

GLOBAL MINIMIZERS OF COEXISTENCE FOR COMPETING SPECIES

MONICA CONTI

Politecnico di Milano - Dipartimento di Matematica "F. Brioschi"
Via Bonardi 9, 20133 Milano, Italy

VERONICA FELLI

Università di Milano Bicocca - Dipartimento di Matematica e Applicazioni
Via Cozzi 53, 20125 Milano, Italy

ABSTRACT. A class of variational models describing ecological systems of k species competing for the same resources is investigated. The occurrence of coexistence in minimal energy solutions is discussed and positive results are proven for suitably differentiated internal dynamics.

1. INTRODUCTION

This paper is focused on a class of variational problems suitable for studying the dynamic of segregation of k organisms which share the same territory $\Omega \subset \mathbb{R}^N$. Calling u_i the density of the i -th population and $F_i(u_i)$ its internal potential, the *free energy* of the system is

$$(1.1) \quad \mathcal{E}(u_1, u_2, \dots, u_k) = \sum_{i=1}^k \int_{\Omega} \left(\frac{1}{2} |\nabla u_i(x)|^2 - F_i(u_i(x)) \right) dx,$$

given by the sum of the internal energies of each species. In this context, the question of finding a global minimizer of the energy in the class of segregated states arises in a natural way. More precisely, if we define

$$\mathcal{U} = \left\{ U = (u_1, u_2, \dots, u_k) \in [H^1(\Omega)]^k : u_i \geq 0, u_i \cdot u_j = 0 \text{ if } i \neq j, \text{ a.e. in } \Omega \right\},$$

we are led to the following *optimal partition problem*:

$$(1.2) \quad \text{finding } U \in \mathcal{U} \text{ such that } \mathcal{E}(U) = \min_{V \in \mathcal{U}} \mathcal{E}(V).$$

This problem has been recently settled in [7], in connection with strongly competing variational systems of Lotka-Volterra type

$$(1.3) \quad -\Delta u_i = f_i(u_i) - \varkappa u_i \sum_{j \neq i} u_j^2, \quad \text{in } \Omega,$$

which, since the pioneering work of Volterra, constitute one of the most studied theoretical models of population ecology, see [14]. As a matter of fact, as the competition rate \varkappa grows indefinitely, the components of any (nonnegative) solution of

the system tend to separate their supports, leading to an element of \mathcal{U} ; in particular, the problem of finding minimal energy solutions of (1.3) formally translates, as $\varkappa \rightarrow \infty$, into (1.2), see also [6, 8, 11].

In the understanding of the spatial behavior of interacting species, a central problem is to establish whether *coexistence* of all the species occurs, or the internal growth leads to *extinction*, that is, configurations where one or more densities are null: in this paper we address the question in the two different theoretical settings, endowing the models with null Dirichlet boundary conditions:

$$(1.4) \quad u_i = 0 \quad \text{on } \partial\Omega, \quad i = 1, \dots, k.$$

At a first insight, extinction has to be expected for competing systems which are, in a sense, too uniform. For instance, in the case of null Neumann boundary conditions, the global minimum of \mathcal{E} on \mathcal{U} is in general achieved by configurations where only one species is alive, see [5, Proposition 2.1]. Nonetheless, a mechanism to avoid extinction can be found in the spatial inhomogeneity of the territory. Indeed, working in a special class of non-convex domains close to a union of k disjoint balls, the existence of *local* minima of \mathcal{E} where all the species are present is proven in [5], (see also [4]).

As a matter of fact, if the internal energies f_i are not differentiated, extinction of global minimizers under null boundary conditions occurs in any domain, see Section 4.1:

Theorem 1.1. *Let Ω be a bounded Lipschitz domain and f be a Lipschitz continuous function. If $f_i = f$ for all $i = 1, \dots, k$ and $F_i(s) = \int_0^s f_i(t) dt$, then any global minimizer of \mathcal{E} on $\mathcal{U} \cap H_0^1(\Omega)$ has at most one nonzero component.*

This motivates the question whether different internal laws might produce a mechanism to ensure coexistence. With the aim of providing a first answer to this conjecture, in this note we consider the special situation when the internal energies f_i are of the same type but act at different density scales, see assumption (2.2).

We first investigate global minimizers for systems in the form (1.3), namely solutions of the energy minimization problem

$$\min \left\{ (u_1, u_2, \dots, u_k) \in [H^1(\Omega)]^k : \sum_{i=1}^k \int_{\Omega} \left(\frac{1}{2} |\nabla u_i(x)|^2 - F_i(u_i(x)) \right) dx + \frac{1}{2} \varkappa \sum_{\substack{i,j=1 \\ i \neq j}}^k u_i(x)^2 u_j(x)^2 \right\}.$$

We prove in Theorem 2.1 that any global minimizer is a coexistence state of (1.3) where all the k species are present, provided the internal growths f_i of $k - 1$ populations act at a small density scale, depending on \varkappa . This is done with a great deal of generality both with respect to the domain and to the competing interaction term appearing in (1.3), see (H1)–(H3) below, but the dependence of f_i 's on \varkappa does not allow recovering a meaningful coexistence result for the corresponding optimal partition problem (1.2), see Remark 2.2. It is worth pointing out that the investigation of positive solutions to competitive systems in the case of $k \geq 3$ densities is a

challenging task and only partial results are known, see e.g. [4, 5, 6, 9, 10, 12] and the discussions therein for more references.

To investigate the possibility of coexistence for solutions to the optimal partition problem (1.2), following an idea developed in [16] to show the existence of sign-changing solutions to some elliptic equations, we focus on a certain class of (possibly convex) domains characterized by the presence of an angle. In this framework we prove, in Theorem 2.3, that any global minimizer of \mathcal{E} among segregated states has *at least two* nontrivial positive components. Although the result is not exhaustive for an arbitrary number of species, for systems of two populations it allows us to provide the full picture of the coexistence phenomenon (Theorem 2.4). Namely, we first prove that any minimal solution of system (1.3) with $k = 2$ is an equilibrium configuration where both the species are present, provided the scales of their internal energies are different but *independent* from the competition rate \varkappa . Hence we perform the asymptotic analysis as $\varkappa \rightarrow \infty$ and prove that both components survive as the interspecific competition becomes larger and larger. As a result, any minimal state of system (1.3) converges to a spatially segregated distribution where the two densities coexist and solve the optimal partition problem (1.2).

In conclusion, our results suggest that in ecological systems with strong competition between the species, suitably differentiated internal energies may ensure coexistence in minimal energy configurations.

2. ASSUMPTIONS AND MAIN RESULTS

Let $k \geq 2$, $\varepsilon = (\varepsilon_2, \dots, \varepsilon_k) \in (0, 1)^{k-1}$, $\lambda > 0$, $\varkappa > 0$, and Ω be an open bounded set in \mathbb{R}^N ($N \geq 2$). We shall consider a class of competitive systems of the form

$$(2.1) \quad \begin{cases} -\Delta u_i(x) = \lambda f_{i,\varepsilon}(u_i(x)) - \varkappa \frac{\partial H}{\partial u_i}(u_1(x), u_2(x), \dots, u_k(x)), & \text{in } \Omega, \\ u_i \in H_0^1(\Omega), & i = 1, \dots, k, \end{cases}$$

where $f_{i,\varepsilon}$ and H satisfy the following sets of assumptions.

Assumptions on $f_{i,\varepsilon}$. Let $g \in C^0(\mathbb{R})$ satisfying

(F1) $g(s) = 0$ for all $s \in (-\infty, 0]$ and g is right differentiable at 0 with $g'_+(0) = 1$;

(F2) there exists $\beta > 0$ such that

$$g(t) < 0 \text{ for all } t > \beta \text{ and } g(t) \geq 0 \text{ for all } t \in (0, \beta);$$

(F3) $\int_0^\beta g(s) ds = \alpha > 0$.

A typical example is given by the classical logistic nonlinearity (see e.g. [9]), namely $g(s) = s - s^2$ for $s \geq 0$. We set

$$(2.2) \quad f_{i,\varepsilon}(s) = \begin{cases} g(s), & \text{if } i = 1, \\ \frac{1}{\sqrt{k\varepsilon_i}} g\left(\frac{\sqrt{k}t}{\varepsilon_i}\right), & \text{if } i = 2, \dots, k. \end{cases}$$

It is immediate to check that, for all $i \geq 2$, $f_{i,\varepsilon}$ satisfies

$$(2.3) \quad f_{i,\varepsilon}(s) < 0 \text{ for all } s > \beta_i, \quad \int_0^{\beta_i} f_{i,\varepsilon}(s) ds = \frac{\alpha}{k},$$

where, for $i \geq 2$, $\beta_i = \frac{\beta \varepsilon_i}{\sqrt{k}}$. For the sake of convenience, we shall refer to β as β_1 .

Assumptions on H . Let $H \in C^1(\mathbb{R}^k)$ satisfy, for all $(s_1, s_2, \dots, s_k) \in \mathbb{R}^k$,

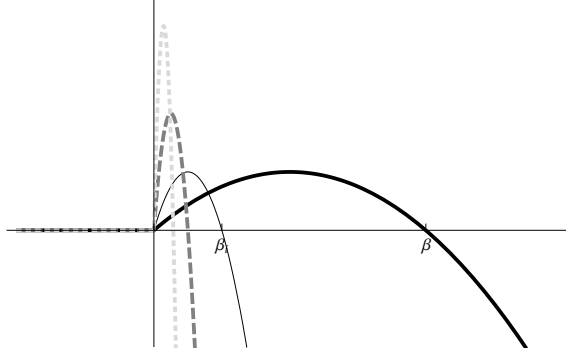


FIGURE 1. The nonlinearities $f_{i,\varepsilon}$ in the case $g(s) = s - s^2$ with $k = 4$ and $\varepsilon = (\frac{1}{2}, \frac{1}{4}, \frac{1}{7})$.

$$(H1) \quad H(s_1, s_2, \dots, s_k) \geq 0,$$

$$(H2) \quad s_i \frac{\partial H}{\partial s_i}(s_1, s_2, \dots, s_k) \geq 0 \quad \text{for all } i = 1, \dots, k,$$

$$(H3) \quad \begin{cases} H(s_1, s_2, \dots, s_k) = 0 & \text{if } s_i = 0 \text{ for at least } k-1 \text{ variables,} \\ \frac{\partial H}{\partial s_i}(s_1, s_2, \dots, s_k) = 0 & \text{implies that either } s_i = 0 \text{ or } s_j = 0 \text{ for all } j \neq i. \end{cases}$$

The first assumption states the competitive character of the interaction term; a typical example which fits all the above assumptions is

$$(2.4) \quad H(s_1, s_2, \dots, s_k) = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^k s_i^2 s_j^2,$$

which is widely used in modeling population dynamics, nonlinear optics (see e.g. [1, 13]), and Bose-Einstein condensation (see [3, 15, 17]).

Setting $F_{i,\varepsilon}(t) = \int_0^t f_{i,\varepsilon}(s) ds$, we define the internal energy of the system

$$I_\varepsilon^{\lambda, \varkappa} : [H_0^1(\Omega)]^k \rightarrow (-\infty, \infty]$$

as

$$(2.5) \quad I_\varepsilon^{\lambda, \varkappa}(u_1, \dots, u_k) = \sum_{i=1}^k \int_{\Omega} \left(\frac{1}{2} |\nabla u_i(x)|^2 - \lambda F_{i,\varepsilon}(u_i(x)) \right) dx \\ + \varkappa \int_{\Omega} H(u_1(x), u_2(x), \dots, u_k(x)) dx.$$

Our first result states that the global minimizers of $I_\varepsilon^{\lambda, \varkappa}$ are configurations of coexistence if the range ε of the internal growths of $k-1$ species is suitably related to λ and \varkappa .

Theorem 2.1. *Let $g \in C^0(\mathbb{R})$ satisfy (F1–3), $H \in C^1(\mathbb{R}^k)$ satisfy (H1)–(H2), and $\Omega \subset \mathbb{R}^N$ be a bounded domain. There exists $\lambda_0 > 0$ such that, for every $\lambda > \lambda_0$ and $\varkappa \geq 0$, there exists $\varepsilon_{\lambda, \varkappa} > 0$ with the following property: for all $\varepsilon \in (0, \varepsilon_{\lambda, \varkappa})^{k-1}$, the competing system (2.1) with $f_{i, \varepsilon}$ as in (2.2) has a solution $U = (u_1, \dots, u_k) \in [H_0^1(\Omega)]^k$ satisfying*

- (i) $u_i \neq 0$ for all $i = 1, \dots, k$;
- (ii) $0 \leq u_i \leq \beta_i$ for all $i = 1, \dots, k$;
- (iii) U is a global minimizer of $I_\varepsilon^{\lambda, \varkappa}$, namely

$$I_\varepsilon^{\lambda, \varkappa}(U) = \min \left\{ I_\varepsilon^{\lambda, \varkappa}(V), V \in [H_0^1(\Omega)]^k \right\}.$$

Furthermore, $\varepsilon_{\lambda, \varkappa}$ depends on the ratio λ/\varkappa and tends to 0 if $\lambda/\varkappa \rightarrow 0$.

Some remarks are in order.

Remark 2.2.

a) It will be clear from the proof how $\varepsilon_{\lambda, \varkappa}$ depends on the data of the problem, see (4.8). For instance, for the Lotka-Volterra model in a ball, H as in (2.4) and $g(u) = u - u^2$, we can choose

$$\varepsilon_{\lambda, \varkappa}^2 = \frac{1}{6k^2} \frac{\lambda}{\varkappa}.$$

b) If the interspecific competition rate \varkappa grows (at λ fixed), then every ε_i becomes smaller and smaller. Hence by (ii) we learn that $k - 1$ components annihilate uniformly in Ω , implying that in the limit configuration as $\varkappa \rightarrow \infty$ only the first component is alive.

Concerning the optimal partition problem stated in the introduction, we shall focus on a special class of domains.

Description of the domain. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded Lipschitz domain with $\mathbf{0} \in \partial\Omega$ such that

- (D1) $\Omega \subset T$, where $T = \left\{ x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : m\sqrt{\sum_{i=2}^N x_i^2} \leq x_1 \leq 1 \right\}$ and $m > 1$;
- (D2) there exists $\delta_0 \in (0, 1)$ such that

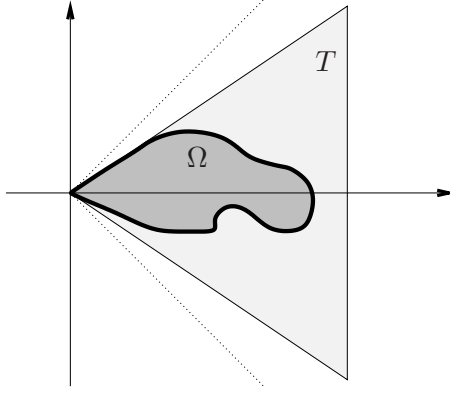
$$\delta\Omega = \{x \in \mathbb{R}^N : \delta^{-1}x \in \Omega\} \subset \Omega,$$

for every $\delta \in (0, \delta_0)$.

Theorem 2.3. *Let $g \in C^0(\mathbb{R})$ satisfy (F1–3) and Ω be a bounded Lipschitz domain satisfying (D1–2). There exist $\lambda_0 > 0$ and $\varepsilon_0 > 0$ such that, if $\varepsilon \in (0, \varepsilon_0)^{k-1}$ and $\lambda > \lambda_0$, then every global minimizer of*

$$\mathcal{E}_\varepsilon^\lambda(u_1, \dots, u_k) = \sum_{i=1}^k \int_\Omega \left(\frac{1}{2} |\nabla u_i(x)|^2 - \lambda F_{i, \varepsilon}(u_i(x)) \right) dx$$

on $\mathcal{U} \cap H_0^1(\Omega)$ has at least two nonnegative and nonzero components.

FIGURE 2. An example of domain $\Omega \subset \mathbb{R}^2$.

In the case of two species we have a better understanding of the phenomenon. In particular, we establish the link between limit configurations of system (2.1) as $\varkappa \rightarrow \infty$ and solutions to the optimal partition problem.

Theorem 2.4. *Let $k = 2$. Let $g \in C^0(\mathbb{R})$ satisfy (F1–3), $H \in C^1(\mathbb{R}^2)$ satisfy (H1–3) and Ω be a bounded Lipschitz domain satisfying (D1–2). There exist $\lambda_0 > 0$ and $\varepsilon_0 > 0$ such that, if $\varepsilon \in (0, \varepsilon_0)^{k-1}$ and $\lambda > \lambda_0$, then the global minimum of $I_\varepsilon^{\lambda, \varkappa}$ is achieved for all $\varkappa > 0$ and every global minimizer is a nontrivial configuration $(u_1^\varkappa, u_2^\varkappa)$ with both $u_i^\varkappa \geq 0$, $u_i^\varkappa \not\equiv 0$, $i = 1, 2$. Moreover, for every fixed $\varepsilon \in (0, \varepsilon_0)^{k-1}$ and $\lambda > \lambda_0$, there exists $U = (u_1, u_2) \in [H_0^1(\Omega)]^2$ such that*

- (1) $u_i \not\equiv 0$ for $i = 1, 2$,
- (2) $U = (u_1, u_2) \in \mathcal{U}$,
- (3) U is a global minimizer of $\mathcal{E}_\varepsilon^\lambda$ on $\mathcal{U} \cap H_0^1(\Omega)$,

and, up to subsequences, u_i^\varkappa converges strongly to u_i in $H^1(\Omega)$.

3. PRELIMINARY RESULTS

Let Ω be a bounded open set in \mathbb{R}^N and $g \in C^0(\mathbb{R})$ satisfy (F1–3). For every $\lambda > 0$, $\varepsilon \in (0, 1)^{k-1}$, and $i \geq 2$, let us define $J_1^\lambda, J_{i, \varepsilon}^\lambda : H_0^1(\Omega) \rightarrow (-\infty, +\infty]$

$$J_1^\lambda(u) = \int_\Omega \left(\frac{1}{2} |\nabla u(x)|^2 - \lambda \int_0^{u(x)} g(s) ds \right) dx,$$

$$J_{i, \varepsilon}^\lambda(u) = \int_\Omega \left(\frac{1}{2} |\nabla u(x)|^2 - \lambda F_{i, \varepsilon}(u(x)) \right) dx.$$

If $\lambda > \lambda_1(\Omega)$, with $\lambda_1(\Omega)$ being the first eigenvalue of the Laplace operator with null Dirichlet boundary conditions, it is easy to prove that the infima

$$m_1^\lambda := \inf \left\{ J_1^\lambda(u), u \in H_0^1(\Omega) \right\}, \quad m_{i, \varepsilon}^\lambda := \inf \left\{ J_{i, \varepsilon}^\lambda(u), u \in H_0^1(\Omega) \right\}$$

are achieved and any minimizer is a positive weak solution to the elliptic equation

$$\begin{cases} -\Delta u = \lambda f_{i, \varepsilon}(u), & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

see e.g. [2].

Lemma 3.1. *There hold*

$$\begin{aligned} m_1^\lambda &\geq -\alpha\lambda|\Omega|, & \lim_{\lambda \rightarrow +\infty} \lambda^{-1}m_1^\lambda &= -\alpha|\Omega|, \\ m_{i,\varepsilon}^\lambda &\geq -\lambda\frac{\alpha}{k}|\Omega|, & \lim_{\lambda \rightarrow +\infty} \lambda^{-1}m_{i,\varepsilon}^\lambda &= -\frac{\alpha}{k}|\Omega|, \quad i = 2, \dots, k, \end{aligned}$$

where $|\cdot|$ denotes the Lebesgue measure in \mathbb{R}^N .

Proof. We prove the result for $m_{i,\varepsilon}^\lambda$, $i \geq 2$; similar computations hold for m_1^λ . By (2.3) we have

$$m_{i,\varepsilon}^\lambda \geq -\lambda \int_{\Omega} \left(\int_0^{v_i(x)} f_{i,\varepsilon}(s) ds \right) dx \geq -\lambda \frac{\alpha}{k} |\Omega|.$$

We are left to show that for any $\delta > 0$ there exists $\phi \in H_0^1(\Omega)$ such that

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1} J_{i,\varepsilon}^\lambda(\phi) \leq -\frac{\alpha}{k} |\Omega| + \delta.$$

Let $\phi \in H_0^1(\Omega)$ such that $\phi \geq 0$ a.e. in Ω and $\phi = \beta/\sqrt{k}$ on a set $\Omega' \subset \Omega$ satisfying $|\Omega'| > |\Omega| - \delta$. Then

$$\lambda^{-1} J_{i,\varepsilon}^\lambda(\varepsilon_i \phi) \leq \lambda^{-1} \frac{\varepsilon_i^2}{2} \int_{\Omega} |\nabla \phi|^2 - \frac{\alpha}{k} |\Omega'| \leq C\lambda^{-1} + \frac{\alpha}{k} (-|\Omega| + \delta)$$

for some $C > 0$, and the result follows for λ large. \square

According to the lemma above, we define λ_0 as the smallest positive number which is greater than $\lambda_1(\Omega)$ and for which the following inequality holds

$$(3.1) \quad \lambda^{-1} m_1^\lambda < -\alpha|\Omega| \left(1 - \frac{1}{2k} \right), \quad \forall \lambda > \lambda_0.$$

Lemma 3.2. *Let $H \in C^1(\mathbb{R}^k)$ satisfy (H1–2). Let (u_1, \dots, u_k) be a weak $[H_0^1(\Omega)]^k$ -solution of the system (2.1). Then*

$$0 \leq u_i(x) \leq \beta_i, \quad \text{for a.e. } x \in \Omega \text{ and all } i = 1, \dots, k.$$

Proof. Testing (1.3) with $-u_i^-$ and using (F1) and (H2), we obtain that $u_i \geq 0$ a.e. in Ω for all $i = 1, 2, \dots, k$. On the other hand, by testing (1.3) with $(u_i - \beta_i)^+$ and using (2.3) and (H2), we deduce the required inequality. \square

Lemma 3.3. *Let Ω be a bounded Lipschitz domain satisfying (D1–2). If $u \in H_0^1(\Omega)$ weakly solves*

$$-\Delta u \leq \lambda g(u), \quad \text{in } \Omega,$$

then, for a.e. $x \in \Omega$,

$$u(x_1, \dots, x_N) \leq \gamma \left(x_1^2 - m^2 \sum_{i=2}^N x_i^2 \right).$$

with $\gamma = \frac{\lambda \max_{[0,\beta]} g}{2(m^2(N-1)-1)}$.

Proof. Let us denote the right hand side of the inequality by $\tilde{u}(x_1, \dots, x_N)$. By simple computations, noticing that \tilde{u} is nonnegative on the boundary of T and $\Omega \subset T$, it is easy to verify that

$$\begin{cases} -\Delta(\tilde{u} - u) \geq \lambda \max_{[0, \beta]} g - \lambda g(u) \geq 0, & \text{in } \Omega, \\ \tilde{u} - u \geq 0, & \text{on } \partial\Omega. \end{cases}$$

Testing the above inequality with $-(\tilde{u} - u)^-$ we deduce that $u(x) \leq \tilde{u}(x)$ for a.e. $x \in \Omega$. \square

4. PROOF OF THE MAIN RESULTS.

4.1. Proof of Theorem 1.1. For $U \in \mathcal{U}$, let $\mathcal{E}(U)$ be defined as in (1.1) with $F_i(s) = \int_0^s f(t) dt$ for all i and let

$$\mu = \inf_{u \in H_0^1(\Omega)} \int_{\Omega} \left(\frac{1}{2} |\nabla u(x)|^2 - F(u) \right) dx,$$

By taking k -tuples of the form $(u, 0, \dots, 0)$, we realize that

$$\inf_{U \in \mathcal{U} \cap H_0^1(\Omega)} \mathcal{E}(U) \leq \mu.$$

Assume there exists a minimizing k -tuple $V = (v_1, \dots, v_k) \in \mathcal{U} \cap H_0^1(\Omega)$ and define $\tilde{V} = (\tilde{v}_1, \dots, \tilde{v}_k)$ where $\tilde{v}_1 = \sum_i v_i$ and $\tilde{v}_i = 0$ for all $i > 1$. Then

$$\mu \geq \mathcal{E}(V) = \mathcal{E}(\tilde{V}) = \int_{\Omega} \left(\frac{1}{2} |\nabla \tilde{v}_1(x)|^2 - F(\tilde{v}_1(x)) \right) dx \geq \mu$$

implying in particular that \tilde{v}_1 is a weak solution of $-\Delta u = f(u)$ (see e.g. [2]). By the Strong Maximum Principle, we deduce that either $\tilde{v}_1 \equiv 0$ (and then $v_i \equiv 0$ for all $i = 1, \dots, k$) or $\tilde{v}_1(x) > 0$ for a.e. $x \in \Omega$ and then $k - 1$ components of V must be null.

4.2. Proof of Theorem 2.1. Let λ_0 as in (3.1) and, for every fixed $\lambda > \lambda_0$, let us consider the minimization problem

$$\Lambda = \inf_{U \in [H_0^1(\Omega)]^k} I_{\varepsilon}^{\lambda, \varkappa}(U),$$

where $I_{\varepsilon}^{\lambda, \varkappa} : [H_0^1(\Omega)]^k \rightarrow (-\infty, +\infty]$ is defined in (2.5).

Step 1. Λ is achieved. We first observe that, by (H1) and Lemma 3.1,

$$I_{\varepsilon}^{\lambda, \varkappa}(U) \geq m_1^{\lambda} + \sum_{i=2}^k m_{i, \varepsilon}^{\lambda} \geq -\lambda \alpha |\Omega| \left(1 + \frac{k-1}{k} \right)$$

for all $U \in [H_0^1(\Omega)]^k$, hence $\Lambda > -\infty$. Let $\{V_n = (v_1^n, \dots, v_k^n)\}_{n \in \mathbb{N}}$ be a minimizing sequence, i.e. $\lim_{n \rightarrow +\infty} I_{\varepsilon}^{\lambda, \varkappa}(V_n) = \Lambda$. Notice that we can choose V_n such that $v_i^n \geq 0$ a.e. in Ω for all $i = 1, \dots, k$; otherwise we take $((v_1^n)^+, \dots, (v_k^n)^+)$ with $(v_i^n)^+ := \max\{v_i^n, 0\}$, which is another minimizing sequence. Indeed, in view of (2.3) and (H2), the function $t \mapsto H(s_1, \dots, s_{i-1}, t, s_{i+1}, \dots, s_k)$ has a global minimum in $t = 0$ thus yielding $H(v_1^+, \dots, v_k^+) \leq H(v_1, \dots, v_k)$. Besides, since the function $t \mapsto H(s_1, \dots, s_{i-1}, t, s_{i+1}, \dots, s_k)$ is non decreasing in $(0, +\infty)$ from (F2) and (H2),

letting $U_n = (u_1^n, \dots, u_k^n)$ with $u_i^n = \min\{v_i^n, \beta_i\}$, we have that $I_\varepsilon^{\lambda, z}(U_n) \leq I_\varepsilon^{\lambda, z}(V_n)$. Then also $\{U_n\}_{n \in \mathbb{N}}$ is a minimizing sequence.

Since $\{U_n\}_{n \in \mathbb{N}}$ is a minimizing sequence and it is uniformly bounded, it is easy to realize that $\{U_n\}_{n \in \mathbb{N}}$ is bounded in $[H_0^1(\Omega)]^k$; hence there exists a subsequence, still denoted as $\{U_n\}_{n \in \mathbb{N}}$, which converges to some $U = (u_1, \dots, u_k) \in [H_0^1(\Omega)]^k$ weakly in $[H_0^1(\Omega)]^k$, strongly in $[L^2(\Omega)]^k$ and a.e. in Ω . A.e. convergence implies that $0 \leq u_i \leq \beta_i$ a.e. in Ω . From the Dominated Convergence Theorem, it follows that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{\Omega} F_{i, \varepsilon}(u_i^n(x)) dx &= \int_{\Omega} F_{i, \varepsilon}(u_i(x)) dx, \quad \text{for every } i = 1, \dots, k, \\ \lim_{n \rightarrow +\infty} \int_{\Omega} H(u_1^n(x), \dots, u_k^n(x)) dx &= \int_{\Omega} H(u_1(x), \dots, u_k(x)) dx, \end{aligned}$$

which, together with weak lower semi-continuity, yields

$$\Lambda \leq I_\varepsilon^{\lambda, z}(U) \leq \liminf_{n \rightarrow +\infty} I_\varepsilon^{\lambda, z}(U_n) = \lim_{n \rightarrow +\infty} I_\varepsilon^{\lambda, z}(U_n) = \Lambda,$$

thus proving that U attains Λ .

Step 2. If $U = (u_1, \dots, u_k)$ is a minimizer attaining Λ , then

$$(4.1) \quad u_1 \not\equiv 0, \quad \text{and} \quad J_1^\lambda(u_1) < -\frac{\alpha\lambda|\Omega|}{2k} < 0.$$

Indeed, letting $u \in H_0^1(\Omega)$ such that $m_1^\lambda = J_1^\lambda(u)$, we have

$$\Lambda \leq I_\varepsilon^{\lambda, z}(u, 0, \dots, 0) = J_1^\lambda(u) = m_1^\lambda.$$

Besides, appealing to Lemma 3.1 we learn that

$$\sum_{i=2}^k J_{i, \varepsilon}^\lambda(u_i) \geq \sum_{i=2}^k m_{i, \varepsilon}^\lambda \geq -\alpha\lambda|\Omega| \frac{k-1}{k}.$$

Since $\Lambda = I_\varepsilon^{\lambda, z}(U) \geq J_1^\lambda(u_1) + \sum_{i=2}^k J_{i, \varepsilon}^\lambda(u_i)$ by (H1), choosing λ as in (3.1) we finally have

$$J_1^\lambda(u_1) \leq m_1^\lambda - \sum_{i=2}^k J_{i, \varepsilon}^\lambda(u_i) < -\alpha\lambda|\Omega| \left(1 - \frac{1}{2k} - \frac{k-1}{k}\right) = -\frac{\alpha\lambda|\Omega|}{2k} < 0,$$

thus proving the estimate in (4.1), which in particular ensures $u_1 \not\equiv 0$.

Step 3. If $U = (u_1, \dots, u_k)$ is a minimizer attaining Λ , then $u_i \not\equiv 0$ for all $i = 1, \dots, k$. We already know by step 2 that $u_1 \not\equiv 0$. Moreover, by standard Critical Point Theory, $U = (u_1, \dots, u_k)$ is a weak solution to (2.1), and hence, by Lemma 3.2, $0 \leq u_i(x) \leq \beta_i$ for all $i = 1, \dots, k$ and a.e. $x \in \Omega$. Assume by contradiction that

$$(4.2) \quad u_i \equiv 0 \quad \text{for some } i > 1.$$

Let us fix $x_0 \in \Omega$ and $r, R > 0$ such that $B(x_0, r) \subset \Omega \subset B(x_0, R)$ and define

$$w_i(x) = \frac{\varepsilon_i}{\sqrt{k}} u_1(\varepsilon_i^{-1}(x - x_0)).$$

Notice that $w_i \in H_0^1(A_i)$ where

$$A_i = \left\{ x \in \mathbb{R}^N : \frac{x - x_0}{\varepsilon_i} \in \Omega \right\}.$$

Moreover $|A_i| = \varepsilon_i^N |\Omega|$ and $A_i \subset B(x_0, R\varepsilon_i)$, which implies $A_i \subset \Omega$ if $\varepsilon_i < \frac{r}{R}$. In particular $w_i \in H_0^1(\Omega)$ if $\varepsilon_i < \frac{r}{R}$. From Lemma 3.2,

$$(4.3) \quad 0 \leq w_i(x) \leq \frac{\beta \varepsilon_i}{\sqrt{k}} \quad \text{for a.e. } x \in A_i.$$

Since

$$\int_0^{w_i(x)} f_{i,\varepsilon}(s) ds = \frac{1}{\sqrt{k}\varepsilon_i} \int_0^{w_i(x)} g(\sqrt{k}\varepsilon_i^{-1}s) ds = \frac{1}{k} \int_0^{u_1(\varepsilon_i^{-1}(x-x_0))} g(t) dt,$$

from (4.1) it follows

$$(4.4) \quad \begin{aligned} J_{i,\varepsilon}^\lambda(w_i) &= \int_{A_i} \left(\frac{1}{2} |\nabla w_i(x)|^2 - \lambda \int_0^{w_i(x)} f_{i,\varepsilon}(s) ds \right) dx \\ &= \int_{A_i} \left(\frac{1}{2k} \left| \nabla u_1 \left(\frac{x - x_0}{\varepsilon_i} \right) \right|^2 - \frac{\lambda}{k} \int_0^{u_1(\varepsilon_i^{-1}(x-x_0))} g(s) ds \right) dx \\ &= \frac{\varepsilon_i^N}{k} \int_\Omega \left(\frac{1}{2} |\nabla u_1(x)|^2 - \lambda \int_0^{u_1(x)} g(s) ds \right) dx \\ &= \frac{\varepsilon_i^N}{k} J_1^\lambda(u_1) < -\varepsilon_i^N \frac{\alpha \lambda |\Omega|}{2k^2}. \end{aligned}$$

We now claim that, letting

$$W = (u_1, \dots, u_{i-1}, w_i, u_{i+1}, \dots, u_k),$$

there holds $I_\varepsilon^{\lambda,\varkappa}(W) < I_\varepsilon^{\lambda,\varkappa}(U)$ provided ε_i is small enough. Indeed

$$(4.5) \quad I_\varepsilon^{\lambda,\varkappa}(W) - I_\varepsilon^{\lambda,\varkappa}(U) = J_{i,\varepsilon}^\lambda(w_i) + \varkappa \int_{A_i} (H(W(x)) - H(U(x))) dx.$$

From (H2) it follows that the function $t \mapsto H(s_1, \dots, s_{i-1}, t, s_{i+1}, \dots, s_k)$ is non decreasing in $(0, +\infty)$, hence, in view of (4.3),

$$(4.6) \quad \begin{aligned} 0 &\leq H(W(x)) - H(U(x)) \\ &= H(u_1(x), \dots, u_{i-1}(x), w_i(x), u_{i+1}(x), \dots, u_k(x)) \\ &\quad - H(u_1(x), \dots, u_{i-1}(x), 0, u_{i+1}(x), \dots, u_k(x)) \\ &\leq H\left(u_1(x), \dots, u_{i-1}(x), \frac{\beta \varepsilon_i}{\sqrt{k}}, u_{i+1}(x), \dots, u_k(x)\right) \\ &\quad - H(u_1(x), \dots, u_{i-1}(x), 0, u_{i+1}(x), \dots, u_k(x)). \end{aligned}$$

Since the restriction of H to the cube $[0, \beta]^k$ is uniformly continuous, it admits a modulus of continuity, i.e. there exists a function $\omega : [0, +\infty) \rightarrow [0, +\infty]$ such that

$\lim_{t \rightarrow 0^+} \omega(t) = 0$ and $|H(X) - H(Y)| \leq \omega(|X - Y|)$ for all $X, Y \in [0, \beta]^k$. Hence, from (4.6) we derive

$$(4.7) \quad 0 \leq H(W(x)) - H(U(x)) \leq \omega\left(\frac{\beta\varepsilon_i}{\sqrt{k}}\right).$$

Combining (4.4), (4.5), and (4.7), we obtain that

$$I_\varepsilon^{\lambda, \varkappa}(W) - I_\varepsilon^{\lambda, \varkappa}(U) < -\varepsilon_i^N \frac{\alpha \lambda |\Omega|}{2k^2} + \varkappa \varepsilon_i^N |\Omega| \omega\left(\frac{\beta\varepsilon_i}{\sqrt{k}}\right)$$

which is strictly negative if ε_i is small enough to satisfy

$$(4.8) \quad \omega\left(\frac{\beta\varepsilon_i}{\sqrt{k}}\right) < \frac{\alpha \lambda}{2k^2 \varkappa}.$$

This concludes the proof.

4.3. Proof of Theorem 2.3. Let $k \geq 2$ and $\lambda \geq \lambda_0$ be fixed as in (3.1).

Assume by contradiction that the minimum of $\mathcal{E}_\varepsilon^\lambda$ on $\mathcal{U} \cap H_0^1(\Omega)$ is achieved by a k -tuple $U = (u_1, \dots, u_k)$ with only one nontrivial component. Reasoning as in (4.1) it is easy to prove that $u_1 \not\equiv 0$, hence we can assume $u_j \equiv 0$ for all $j > 1$; notice that u_1 is in particular a global minimizer of J_1^λ . The strategy leading a contradiction consists in modifying u_1 near the origin in order to create a new k -tuple $V \in \mathcal{U}$ with a second non-vanishing component and

$$\mathcal{E}_\varepsilon^\lambda(V) < \mathcal{E}_\varepsilon^\lambda(U)$$

for ε small. To this aim, let $\phi \in C^2(\mathbb{R})$ be a cut-off function such that $0 \leq \phi \leq 1$ and

$$\phi(s) = \begin{cases} 0 & s \leq 1, \\ 1 & s \geq 2. \end{cases}$$

Given $\delta \in (0, \delta_0)$ we set

$$\Omega_\delta = \{x \in \Omega : x_1 < \delta\}, \quad \Omega'_\delta = \{x \in \Omega : \delta < x_1 < 2\delta\}.$$

Let us define

$$u_{1,\delta}(x) = \phi(\delta^{-1}x_1)u_1(x),$$

which vanishes on Ω_δ and belongs to $H_0^1(\Omega)$ since, for $x \in \Omega_{2\delta}$,

$$\nabla u_{1,\delta}(x) = \delta^{-1} \phi'(\delta^{-1}x_1)u_1(x)(1, 0, \dots, 0) + \phi(\delta^{-1}x_1)\nabla u_1(x).$$

The growth of energy occurring when substituting u_1 in the minimizing k -tuple with $u_{1,\delta}$ can be estimated as follows. Observing first that $F_1(u_{1,\delta}) > 0$ by Lemma 3.2,

and $F_1(s) \leq \Gamma s^2$ for some $\Gamma > 0$ by (F1-2), we have

$$\begin{aligned}
& J_1^\lambda(u_{1,\delta}) - J_1^\lambda(u_1) \\
& \leq \int_{\Omega_{2\delta}} \frac{1}{2} \left(|\nabla u_{1,\delta}(x)|^2 - |\nabla u_1(x)|^2 \right) dx + \lambda \int_{\Omega_{2\delta}} F_1(u_1(x)) dx \\
& \leq \int_{\Omega'_\delta} \left(\frac{1}{2} \delta^{-2} (\phi'(\delta^{-1}x_1))^2 u_1^2(x) + \delta^{-1} \phi(\delta^{-1}x_1) \phi'(\delta^{-1}x_1) u_1(x) \frac{\partial}{\partial x_1} u_1(x) \right) dx \\
& \quad + \Gamma \lambda \int_{\Omega_{2\delta}} u_1^2(x) dx.
\end{aligned}$$

An integration by parts provides

$$\begin{aligned}
& 2 \int_{\Omega'_\delta} \phi(\delta^{-1}x_1) \phi'(\delta^{-1}x_1) u_1(x) \frac{\partial}{\partial x_1} u_1(x) dx = \int_{\Omega'_\delta} \phi(\delta^{-1}x_1) \phi'(\delta^{-1}x_1) \frac{\partial}{\partial x_1} u_1^2(x) dx \\
& = \int_{\partial\Omega'_\delta} \phi(\delta^{-1}x_1) \phi'(\delta^{-1}x_1) u_1^2(x) d\sigma \\
& \quad - \frac{1}{\delta} \int_{\Omega'_\delta} \left((\phi'(\delta^{-1}x_1))^2 + \phi(\delta^{-1}x_1) \phi''(\delta^{-1}x_1) \right) u_1^2(x) dx \\
& \leq \|\phi'\|_{L^\infty(\mathbb{R})} \int_{\partial\Omega'_\delta} u_1^2(x) d\sigma + \frac{1}{\delta} (\|\phi'\|_{L^\infty(\mathbb{R})}^2 + \|\phi''\|_{L^\infty(\mathbb{R})}) \int_{\Omega'_\delta} u_1^2(x) dx \\
& \leq M \int_{\partial\Omega'_\delta} u_1^2(x) d\sigma + \frac{M}{\delta} \int_{\Omega'_\delta} u_1^2(x) dx,
\end{aligned}$$

for some $M = M(\phi) > 0$. Appealing to Lemma 3.3 we have that $u_1(x) \leq \gamma x_1^2 \leq 4\gamma\delta^2$ for all $x \in \Omega_{2\delta}$. Hence we have the following estimate:

$$\begin{aligned}
& J_1^\lambda(u_{1,\delta}) - J_1^\lambda(u_1) \\
& \leq |\Omega_{2\delta}| \left(\frac{1}{2} \delta^{-2} M (4\gamma\delta^2)^2 + \Gamma \lambda (4\gamma\delta^2)^2 + \frac{1}{2} \delta^{-2} M (4\gamma\delta^2)^2 \right) \\
& \quad + \frac{1}{2} \delta^{-1} M |\partial\Omega'_\delta|_{N-1} (4\gamma\delta^2)^2 \\
& \leq C \delta^{N+2},
\end{aligned}$$

for some $C > 0$. Now fix any $i > 1$, set

$$(4.9) \quad v_i(x) = \frac{\varepsilon_i}{\sqrt{k}} u_1(\varepsilon_i^{-1}x),$$

and define $V = (v_1, \dots, v_k)$ where

$$(4.10) \quad v_1(x) = u_{1,\varepsilon_i}(x) = \phi(\varepsilon_i^{-1}x_1) u_1(x),$$

and $v_j \equiv 0$ if $j \neq 1, i$. Notice that $v_1 \cdot v_i = 0$ by construction, so that $V \in \mathcal{U}$ and hence $H(v_1, \dots, v_k) = 0$ by (H3). Besides, by the above computations with $\delta = \varepsilon_i$

and arguing as in (4.4) to estimate $J_{i,\varepsilon}^\lambda(v_i)$, we have

$$(4.11) \quad \begin{aligned} \mathcal{E}_\varepsilon^\lambda(V) - \mathcal{E}_\varepsilon^\lambda(U) &= J_1^\lambda(v_1) - J_1^\lambda(u_1) + J_{i,\varepsilon}^\lambda(v_i) \\ &\leq \frac{\varepsilon_i^N}{k} J_1^\lambda(u_1) + C\varepsilon_i^{N+2} \\ &\leq \varepsilon_i^N \left(-\frac{\alpha \lambda |\Omega|}{2k^2} + C\varepsilon_i^2 \right), \end{aligned}$$

which is strictly negative for ε_i small and provides a contradiction.

4.4. Proof of Theorem 2.4.

Coexistence. Let $k = 2$ and $\varkappa > 0$ be fixed. Arguing as in the proof of Theorem 2.1, we immediately obtain that the global minimum of $I_\varepsilon^{\lambda,\varkappa}$ is achieved for all $\varkappa > 0$. Assume by contradiction that the global minimum of $I_\varepsilon^{\lambda,\varkappa}$ on $[H_0^1(\Omega)]^2$ is achieved by a pair of the form $U^\varkappa = (u_1^\varkappa, 0)$ (recall that by (4.1) the first component of any minimizer must be nontrivial). Arguing exactly as in the proof of Theorem 2.3, we define a new pair $V^\varkappa = (v_1^\varkappa, v_2^\varkappa)$ by setting

$$v_1^\varkappa(x) = \phi(\varepsilon_2^{-1}x_1)u_1^\varkappa(x), \quad v_2^\varkappa(x) = \frac{\varepsilon_2}{\sqrt{2}}u_1^\varkappa(\varepsilon_2^{-1}x),$$

as in (4.10) and (4.9) respectively. Since v_1^\varkappa and v_2^\varkappa have disjoint supports, by (H3) it holds $H(v_1^\varkappa, v_2^\varkappa) = 0$ and no interaction term appears in the evaluation of $I_\varepsilon^{\lambda,\varkappa}(V^\varkappa)$. Hence

$$I_\varepsilon^{\lambda,\varkappa}(V^\varkappa) - I_\varepsilon^{\lambda,\varkappa}(U^\varkappa) = J_1^\lambda(v_1^\varkappa) - J_1^\lambda(u_1^\varkappa) + J_{2,\varepsilon}^\lambda(v_2^\varkappa),$$

and estimating as in (4.11), we find that $I_\varepsilon^{\lambda,\varkappa}(V^\varkappa) - I_\varepsilon^{\lambda,\varkappa}(U^\varkappa)$ is strictly negative for ε_2 sufficiently small, a contradiction. Hence, if ε_2 is small enough, for any positive value of \varkappa the competing system (2.1) has a solution $U^\varkappa = (u_1^\varkappa, u_2^\varkappa)$ with both nonzero components which minimizes the energy $I_\varepsilon^{\lambda,\varkappa}$.

Asymptotic analysis. Let λ and ε be fixed and consider $U^\varkappa = (u_1^\varkappa, u_2^\varkappa)$ such that

$$I_\varepsilon^{\lambda,\varkappa}(U^\varkappa) = \Lambda^\varkappa = \inf_{U \in [H_0^1(\Omega)]^2} I_\varepsilon^{\lambda,\varkappa}(U).$$

The convergence of U^\varkappa to a minimizer of $\mathcal{E}_\varepsilon^\lambda$ on $\mathcal{U} \cap H_0^1(\Omega)$ can be proven as in [5, Theorem 2.3], with minor changes. For the reader's convenience, we report some details. Notice first that evaluating $I_\varepsilon^{\lambda,\varkappa}(U)$ for all $U \in \mathcal{U}$ annihilates the interaction term in light of (H3), so that

$$(4.12) \quad \Lambda^\varkappa \leq \min\{\mathcal{E}_\varepsilon^\lambda(U), U \in \mathcal{U}\} =: c,$$

hence we have

$$\|U^\varkappa\|_{[H_0^1(\Omega)]^2}^2 \leq 2\Lambda^\varkappa + 2|\Omega|\lambda \sum_i \int_0^{\beta_i} f_{i,\varepsilon}(s)ds \leq 2c + 3|\Omega|\lambda\alpha.$$

Then u_i^\varkappa is bounded in $H_0^1(\Omega)$ uniformly with respect to \varkappa , and there exists $u_i \geq 0$ such that, up to subsequences, $u_i^\varkappa \rightharpoonup u_i$ weakly in $H_0^1(\Omega)$ and $u_i^\varkappa(x) \rightarrow u_i$ for almost

every x as $\varkappa \rightarrow +\infty$. Let us now multiply the equation of u_i^\varkappa times u_i^\varkappa on account of the boundary conditions: then $\varkappa \int_\Omega u_i^\varkappa \frac{\partial H}{\partial s_i}(u_1^\varkappa, u_2^\varkappa)$ is bounded uniformly in \varkappa , giving

$$\int_\Omega u_i^\varkappa(x) \frac{\partial H}{\partial s_i}(u_1^\varkappa(x), u_2^\varkappa(x)) dx \rightarrow 0, \quad \text{as } \varkappa \rightarrow \infty.$$

By (H2) and the Dominated Convergence Theorem (recall that $0 \leq u_i^\varkappa \leq \beta_i$) we infer that

$$u_i(x) \frac{\partial H}{\partial u_i}(u_1(x), u_2(x)) = 0 \quad \text{a.e. } x \in \Omega,$$

implying in light of (H3) that $u_1(x) \cdot u_2(x) = 0$ and hence $U = (u_1, u_2) \in \mathcal{U}$. Now notice that for $\varkappa \leq \varkappa'$ it holds $\Lambda^\varkappa \leq \Lambda^{\varkappa'} \leq c$, hence the following chain of inequalities holds:

$$\begin{aligned} c &\geq \lim_{\varkappa \rightarrow \infty} \Lambda^\varkappa = \lim_{\varkappa \rightarrow \infty} I_\varepsilon^{\lambda, \varkappa}(U^\varkappa) \\ &= \limsup_{\varkappa \rightarrow \infty} \left[\sum_{i=1}^2 \left\{ \frac{1}{2} \int_\Omega |\nabla u_i^\varkappa(x)|^2 dx - \lambda \int_\Omega F_{i, \varepsilon}(u_i^\varkappa(x)) dx \right\} \right. \\ &\quad \left. + \varkappa \int_\Omega H(u_1^\varkappa(x), u_2^\varkappa(x)) dx \right] \\ &\geq \limsup_{\varkappa \rightarrow \infty} \sum_{i=1}^2 \left\{ \frac{1}{2} \int_\Omega |\nabla u_i^\varkappa(x)|^2 dx - \lambda \int_\Omega F_{i, \varepsilon}(u_i^\varkappa(x)) dx \right\} \\ &\geq \liminf_{\varkappa \rightarrow \infty} \sum_{i=1}^2 \left\{ \frac{1}{2} \int_\Omega |\nabla u_i^\varkappa(x)|^2 dx - \lambda \int_\Omega F_{i, \varepsilon}(u_i^\varkappa(x)) dx \right\} \\ &\geq \sum_{i=1}^2 \left\{ \frac{1}{2} \int_\Omega |\nabla u_i(x)|^2 dx - \lambda \int_\Omega F_{i, \varepsilon}(u_i(x)) dx \right\} = \mathcal{E}_\varepsilon^\lambda(U) \geq c. \end{aligned}$$

Therefore all the above inequalities are indeed equalities. In particular $\mathcal{E}_\varepsilon^\lambda(U) = c$, meaning that U is a global minimizer of $\mathcal{E}_\varepsilon^\lambda$.

Moreover, we learn that $\lim_{\varkappa \rightarrow +\infty} \|U^\varkappa\|_{[H_0^1(\Omega)]^2} = \|U\|_{[H_0^1(\Omega)]^2}$ which implies that the weak convergence of U^\varkappa to U is actually strong in $[H_0^1(\Omega)]^2$.

Finally, to prove that both the components of V are positive, we appeal to Theorem 2.3 in the case $k = 2$, ensuring that for ε_2 small any global minimizer of $\mathcal{E}_\varepsilon^\lambda$ on $\mathcal{U} \cap [H_0^1(\Omega)]^2$ has two nontrivial components.

REFERENCES

- [1] A. Ambrosetti, E. Colorado, *Standing waves of some coupled nonlinear Schrödinger equations*, J. Lond. Math. Soc. (2) 75 (2007), no. 1, 67–82.
- [2] A. Ambrosetti, A. Malchiodi, *Nonlinear analysis and semilinear elliptic problems*, Cambridge Studies in Advanced Mathematics, 104. Cambridge University Press, Cambridge, 2007.
- [3] S.M. Chang, C. S. Lin, T.C. Lin, W.W. Lin, *Segregated nodal domains of two-dimensional multispecies Bose-Einstein condensates*, Phys. D 196 (2004), no. 3-4, 341–361.
- [4] M. Conti, V. Felli, *Coexistence and segregation for strongly competing species in special domains*, Interfaces Free Bound. 10 (2008), 173–195.

- [5] M. Conti, V. Felli, *Minimal coexistence configurations for multispecies systems*, *Nonlinear Anal.* 71 (2009), no. 7-8, 3163–3175.
- [6] M. Conti, S. Terracini, G. Verzini, *An optimal partition problem related to nonlinear eigenvalues*, *Journal of Funct. Anal.* 198 (2003), no. 1, 160–196.
- [7] M. Conti, S. Terracini, G. Verzini, *A variational problem for the spatial segregation of reaction-diffusion systems*, *Indiana Univ. Math. J.* 54 (2005), no. 3, 779–815.
- [8] E.C.M. Crooks, E.N. Dancer, D. Hilhorst, M. Mimura, H. Ninomiya, *Spatial segregation limit for a competition-diffusion system with Dirichlet boundary conditions*, *Nonlinear Analysis: Real World Applications* 5 (2004), no. 4, 645–665.
- [9] E.N. Dancer, Y.H. Du, *Positive solutions for a three-species competition system with diffusion. I. General existence results*, *Nonlinear Anal.* 24 (1995), no. 3, 337–357.
- [10] E.N. Dancer, Y.H. Du, *Positive solutions for a three-species competition system with diffusion. II. The case of equal birth rates*, *Nonlinear Anal.* 24 (1995), no. 3, 359–373.
- [11] E.N. Dancer, D. Hilhorst, M. Mimura, L.A. Peletier, *Spatial segregation limit of a competition-diffusion system*, *European J. Appl. Math.* 10 (1999), 97–115.
- [12] Y. Lou, S. Martínez, W-M. Ni, *On 3×3 Lotka-Volterra competition systems with cross-diffusion*, *Discrete Contin. Dynam. Systems* 6 (2000), no. 1, 175–190.
- [13] L. Maia, E. Montefusco, B. Pellacci, *Positive solutions for a weakly coupled nonlinear Schrödinger system*, *J. Diff. Eq.* 229 (2006), no. 2, 743–767.
- [14] J. Mawhin, *The legacy of Pierre-Francois Verhulst and Vito Volterra in population dynamics, The first 60 years of nonlinear analysis of Jean Mawhin*, *World Sci. Publ.*, River Edge, NJ (2004), 147–160.
- [15] B. Noris, M. Ramos, *Existence and bounds of positive solutions for a nonlinear Schrödinger system*, [arxiv/math:0912.0150](https://arxiv.org/abs/math/0912.0150).
- [16] G. Sweers, *A sign-changing global minimizer on a convex domain*, *Progress in partial differential equations: elliptic and parabolic problems (Pont-à-Mousson, 1991)*, 251–258, *Pitman Res. Notes Math. Ser.*, 266, Longman Sci. Tech., Harlow, 1992.
- [17] S. Terracini, G. Verzini, *Multipulse Phases in k -Mixtures of Bose-Einstein Condensates*, *Arch. Rat. Mech. Anal.* 194 (2009), no. 3, 717–741.

E-mail address: monica.conti@polimi.it

E-mail address: veronica.felli@unimib.it