mathbb{I}_n$  at the multiplicatively closed subset  $S := \{\prod_{i=1}^n (H_i + \alpha_i)_*^{n_i} \mid (\alpha_i) \in \mathbb{Z}^n, (n_i) \in \mathbb{N}^n\}$ of  $\mathbb{I}_n$  where  $(H_i + \alpha_i)_* := \begin{cases} H_i + \alpha_i & \text{if } \alpha_i \ge 0, \\ (H_i + \alpha_i)_1 & \text{if } \alpha_i < 0, \end{cases}$  since, for all elements  $\beta \in \mathbb{Z}^n$ ,

$$v_{\beta} \prod_{i=1}^{n} (H_i + \alpha_i)_*^{n_i} = \prod_{i=1}^{n} (H_i + \alpha_i - \beta_i)_*^{n_i} v_{\beta}.$$
 (32)

The left (resp. right) localization of the Jacobian algebra

$$\mathbb{A}_n = K\langle y_1, \dots, y_n, H_1^{\pm 1}, \dots, H_n^{\pm 1}, x_1, \dots, x_n \rangle, \quad (\text{where } y_i := H_i^{-1} x_i)$$

at the multiplicatively closed set  $S_{y_1,\ldots,y_n} := \{y^{\alpha} \mid \alpha \in \mathbb{N}^n\}$  (resp.  $S_{x_1,\ldots,x_n} := \{x^{\alpha} \mid \alpha \in \mathbb{N}^n\}$ ) is the algebra

$$\mathcal{A}_n \simeq S_{y_1,\dots,y_n}^{-1} \mathbb{A}_n \simeq \mathbb{A}_n S_{x_1,\dots,x_n}^{-1}.$$
(33)

The algebra  $\mathbb{A}_n$  has the involution \*. The algebra  $\mathcal{A}_n \simeq \mathbb{A}_n/\mathfrak{a}^e$  inherits the involution \* since  $(\mathfrak{a}_n^e)^* = \mathfrak{a}_n^e$ , and so do the algebras  $\mathbb{A}_{n,m} := \mathcal{A}_{n-m} \otimes \mathbb{A}_m$  where  $m = 0, 1, \ldots, n$  and  $\mathbb{A}_0 = \mathcal{A}_0 := K$ . Therefore, the algebras  $\mathbb{A}_{n,m}$  are self-dual, and so l.gldim $(\mathbb{A}_{n,m}) = r.gldim(\mathbb{A}_{n,m}) := gldim(\mathbb{A}_{n,m})$ .

**Theorem 7.2** Let  $\mathbb{A}_{n,m} := \mathcal{A}_{n-m} \otimes \mathbb{A}_m$  where m = 0, 1, ..., n and  $\mathbb{A}_0 = \mathcal{A}_0 := K$ . Then  $\operatorname{wdim}(\mathbb{A}_{n,m}) = n$  for all m = 0, 1, ..., n. In particular,  $\operatorname{wdim}(\mathbb{A}_n) = n$ .

*Proof.* By Theorem 7.1.(1,3) and (33),

 $n = \text{gldim}(\mathcal{A}_n) = \text{wdim}(\mathcal{A}_n) = \text{l.gldim}(S_{y_1,\dots,y_n}^{-1}\mathbb{A}_n) \leq \text{wdim}(S_{y_1,\dots,y_{n-m}}^{-1}\mathbb{A}_n) = \text{wdim}(\mathbb{A}_{n,m})$  $\leq \text{wdim}(\mathbb{I}_{n,m}) = n \quad \text{(by (31) and Theorem 6.2)}.$ 

Therefore, wdim $(\mathbb{A}_{n,m}) = n$  for all n and m.  $\Box$ 

**Corollary 7.3** Let A be a prime factor algebra of the algebra  $A_n$ . Then wdim(A) = n.

*Proof.* By Corollary 3.5, [11], the algebra A is isomorphic to the algebra  $A_{n,m}$  for some m. Now, the corollary follows from Theorem 7.2.  $\Box$ 

The next theorem is an analogue of Hilbert's Syzygy Theorem for the Jacobian algebras and their prime factor algebras.

**Theorem 7.4** Let K be an algebraically closed uncountable field of characteristic zero. Let A be a prime factor algebra of  $\mathbb{A}_n$  (for example,  $A = \mathbb{A}_n$ ) and B be a Noetherian finitely generated algebra over K. Then wdim $(A \otimes B) =$ wdim(A) +wdim(B) = n +wdim(B).

*Proof.* Recall that  $A \simeq \mathbb{A}_{n,m}$  for some  $m \in \{0, 1, ..., n\}$  and wdim $(\mathbb{A}_{n,m}) = n$  (Theorem 7.2). Since

$$n + \operatorname{wdim}(B) = \operatorname{wdim}(\mathbb{A}_{n,m}) + \operatorname{wdim}(B) \leq \operatorname{wdim}(\mathbb{A}_{n,m} \otimes B)$$
$$\leq \operatorname{wdim}(S^{-1}\mathbb{I}_n \otimes B) \leq \operatorname{wdim}(\mathbb{I}_n \otimes B)$$
$$= n + \operatorname{wdim}(B) \text{ (by Theorem 6.5).}$$

Therefore, wdim $(\mathbb{A}_{n,m} \otimes B) = n + l.gldim(B)$ . The proof of the theorem is complete.  $\Box$ 

**Proposition 7.5**  $n \leq \text{gldim}(\mathbb{A}_{n,m}) \leq n + m$  for all  $n \in \mathbb{N}$  and m = 0, 1, ..., n. In particular,  $n \leq \text{gldim}(\mathbb{A}_n) \leq 2n$ .

Proof. By Theorem 7.1.(3), Proposition 6.7, (31) and (33),  $n = \text{gldim}(\mathcal{A}_n) \leq \text{gldim}(\mathbb{A}_{n,m}) \leq \text{gldim}(\mathbb{I}_{n,m}) \leq n+m$ .  $\Box$ 

Conjecture. gldim $(\mathbb{A}_n) = n$ .

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Department of Pure Mathematics University of Sheffield Hicks Building Sheffield S3 7RH UK email: v.bavula@sheffield.ac.uk