

# EIGENVALUE ASYMPTOTIC OF ROBIN LAPLACE OPERATORS ON TWO-DIMENSIONAL DOMAINS WITH CUSPS

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ABSTRACT. We consider Robin Laplace operators on a class of two-dimensional domains with cusps. Our main results include the formula for the asymptotic distribution of the eigenvalues of such operators. In particular, we show how the eigenvalue asymptotic depends on the geometry of the cusp and on the boundary conditions.

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## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^2$  be an open domain such that the spectrum of the Dirichlet Laplacian  $-\Delta_\Omega^D$  on  $\Omega$  is discrete. Denote by  $N_\lambda(-\Delta_\Omega^D)$  the counting function of  $-\Delta_\Omega^D$ , i.e. the number of eigenvalues of  $-\Delta_\Omega^D$  less than  $\lambda$ . The classical result by H. Weyl, [We], states that if  $\Omega$  is bounded, then

$$N_\lambda(-\Delta_\Omega^D) = \frac{\lambda}{4\pi} |\Omega| + o(\lambda) \quad \lambda \rightarrow \infty, \quad (1.1)$$

where  $|\Omega|$  denotes the volume of  $\Omega$ . The proof of (1.1) for unbounded domains with finite volume is due to M. Birman, M. Solomyak and B. Boyarski, see e.g. [BiSo]. The situation is different for the Neumann Laplacian  $-\Delta_\Omega^N$ . In this case equation (1.1), with  $N_\lambda(-\Delta_\Omega^N)$  in place of  $N_\lambda(-\Delta_\Omega^D)$ , holds whenever  $\Omega$  is bounded and has sufficiently regular boundary, see e.g. [Iv1, N, NS] for the estimates on the rest term in (1.1). However, the Neumann Laplacian might not satisfy (1.1) (its spectrum might even not be discrete) if  $\Omega$  has rough boundary or if  $\Omega$  is unbounded, [Ber, DS, HSS, JMS, NS, Sol].

Here we will focus on unbounded domains with regular boundary and we will consider two-dimensional domains of the form

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x > 1, |y| < f(x)\}, \quad (1.2)$$

where  $f : (1, \infty) \rightarrow \mathbb{R}$  is a positive function such that  $f(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Then the counting function  $N_\lambda(-\Delta_\Omega^D)$  of the Dirichlet Laplacian satisfies (1.1) as long as  $f$  is integrable. If  $f$  decays too slowly, so that  $|\Omega| = \infty$ , then the spectrum of  $-\Delta_\Omega^D$  is still discrete, but  $N_\lambda(-\Delta_\Omega^D)$  grows super-linearly in  $\lambda$ , see [Be, Da, Ro, Si]. On the other hand, the spectrum of the Neumann Laplacian  $-\Delta_\Omega^N$  is discrete *if and only if*

$$\lim_{x \rightarrow \infty} \left( \int_1^x \frac{dt}{f(t)} \right) \left( \int_x^\infty f(t) dt \right) = 0. \quad (1.3)$$

This remarkable fact was proved in [EH], see also [Ma]. Asymptotic behaviour of  $N_\lambda(-\Delta_\Omega^N)$  on domains of this type was studied in [Be, Iv2, JMS, N, Sol]. We would like to point out that  $f$  must decay faster than any power function for (1.3) to hold. We thus notice a huge difference between the spectral properties of  $-\Delta_\Omega^D$  and  $-\Delta_\Omega^N$  on such domains.

Motivated by this discrepancy, we want to study the gap between Dirichlet and Neumann Laplacians. To do so we consider a family of Laplace operators on  $\Omega$  which formally correspond to the so-called Robin boundary conditions

$$\frac{\partial u}{\partial n}(x, y) + h(x)u(x, y) = 0, \quad x > 1, y = \pm f(x), \quad (1.4)$$

where  $\frac{\partial u}{\partial n}$  denotes the normal derivative of  $u$  and  $h : (1, \infty) \rightarrow \mathbb{R}_+$  is a sufficiently smooth bounded function. The extreme cases  $h \equiv 0$  and  $h \equiv \infty$  correspond to Neumann and Dirichlet Laplacians respectively. First question that arises is under what conditions on  $h$  and  $f$  is the spectrum of the associated Robin Laplacian discrete. Next we would like to know how the coefficient  $h(x)$  of the boundary conditions affects the asymptotic distribution of eigenvalues of the Robin Laplacian.

The paper is organised as follows. In section 3 we formulate our main results, see Theorems 3.3 and 3.6. Similarly as in [Ber, JMS], we show that the leading term of the eigenvalue asymptotic has two contributions, one of which results from an auxiliary one-dimensional Schrödinger operator. The boundary conditions affect the eigenvalue asymptotic through the term  $h(x)\sqrt{1 + f'(x)^2}/f(x)$  which enters into the potential of this operator, see equations (2.2) and (2.7). For some particular choices of  $h$  and  $f$  this contribution can be calculated explicitly, the corresponding results are given in section 3.1.

The proofs of the main results are given in section 5. Our strategy is to treat separately the contribution to the counting function from a finite part of  $\Omega$  and from the tail. In section 5.1 it is shown that the contribution from the finite part satisfies the Weyl law (1.1). The key point of the proof is to transform, in the remaining part of  $\Omega$ , the problem to a Neumann Laplacian plus a positive potential that reflects the boundary term, see section 5.2. To this end we employ the technique known as ground state representation, which has been recently used e.g. in [FSW] to derive eigenvalue estimates for Schrödinger operators with regular ground states, see also [FLS]. Once this transformation is done, we show, by rather standard arguments, that one part of the eigenvalue distribution of such Neumann Laplacian with additional potential is asymptotically (i.e. for  $\lambda \rightarrow \infty$ ) equivalent to eigenvalue distribution of a direct sum of certain one-dimensional Schrödinger operators, see section 5.3. This enables us to prove Theorem 3.3. Finally, in the closing section 6 we discuss some generalisations for Robin Laplacians with non symmetric boundary conditions.

## 2. PRELIMINARIES AND NOTATION

Given a self-adjoint operator  $T$  with a purely discrete spectrum we denote by  $N_\lambda(T)$  the number of its eigenvalues, counted with multiplicities, less than  $\lambda$ . We will write  $A \simeq B$  if the operators  $A$  and  $B$  are unitarily equivalent and we will use the notation

$$f_1(\lambda) \sim f_2(\lambda) \quad \lambda \rightarrow \infty \quad \iff \quad \lim_{\lambda \rightarrow \infty} \frac{f_1(\lambda)}{f_2(\lambda)} = 1.$$

We will consider the eigenvalue behaviour of the Robin boundary value problem in a weak sense. Therefore the main object of our interest is the self-adjoint operator  $A_\sigma$  in  $L^2(\Omega)$

associated with the closure of the quadratic form

$$Q_\sigma[u] = \int_\Omega |\nabla u|^2 dx dy + \int_1^\infty \sigma(x) (|u(x, f(x))|^2 + |u(x, -f(x))|^2) dx \quad (2.1)$$

on  $C_0^2(\bar{\Omega})$ . Here  $C_0^2(\bar{\Omega})$  denotes the restriction to  $\Omega$  of functions from  $C^2(\mathbb{R}^2)$  such that for each  $y$  the support of  $u(\cdot, y)$  is a compact subset of  $(1, \infty)$ . The operator  $A_\sigma$  formally corresponds to the Laplace operator on  $\Omega$  with Dirichlet boundary condition at  $\{x = 1\}$  and mixed boundary conditions (1.4) at the rest of the boundary, if we chose  $\sigma$  such that

$$\sigma(x) = h(x) \sqrt{1 + f'(x)^2}.$$

**Remark 2.1.** Since we work under the assumption that  $f'(x) \rightarrow 0$  as  $x \rightarrow \infty$ , see below, and since the asymptotic of  $N_\lambda(A_\sigma)$  depends only on the behaviour of  $\sigma$  at infinity, from now on we will work with the function  $\sigma$  instead of  $h$ .

We will also need the following auxiliary potentials:

$$V(x) = \frac{1}{4} \left( \frac{f'}{f} \right)^2 + \frac{1}{2} \left( \frac{f'}{f} \right)', \quad W_\sigma(x) = V(x) + \frac{\sigma(x)}{f(x)}. \quad (2.2)$$

Throughout the whole paper we will suppose that  $f$  satisfies

**Assumption 2.2.**  $f \in C^\infty(1, \infty)$  is positive and such that  $f'(x) \leq 0$  for all  $x$  large enough. Moreover,

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} f''(x) = 0. \quad (2.3)$$

Note that (2.3) implies  $f'(x) \rightarrow 0$  as  $x \rightarrow \infty$ , see Lemma 2.3 below.

**Lemma 2.3.** Let  $f \in C^2(1, \infty)$  be a nonnegative function. Assume that  $f$  and  $|f''|$  are bounded on  $(1, \infty)$ . For a given  $x > 1$  define  $M_x = \sup_{s \geq x} f(s)$  and  $M_x'' = \sup_{s \geq x} |f''(s)|$ . Then

$$(f'(x))^2 \leq 2 M_x M_x''. \quad (2.4)$$

*Proof.* Let  $s > x$ . The Taylor expansion of  $f$  at the point  $x$  gives

$$f(s) - f(x) = t f'(x) + \frac{t^2}{2} f''(y), \quad y \in [x, s], \quad t = s - x. \quad (2.5)$$

On the other hand,  $f \geq 0$  ensures that  $|f(x) - f(s)| \leq M_x$  for all  $s > x$ . This together with (2.5) implies that the inequality

$$|f'(x)| \leq \frac{M_x}{t} + \frac{t M_x''}{2}$$

holds for all  $t > 0$ . Optimization with respect to  $t$  then gives the result.  $\square$

**Remark 2.4.** Note that if we leave out the assumption  $f \geq 0$ , then the above proof still works with the modification that now  $|f(x) - f(s)| \leq 2M_x$ . This results into the Landau inequality, i.e. inequality (2.4) with the factor 2 on the right hand side replaced by 4.

The hypothesis on  $\sigma$  are the following:

**Assumption 2.5.** The function  $\sigma \in C^2(1, \infty)$  is non negative. Moreover,  $\sigma$ ,  $\sigma'$  and  $\sigma''$  are bounded and

$$\lim_{x \rightarrow \infty} W_\sigma(x) = \infty. \quad (2.6)$$

In order to formulate our next assumption, we introduce the operator

$$\mathcal{H}_\sigma = -\frac{d^2}{dx^2} + W_\sigma(x) \quad \text{in } L^2(1, \infty) \quad (2.7)$$

with Dirichlet boundary condition at  $x = 1$ . More precisely,  $\mathcal{H}_\sigma$  is the operator generated by the closure of the quadratic form

$$\int_1^\infty (|\psi'|^2 + W_\sigma \psi^2) dx, \quad \psi \in C_0^2(1, \infty).$$

Alongside with  $\mathcal{H}_\sigma$  we will also consider the auxiliary operator

$$\mathcal{B} = -\partial_x^2 - \frac{1}{f^2(x)} \partial_y^2 \quad \text{in } L^2((1, \infty) \times (-1, 1)) \quad (2.8)$$

with Dirichlet boundary conditions.

**Assumption 2.6.** For  $0 < \varepsilon < 1$  we have

$$N_\lambda((1 \pm \varepsilon) \mathcal{H}_\sigma) = N_\lambda(\mathcal{H}_\sigma)(1 + \mathcal{O}(\varepsilon)), \quad (2.9)$$

$$N_\lambda((1 \pm \varepsilon) \mathcal{B}) = N_\lambda(\mathcal{B})(1 + \mathcal{O}(\varepsilon)) \quad (2.10)$$

**Remark 2.7.** A similar assumption was made in [JMS]. Although this assumption is essential for the approach used in the proof of Theorem 3.3 below, it is natural to believe that the statement holds under more general conditions. Note also that for domains with finite volume (2.10) holds automatically.

### 3. MAIN RESULTS

**Theorem 3.1.** *If 2.2 and 2.5 are satisfied, then the spectrum of  $A_\sigma$  is discrete.*

**Remark 3.2.** Contrary to the case of Neumann Laplacian, the spectrum of  $A_\sigma$  can be discrete also if the volume of  $\Omega$  is infinite. For example if  $\sigma$  is constant, then (2.6) is automatically satisfied in view of the fact that  $f(x)V(x) \rightarrow 0$  as  $x \rightarrow \infty$ , see equation (5.13). On the other hand, condition (2.6) is, unlike (1.3), only sufficient.

**Theorem 3.3.** *Suppose that assumptions 2.2, 2.5 and 2.6 are satisfied. Then*

$$N_\lambda(A_\sigma) \sim N_\lambda(-\Delta_\Omega^D) + N_\lambda(\mathcal{H}_\sigma) \quad \lambda \rightarrow \infty. \quad (3.1)$$

**Remark 3.4.** The second term in (3.1) is a contribution from the eigenvalues of the operator  $A_\sigma$  restricted to the space of functions which depend only on  $x$ . This is analogous to the case of Neumann Laplacian, [DS, JMS, Sol]. On the other hand, the presence of the boundary term  $\sigma(x)$  enables us to apply (3.1) also in the situation in which the Neumann Laplacian does not have purely discrete spectrum.

**Remark 3.5.** Theorem 3.3 allows a straightforward generalisation to Robin Laplacians with different boundary conditions on the upper and lower boundary of  $\Omega$ , say given through functions  $\sigma_1(x)$  and  $\sigma_2(x)$ . In that case we only have to replace  $\sigma(x)$  in (2.2) by  $(\sigma_1(x) + \sigma_2(x))/2$ , see section 6.1 for details.

For domains with finite volume Theorem 3.3 and the Weyl formula (1.1) give

**Theorem 3.6.** *Let  $|\Omega| < \infty$  and suppose that assumptions 2.2, 2.5 and (2.9) are satisfied. Then*

$$N_\lambda(A_\sigma) \sim \frac{\lambda}{4\pi} |\Omega| + N_\lambda(\mathcal{H}_\sigma) \quad \lambda \rightarrow \infty. \quad (3.2)$$

**Remark 3.7.** Note that if  $\sigma \equiv 0$ , then the condition  $|\Omega| < \infty$  is necessary for the spectrum of  $A_0 = -\Delta_\Omega^N$  to be discrete, see (1.3). Hence in that case there is no difference between Theorems 3.3 and 3.6 and the resulting eigenvalue asymptotic agrees with the one obtained in [Ber, JMS].

**Corollary 3.8.** *Let  $|\Omega| < \infty$  and let  $\sigma(x) = \sigma$  be constant. Assume that  $f$  satisfies 2.2. Then*

$$\limsup_{x \rightarrow \infty} x^2 f(x) = 0 \quad \implies \quad N_\lambda(A_\sigma) \sim \frac{|\Omega|}{4\pi} \lambda \quad \lambda \rightarrow \infty \quad (3.3)$$

$$\lim_{x \rightarrow \infty} x^2 f(x) = a^2 \quad \implies \quad N_\lambda(A_\sigma) \sim \left( \frac{|\Omega|}{4\pi} + \frac{|a|}{4\sqrt{\sigma}} \right) \lambda \quad \lambda \rightarrow \infty. \quad (3.4)$$

**Remark 3.9.** Equation (3.3) provides a sufficient condition on the decay of  $f$  for the Weyl's law in the case of constant  $\sigma$ . Notice that the borderline decay behaviour is  $f(x) \sim x^{-2}$  which is in contrast to  $f(x) \sim x^{-1}$  in the case of Dirichlet Laplacian. The reason behind this is that the principle eigenvalues of Robin and Dirichlet Laplacians on an interval of the width  $2f(x)$  scale in a different way as  $f(x) \rightarrow 0$ . Observe also that (3.4) turns into (3.3) when  $\sigma \rightarrow \infty$ , as expected.

If the volume of  $\Omega$  is infinite, then we confine ourselves to situations when  $f$  is a power function. The asymptotic distribution of the Dirichlet-Laplacian on such region is known, see [Ro], [Si]. These results together with Theorem 3.3 yield

**Corollary 3.10.** *Let  $f(x) = x^{-\alpha}$ ,  $0 < \alpha \leq 1$ . If  $\mathcal{H}_\sigma$  satisfies (2.9), then as  $\lambda \rightarrow \infty$  we have*

$$N_\lambda(A_\sigma) \sim \frac{1}{\pi} \left( \frac{2}{\pi} \right)^{\frac{1}{\alpha}} \zeta \left( \frac{1}{\alpha} \right) B \left( 1 + \frac{1}{2\alpha}, \frac{1}{2} \right) \lambda^{\frac{1}{2} + \frac{1}{2\alpha}} + N_\lambda(\mathcal{H}_\sigma) \quad \alpha < 1,$$

$$N_\lambda(A_\sigma) \sim \frac{1}{\pi} \lambda \log \lambda + N_\lambda(\mathcal{H}_\sigma) \quad \alpha = 1,$$

where  $\zeta(\cdot)$  and  $B(\cdot, \cdot)$  denote the Riemann zeta and the Euler beta function respectively.

**3.1. Examples.** We give the asymptotic of  $N_\lambda(A_\sigma)$  for some concrete choices of  $f$  and  $\sigma$ .

**3.1.1.**  $f(x) = x^{-\alpha}$ ,  $\alpha > 1$ ,  $\sigma(x) = \sigma = \text{const}$ . Here

$$W_\sigma(x) = \left( \frac{\alpha^2}{4} + \frac{\alpha}{2} \right) x^{-2} + \sigma x^\alpha$$

is convex and increasing at infinity so that assumption 2.6 is satisfied, see [Ti, Chap. 7]. Theorem 3.6 in combination with Theorem 4.2, see Section 4, gives

$$N_\lambda(A_\sigma) \sim \frac{|\Omega|}{4\pi} \lambda + \frac{1}{\alpha\pi} \sigma^{-\frac{1}{\alpha}} B \left( \frac{1}{\alpha}, \frac{3}{2} \right) \lambda^{\frac{1}{2} + \frac{1}{\alpha}}, \quad \lambda \rightarrow \infty. \quad (3.5)$$

Note that, in agreement with Corollary 3.8,  $N_\lambda(A_\sigma)$  obeys Weyl's law as long as  $\alpha > 2$  and for  $\alpha = 2$  the order of  $\lambda$  is linear, but the coefficient is different from the one in the Weyl asymptotic. When  $\alpha < 2$ , then the behaviour of  $N_\lambda(A_\sigma)$  for  $\lambda \rightarrow \infty$  is fully determined by the second term on the right hand side of (3.5).

3.1.2.  $f(x) = x^{-\alpha}$ ,  $0 < \alpha \leq 1$ ,  $\sigma(x) = \sigma x^{-\beta}$ . Assumptions 2.5 is satisfied if and only if  $0 \leq \beta < \alpha$ . For these values of  $\beta$  Corollary 3.10 and Theorem 4.2 give

$$N_\lambda(A_\sigma) \sim \frac{\sigma^{-\frac{1}{\alpha-\beta}}}{(\alpha-\beta)\pi} B\left(\frac{1}{\alpha-\beta}, \frac{3}{2}\right) \lambda^{\frac{1}{2} + \frac{1}{\alpha-\beta}}, \quad \lambda \rightarrow \infty.$$

#### 4. AUXILIARY MATERIAL

In this section we collect some auxiliary material, which will be used in the proof of the main results. First we fix some necessary notation. Given a continuous function  $q : (1, \infty) \rightarrow \mathbb{R}$  such that  $q(x) \rightarrow \infty$  as  $x \rightarrow \infty$ , we denote by  $T_{(a,b)}^{D,D}$  the operator in  $L^2(a, b)$  acting as

$$T_{(a,b)}^{D,D} = -\frac{d^2}{dx^2} + q(x), \quad 1 \leq a < b < \infty$$

with Dirichlet boundary conditions at  $x = a$  and  $x = b$ . Operators  $T_{(a,b)}^{D,N}$ ,  $T_{(a,b)}^{N,N}$  and  $T_{(a,b)}^{N,D}$  are defined accordingly. For  $b = \infty$  we use the simplified notation  $T_{(a,\infty)}^D$  etc. to indicate the corresponding boundary condition at  $x = a$ . It is well known that imposing Dirichlet boundary condition at  $x = a$  is a rank one perturbation. Variational principle thus implies that

$$0 \leq N_\lambda(T_{(a,\infty)}^N) - N_\lambda(T_{(a,\infty)}^D) \leq 1 \quad \forall a. \quad (4.1)$$

**Lemma 4.1.** *Suppose that  $q(x)$  is a continuous function such that  $q(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . Then for any  $s > 1$  it holds*

$$N_\lambda(T_{(1,\infty)}^N) \sim N_\lambda(T_{(1,\infty)}^D) \sim N_\lambda(T_{(s,\infty)}^D) \sim N_\lambda(T_{(s,\infty)}^N) \quad \lambda \rightarrow \infty. \quad (4.2)$$

*Proof.* In view of (4.1) it suffices to consider the Dirichlet operator only. Let  $I_\lambda := \{x > s : q(x) < \lambda/2\}$ . Then

$$N_\lambda(T_{(s,\infty)}^D) \geq N_{\frac{\lambda}{2}}\left(-\frac{d^2}{dx^2}\right)_{L^2(I_\lambda)}^{Dir} \geq \frac{\sqrt{\lambda}}{\pi\sqrt{2}} |I_\lambda| (1 + o(1)) \quad \lambda \rightarrow \infty,$$

where the superscript *Dir* indicates Dirichlet boundary conditions at the end points of  $I_\lambda$ . Since  $|I_\lambda| \rightarrow \infty$  as  $\lambda \rightarrow \infty$ , this shows that  $\liminf_{\lambda \rightarrow \infty} \lambda^{-1/2} N_\lambda(T_{(s,\infty)}^D) = \infty$ . In view of the equation

$$N_\lambda(T_{(1,s)}^{D,N}) \sim N_\lambda(T_{(1,s)}^{D,D}) = \mathcal{O}(\sqrt{\lambda}) \quad \lambda \rightarrow \infty \quad \forall s > 1,$$

the result follows from the Dirichlet-Neumann bracketing (by putting additional boundary conditions at  $x = s$ ), see e.g. [RS, Chap.13].  $\square$

Under certain additional assumptions one can recover the eigenvalue distribution of such operators from the potential  $q$ . The following theorems are due to [Ti, Chap. 7]:

**Theorem 4.2** (Titchmarsh). *Suppose that  $q(x)$  is continuous increasing unbounded function, that  $q'(x)$  is continuous and  $x^3 q'(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . Then*

$$N_\lambda\left(T_{(s,\infty)}^D\right) \sim \frac{1}{\pi} \int_s^\infty (\lambda - q(x))_+^{\frac{1}{2}} dx, \quad \lambda \rightarrow \infty. \quad (4.3)$$

**Theorem 4.3** (Titchmarsh). *Suppose that  $q(x)$  is continuous increasing and convex at infinity. Then (4.3) holds true.*

A simple combination of the above results gives

**Lemma 4.4.** *Assume that there exists some  $x_c$  such that  $q$  satisfies the hypothesis of Theorem 4.2 or 4.3 for all  $x > x_c$ . Then for any  $s \geq x_c$  we have*

$$N_\lambda \left( T_{(1,\infty)}^D \right) \sim \frac{1}{\pi} \int_1^\infty (\lambda - q(x))_+^{\frac{1}{2}} dx \sim \frac{1}{\pi} \int_s^\infty (\lambda - q(x))_+^{\frac{1}{2}} dx \quad \lambda \rightarrow \infty. \quad (4.4)$$

Next we consider the operators

$$\mathcal{B}_n^{N/D} = -\partial_x^2 - \frac{1}{f^2(x)} \partial_y^2 \quad \text{in} \quad L^2((n, \infty) \times (-1, 1))$$

subject to Dirichlet boundary conditions on  $(n, \infty) \times (\{1\} \cup \{-1\})$  and Neumann/Dirichlet boundary condition on  $\{n\} \times (-1, 1)$  respectively. We have

**Lemma 4.5.** *For any  $n \in \mathbb{N}$  it holds*

$$N_\lambda(\mathcal{B}_n^N) \sim N_\lambda(\mathcal{B}_n^D) \sim \frac{\lambda}{2\pi} \int_n^\infty f(x) dx \quad \lambda \rightarrow \infty \quad \text{if} \quad |\Omega| < \infty, \quad (4.5)$$

$$N_\lambda(\mathcal{B}_n^N) \sim N_\lambda(\mathcal{B}_n^D) \sim N_\lambda(\mathcal{B}) \quad \lambda \rightarrow \infty \quad \text{if} \quad |\Omega| = \infty. \quad (4.6)$$

*Proof.* Equation (4.5) for  $\mathcal{B}_n^D$  follows directly from [SV, Thm. 1.2.1]. Hence it remains to prove (4.5) for  $\mathcal{B}_n^N$  and (4.6). Note that

$$N_\lambda(\mathcal{B}_n^D) = \sum_{k=1}^\infty N_\lambda(L_{k,n}^D), \quad N_\lambda(\mathcal{B}_n^N) = \sum_{k=1}^\infty N_\lambda(L_{k,n}^N), \quad \text{where} \quad L_{k,n}^{N/D} = -\frac{d^2}{dx^2} + \frac{\pi^2 k^2}{4f(x)^2}$$

are one-dimensional operators acting in  $L^2(n, \infty)$  with Neumann/Dirichlet boundary conditions on  $x = n$ . Obviously there exists a positive constant  $c$  such that for any  $n$  and any  $k$  the operator inequality  $L_{k,n}^D \geq L_{k,n}^N \geq c k^2$  holds. This means that there exists some  $K(\lambda)$  with  $K(\lambda) = \mathcal{O}(\sqrt{\lambda})$  as  $\lambda \rightarrow \infty$  and such that

$$\sum_{k \geq 1} N_\lambda(L_{k,n}^N) = \sum_{k \geq 1}^{K(\lambda)} N_\lambda(L_{k,n}^N), \quad \sum_{k \geq 1} N_\lambda(L_{k,n}^D) = \sum_{k \geq 1}^{K(\lambda)} N_\lambda(L_{k,n}^D)$$

Moreover, since  $0 \leq N_\lambda(L_{k,n}^N) - N_\lambda(L_{k,n}^D) \leq 1$  holds for all  $n \in \mathbb{N}$  and for all  $k \geq 1$ , see (4.1),

$$\sum_{k \geq 1} N_\lambda(L_{k,n}^D) = \sum_{k \geq 1}^{K(\lambda)} N_\lambda(L_{k,n}^D) \leq \sum_{k \geq 1}^{K(\lambda)} N_\lambda(L_{k,n}^N) \leq \sum_{k \geq 1} N_\lambda(L_{k,n}^D) + \mathcal{O}(\sqrt{\lambda}). \quad (4.7)$$

The latter implies (4.5) since  $N_\lambda(\mathcal{B}_n^D)$  grows linearly in  $\lambda$  when  $|\Omega| < \infty$  as mentioned above. To prove (4.6) we consider the operators  $\mathcal{B}_{n,m}^D$  obtained from  $\mathcal{B}_n^D$  by putting additional Dirichlet boundary condition at  $\{x = m\}$ ,  $m > n$ . From [SV, Thm. 1.2.1] we get

$$\liminf_{\lambda \rightarrow \infty} \lambda^{-1} N_\lambda(\mathcal{B}_n^D) \geq \liminf_{\lambda \rightarrow \infty} \lambda^{-1} N_\lambda(\mathcal{B}_{n,m}^D) = \frac{1}{2\pi} \int_n^m f(x) dx \quad \forall m > n,$$

which implies, by letting  $m \rightarrow \infty$ , that  $\liminf_{\lambda \rightarrow \infty} \lambda^{-1} N_\lambda(\mathcal{B}_n^D) = \infty$ . In view of (4.7) we obtain  $N_\lambda(\mathcal{B}_n^N) \sim N_\lambda(\mathcal{B}_n^D)$ . Finally, from the Dirichlet-Neumann bracketing we deduce that  $N_\lambda(\mathcal{B}) \sim N_\lambda(\mathcal{B}_n^D)$ .  $\square$

## 5. PROOFS OF THE MAIN RESULTS

As mentioned in the introduction, the idea of the proof is to split  $N_\lambda(A_\sigma)$  into two parts corresponding to the contribution from a finite part of  $\Omega$  and from the tail.

**5.1. Step 1.** Here we show that the contribution from the part of  $\Omega$  where  $x < n$  obeys the Weyl asymptotic irrespectively of the boundary conditions. Let us define

$$\Omega_n := \{(x, y) \in \Omega : 1 < x < n\}, \quad E_n := \Omega \setminus \Omega_n.$$

We denote by  $Q_{n,l}^N$  and  $Q_{n,r}^N$  the quadratic forms defined by the reduction of  $Q_\sigma$  on  $\Omega_n$  and  $E_n$  and acting on the functions from  $C^2(\overline{\Omega_n})$  and  $C_0^2(\overline{E_n})$  respectively. Moreover, let  $T_n^N$  and  $S_n^N$  be the operators associated with the closures of the forms  $Q_{n,l}^N$  and  $Q_{n,r}^N$ .

Similarly we denote by  $Q_{n,l}^D$  and  $Q_{n,r}^D$  the respective quadratic forms which are defined in the same way as  $Q_{n,l}^N$  and  $Q_{n,r}^N$  but with the additional Dirichlet boundary condition at  $\{x = n\}$ . We then denote by  $T_n^D$  and  $S_n^D$  the operators associated with the closures of the forms  $Q_{n,l}^D$  and  $Q_{n,r}^D$ . From the Dirichlet-Neumann bracketing we obtain the operator inequality

$$T_n^N \oplus S_n^N \leq A_\sigma \leq T_n^D \oplus S_n^D, \quad n \in \mathbb{N}, \quad (5.1)$$

which implies that

$$N_\lambda(T_n^D) + N_\lambda(S_n^D) \leq N_\lambda(A_\sigma) \leq N_\lambda(T_n^N) + N_\lambda(S_n^N), \quad n \in \mathbb{N}, \quad \lambda > 0. \quad (5.2)$$

**Lemma 5.1.** *For any  $n \in \mathbb{N}$  it holds*

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1} N_\lambda(T_n^D) = \lim_{\lambda \rightarrow \infty} \lambda^{-1} N_\lambda(T_n^N) = \frac{1}{2\pi} \int_1^n f(x) dx. \quad (5.3)$$

*Proof.* Fix  $n \in \mathbb{N}$ . Since  $\sigma$  is bounded and  $H^1(-f(x), f(x))$  is for every  $x \in (1, n)$  continuously embedded into  $L^\infty(-f(x), f(x))$ , it follows that there exists a constant  $c_n$  such that

$$\|\nabla u\|_{L^2(\Omega_n)}^2 + \|u\|_{L^2(\Omega_n)}^2 \leq Q_{n,l}^N[u] + \|u\|_{L^2(\Omega_n)}^2 \leq c_n \left( \|\nabla u\|_{L^2(\Omega_n)}^2 + \|u\|_{L^2(\Omega_n)}^2 \right)$$

holds for all  $u \in C^2(\overline{\Omega_n})$ . Hence the domain of the closure of the quadratic form  $Q_{n,l}^N$  is a subset of  $H^1(\Omega_n)$ . The same reasoning shows that the domain of the closure of  $Q_{n,l}^D$  contains the space  $H_0^1(\Omega_n)$ . From the fact that  $\sigma \geq 0$  and from the variational principle we thus conclude that

$$N_\lambda(-\Delta_{\Omega_n}^D) \leq N_\lambda(T_n^D) \leq N_\lambda(T_n^N) \leq N_\lambda(-\Delta_{\Omega_n}^N), \quad (5.4)$$

where  $-\Delta_{\Omega_n}^D$  and  $-\Delta_{\Omega_n}^N$  denote the Dirichlet and Neumann Laplacian on  $\Omega_n$  respectively. Since  $\Omega_n$  has the  $H^1$ -extension property, the Weyl formula

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1} N_\lambda(-\Delta_{\Omega_n}^D) = \lim_{\lambda \rightarrow \infty} \lambda^{-1} N_\lambda(-\Delta_{\Omega_n}^N) = \frac{|\Omega_n|}{4\pi}$$

holds for both  $-\Delta_{\Omega_n}^D$  and  $-\Delta_{\Omega_n}^N$ , see [Me, BiSo], [NS]. In view of (5.4), this completes the proof.  $\square$

**5.2. Step 2.** Next we will treat the contribution to the counting function  $N_\lambda(A_\sigma)$  from the tail of  $\Omega$ . Our first aim is to transform the boundary term in (2.1) into an effective additional potential. To this end we use a ground state representation for the test functions  $\psi$ . Let  $\mu(x)$  be the first eigenvalue of the one-dimensional problem

$$\begin{aligned} -\partial_y^2 v(x, y) &= \mu(x) v(x, y), \\ \partial_y v(x, -f(x)) &= \sigma(x) v(x, -f(x)), \quad \partial_y v(x, f(x)) = -\sigma(x) v(x, f(x)) \end{aligned} \quad (5.5)$$



with the corresponding eigenfunction  $v$ . By lemma A.2  $0 < v \leq 1$  and  $v(x, y) \rightarrow 1$  as  $x \rightarrow \infty$  uniformly in  $y$ . Moreover,  $v \in C^2(\overline{E}_n)$ . Thus every function  $\psi \in D(\mathcal{Q}_n^N)$  can be written as

$$\psi(x, y) = v(x, y) \varphi(x, y), \quad \varphi \in C_0^2(\overline{E}_n). \quad (5.6)$$

Similarly, for every function  $\psi \in D(\mathcal{Q}_n^D)$  we have

$$\psi(x, y) = v(x, y) \varphi(x, y), \quad \varphi \in C_0^2(\overline{E}_n) \cap \{\varphi : \varphi(n, \cdot) = 0\} \quad (5.7)$$

In view of (5.6) and (5.7) we can thus identify  $\mathcal{Q}_n^N[\psi]$  and  $\mathcal{Q}_n^D[\psi]$  with quadratic forms  $\mathcal{Q}_n^N[\varphi]$  and  $\mathcal{Q}_n^D[\varphi]$  given by

$$\mathcal{Q}_n^{N/D}[\varphi] = \mathcal{Q}_n^{N/D}[v \varphi].$$

and acting in the weighted space  $L^2(E_n, v^2 dx dy)$ . The forms  $\mathcal{Q}_n^N[\varphi]$  and  $\mathcal{Q}_n^D[\varphi]$  are defined on  $D(\mathcal{Q}_n^N) = C_0^2(\overline{E}_n)$  and  $D(\mathcal{Q}_n^D) = C_0^2(\overline{E}_n) \cap \{\varphi : \varphi(n, \cdot) = 0\}$  respectively. A straightforward calculation based on integration by parts in  $y$  then gives

$$\mathcal{Q}_n^{N,D}[\varphi] = \int_{E_n} (|\partial_x(v\varphi)|^2 + \mu(x)v^2|\varphi|^2 + v^2|\partial_y\varphi|^2) dx dy. \quad (5.8)$$

Since  $v \rightarrow 1$  and  $\mu(x) \sim \sigma(x)/f(x)$  as  $x \rightarrow \infty$ , see appendix, it is natural to compare  $\mathcal{Q}_n^{N,D}$  with the quadratic form

$$q_n[\varphi] = \int_{E_n} (|\partial_x\varphi|^2 + |\partial_y\varphi|^2 + \frac{\sigma(x)}{f(x)}|\varphi|^2) dx dy.$$

Let  $\mathfrak{S}_n^N$  and  $\mathfrak{S}_n^D$  be the operators in  $L^2(E_n)$  generated by the closures of the quadratic form  $q_n[u]$  on  $D(\mathcal{Q}_n^N)$  and  $D(\mathcal{Q}_n^D)$  respectively.

**Lemma 5.2.** *Suppose that assumptions 2.2 and 2.5 are satisfied. For any  $\varepsilon$  there exists an  $N_\varepsilon$  such that for all  $n > N_\varepsilon$*

$$N_\lambda(S_n^N) \leq N_\lambda((1 - \varepsilon)\mathfrak{S}_n^N - \varepsilon), \quad N_\lambda(S_n^D) \geq N_\lambda((1 + \varepsilon)\mathfrak{S}_n^D + \varepsilon) \quad \lambda > 0. \quad (5.9)$$

*Proof.* Let  $\varepsilon > 0$  and let  $\varphi$  belong to the domain of the quadratic forms  $\mathcal{Q}_n^N$  ( $\mathcal{Q}_n^D$ ). From the fact that

$$\lim_{x \rightarrow \infty} v(x, y) = 1, \quad \lim_{x \rightarrow \infty} \partial_x v(x, y) = 0 \quad (\text{uniformly in } y), \quad \lim_{x \rightarrow \infty} \frac{\mu(x)f(x)}{\sigma(x)} = 1,$$

see Lemma A.2, and from the estimate

$$|2v\partial_x v \varphi \partial_x \varphi| \leq \varepsilon |\partial_x \varphi|^2 v^2 + \varepsilon^{-1} |\varphi|^2 |\partial_x v|^2$$

we conclude that for  $n$  large enough

$$\begin{aligned} \mathcal{Q}_n^N[\varphi] &\geq (1 - \varepsilon) \int_{E_n} \left( |\partial_x \varphi|^2 + |\partial_y \varphi|^2 + \frac{\sigma(x)}{f(x)} |\varphi|^2 \right) dx dy - \varepsilon \|\varphi\|_{L^2(E_n)}^2 \\ \mathcal{Q}_n^D[\varphi] &\leq (1 + \varepsilon) \int_{E_n} \left( |\partial_x \varphi|^2 + |\partial_y \varphi|^2 + \frac{\sigma(x)}{f(x)} |\varphi|^2 \right) dx dy + \varepsilon \|\varphi\|_{L^2(E_n)}^2. \end{aligned} \quad (5.10)$$

Moreover, by (A.6) we also have  $|v| \leq 1$  so that (still for  $n$  large enough)

$$(1 - \varepsilon) \|\varphi\|_{L^2(E_n)}^2 \leq \int_{E_n} |\varphi|^2 v^2 dx dy \leq \|\varphi\|_{L^2(E_n)}^2.$$

Equation (5.9) then follows from the variational principle by choosing  $\varepsilon$  in appropriate way (depending on  $\varepsilon$ ).  $\square$

**5.3. Step 3.** We transform the problem of studying the Laplace operator on  $E_n$  to the problem of studying a modified operator on the simpler domain

$$D_n = (n, \infty) \times (-1, 1).$$

To this end we introduce the transformation  $U : L^2(E_n) \rightarrow L^2(D_n)$  defined by

$$(U\varphi)(x, t) = \sqrt{f(x)} \varphi(x, f(x)t), \quad (x, t) \in D_n.$$

Let  $\mathcal{A}_n^N$  and  $\mathcal{A}_n^D$  be the operators associated with the closure of the form

$$\widehat{Q}_n[u] := q_n[U^{-1}u], \quad (5.11)$$

on  $C_0^2(\overline{D}_n)$  and  $C_0^2(\overline{D}_n) \cap \{u : u(n, \cdot) = 0\}$  respectively. Since  $U$  maps  $L^2(E_n)$  unitarily onto  $L^2(D_n)$  and  $U C_0^2(\overline{E}_n) = C_0^2(\overline{D}_n)$ , the variational principle gives

$$N_\lambda(\mathcal{A}_n^N) = N_\lambda(\mathfrak{S}_n^N), \quad N_\lambda(\mathcal{A}_n^D) = N_\lambda(\mathfrak{S}_n^D). \quad (5.12)$$

By a direct calculation

$$\widehat{Q}_n[u] = \int_{D_n} (|\partial_x u|^2 + W_\sigma u^2 - 2t \frac{f'}{f} \partial_x u \partial_t u + \frac{f'^2}{f^2} (t u \partial_t u + t^2 |\partial_t u|^2) + \frac{1}{f^2} |\partial_t u|^2) dx dt.$$

Now Let  $\eta \in (0, 1)$  be arbitrary. Since  $|t| \leq 1$  we get

$$|2t \frac{f'}{f} \partial_x u \partial_t u| \leq \eta |\partial_x u|^2 + \eta^{-1} \frac{f'^2}{f^2} |\partial_t u|^2, \quad \frac{f'^2}{f^2} |t u \partial_t u| \leq \eta^{-1} \frac{f'^4}{f^2} |u|^2 + \frac{\eta}{f^2} |\partial_t u|^2.$$

Moreover, from (2.4) and from the fact  $f$  is decreasing at infinity, by assumption 2.2, it follows that

$$f'(x)^2 \leq 2f(x) \sup_{s \geq x} |f''(s)|, \quad (5.13)$$

for all  $x$  large enough. Since  $f'' \rightarrow 0$  as  $x \rightarrow \infty$ , for any  $\eta \in (0, 1)$  there clearly exists an  $N_\eta$  such that for any  $n > N_\eta$  it holds

$$\widehat{Q}_n[u] \leq \int_{D_n} ((1 \pm \eta) |\partial_x u|^2 + W_\sigma u^2 + \frac{1 \pm 2\eta}{f^2} |\partial_t u|^2 \pm \eta u^2) dx dt. \quad (5.14)$$

We denote by  $H_n^N$  and  $H_n^D$  the operators acting in  $L^2(\Omega_{n,r})$  associated with the closures of the quadratic form

$$\int_{D_n} (|\partial_x u|^2 + \frac{|\partial_t u|^2}{f^2(x)} + W_\sigma(x) u^2) dx dt$$

defined on  $C_0^2(\overline{D}_n)$  and  $C_0^2(\overline{D}_n) \cap \{u : u(n, \cdot) = 0\}$  respectively.

**Lemma 5.3.** *Suppose that assumptions 2.2 and 2.5 are satisfied. For any  $\varepsilon$  there exists an  $N_\varepsilon$  such that for all  $n > N_\varepsilon$  and any  $\lambda > 0$  it holds*

$$N_\lambda(\mathcal{A}_n^N) \leq N_\lambda((1 - \varepsilon)H_n^N), \quad N_\lambda(\mathcal{A}_n^D) \geq N_\lambda((1 + \varepsilon)H_n^D). \quad (5.15)$$

*Proof.* In view of the fact that  $W_\sigma(x) \rightarrow \infty$  the statement follows from (5.14).  $\square$

Next we observe that since  $W_\sigma$  depends only on  $x$ , the matrix representations of the operators  $H_n^N$  and  $H_n^D$  in the basis of (normalised) eigenfunctions of the operator  $-f(x)^{-2} \frac{d^2}{dt^2}$  on the

interval  $(-1, 1)$  with Neumann boundary conditions are diagonal. We thus have the following unitary equivalence:

$$H_n^N \simeq \bigoplus_{k=0}^{\infty} \mathcal{H}_{k,n}^N, \quad H_n^D \simeq \bigoplus_{k=0}^{\infty} \mathcal{H}_{k,n}^D, \quad \mathcal{H}_{k,n}^{N/D} = -\frac{d^2}{dx^2} + W_{\sigma}(x) + \frac{k^2\pi^2}{4f(x)^2}, \quad (5.16)$$

where  $\mathcal{H}_{k,n}^{N/D}$  are operators in  $L^2(n, \infty)$  with Neumann/Dirichlet boundary condition at  $x = n$ . We denote

$$\mathcal{H}_{0,n}^N = \mathcal{H}_n^N, \quad \mathcal{H}_{0,n}^D = \mathcal{H}_n^D.$$

As a consequence of (5.16) we get

*Proof of Theorem 3.1.* We make use of inequality (5.1) for some fixed  $n$  and show that the operator on the left hand side of (5.1) has purely discrete spectrum. By general arguments of the spectral theory this will imply the statement. Since the spectrum of  $T_n^N$  is discrete, it suffices to show that the same is true for  $S_n^N$ . In view of Lemma 5.2 and equations (5.12), (5.15) it is enough to prove the discreteness of the spectrum of  $H_n^N$ . By (5.16) we have

$$\text{spect}(H_n^N) = \bigcup_{k=0}^{\infty} \text{spect}(\mathcal{H}_{k,n}^N),$$

First we notice that  $\text{spect}(\mathcal{H}_{k,n}^N)$  is purely discrete for each  $k$  and  $n$ . Indeed, a sufficient condition for the spectrum of  $\mathcal{H}_{k,n}^N$  to be purely discrete is that

$$W_{\sigma}(x) + \frac{k^2\pi^2}{4f(x)^2} \rightarrow \infty \quad \text{as } x \rightarrow \infty, \quad (5.17)$$

see e.g. [RS, Thm. 13.67], which is a direct consequence of assumption (2.5). Hence the spectrum of  $H_n^N$  is pure point, i.e. consists only of eigenvalues. Moreover, since  $f^2(x)W_{\sigma}(x) \rightarrow 0$  as  $x \rightarrow \infty$  by (5.13) and boundedness of  $\sigma$ , it is easy to see that

$$\forall n \quad \inf \text{spect}(\mathcal{H}_{k,n}^N) \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

Hence all the eigenvalues in the spectrum of  $H_n^N$  have finite multiplicity and  $\text{spect}(H_n^N)$  contains no finite point of accumulation. This means that  $\text{spect}(H_n^N)$  is discrete.  $\square$

*Proof of Theorem 3.3.* Case  $|\Omega| < \infty$ . By Lemma 4.1 the asymptotic behaviour of  $N_{\lambda}(\mathcal{H}_n^{N,D})$  does not depend on the boundary condition at  $x = n$ , nor on  $n$  itself:

$$N_{\lambda}(\mathcal{H}_{\sigma}) \sim N_{\lambda}(\mathcal{H}_n^N) \sim N_{\lambda}(\mathcal{H}_n^D) \quad \lambda \rightarrow \infty, \quad \forall n \in \mathbb{N}. \quad (5.18)$$

Now fix an  $\varepsilon > 0$ . From Lemma 5.2, (5.12) and (5.15) we see that for all  $n$  large enough it holds

$$N_{\lambda}(S_n^N) \leq N_{\lambda}((1 - \varepsilon)H_n^N), \quad N_{\lambda}(S_n^D) \geq N_{\lambda}((1 + \varepsilon)H_n^D) \quad (5.19)$$

Moreover,  $f^2(x)W_{\sigma}(x) \rightarrow 0$  at infinity so that

$$(1 - \varepsilon) \frac{k^2\pi^2}{4f(x)^2} \leq W_{\sigma}(x) + \frac{k^2\pi^2}{4f(x)^2} \leq (1 + \varepsilon) \frac{k^2\pi^2}{4f(x)^2} \quad \forall k \geq 1 \quad (5.20)$$

for all  $x$  large enough uniformly in  $k$ . Now observe that the sequence  $\{k^2\pi^2/4f(x)^2\}_{k \geq 1}$  enlists *all* the eigenvalues of the operator  $-f(x)^{-2} \frac{d^2}{dx^2}$  on the interval  $(-1, 1)$  with Dirichlet

boundary conditions. Hence it follows from (5.16) and (5.20) that for  $n$  large enough

$$\begin{aligned} N_\lambda((1 + \varepsilon) H_n^D) &\geq N_\lambda((1 + \varepsilon)^2 \mathcal{B}_n^D) + N_\lambda((1 + \varepsilon) \mathcal{H}_n^D) \\ N_\lambda((1 - \varepsilon) H_n^N) &\leq N_\lambda((1 - \varepsilon)^2 \mathcal{B}_n^N) + N_\lambda((1 - \varepsilon) \mathcal{H}_n^N), \end{aligned} \quad (5.21)$$

where  $\mathcal{B}_n^{N/D}$  are the operators defined in Section 4. Note that  $\mathcal{B}_n^{N/D}$  and  $\mathcal{H}_n^{N/D}$  satisfy assumption (2.10) by Lemma 4.5 and equation (5.18). In view of (5.2) we then conclude that for  $n$  large enough

$$N_\lambda(A_\sigma) \leq (1 + \mathcal{O}(\varepsilon)) (N_\lambda(T_n^N) + N_\lambda(\mathcal{B}_n^N) + N_\lambda(\mathcal{H}_n^N)) \quad (5.22)$$

$$N_\lambda(A_\sigma) \geq (1 + \mathcal{O}(\varepsilon)) (N_\lambda(T_n^D) + N_\lambda(\mathcal{B}_n^D) + N_\lambda(\mathcal{H}_n^D)), \quad (5.23)$$

If the volume of  $\Omega$  is finite then it follows from Lemmas 4.5, 5.1 and equations (5.18), (5.22), (5.23) that for any  $\varepsilon > 0$

$$1 + \mathcal{O}(\varepsilon) \leq \liminf_{\lambda \rightarrow \infty} \frac{N_\lambda(A_\sigma)}{\frac{\lambda}{4\pi} |\Omega| + N_\lambda(\mathcal{H}_\sigma)} \leq \limsup_{\lambda \rightarrow \infty} \frac{N_\lambda(A_\sigma)}{\frac{\lambda}{4\pi} |\Omega| + N_\lambda(\mathcal{H}_\sigma)} \leq 1 + \mathcal{O}(\varepsilon).$$

By letting  $\varepsilon \rightarrow 0$  we arrive at (3.1).

Case  $|\Omega| = \infty$ . If the volume of  $\Omega$  is infinite, then Lemma 4.5 gives

$$N_\lambda(T_n^N) + N_\lambda(\mathcal{B}_n^N) \sim N_\lambda(\mathcal{B}_n^N) \sim N_\lambda(\mathcal{B}_n^D) \sim N_\lambda(T_n^D) + N_\lambda(\mathcal{B}_n^D) \sim N_\lambda(\mathcal{B}) \quad (5.24)$$

as  $\lambda \rightarrow \infty$ . Moreover, mimicking all the above estimates for the Dirichlet-Laplacian  $-\Delta_\Omega^D$  instead of  $A_\sigma$  it is straightforward to verify that for any  $\varepsilon > 0$  and  $n$  large enough, depending on  $\varepsilon$ , it holds

$$N_\lambda((1 - \varepsilon) \mathcal{B}_n^N) \leq N_\lambda(-\Delta_\Omega^D) \leq N_\lambda((1 + \varepsilon) \mathcal{B}_n^D).$$

This together with (2.10) and (5.24) implies that  $N_\lambda(\mathcal{B}) \sim N_\lambda(-\Delta_\Omega^D)$  as  $\lambda \rightarrow \infty$ . Equation (3.1) thus follows again from (5.22) and (5.23).  $\square$

*Proof of Corollary 3.8.* Note that the assumption 2.5 is fulfilled. Indeed, equation (5.13) shows that  $f(x)V(x) \rightarrow 0$ . Consequently (2.6) holds true since  $f \rightarrow 0$  and

$$W_\sigma(x) \sim \frac{\sigma}{f(x)} \quad x \rightarrow \infty. \quad (5.25)$$

To prove (3.3) we recall that

$$\liminf_{\lambda \rightarrow \infty} \lambda^{-1} N_\lambda(A_\sigma) \geq \liminf_{\lambda \rightarrow \infty} \lambda^{-1} N_\lambda(-\Delta_\Omega^D) = \frac{|\Omega|}{4\pi}.$$

On the other hand, if  $\limsup_{x \rightarrow \infty} x^2 f(x) = 0$ , then (5.25) says for any  $\varepsilon > 0$  exists an  $x_\varepsilon$  such that  $W_\sigma(x) \geq \frac{x^2}{\varepsilon^2}$  holds for all  $x \geq x_\varepsilon$ . Lemma 4.4 gives

$$\limsup_{\lambda \rightarrow \infty} \lambda^{-1} N_\lambda(\mathcal{H}_\sigma) \leq \limsup_{\lambda \rightarrow \infty} \lambda^{-1} N_\lambda \left( -\frac{d^2}{dx^2} + \frac{x^2}{\varepsilon^2} \right)_{L^2(x_\varepsilon, \infty)} = \frac{\varepsilon}{4}.$$

From Lemma 4.5 and the proof of Theorem 3.3, see equations (5.2), (5.3), (5.19) and (5.21) we then get

$$\limsup_{\lambda \rightarrow \infty} \frac{N_\lambda(A_\sigma)}{\lambda} \leq (1 + \mathcal{O}(\varepsilon)) \frac{|\Omega|}{4\pi} + \limsup_{\lambda \rightarrow \infty} \frac{N_\lambda((1 - \varepsilon) \mathcal{H}_\sigma)}{\lambda} \leq (1 + \mathcal{O}(\varepsilon)) \frac{|\Omega|}{4\pi} + \mathcal{O}(\varepsilon).$$

Equation (3.3) now follows by letting  $\varepsilon \rightarrow 0$ . In order to prove (3.4) we note that  $W_\sigma(x) \sim \sigma a^{-2} x^2$  as  $x \rightarrow \infty$ , see (5.25). From Lemma 4.4 we thus deduce that

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1} N_\lambda(\mathcal{H}_\sigma) = \frac{|a|}{4\sqrt{\sigma}},$$

so that (2.9) is satisfied and (3.4) follows from Theorem 3.6.  $\square$

## 6. GENERALISATIONS

**6.1. Non symmetric boundary conditions.** As mentioned in Remark 3.5, the above approach can be applied also to Robin Laplacians with different boundary conditions on the upper and lower boundary of  $\Omega$ . More precisely, to operators  $A_{\sigma_1, \sigma_2}$  generated by the closure of the form

$$Q_{\sigma_1, \sigma_2}[u] = \int_{\Omega} |\nabla u|^2 dx dy + \int_1^\infty (\sigma_1(x) u(x, f(x))^2 + \sigma_2(x) u(x, -f(x))^2) dx \quad (6.1)$$

on  $C_0^2(\overline{\Omega})$ . We can proceed in the same way as in section 5 replacing the function  $v(x, y)$  in step 2 by the function  $w(x, y)$ , which solves the eigenvalue problem

$$\begin{aligned} -\partial_y^2 w(x, y) &= \bar{\mu}(x) w(x, y), \\ \partial_y w(x, -f(x)) &= \sigma_1(x) w(x, -f(x)), \quad \partial_y w(x, f(x)) = -\sigma_2(x) w(x, f(x)), \end{aligned} \quad (6.2)$$

with  $\bar{\mu}(x)$  being the principle eigenvalue. From equation (A.11), see appendix, we then get a generalisation of Theorem 3.3.

**Proposition 6.1.** *Suppose that assumptions 2.2, 2.5 and 2.6 for  $\sigma_1, \sigma_2$  are satisfied. Then*

$$N_\lambda(A_{\sigma_1, \sigma_2}) \sim N_\lambda(-\Delta_\Omega^D) + N_\lambda(\mathcal{H}_{\bar{\sigma}}) \quad \lambda \rightarrow \infty, \quad \bar{\sigma}(x) = \frac{\sigma_1(x) + \sigma_2(x)}{2}. \quad (6.3)$$

**6.2. Dirichlet-Neumann Laplacian.** Our second remark concerns the case in which we impose Dirichlet boundary condition on one of the boundaries of  $\Omega$ . We confine ourselves to the special situation when we have Dirichlet boundary condition on one boundary and Neumann on the other. We denote the resulting operator by  $A_{0, \infty}$ .

**Proposition 6.2.** *Let  $|\Omega| < \infty$  and assume that  $f$  is decreasing at infinity. Then*

$$N_\lambda(A_{0, \infty}) \sim \frac{|\Omega|}{4\pi} \lambda, \quad \lambda \rightarrow \infty. \quad (6.4)$$

*Proof.* First we observe that by the variational principle.

$$\liminf_{\lambda \rightarrow \infty} \lambda^{-1} N_\lambda(A_{0, \infty}) \geq \liminf_{\lambda \rightarrow \infty} \lambda^{-1} N_\lambda(-\Delta_\Omega^D) = \frac{|\Omega|}{4\pi}. \quad (6.5)$$

Assume that  $f$  is decreasing on  $(a, \infty)$  and that  $\lambda$  is large enough so that there exists a unique point  $x_\lambda > a$  such that  $f(x_\lambda) = \pi/(4\sqrt{\lambda})$ . We impose additional Neumann boundary condition at  $\{x = x_\lambda\}$  dividing thus  $\Omega$  into the finite part  $\Omega_\lambda := \{(x, y) \in \Omega : x < x_\lambda\}$  and its complement  $\Omega_\lambda^c$ . It is then easy to see that the quadratic form of the corresponding operator acting on  $\Omega_\lambda^c$  is bounded from below by

$$\int_{x_\lambda}^\infty \int_{-f(x)}^{f(x)} \left( \frac{\pi^2}{16 f^2(x)} u^2 + |\partial_x u|^2 \right) dy dx \geq \lambda \int_{x_\lambda}^\infty \int_{-f(x)}^{f(x)} u^2 dy dx$$

for all functions  $u$  from its domain. Consequently, this operator does not have any eigenvalues below  $\lambda$ . To estimate the number of eigenvalues of the operator acting on  $\Omega_\lambda$ , we cover  $\Omega_\lambda$  with a finite collection of disjoint cubes of size  $L = 1/(\varepsilon\sqrt{\lambda})$  with  $\varepsilon > 0$ . Since  $\Omega_\lambda$  has the extension property, the standard technique of Neumann bracketing gives

$$\begin{aligned} \lambda^{-1}N_\lambda(A_{0,\infty}) &\leq \lambda^{-1}N_\lambda(-\Delta_{\Omega_\lambda}^N) \leq \frac{|\Omega_\lambda|}{4\pi} (1 + \mathcal{O}(\varepsilon)) + c \frac{|\partial\Omega_\lambda|}{\sqrt{\lambda}} (1 + \varepsilon^{-1}) \\ &\leq \lambda^{-1}N_\lambda(-\Delta_{\Omega_\lambda}^N) \leq \frac{|\Omega_\lambda|}{4\pi} (1 + \mathcal{O}(\varepsilon)) + \tilde{c} \frac{x_\lambda}{\sqrt{\lambda}} (1 + \varepsilon^{-1}), \end{aligned} \quad (6.6)$$

where  $\tilde{c}$  is independent of  $\lambda$ . However, since  $f$  is integrable and decreasing at infinity it is easily seen that  $x_\lambda f(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Hence

$$\limsup_{\lambda \rightarrow \infty} \frac{x_\lambda}{\sqrt{\lambda}} = \frac{4}{\pi} \limsup_{\lambda \rightarrow \infty} x_\lambda f(x_\lambda) = 0.$$

Letting first  $\lambda \rightarrow \infty$  and then  $\varepsilon \rightarrow 0$  in (6.6) we obtain  $\limsup_{\lambda \rightarrow \infty} \lambda^{-1}N_\lambda(A_{0,\infty}) \leq |\Omega|/4\pi$ , which together with (6.5) implies the statement.  $\square$

#### APPENDIX A.

**Lemma A.1.** *Let  $\mu(x)$  be the function defined by the problem (5.5). Then*

$$\mu(x) \leq \frac{\sigma(x)}{f(x)} \quad \forall x > 1. \quad (A.1)$$

*Proof.* For each fixed  $x \in (1, \infty)$  we define the quadratic form

$$a_x[u] = \int_{-f(x)}^{f(x)} |u'(y)|^2 dy + \sigma(x) (|u(f(x))|^2 + |u(-f(x))|^2), \quad u \in D(a_x), \quad (A.2)$$

where  $D(a_x) = H^1(-f(x), f(x))$ . The variational definition of  $\mu$  says that

$$\mu(x) = \inf_{u \in D(a_x)} \frac{a_x[u]}{\|u\|_{L^2(-f(x), f(x))}^2} \leq \frac{a_x[1]}{\|1\|_{L^2(-f(x), f(x))}^2} = \frac{\sigma(x)}{f(x)}.$$

$\square$

In the next Lemma we use the notation  $\kappa(x) := \sqrt{\mu(x)}$ .

**Lemma A.2.** *Let the assumption 2.5 be satisfied. Then the eigenfunction  $v(x, y)$  of the problem (5.5) associated to the eigenvalue  $\mu(x)$  is twice continuously differentiable in  $x$ . Moreover we have*

$$\lim_{x \rightarrow \infty} \frac{f(x)\mu(x)}{\sigma(x)} = 1 \quad (A.3)$$

$$\lim_{x \rightarrow \infty} v(x, y) = 1 \quad \text{uniformly in } y, \quad (A.4)$$

$$\lim_{x \rightarrow \infty} \partial_x v(x, y) = 0 \quad \text{uniformly in } y. \quad (A.5)$$

*Proof.* It is easy to see that

$$v(x, y) = \cos(\kappa(x)y), \quad (A.6)$$

where  $\kappa(x)$  is the first positive solution to the implicit equation

$$F(x, \kappa) := \kappa \tan(\kappa f(x)) - \sigma(x) = 0. \quad (A.7)$$

Since  $f(x)\kappa(x) \rightarrow 0$  as  $x \rightarrow \infty$  by Lemma A.1 (recalling that  $\sigma(x)f(x) \rightarrow 0$ ), we easily deduce from (A.7) that

$$\lim_{x \rightarrow \infty} \frac{f(x)\kappa^2(x)}{\sigma(x)} = 1, \quad (\text{A.8})$$

which proves (A.3). Equation (A.4) thus follows directly from (A.6) and the fact that  $f(x)\kappa(x) \rightarrow 0$ . Next we note that (A.7) implies

$$0 < \kappa(x) < \frac{\pi}{2f(x)} \quad \forall x > 1,$$

and hence

$$\partial_\kappa F(x, \kappa) = \tan(f(x)\kappa) + \frac{f(x)\kappa}{\cos^2(f(x)\kappa)} > 0. \quad (\text{A.9})$$

Since  $\sigma \in C^2(1, \infty)$ , the implicit function theorem shows that  $\kappa$  is of the class  $C^2$  and in view of (A.6) we see that  $v$  is twice continuously differentiable in  $x$ .

In order to prove (A.5) we need some information about the behaviour of  $\kappa'$  for large  $x$ . From the positivity of  $f$  and  $\sigma$  and from the Taylor theorem we conclude that  $\sigma'/\sqrt{\sigma}$  is bounded and that  $f'/\sqrt{f} \rightarrow 0$ , see equation (5.13). Equations (A.9) and (A.8) then give

$$\kappa'(x) = -\frac{\partial_x F}{\partial_\kappa F} \sim \frac{\sqrt{\sigma(x)}}{2\sqrt{f(x)}} (f'(x) - \sigma'(x)) \quad x \rightarrow \infty. \quad (\text{A.10})$$

On the other hand, a direct calculation shows that

$$|\partial_x v(x, y)| \leq |\kappa'(x)| f^{3/2}(x) \sqrt{\sigma(x)} \quad \forall x > 1.$$

This implies (A.5). □

Notice that if we replace the eigenvalue problem (5.5) by (6.2), then a straightforward analysis of the associated implicit equation shows that

$$\lim_{x \rightarrow \infty} \frac{f(x)\bar{\mu}(x)}{\bar{\sigma}(x)} = 1, \quad \bar{\sigma}(x) = \frac{\sigma_1(x) + \sigma_2(x)}{2}. \quad (\text{A.11})$$

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