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On the complexity of the $\{k\}$-packing function problem

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Abstract

Given a positive integer k, the "$\{k\}$-packing function problem" ($\{k\}$PF) is to find in a given graph G, a function f that assigns a nonnegative integer to the vertices of G in such a way that the sum of $f(v)$ over each closed neighborhood is at most k and over the whole vertex set of G (weight of f) is maximum. It is known that $\{k\}$PF is linear time solvable in strongly chordal graphs and in graphs with clique-width bounded by a constant. In this paper we prove that $\{k\}$PF is NP-complete, even when restricted to chordal graphs that constitute a superclass of strongly chordal graphs. To find other subclasses of chordal graphs where $\{k\}$PF is tractable, we prove that it is linear time solvable for doubly chordal graphs, by proving that it is so in the superclass of dually chordal graphs, which are graphs that have a maximum neighborhood ordering.

Keywords: chordal graph; NP-completeness; polynomial instances

1. Introduction

All the graphs in this paper are simple (without self-loops or multiple edges), finite, and undirected. For a graph G, $V(G)$ and $E(G)$ denote, respectively, its vertex and edge sets. For any $v \in V(G)$, $N_G[v]$ is the "closed neighborhood" and $N_G(v)$, the "open neighborhood" of v in G. For a given graph G and a function $f : V(G) \to \mathbb{R}$, we denote $f(A) = \sum_{v \in A} f(v)$, where $A \subseteq V(G)$. The "weight" of f is $f(V(G))$. A "split" graph H with partition (Q, S) is a graph for which $V(H)$ can be divided into a clique Q and stable set S.

A "chord" of a cycle C in a graph G is an edge uv not in C such that u and v lie in C. A graph G is "chordal" if it does not contain an induced chordless cycle on n vertices for any $n \geq 4$. A graph G is "strongly chordal" if it is chordal and every cycle of even length in G has an odd chord, that is, a chord that connects two vertices that are at odd distance apart from each other in the cycle.

A vertex u in $N_G[v]$ is a "maximum neighbor" of v if for all $w \in N_G[v]$, $N_G[w] \subseteq N_G[u]$. A vertex ordering $v_1 \dots v_n$ is a "maximum neighborhood ordering of G" if for each $i < n$, v_i has a maximum neighbor in G_i, where G_i is the subgraph of G induced by $\{v_i, \dots, v_n\}$. A graph is "dually chordal" if it has a maximum neighborhood ordering. A graph is "doubly chordal" if it is chordal and dually chordal.

Given a graph G and a positive integer k, a set $B \subseteq V(G)$ is a "k-limited packing" in G if each closed neighborhood has at most k vertices of B (Gallant et al., 2010). Observe that a k-limited packing in G can be considered as a function $f : V(G) \to \{0, 1\}$ such that $f(N_G[v]) \leq k$ for all $v \in V(G)$. The maximum possible weight of a k-limited packing in G is denoted by $L_k(G)$. When $k = 1$, a k-limited packing in G is a "2-packing" in G and $L_k(G)$ is the known "packing number" of G, $\rho(G)$.

This concept is a good model for many utility location problems in operations research, for example, the problem of locating garbage dumps in a city. In most of them, the utilities—garbage dumps—are necessary but probably obnoxious. That is why it is of interest to place the maximum number of utilities in such a way that no more than a given number of them is near to each agent in a given scenario. The above definition induces the study of the problem of deciding if a given graph G has a k-limited packing of weight $L_k(G)$. The study of the computational complexity of this problem was started in Dobson et al. (2010). Regarding NP-completeness results, the graph is considered NP-complete even for instances given by split graphs (Dobson et al., 2010), and also for bipartite graphs (Dobson et al., 2011).

In order to expand the set of utility location problems to be modeled, in Leoni and Hinrichsen (2014) the concept of $\{k\}$-packing function ($\{k\}$PF) of a graph was introduced as a variation of a k-limited packing. Recalling the problem of locating garbage dumps in a given city, if a graph G and positive integer k model the scenario, when dealing with $\{k\}$PFs we are allowed to locate more than one garbage dump in any vertex of G subject to there are at most k garbage dumps in each closed neighborhood. Formally:

Definition 1. *Given a graph G and positive integer k, a $\{k\}$ PF of G is a function $f : V(G) \to \mathbb{N}_0$ such that for all $v \in V(G)$,*

$$f(N_G[v]) \leq k.$$

The maximum possible weight of a $\{k\}$PF of G is denoted by $L_{\{k\}}(G)$.

Since any k-limited packing in G can be seen as a $\{k\}$PF of G, it is clear that

$$L_k(G) \leq L_{\{k\}}(G).$$

Nevertheless, Figs. 1 and 2 show that this inequality may or may not be strict.

This definition induces the study of the following decision problem for fixed $k \in \mathbb{N}$:

$\{k\}$**PF (fixed** $k \geq 1$)
Instance: (G, α), where G is a graph and $\alpha \in \mathbb{N}$.
Question: Does G have a $\{k\}$PF of weight at least α?

It is known that $\{k\}$PF is linear time solvable in strongly chordal graphs and also in graphs with clique-width bounded by a constant (Leoni and Hinrichsen, 2014). The complexity of $\{k\}$PF for a general graph remained open. In Section 2, we prove that $\{k\}$PF is NP-complete, even for split graphs.

Fig. 1. A graph G with $L_3(G) = 4$ and $L_{\{3\}}(G) = 6$.

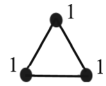

Fig. 2. $L_3(K_3) = L_{\{3\}}(K_3) = 3$.

A split graph is a chordal graph. Looking for other subclasses of chordal graphs where $\{k\}$PF is tractable apart from strongly chordal graphs, in Section 3 we prove that it is linear time solvable for doubly chordal graphs, by proving that it is so in the superclass of dually chordal graphs. Theorem 2 and the results of Section 3 have been already published without proofs in an electronic version (Dobson et al., 2015).

2. NP-completeness results

In this section, we solve the complexity of $\{k\}$PF for a general graph. In particular, we prove that it is NP-complete even when restricted to a split graph.

For this purpose, let us first recall a well-known concept in set theory and its associated decision problem. Given a nonempty set X and a family S of subsets of X, a "set packing" (SP) of X is a subfamily T of S such that all the members in T are pairwise disjoint.

SET PACKING (SP)
Instance: (X, S, q), where S is a collection of subsets of X and $q \in \mathbb{N}$ with $q \leq |S|$.
Question: Does S contain an SP of cardinality at least q?

It is known that SP is NP-complete (Karp, 1972). We will reduce SP to $\{k\}$PF in a general graph in polynomial time. This reduction follows some ideas given in Gairing et al. (2003) in the context of $\{k\}$-domination.

Theorem 2. *$\{k\}$PF is NP-complete for all integer k fixed.*

Proof. Clearly, $\{k\}$PF is in NP. To prove that it is NP-complete, we will reduce SP to $\{k\}$PF in a general graph in polynomial time.

Given an instance (X, S, q) of SP, where $X = \{x_1, \ldots, x_n\}$ and $S = \{S_1, \ldots, S_m\}$, we create an instance (G, α) of $\{k\}$PF in the following way: for each $S_j \in S$, there exists a subgraph G_j of G consisting of k paths on three vertices P_r^j, $r = 1, \ldots, k$, and a vertex d_j. We label $a_{j,r}, b_{j,r}, c_{j,r}$ the vertices of path P_r^j with $N_G(b_{j,r}) = \{a_{j,r}, c_{j,r}\}$, for $1 \leq r \leq k$. In addition, the vertex d_j is adjacent to each of the vertices $a_{j,1}, \ldots, a_{j,k}$. For each element $x_i \in X$ there is a vertex v_i that is adjacent to $c_{j,1}, \ldots, c_{j,k}$ if and only if $x_i \in S_j$. In addition, there exists a vertex v that is adjacent to each v_i and a 1° vertex w (e.g., see Fig. 3). The transformation is completed by taking $\alpha = k^2|S| + k + q$.

Assume T is an SP of X with $|T| \geq q$. We construct a $\{k\}$PF f of G as follows: for each $j = 1, \ldots, m$, let $f(a_{j,r}) = 0$, $f(b_{j,r}) = k - 1$, $f(c_{j,r}) = 1$, $f(d_j) = 1$ if $S_j \in T$, and $f(a_{j,r}) = 0$, $f(b_{j,r}) = k$, $f(c_{j,r}) = 0$, $f(d_j) = 0$ otherwise. Also let $f(v_i) = 0$ for $i = 1, \ldots, n$, $f(v) = 0$ and $f(w) = k$. From the way f was defined, it is clear that for every vertex $x \in V(G_j)$, $f(N_G[x]) \leq k$.

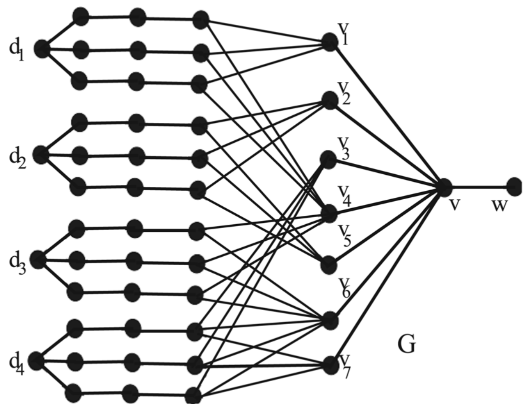

Fig. 3. Transformation in Theorem 2 for $X = \{x_1, \ldots, x_7\}$, $S_1 = \{x_1, x_4\}$, $S_2 = \{x_2, x_5\}$, $S_3 = \{x_4, x_6\}$, and $S_4 = \{x_3, x_6, x_7\}$ and $k = 3$.

Since T is an SP of X, each vertex v_i is adjacent in G to at most k c-vertices with value 1 and therefore $f(N_G[v_i]) \leq k$. Besides, $f(N_G[v]) = f(N_G[w]) = k$. We have $f(V(G_j)) = k^2 + 1$ for each subgraph G_j with $S_j \in T$ and $f(V(G_j)) = k^2$, for each subgraph G_j with $S_j \notin T$. Since $|T| \geq q$, $f(V(G)) \geq q(k^2 + 1) + (|S| - q)k^2 + k = \alpha$ and therefore f is a $\{k\}$PF of weight at least α.

Conversely, let f be a $\{k\}$PF of G with $f(V(G)) \geq \alpha$. In the sequel, we will show the existence of at least q subgraphs G_j with $f(V(G_j)) = k^2 + 1$, and that each v_i is adjacent to at most one such subgraph (via c-vertices). Those subgraphs G_j with $f(V(G_j)) = k^2 + 1$ will determine an SP of X with cardinality at least q.

First, let us show that $f(V(G_j)) \leq k^2 + 1$ for every j: if this is not the case, $k^2 + 2 \leq f(V(G_j)) = f(d_j) + \sum_{r=1}^{k} f(V(P_r^j)) \leq f(d_j) + k^2$ and therefore $2 \leq f(d_j)$. Besides, since $\sum_{r=1}^{k} f(c_{j,r}) \leq k$ and $f(a_{j,r}) + f(b_{j,r}) \leq k - f(d_j) \leq k - 2$ for $1 \leq r \leq k$, we have

$$
\begin{aligned}
f(V(G_j)) &= f(d_j) + \sum_{r=1}^{k}(f(a_{j,r}) + f(b_{j,r})) + \sum_{r=1}^{k} f(c_{j,r}) \\
&\leq f(d_j) + k(k-2) + k = f(d_j) + k^2 - k.
\end{aligned}
$$

Then $k^2 + 2 \leq f(V(G_j)) \leq f(d_j) + k^2 - k$. It follows that $k + 2 \leq f(d_j)$, a contradiction since $f(d_j) \leq k$.

Second, there exists j' with $1 \leq j' \leq |S|$ such that $f(V(G_{j'})) = k^2 + 1$: if $f(V(G_j)) \leq k^2$ for every j, then $f(V(G)) = \sum_{j=1}^{|S|} f(V(G_j)) + f(N_G[v]) \leq |S|k^2 + k$. But $f(V(G)) \geq |S|k^2 + k + q$. Thus $q \leq 0$, a contradiction.

Note that $f(d_{j'}) = 1$. If not, that is, if $f(d_{j'}) = t > 1$, we would have $k^2 + 1 = f(V(G_{j'})) = t + \sum_{r=1}^{k}(f(a_{j',r}) + f(b_{j',r})) + \sum_{r=1}^{k} f(c_{j',r}) \leq t + k(k - t) + k$ (since $f(a_{j',r}) + f(b_{j',r}) \leq k - t$ for each r and clearly $\sum_{r=1}^{k} f(c_{j',r}) \leq k$) concluding that $1 - t \leq k(1 - t)$, a contradiction since $k \geq 1$. Besides, since $f(V(P_r^{j'})) \leq k$ for $r = 1, \ldots, k$ and $f(V(G_{j'})) = k^2 + 1$, we have $f(V(P_r^{j'})) = k$ for

$r = 1, \ldots, k$. In addition, for each $1 \leq r \leq k$, $f(a_{j',r}) + f(b_{j',r}) \leq k - 1$ and therefore $f(c_{j',r}) = k - (f(a_{j',r}) + f(b_{j',r})) \geq k - (k-1) = 1$. Moreover, since $f(N_G[v_i]) \leq k$ for each $i = 1, \ldots, n$, we have $f(c_{j',r}) = 1$ for each $r = 1, \ldots, k$.

On the one hand, we conclude that for every x_i there exists at most one j, $1 \leq j \leq |S|$ such that $f(V(G_j)) = k^2 + 1$ and its c-vertices adjacent to v_i.

On the other hand, let p be the number of subgraphs G_j for which $f(V(G_j)) = k^2 + 1$, where $1 \leq p \leq |S|$. Then

$$\begin{aligned}|S|k^2 + k + q \leq f(V(G)) &= f\left(\bigcup_{j=1}^{|S|} V(G_j)\right) + f(N_G[v]) \\ &\leq p(k^2+1) + (|S| - p)k^2 + k = |S|k^2 + k + p.\end{aligned}$$

Then $q \leq p$.

Since $f(N_G[v_i]) \leq k$, there is at most one G_j with $f(V(G_j)) = k^2 + 1$ adjacent to v_i. The subsets corresponding to those G_j with $f(V(G_j)) = k^2 + 1$ constitute an SP of cardinality at least q. □

Next, we present a useful property that allows us to prove in Theorem 4 that $\{k\}$PF is NP-complete even on split graphs.

Lemma 3. *Let G be a graph and k be a fixed positive integer. For $u, v \in V(G)$ such that $N_G[u] \subseteq N_G[v]$, there exists a maximum $\{k\}$PF of G such that $f(v) = 0$.*

Proof. Let $u, v \in V(G)$ such that $N_G[u] \subseteq N_G[v]$.

Let f^* be a maximum $\{k\}$PF of G. Observe

$$f^*(u) + f^*(v) \leq f^*(N_G[u]) \leq f^*(N_G[v]) \leq k.$$

We define the function f over $V(G)$ by $f(v) = 0$, $f(u) = f^*(u) + f^*(v)$, and $f(w) = f^*(w)$ if $w \in V(G) - \{u, v\}$. It turns out that f is a $\{k\}$PF of G since for $w \in N_G[u]$,

$$\begin{aligned}f(N_G[w]) &= f(N_G[w] - \{u, v\}) + f(\{u, v\}) = f^*(N_G[w] - \{u, v\}) + f(u) + f(v) \\ &= f^*(N_G[w] - \{u, v\}) + f^*(u) + f^*(v) = f^*(N_G[w]) \leq k.\end{aligned}$$

Clearly $f(V(G)) = f^*(V(G))$, thus f is maximum and the proof is completed. □

Theorem 4. *For a fixed positive integer k, $\{k\}$PF is NP-complete on split graphs.*

Proof. Clearly, $\{k\}$PF on split graphs is in NP. We will reduce $\{k\}$PF on a general graph to $\{k\}$PF on a split graph in polynomial time. Given a graph G with $V(G) = \{v_1, \ldots, v_n\}$, we construct a split graph H with partition (Q, S) defined as follows:

- $Q = \{q_1, \ldots, q_n\}$ is a clique and $S = \{s_1, \ldots, s_n\}$ is a stable set,
- for all $i, j = 1, \ldots, n$, $q_i s_j \in E(H)$ if and only if $v_i v_j \in E(G)$ or $i = j$.

We will prove that $L_{\{k\}}(G) = L_{\{k\}}(H)$.

Note that for all $i \in \{1, \dots, n\}$, $N_H[s_i] \subseteq N_H[q_i]$. Then, from Lemma 3, there exists a maximum $\{k\}$PF f of H such that $f(Q) = 0$. We define the function f' over $V(G)$ such that $f'(v_i) = f(s_i)$ for all $i \in \{1, \dots, n\}$. From its definition, f' is a $\{k\}$PF of G. Hence,

$$L_{\{k\}}(G) \le L_{\{k\}}(H).$$

Conversely, let f be a maximum $\{k\}$PF of G. Let us define the function f^* over $V(H)$ such that $f^*(s_i) = f(v_i)$ and $f^*(q_j) = 0$. Note that

$$\begin{aligned} f^*(N_H[q_i]) &= f^*(Q \cap \{s_j : v_iv_j \in E(G) \vee\ i = j\}) = f^*(\{v_j : v_iv_j \in E(G) \vee\ i = j\}) \\ &= f(\{s_j : v_iv_j \in E(G) \vee\ i = j\}) \le f(N_G[v_i]) \le k, \end{aligned}$$

thus f^* is a $\{k\}$PF of H. Hence,

$$L_{\{k\}}(G) \ge L_{\{k\}}(H).$$ □

As a split graph is also a chordal graph, as a corollary we obtain that $\{k\}$PF is NP-complete on chordal graphs, for fixed positive integer k.

3. Polynomial instances: dually chordal graphs

In this section, we obtain another graph class where $\{k\}$PF is linear time solvable, namely the class of dually chordal graphs. In particular, we show how to construct in linear time a $\{k\}$PF of maximum cardinality of a dually chordal graph from a given 1-limited packing of maximum cardinality.

For this purpose, let us first show the following relation between $L_{\{k\}}(G)$ and $L_1(G)$ of a given graph G.

Lemma 5. *Given a graph G and a positive integer k, $L_{\{k\}}(G) \ge k \cdot L_1(G)$.*

Proof. Let B be a 1-limited packing of G such that $|B| = L_1(G)$. Let f be the function defined over $V(G)$ by $f(v) = k$ if $v \in B$ and $f(v) = 0$ if $v \in V(G) - B$. From the way f was defined, we have $f(N_G[v]) = k \cdot |N_G[v] \cap B|$ for each $v \in V(G)$. Since B is a 1-limited packing of G, $|N_G[v] \cap B| \le 1$ for each $v \in V(G)$. Thus $f(N_G[v]) \le k$ for each $v \in V(G)$, implying that f is a $\{k\}$PF of G. Clearly, the weight of f is $k \cdot L_1(G)$ and therefore $L_{\{k\}}(G) \ge k \cdot L_1(G)$. □

The inequality stated by Lemma 5 becomes an equality under certain circumstances. In order to go toward this direction, let us recall the following well-known definitions.

A "dominating set" of a graph G is a subset D of $V(G)$ such that $|N_G[v] \cap D| \ge 1$ for all $v \in V(G)$. The "dominating number" $\gamma(G)$ is the minimum cardinality of a dominating set of G.

Recall that $L_1(G)$ is also known as the packing number of G and denoted by $\rho(G)$. It is known that $L_1(G) \le \gamma(G)$ for every graph G. There are many graph classes for which $L_1(G) = \gamma(G)$ (for instance, see Rubalcaba et al., 2006). For those classes, we can prove the following interesting fact that allows us to obtain the value of the $\{k\}$PF number of a graph in such a class.

Theorem 6. *Given a graph G and positive integer k, if $\gamma(G) = L_1(G)$ then $L_{\{k\}}(G) = k \cdot L_1(G)$.*

Proof. Let G be a graph with $\gamma(G) = L_1(G)$. From Lemma 5, we have $L_{\{k\}}(G) \ge k \cdot L_1(G)$.

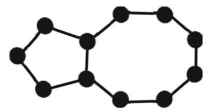

Fig. 4. A graph G with $\gamma(G) = 4$, $L_1(G) = 3$, and $L_{\{3\}}(G) = 9 = 3L_1(G)$.

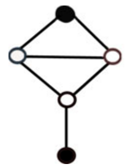

Fig. 5. A minimum dominating set and maximum 1-limited packing (2-packing) of a dually chordal graph G.

Now let f be a $\{k\}$PF of G of weight $f(V(G)) = L_{\{k\}}(G)$. Take a dominating set D of G with $|D| = \gamma(G)$. Since

$$V(G) = \cup_{w \in D} N_G[w],$$

it is clear that $f(V(G)) \leq \sum_{w \in D} f(N_G[w])$. Moreover, since f is a $\{k\}$PF of G, $f(N_G[w]) \leq k$ for every w, implying $f(V(G)) \leq k \cdot |D|$. In particular, $L_{\{k\}}(G) \leq k \cdot \gamma(G)$. Since $\gamma(G) = L_1(G)$, the proof is complete. □

The example in Fig. 4 shows that the sufficient condition in Theorem 6 is not necessary.

The hypothesis of Theorem 6 is satisfied by dually chordal graphs (Brandstädt et al., 1988) (e.g., see Fig. 5).

It is known that a maximum 1-limited packing P of a dually chordal graph G can be obtained in linear time (Brandstädt et al., 1988). Following the ideas in this section, a maximum $\{k\}$PF f of G can be obtained by setting $f(v) = k$ if $v \in P$ and $f(v) = 0$ if $v \in V(G) - P$. Hence, we have proved:

Theorem 7. *$\{k\}$PF on dually chordal graphs can be solved in linear time.*

4. Final remarks

In this paper, we proved that the complexity of the problem studied is NP-complete for a general graph, even when restricted to chordal graphs. In Leoni and Hinrichsen (2014), it is proved that $\{k\}$PF is linear time solvable in strongly chordal graphs. Now we proved that the linearity of this problem can be extended to the superclass given by dually chordal graphs (Theorem 7). Hence, it remains linear time solvable for doubly chordal (chordal and dually chordal) graphs.

Finally, note that the result of Theorem 6 can be used to obtain more tractable instances of $\{k\}$PF. It is interesting to explore additional classes of graphs for which the dominating and packing numbers are equal. Actually, Theorem 6 implies that the tractability of $\{k\}$PF for a given graph in such a class will be bounded by the maximum between the times in which the dominating and packing numbers of the graph can be obtained.

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