EUROGRAPHICS 2015 / O. Sorkine-Hornung and M. Wimmer
(Guest Editors)

Volume 34 (2015), Number 2

Selective Degree Elevation for Multi-Sided Bézier Patches

J. Smith[1] and S. Schaefer[1]

[1]Texas A&M University

Abstract

This paper presents a method to selectively elevate the degree of an S-Patch of arbitrary dimension. We consider not only S-Patches with 2D domains but 3D and higher-dimensional domains as well, of which volumetric cage deformations are a subset. We show how to selectively insert control points of a higher degree patch into a lower degree patch while maintaining the polynomial reproduction order of the original patch. This process allows the user to elevate the degree of only one portion of the patch to add new degrees of freedom or maintain continuity with adjacent patches without elevating the degree of the entire patch, which could create far more degrees of freedom than necessary. Finally we show an application to cage-based deformations where we increase the number of control points by elevating the degree of a subset of cage faces. The result is a cage deformation with higher degree triangular Bézier functions on a subset of cage faces but no interior control points.

Categories and Subject Descriptors (according to ACM CCS): I.3.3 [Computer Graphics]: Picture/Image Generation—Line and curve generation

1. Introduction

The Bézier form of curves, surfaces, and volumes has many applications in Computer Graphics. Curves in Bézier form are used in applications from vector graphics to animation curves. Surfaces in Bézier form are used to model 3D surfaces or even as a basis for finite element methods. Volumes have found applications in surface deformation [SP86].

The Bézier form for curves is defined via Bernstein basis functions and is given by

$$F(t) = \sum_{m=1}^{d} \binom{d}{m} (1-t)^{d-m} t^m f_m, \tag{1}$$

where d is the degree of the curve and f_k are the control points. These 1D parametric curves are related to a binomial expansion of univariate barycentric basis functions $(1-t), t$.

Extending Bézier curves from 1D to Bézier patches in higher dimensions requires barycentric coordinates defined over higher dimensional domains, which are not unique for non-simplicial shapes. Given a domain polygon (or, more generally, a higher dimensional domain polytope) P with n vertices p_k, generalized barycentric coordinates are functions $w_k(x)$ associated with each vertex p_k of an evaluation point x such that

$$\begin{aligned} \textstyle\sum_{k=1}^{n} w_k(x) &= 1 \\ \textstyle\sum_{k=1}^{n} w_k(x) p_k &= x. \end{aligned} \tag{2}$$

While triangular and tensor-product Bézier patches are common, both are special cases of a more general construction called S-Patches [LD89]. S-Patches directly generalize Equation 1 by replacing the binomial expansion of 1D barycentric coordinates with a multinomial expansion of generalized barycentric coordinates. Given a polytope P, a degree d S-Patch is given by

$$F(x) = \sum_{|\vec{i}|=d} B^d_{\vec{i}}(x) f_{\vec{i}},$$

where the index $\vec{i}$ is a vector of n non-negative integers, $|\vec{i}|$ is the sum of the entries of $\vec{i}$, and $f_{\vec{i}}$ are the control points. The basis functions $B^d_{\vec{i}}(x)$ are the terms in the multinomial expansion of $(\sum_{k=1}^{n} w_k(x))^d$

$$B^d_{\vec{i}}(x) = \binom{d}{\vec{i}} \prod_{k=1}^{n} (w_k(x))^{\vec{i}_k} \tag{3}$$

where $w_k(x)$ are the generalized barycentric basis functions with respect to the polytope P and $\binom{d}{\vec{i}}$ is the multinomial coefficient $\binom{d}{\vec{i}} = \frac{d!}{i_1! \cdots i_n!}$. Figure 1 shows an example of the

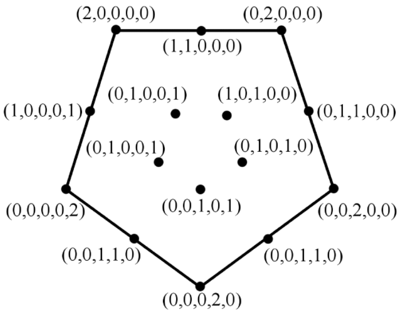

Figure 1: *An example of the indexing of a 5-sided quadratic S-Patch.*

indexing of a quadratic 5-sided S-Patch. We refer to control points whose basis functions have no influence on the boundary of the domain as interior points, and the remaining control points are referred to as boundary points. For example, in Figure 1, $f_{(0,0,1,1,0)}$ is a boundary control point and $f_{(0,1,0,0,1)}$ is an interior control point.

While the original definition of S-Patches was restricted to convex, multi-sided polygon domains, this limitation was due to the barycentric coordinates available at that time [Wac75] rather than any limitation of the S-Patch construction. Today more general forms of barycentric coordinates exist for non-convex domains and in arbitrary dimensions [JSW05, FKR05]. For this paper, we consider the fully general form of S-Patches for arbitrary-sided, closed domains with or without holes in any dimension. For example, though seldom viewed in this light, cage-based deformations [JSW05] with generalized barycentric coordinates are really deformations using S-Patch volumes of degree one.

However, S-Patches are limited by the number of degrees of freedom in the representation (i.e.; control points). In some cases the user may need more degrees of freedom to manipulate a surface or a deformation than the initial set of control points allows for. The solution is to either subdivide the representation (only possible for simplicial/tensor product representations), redesign the S-Patch (requires user intervention and does not preserve the original surface/deformation), or to perform degree elevation.

Unfortunately, degree elevation increases the number of degrees of freedom far too rapidly, especially in higher dimensions. For cage-based deformation, raising the degree of a modest control cage with 100 vertices to a quadratic generates over 5000 control points, most of which will be interior points. This explosive growth in control points can quickly become intractable for a user to manipulate. Moreover, one of the strengths of cage-based deformations is that the user only needs to manipulate the boundary of the shape to control the deformation rather than any interior points, and any S-Patch of degree greater than one will have interior points.

We present a method to partially elevate the degree of an S-Patch by inserting any control point from a higher degree S-Patch into an S-Patch containing lower degree control points. This process will modify the basis functions of the existing low degree control points of the S-Patch and give the user the ability to elevate the degree of a single boundary edge (or facet in higher dimensions) without fully elevating the degree of the entire patch. For applications such as cage-based deformation, this process allows the user to refine portions of the boundary control cage without introducing interior points caused by fully elevating the degree of the cage. In applications such as finite element methods, local regions can be degree elevated and adjacent elements need to only elevate the degree of the shared facets to maintain continuity.

2. Related Work

As previously stated, generalized barycentric coordinates are intimately connected to S-Patches [LD89]. Wachspress created an interpolant over 2D, convex polygons for solving finite-element problems [Wac75], which Warren et al. [WSHD07] generalized to convex shapes, smooth or discrete, in arbitrary dimension. Mean value coordinates in 2D [Flo03, HF06] and 3D [JSW05, FKR05] alleviate the convexity constraint of Wachspress coordinates and are easy to compute. Harmonic coordinates [JMD*07] are always positive but require finite element methods to compute. While many more constructions exist [HS08, MS10, FS08, DF09, LJH13], our method is independent of the choice of barycentric coordinates. We use mean value coordinates in our examples because they are easy to compute.

The most related work to our own comes from the Finite Element literature. Finite element analysis approximates solutions of PDEs projected in to a space spanned by a basis defined by a tessellation of the domain where each portion of the tessellation contains a function (element). These elements typically have triangle or rectangle domains whose basis functions are represented as polynomials of arbitrary degree in the Lagrange basis that meet continuously with adjacent elements. To increase the precision or convergence of the solution, researchers have either refined the elements into multiple sub-elements (h-refinement) or increased the degree of the elements (p-refinement) [Hug12]. P-refinement is a similar strategy to ours except that we operate on multi-sided patches, with arbitrary domains, using barycentric coordinates for which there may be no analytic representation.

Cage refinement [HF06] is also related to our work. In this case, the boundary of the domain is linearly subdivided and the unnormalized barycentric coordinate functions are recomputed for the vertices that make up the refined domain. This method creates new control points for applications such as surface deformation. In contrast to our method, which creates higher degree S-Patches on the boundary dur-

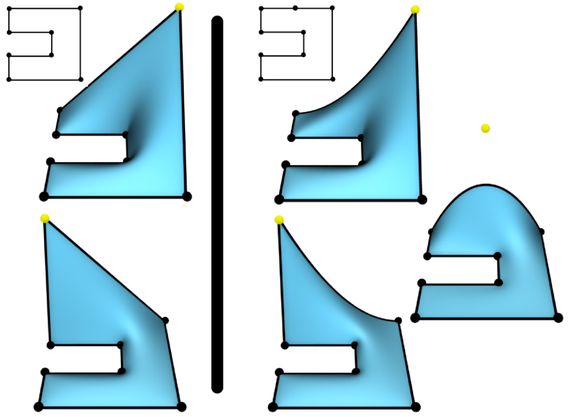

Figure 2: *In this figure and throughout the paper, we show the domain in black and basis functions associated with yellow control points as 3D blue elevations from the domain. An 8-sided, concave linear S-patch illustrating the basis functions associated with vertices on the top edge (left). We use selective degree elevation to increase the degree of the top edge to a quadratic by adding a single control point (right). Full degree elevation would have inserted* 28 *new control points.*

ing refinement, cage refinement maintains piecewise linear functions on the boundary. For application like deformation, cage refinement maps a planar boundary to a piecewise linear deformed boundary, which can create significant distortion at the refined vertices. In addition, cage refinement is not applicable to all forms of barycentric coordinates, such as Wachspress coordinates [Wac75], whereas our method is applicable to all forms of barycentric coordinates.

Finally, the construction of Serendipity Finite Elements [AA11] is somewhat related to our technique. Serendipity elements have rectangular domains and remove as many internal control points as possible while maintaining the same level of approximation order. For example, quadratic serendipity elements are the most common and sacrifice bi-quadratic precision to remove all internal control points while maintaining quadratic precision. Rand et al. [RGB14] have generalized this construction to multi-sided patches in 2D. However, it is not possible to remove all internal control points for higher degree elements. Our method is orthogonal to serendipity constructions in that we could use serendipity elements as opposed to S-Patches as a starting point for selective degree elevation.

3. Selective Degree Elevation

S-Patches have a simple degree elevation formula that can be derived using blossoming [dB87], but degree elevation can also be derived directly from the properties of barycentric coordinates. Using Equation 3, one can see that the basis

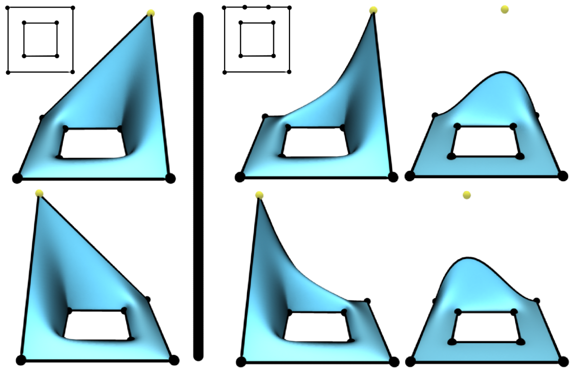

Figure 3: *(left) An 8-sided, linear S-patch with a hole in the domain. (right) The basis functions resulting from selectively elevating the degree of the top edge from linear to cubic.*

functions of various degrees are related to each other via

$$B^{d+1}_{\vec{j}}(x) = B^{d}_{\vec{j}-I^n_k}(x)\frac{d+1}{\vec{j}_k}w_k(x) \tag{4}$$

where $\vec{j}_k > 0$ and I^n_k is the k^{th} row of the $n \times n$ identity matrix. Let $f_{\vec{i}}$ be the values of the control points for an S-Patch of degree $|\vec{i}| = d$ and $\hat{f}_{\vec{j}}$ be control points of the same S-Patch function of degree $|\vec{j}| = d+1$. Then, because degree elevation does not change the S-Patch function and $\sum_{k=1}^{n} w_k(x) = 1$

$$\left(\sum_{|\vec{i}|=d} B^d_{\vec{i}}(x) f_{\vec{i}}\right)\left(\sum_{k=1}^{n} w_k(x)\right) = \sum_{|\vec{j}|=d+1} B^{d+1}_{\vec{j}}(x)\hat{f}_{\vec{j}}. \tag{5}$$

Expanding the left-hand side of Equation 5 and using Equation 4 yields the formula for degree elevation.

$$\hat{f}_{\vec{j}} = \sum_{\vec{j}_k>0} \frac{\vec{j}_k}{d+1} f_{\vec{j}-I^n_k}. \tag{6}$$

While Equation 6 provides a mechanism for elevating the degree of a multi-sided S-Patch in any dimension, it can produce too many degrees of freedom for artists to manipulate. For surface modeling applications where the domain P of the patch is a 2D polygon, it is unlikely that the number of sides in the patch will be high. However, many-sided domains are common in cage-based deformations [JSW05, JMD*07]. For these types of deformations, it is not uncommon to have a control cage with 100 or more vertices since the cage must conform to the shape of the high resolution object. If the artist desires more degrees of freedom, we must either redesign the initial control cage and discard the current deformation or increase the number of degrees of freedom through degree elevation. The problem with degree elevation is that the number of control points grows rapidly with the degree d. An S-Patch of degree d with domain consisting

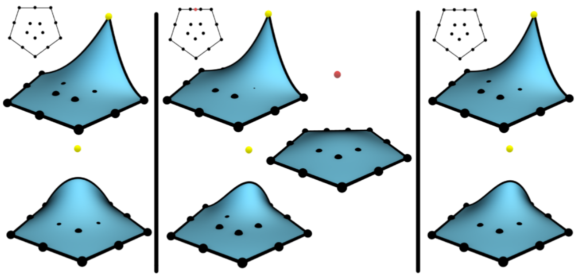

Figure 4: *A quadratic 5-sided S-Patch (left) with the top edge ∂P_ℓ raised to a cubic (middle) creates a ghost point in red. This point has no influence on the boundary but is non-zero over the interior of the patch. The basis functions after deleting the ghost point (right) still span all quadratics.*

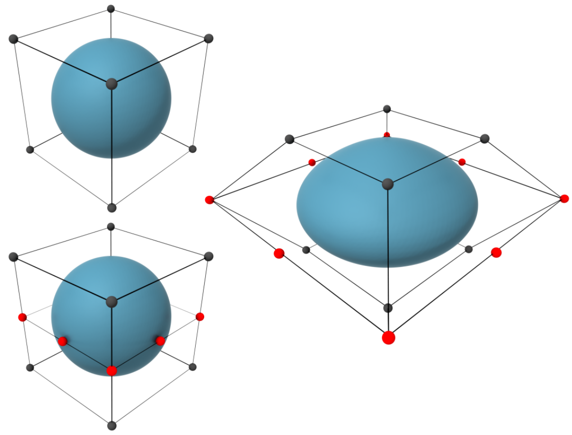

Figure 5: *The top left shows an input sphere with a control cage of black points. The bottom left image shows the insertion of several quadratic control points (red). The right image modifies only the quadratic points to deform the sphere.*

of n vertices will have $\binom{n+d-1}{d}$ control points. Hence, a 100 point control cage will have 171700 control points as a cubic where over 99% of the control points will be interior control points.

Our solution is to allow the user to insert a control point associated with a higher degree basis function into a lower degree patch and appropriately modify the lower degree basis functions. We will refer to a control point $f_{\vec{i}}$ as a degree $d = |\vec{i}|$ control point. Let $\hat{f}_{\vec{j}}$ be a control point of a degree $|\vec{j}| = d+m$ multi-sided S-Patch that we wish to insert into a lower degree S-Patch with control points $f_{\vec{i}}$ where $|\vec{i}| = d$. From Equation 6, we know there exist scalar coefficients $\alpha_{\vec{j},\vec{i}}$ such that

$$\hat{f}_{\vec{j}} = \sum_{|\vec{i}|=d} \alpha_{\vec{j},\vec{i}} f_{\vec{i}}. \tag{7}$$

Therefore,

$$\begin{aligned} F(x) &= \sum_{|\vec{i}|=d} B^d_{\vec{i}}(x) f_{\vec{i}} \\ &= \sum_{|\vec{i}|=d} B^d_{\vec{i}}(x) f_{\vec{i}} + B^{d+m}_{\vec{j}}(x)\left(\hat{f}_{\vec{j}} - \sum_{|\vec{i}|=d} \alpha_{\vec{j},\vec{i}} f_{\vec{i}}\right). \end{aligned}$$

Grouping the common coefficients yields

$$F(x) = \sum_{|\vec{i}|=d} \left(B^d_{\vec{i}}(x) - \alpha_{\vec{j},\vec{i}} B^{d+m}_{\vec{j}}(x)\right) f_{\vec{i}} + B^{d+m}_{\vec{j}}(x)\hat{f}_{\vec{j}}, \tag{8}$$

which creates new basis functions for the control points $f_{\vec{i}}$.

This degree elevation formula has the property that if $\hat{f}_{\vec{j}}$ satisfies Equation 7, the function $F(x)$ is unchanged. Moreover, the values of the old control points $f_{\vec{i}}$ are unchanged. Finally, when all the control points of degree $d+m$ have been inserted, the basis functions associated with the degree d control points will be zero everywhere, and the patch corresponds to the full degree elevation equation in Equation 6.

Figure 2 shows an example of this process for a 2D S-Patch with a concave domain. We start with a linear patch (left) and elevate a single edge to a quadratic function (right) and show the resulting basis functions. The result is a patch with linear functions along all edges except for the top edge, which is a quadratic function in Bézier form. This patch can still reproduce all linear functions, but we can match a quadratic curve exactly on the elevated edge if, for example, the patch met another quadratic patch. However, we are not limited to elevating the degree by one. Figure 3 shows an example with a hole in the domain where we elevate the top edge from a linear directly to a cubic function. In this case we introduce two additional control points, whereas elevating the entire patch to a cubic would produce 112 additional control points.

However, there is a problem with this selective degree elevation process, which we illustrate with Figure 4 showing a quadratic 5-sided S-Patch (left). If we raise the degree of an edge to a cubic function by inserting the two cubic control points on that edge (middle), the basis function for the quadratic control point, shown in red, on the edge is now zero everywhere on the boundary. However, this function is non-zero (although only slightly) over the interior of the domain. The result is a non-intuitive control point for the user, which we call a "ghost point," that previously was a boundary point but now only affects the interior of the S-Patch. This problem only arises when we insert all of the higher degree control points belonging to a lower dimensional facet of the domain.

Our solution is to delete these ghost points and distribute their basis function to the basis functions of the remaining control points while maintaining the polynomial reproduction order of the lower degree patch. For example, in Figure 4 there exist values of the control points that can reproduce any quadratic polynomial over the domain of the patch. When we elevate the degree of an edge to a cubic,

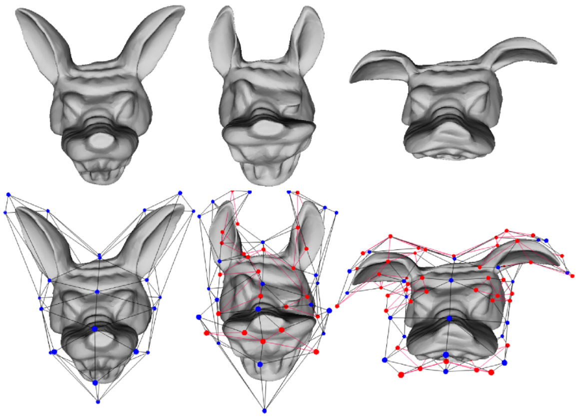

Figure 6: *Degree elevation to deform a complex model. We start with the linear control cage (left) and elevate the degree of selected faces to quadratic or cubic (right) to add more degrees of freedom and create various facial expressions.*

there are still values of control points that will reproduce any quadratic polynomial. Even though the ghost point has no contribution to its own edge, the ghost point must have a specific value to reproduce a given quadratic. Since an S-Patch reduces to a lower dimensional S-Patch on its boundary and a degree $d+m$ S-Patch encompasses the space spanned by a degree d S-Patch, we can delete the ghost point by representing it in terms of the higher degree control points along the patch's lower dimensional boundary face.

Let ∂P_ℓ be a lower dimensional face on the boundary of P whose vertices have indices Γ where each element in Γ is in the range $1 \dots n$. We define a control point $f_{\vec{i}}$ to belong to ∂P_ℓ if $\vec{i}_k \neq 0 \Leftrightarrow k \in \Gamma$. Let $G(x)$ be the function defined by the unmodified S-Patch basis functions over ∂P_ℓ of degree d with control points $f_{\vec{i}}$ where $|\vec{i}| = d$ and $H(x)$ be the function defined by the unmodified S-Patch basis functions over ∂P_ℓ of degree $d+m$ with control points $f_{\vec{j}}$ where $|\vec{j}| = d+m$. If all of the $f_{\vec{i}}$ necessary to define $G(x)$ do not exist, we first use selective degree elevation to insert those control points. Now let β be the set of indices of control points belonging to ∂P_ℓ of degree d. To delete the control points belonging to ∂P_ℓ of degree d, we minimize

$$\min_{f_{\vec{i}\in\beta}} \int_{x\in\partial P_\ell} (G(x) - H(x))^2 dx.$$

The solution gives $f_{\vec{i}\in\beta}$ as a weighted combination of the $f_{\vec{j}}$. For example, if $f_{(1,1,0,0,0)}$ is the red control point in Figure 4 (middle), then minimizing this function gives

$$f_{(1,1,0,0,0)} = \begin{pmatrix} \frac{-1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{-1}{4} \end{pmatrix} \begin{pmatrix} f_{(3,0,0,0,0)} \\ f_{(2,1,0,0,0)} \\ f_{(1,2,0,0,0)} \\ f_{(0,3,0,0,0)} \end{pmatrix}.$$

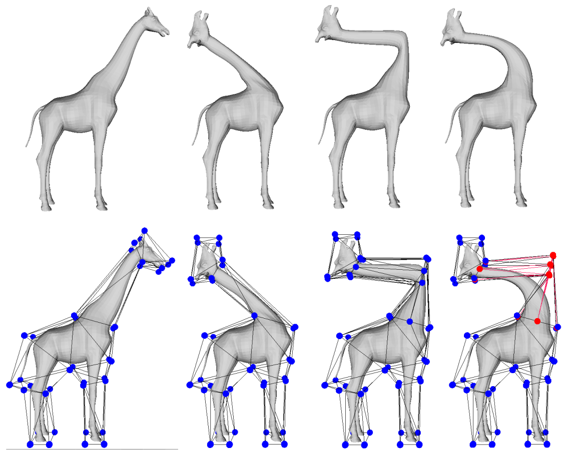

Figure 7: *The top row shows the original shape (left) followed by three deformations using the cages from the bottom row. The bottom row, from left to right, shows the original cage, a deformation using only the original control cage that does not provide sufficient degrees of freedom for the desired deformation, the deformation using cage refinement, and our deformation adding a few quadratic vertices to allow the neck to be bent in an arc.*

Adding this weighted combination of $f_{\vec{j}}$ into the S-Patch and grouping common terms as done in Equation 8 yields a modification of the remaining basis functions. Figure 4 (right) shows the effect on the basis functions after we delete the quadratic control point on the degree elevated edge. The new function still reproduces all quadratic polynomials over P but can also represent cubic functions on a single edge.

Note that this degree elevation procedure and ghost point removal does not alter the continuity of the underlying S-Patch. The S-Patch basis functions derive their continuity directly from the barycentric coordinates used to construct the patch, which is obvious from Equation 3 since the S-Patch basis functions are products of the barycentric basis functions. In our examples, we use mean value coordinates [Flo03], which are C^∞ over the interior of the domain. Our method only takes linear combinations of these basis functions and, therefore, inherits the smoothness of these functions (C^∞ in our examples).

4. Deformation

The selective degree elevation method in Section 3 is general in that it operates on S-Patches of any degree over polytope domains in arbitrary dimension. However, in the case of image or surface deformation, the problem is easier. Cage-based deformations [JSW05] correspond to S-Patches of degree one where the control points $f_{\vec{i}} \in \mathbb{R}^m$, where m is the dimension of the domain P, and are initially set to the p_k

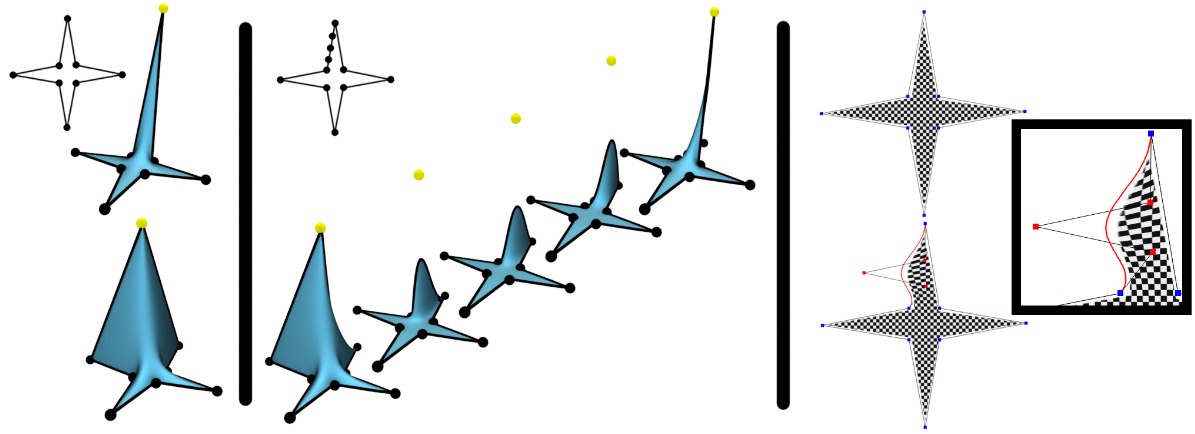

Figure 8: *8-sided, linear concave S-patch (left). The basis functions resulting from selectively raising an edge from linear to quartic along the elevated edge (middle). An example of using this patch for image deformation (right).*

for $\vec{i}_k = 1$. Hence $F(x)$ is a map from $\mathbb{R}^m$ to $\mathbb{R}^m$. In this setting, reproducing linear polynomials is important to provide the identity transformation and translation invariance, but higher-order polynomial reproduction is not necessary. Therefore, the weights $\alpha_{\vec{j},\vec{i}}$ for any degree $|\vec{j}| = d$ in Equation 7 are particularly simple and are given by

$$\hat{f}_{\vec{j}} = \sum_k \frac{\vec{j}_k}{d} p_k.$$

Figure 5 shows an example deformation of a sphere using a cube for the control cage. We partially elevate four of the faces to insert 8 new quadratic control points. On the right, we move the newly inserted control points outward to deform the sphere without modifying the original control points of the cage, which is not possible with only the original control points.

In Figure 6 we show an example of the head of the armadillo man surrounded by a simple deformation cage with 24 vertices. To the right, we show a couple of different deformations where selected faces have been elevated to quadratic or cubic functions depending on the deformation. In all of these cases, we did not create any interior control points and only manipulate the deformation using control points on the boundary of the cage. Had we raised the degree of the entire cage to a quadratic, we would have produced a cage with 300 control points or 2600 control points if elevated to a cubic. In contrast, the maximal number of control points we use in any of these deformations is 68. Figures 7 and Figure 9 show additional 3D deformations where we raise the degree of the cage locally to obtain more degrees of freedom compared to inserting the same control points using cage refinement.

Note that the deformation of the boundary of the domain is no longer piecewise linear when we elevate the degree of selected boundary faces. Instead the boundary of the domain maps to a curved shape. In contrast, cage refinement [HF06] produces piecewise linear deformations of the boundary and is not smooth at the boundary. Figure 10 shows a comparison of cage refinement versus selective degree elevation for 2D image deformation. Figure 8 shows a more extreme example of a non-convex shape used in image deformation where a single edge is raised to a quartic.

5. Basis Function Evaluation

For many applications the evaluation points of an S-patch are known in advance. In cage-based deformations the evaluation points are the vertices of the high resolution shape being deformed. We begin with an S-Patch of uniform degree and calculate the values of each of the basis functions $B^d_{\vec{i}}(x)$ at these evaluation points according to Equation 3. As the user performs selective degree elevation, we perform linear operations on the values of these basis functions according to Equation 8 to modify the values of the existing basis functions. The value of the basis function for the newly inserted control point is simply given by Equation 3. Hence, selective degree elevation requires computing only a small number of linear operations per evaluation point.

6. Conclusions and Future Work

Selective degree elevation provides more degrees of freedom to a multi-sided S-Patch with the granularity of inserting a single control point. Such an operation allows a user to insert exactly the number of degrees of freedom they desire without dramatically increasing the number of control points. While we have only shown examples of inserting boundary control points, Section 3 is general enough that interior control points can be inserted as well. There are fewer scenarios where doing so is advantageous, but one possible application might be to control the derivative(s) of the function along the

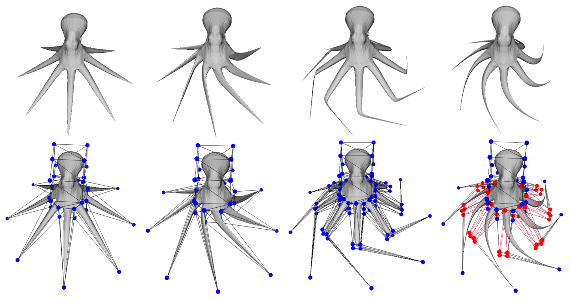

Figure 9: *The top row shows the original shape (left) followed by deformations using the cages on the bottom row. The original cage (middle left) does not contain enough control points to bend the arms. Using cage refinement (middle right) allows the arms to be bent but not with a smooth appearance. Our addition of a few quadratic control points allows the arms to bend smoothly.*

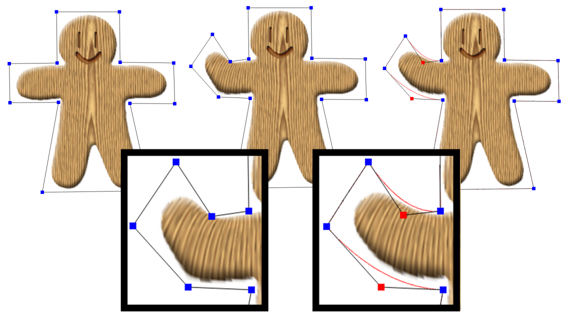

Figure 10: *An example of image deformation. From left to right: the linear control cage, a deformation with cage refinement [HF06], selective degree elevation of two edges to a quadratic. The deformed boundary of the cage is a quadratic Bézier curve in the right image (shown in red).*

boundary to match some higher-dimensional cross-boundary derivative.

Finally, our method can potentially be used in finite element applications, although we have left this application for future work. Similar to p-refinement in finite element methods, our method can control the number of degrees of freedom by raising the degree. However, we can easily raise the degree locally while maintaining continuity to surrounding elements using our method.

Acknowledgements

This work was supported by NSF CAREER award IIS 1148976.

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