

# Impact of Spatial Correlation on the Finite-SNR Diversity-Multiplexing Tradeoff

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**Abstract**—The impact of spatial correlation on the performance limits of multielement antenna (MEA) channels is analyzed in terms of the diversity-multiplexing tradeoff (DMT) at finite signal-to-noise ratio (SNR) values. A lower bound on the outage probability is first derived. Using this bound accurate finite-SNR estimate of the DMT is then derived. This estimate allows to gain insight on the impact of spatial correlation on the DMT at finite SNR. As expected, the DMT is severely degraded as the spatial correlation increases. Moreover, using asymptotic analysis, we show that our framework encompasses well-known results concerning the asymptotic behavior of the DMT.

**Index Terms**—Diversity-Multiplexing tradeoff (DMT), finite SNR, outage probability, spatial correlation.

## I. INTRODUCTION

Multielement antenna (MEA) systems have been used either to increase the diversity gain in order to better combat channel fading [1], or to increase the data rate by means of spatial multiplexing gain [2]. Recently, Zheng and Tse showed that both gains can be achieved with an asymptotic optimal tradeoff at high-SNR regime [3]. This tradeoff is a characterization of the maximum diversity gain that can be achieved at each multiplexing gain. At low to moderate SNR values (typically 3 – 20 dB), this asymptotic tradeoff is in fact an optimistic upper bound on the finite-SNR diversity-multiplexing tradeoff. A framework was introduced by Narasimhan to characterize the diversity performance of rate-adaptive MIMO systems at finite SNR [4]. As expected, the achievable diversity gains at realistic SNR values are significantly lower than for the asymptotic values. In [3] and [4], the authors assume independent and identically distributed (i.i.d.) fading channels. However, in real propagation environments, the fades can be correlated which may be detrimental to the performances of MEA systems [5]. Recently, the impact of spatial correlation on the finite-SNR DMT was studied in [6].

In this paper, we analyze the impact of spatial correlation on the DMT at finite-SNR values using a new framework and we extend to the high-SNR regimes, the results obtained for finite-SNR in [7]. We prove that our framework encompasses the

well-known asymptotic DMT for correlated and uncorrelated channels [3], [8]. Thus, our framework may be seen as a generalization of the asymptotic analysis.

When capacity-achieving codes are used over a quasi-static channel, the errors are mainly caused by atypical deep fades of the channel, that is, the block-error probability is equal to the outage probability of the channel  $P_{out}$  which will be formally defined later in the paper. Therefore,  $P_{out}$  is the key parameter for deriving the performance limits of MEA systems. Since derivation of an exact expression for  $P_{out}$  is difficult, we alternatively derive lower bounds on  $P_{out}$  over both spatially correlated and uncorrelated channels. These bounds are then used to obtain insightful estimates of the related finite-SNR DMT. These estimates allow to characterize the potential limits of MEA systems, in terms of DMT, in a more realistic propagation environment and for practical SNR values. The paper is organized as follows. Section II presents the system model and introduces the related definitions. In section III, we derive lower bounds on  $P_{out}$ . Finite-SNR DMT estimates are given in section IV. An asymptotic analysis of the diversity estimates is investigated in section V. Numerical results are reported in Section VI and Section VII concludes the paper.

## II. CHANNEL MODEL AND RELATED DEFINITIONS

Let an MEA system consist of  $N_t$  transmit antennas and  $N_r$  receive antennas. We restrict our analysis to a Rayleigh flat-fading channel, where the entries of the  $N_r \times N_t$  channel matrix  $\mathbf{H}$  are circularly-symmetric zero mean complex Gaussian distributed and possibly correlated. The channel is assumed to be quasi-static, unknown at the transmitter and completely tracked at the receiver. For the effect of spatial fading correlation, we model  $\mathbf{H}$  as:  $\mathbf{H} = \mathbf{R}_r^{1/2} \mathbf{H}_w \mathbf{R}_t^{1/2}$ , where matrix  $\mathbf{H}_w$  represents the  $N_r \times N_t$  spatially uncorrelated channel, and where matrices  $\mathbf{R}_r$ , of dimension  $N_r \times N_r$  and  $\mathbf{R}_t$ , of dimension  $N_t \times N_t$  are positive-definite Hermitian matrices that specify the receive and transmit correlations respectively [9].

For an SNR-dependent spectral efficiency  $R(SNR)$  in bps/Hz,  $P_{out}$  is defined as [10]:

$$P_{out} = Prob(I < R) \quad (1)$$

where  $I$  is the channel mutual information. Assuming equal power allocation over the transmit antennas,  $I$  is given by:

$$I = \log_2 \left( \det \left( \mathbf{I}_{N_r} + \frac{\eta}{N_t} \mathbf{H} \mathbf{H}^H \right) \right) \text{ bps/Hz} \quad (2)$$

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where  $\mathbf{I}_{N_r}$  is the  $N_r \times N_r$  identity matrix and where the superscript  $H$  indicates for conjugate transposition. The multiplexing and diversity gains are respectively defined as [4]:

$$r = \frac{R}{\log_2(1 + g \cdot \eta)} \quad (3)$$

$$d(r, \eta) = -\eta \frac{\partial \ln P_{out}(r, \eta)}{\partial \eta} \quad (4)$$

where  $g$  is the array gain which is equal to  $N_r$ , and where  $\eta$  is the mean SNR value at each receive antenna. As defined in (3) and (4), the multiplexing gain  $r$  provides an indication of a rate adaptation strategy as the SNR changes, while the diversity gain  $d(r, \eta)$  can be used to estimate the additional SNR required to decrease  $P_{out}$  by a specific amount for a given  $r$ .

### III. LOWER BOUND ON THE OUTAGE PROBABILITY

Let us denote the orthogonal-triangular (QR) decomposition of  $\mathbf{H}_w = \mathbf{Q}\mathbf{R}$ , where  $\mathbf{Q}$  is an  $N_r \times N_r$  unitary matrix and where  $\mathbf{R}$  is an  $N_r \times N_t$  upper triangular matrix with independent entries. The square magnitudes of the diagonal entries of  $\mathbf{R}$ ,  $|R_{l,l}|^2$ , are chi-square distributed with  $2(N_r - l + 1)$  degrees of freedom,  $l = 1, \dots, \min(N_t, N_r)$ . The off-diagonal elements of  $\mathbf{R}$  are i.i.d. Gaussian variables, with zero mean and unit variance. Let the singular value decompositions (SVDs) of  $\mathbf{R}_t^{1/2} = \mathbf{U}\mathbf{D}\mathbf{V}^H$ , and  $\mathbf{R}_r^{1/2} = \mathbf{U}'\mathbf{D}'\mathbf{V}'^H$ , where  $\mathbf{U}$  and  $\mathbf{V}$  (or  $\mathbf{U}'$ ,  $\mathbf{V}'$ ) are  $N_t \times N_t$  (or  $N_r \times N_r$ ) that satisfy  $\mathbf{U}^H\mathbf{U} = \mathbf{V}^H\mathbf{V} = \mathbf{I}_{N_t}$  (or  $\mathbf{U}'^H\mathbf{U}' = \mathbf{V}'^H\mathbf{V}' = \mathbf{I}_{N_r}$ ), and where  $\mathbf{D}$  (or  $\mathbf{D}'$ ) is an  $N_t \times N_t$  (or  $N_r \times N_r$ ) diagonal matrix, whose diagonal elements are the singular values of  $\mathbf{R}_t^{1/2}$  (or  $\mathbf{R}_r^{1/2}$ ). We assume, without loss of generality, that the elements of  $\mathbf{D}$  and  $\mathbf{D}'$  are ordered in descending order of their magnitudes along the diagonal. Using these SVDs, since  $\det(\mathbf{I} + \mathbf{X}\mathbf{Y}) = \det(\mathbf{I} + \mathbf{Y}\mathbf{X})$  and since unitary transformations do not change the statistics of random matrices, we have:

$$\begin{aligned} I &= \log_2 \left( \det \left( \mathbf{I}_{N_r} + \frac{\eta}{N_t} \mathbf{R}_r^{1/2} \mathbf{H}_w \mathbf{R}_t \mathbf{H}_w^H \mathbf{R}_r^{H/2} \right) \right) \\ &\stackrel{d}{=} \log_2 \left( \det \left( \mathbf{I}_{N_r} + \frac{\eta}{N_t} \mathbf{R} \mathbf{D}^2 \mathbf{R}^H \mathbf{D}'^2 \right) \right) \end{aligned} \quad (5)$$

where the symbol  $\stackrel{d}{=}$  means equality in distributions. From (5), it is clear that  $\mathbf{R}_t$  and  $\mathbf{R}_r$  contribute to the channel mutual information through their diagonal matrix representatives  $\mathbf{D}^2$  and  $\mathbf{D}'^2$  respectively. Since  $\mathbf{D}^2$  and  $\mathbf{D}'^2$  have a similar role in (5), without loss of generality, we can focus on the spatial correlation at the transmitter, that is, we assume  $\mathbf{R}_r^{1/2} = \mathbf{D}'^2 = \mathbf{I}_{N_r}$ .

Let  $D_k$ ,  $k = 1, \dots, N_t$ , denote the  $k^{th}$  diagonal element of  $\mathbf{D}$ , and let  $R_{l,k}$  represent the element of  $\mathbf{R}$  at the  $l^{th}$  row and the  $k^{th}$  column,  $l = 1, \dots, N_r$ ,  $k = 1, \dots, N_t$ . Using the fact that  $\det(\mathbf{A}) \leq \prod_l A_{l,l}$ , for any nonnegative-definite matrix  $\mathbf{A}$ , we obtain from (5):

$$I \leq \sum_{l=1}^t \log_2 \left( 1 + \frac{\eta}{N_t} \Delta_l \right), \quad (6)$$

where  $t = \min(N_t, N_r)$  and where  $\Delta_l = \sum_{k=1}^{N_t} D_k^2 |R_{l,k}|^2$ ,  $l = 1, \dots, t$ , is the  $l^{th}$  diagonal entry of  $\mathbf{R}\mathbf{D}^2\mathbf{R}^H$ . Since  $R_{l,k}$  are independent, then  $\Delta_l$  are also independent. In order to derive a lower bound on  $P_{out}$ , the distribution function of  $\Delta_l$  is needed. When all  $D_k^2$ ,  $k = 1, \dots, N_t$  are equal which corresponds to the uncorrelated case, the trace constraint  $\text{trace}(\mathbf{R}_t^{1/2}) = N_t$  imposes  $D_k^2 = 1$ ,  $k = 1, \dots, N_t$ . That is,  $\Delta_l$  is chi-square distributed with  $2(N_r + N_t - 2l + 1)$  degrees of freedom. Otherwise,  $\Delta_l$  may be viewed as a generalized quadratic form of a Gaussian random vector. We first derive the distribution function of  $\Delta_l$ ,  $l = 1, \dots, t$ , in Lemma 1.

*Lemma 1:* Assuming that all  $D_k^2$ 's,  $k = 1, \dots, N_t$ , are distinct, the distribution function of  $\Delta_l$ ,  $l = 1, \dots, t$ , is given by:

$$f_{\Delta_l}(x) = \sum_{k=1}^{N_r-l+1} a_k^{(l)} f_{G(k, D_l^2)}(x) + \sum_{k=1}^{N_t-l} a_1^{(l+k)} f_{G(1, D_{l+k}^2)}(x), \quad (7)$$

where  $G(\alpha, \beta)$  is a Gamma random variable with probability distribution function given by:  $f_{G(\alpha, \beta)}(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-\frac{x}{\beta}}$ ,  $x \geq 0$ ,  $\alpha > 0$ ,  $\beta > 0$ . The coefficients  $a_k^{(l)}$  and  $a_1^{(l+k)}$  are given by:

$$\begin{aligned} a_k^{(l)} &= \frac{(-D_l^2)^{-(N_r-l+1-k)}}{(N_r-l+1-k)!} \\ &\quad \frac{d^{(N_r-l+1-k)}}{d(jv)^{(N_r-l+1-k)}} \left[ (1-jvD_l^2)^{N_r-l+1} \Psi_{\Delta_l}(jv) \right]_{jv=D_l^{-2}}, \\ a_1^{(l+k)} &= \left[ (1-jvD_{l+k}^2) \Psi_{\Delta_l}(jv) \right]_{jv=D_{l+k}^{-2}}, \end{aligned}$$

where  $\frac{d^{(k)}f(x)}{dx^{(k)}}$  is the  $k^{th}$  derivative of  $f(x)$  and where  $\Psi_{\Delta_l}(jv)$  is the moment generating function of  $\Delta_l$  given by:

$$\Psi_{\Delta_l}(jv) = (1-jvD_l^2)^{-(N_r-l+1)} \prod_{k=1}^{N_t-l} (1-jvD_{l+k}^2)^{-1}.$$

*Proof:* The proof can be found in [7]. ■

Note that when all  $D_k^2$ ,  $k = 1, \dots, N_t$ , are not distinct, the distribution of  $\Delta_l$ ,  $l = 1, \dots, t$ , can be derived using the same mechanism. Using (1), (6) and Lemma 1, a lower bound on  $P_{out}$  may be expressed in the following theorem.

*Theorem 1 (Lower Bound):* Lower bounds on the outage probability  $P_{out}$  for the uncorrelated  $D_k^2 = 1$ ,  $k = 1, \dots, N_t$ , and correlated spatial fading channels are respectively given by:

$$P_{out}^{uncorr} \geq \prod_{l=1}^t \Gamma_{inc}(\xi_l, N_r + N_t - 2l + 1), \quad (8)$$

$$\begin{aligned} P_{out}^{corr} &\geq \prod_{l=1}^t \left( \sum_{k=1}^{N_r-l+1} a_k^{(l)} \Gamma_{inc}\left(\frac{\xi_l}{D_l^2}, k\right) \right. \\ &\quad \left. + \sum_{k=1}^{N_t-l} a_1^{(l+k)} \Gamma_{inc}\left(\frac{\xi_l}{D_{l+k}^2}, 1\right) \right), \end{aligned} \quad (9)$$

where  $b_l$ ,  $l = 1, \dots, t$ , are arbitrary positive coefficients that satisfy  $r = \sum_{l=1}^t b_l$ ,  $\Gamma_{inc}$  is the incomplete Gamma function

defined by  $\Gamma_{inc}(x, a) = \frac{1}{(a-1)!} \int_0^x t^{a-1} e^{-t} dt$  and where  $\xi_l$  is given by:  $\xi_l = \frac{N_t}{\eta} \left( (1 + g\eta)^{b_l} - 1 \right)$ .

*Proof:* The proof has been given in [7] but follows along similar lines as [6, Theorem 1]. ■

In order to obtain tighter results, the lower bounds given in Theorem 1 are maximized over the set of coefficients  $b_l$ ,  $l = 1, \dots, t$ , for each multiplexing gain  $r$  and each SNR value  $\eta$ . Clearly, the computational time of this optimization problem is much smaller than that required by Monte Carlo simulations for computing the exact  $P_{out}$ . It is worth noting that since  $\xi_l \geq \alpha_{\rho, l}$ , where  $\alpha_{\rho, l}$  was defined in [6, Theorem 1], and since  $\Gamma_{inc}(x, a)$  is an increasing function in  $x$ , the lower bounds given by (8) is tighter than that given by [6, Theorem 1] for the uncorrelated case. Moreover, (9) appears as a finite product of a weighted sum of  $\Gamma_{inc}$  functions, which is more insightful and easier to compute than the lower bound derived in [6, Theorem 1], which involves weighted infinite series of  $\Gamma_{inc}$  functions.

#### IV. FINITE-SNR DIVERSITY AND CORRELATION

Using Theorem 1 and (4), an estimate of the finite-SNR diversity for a given multiplexing gain is now derived in the following corollary.

*Corollary 1 (Diversity estimate):* An estimate of the diversity for the correlated and uncorrelated spatial fading are respectively given by:

$$\hat{d}^{uncorr}(r, \eta) = \frac{N_t}{\eta} \sum_{l=1}^t \left( (1 + g\eta)^{b_l} - b_l g\eta (1 + g\eta)^{b_l-1} - 1 \right) \frac{\xi_l^{N_t + N_r - 2l} e^{-\xi_l} / (N_t + N_r - 2l)!}{\Gamma_{inc}(\xi_l, N_r + N_t - 2l + 1)} \quad (10)$$

$$\hat{d}^{corr}(r, \eta) = \frac{N_t}{\eta} \sum_{l=1}^t \left( (1 + g\eta)^{b_l} - b_l g\eta (1 + g\eta)^{b_l-1} - 1 \right) \frac{Q_l(\xi_l)}{P_l(\xi_l)}, \quad (11)$$

where  $Q_l(\xi_l)$  and  $P_l(\xi_l)$  are given by:

$$Q_l(\xi_l) = \sum_{k=1}^{N_r - l + 1} \frac{a_k^{(l)}}{(k-1)!} \left( \frac{\xi_l}{D_l^2} \right)^{k-1} e^{-\frac{\xi_l}{D_l^2}} D_l^{-2} + \sum_{k=1}^{N_t - l} a_1^{(l+k)} e^{-\frac{\xi_l}{D_{l+k}^2}} D_{l+k}^{-2}$$

$$P_l(\xi_l) = \sum_{k=1}^{N_r - l + 1} a_k^{(l)} \Gamma_{inc} \left( \frac{\xi_l}{D_l^2}, k \right) + \sum_{k=1}^{N_t - l} a_1^{(l+k)} \Gamma_{inc} \left( \frac{\xi_l}{D_{l+k}^2}, 1 \right).$$

Note that (10) and (11) have similar closed forms. Clearly, (10) can be obtained from (11) by replacing  $P_l(\xi_l)$  and  $Q_l(\xi_l)$  by  $\Gamma_{inc}(\xi_l, N_r + N_t - 2l + 1)$  and  $\left( \xi_l^{N_t + N_r - 2l} e^{-\xi_l} \right) / (N_t + N_r - 2l)!$  respectively. It should be pointed out that (10) and (11) are simpler and more insightful than the diversity

estimates given in [6, Theorem 3], which again involve infinite series.

#### V. ASYMPTOTIC BEHAVIOR OF THE DIVERSITY ESTIMATES

In order to examine whether the diversity estimates given in Corollary 1 match the well-known asymptotic DMT at high-SNR given in [3], we analyze the asymptotic behavior of the diversity estimates we derived, as  $\eta \rightarrow \infty$  or as  $r \rightarrow 0$ . First, we present the following lemma.

*Lemma 2:* Assuming full-rank transmit spatial correlation, we can write:

$$\lim_{\eta \rightarrow \infty} \hat{d}^{uncorr}(r, \eta) = \lim_{\eta \rightarrow \infty} \hat{d}^{corr}(r, \eta). \quad (12)$$

*Proof:* For convenience, the proof is presented in Appendix I. ■

The result in Lemma 2 is very insightful. It states that, at a high-SNR regime and at a given multiplexing gain  $r$ , the diversity estimate is independent of the spatial correlation. More interestingly, our asymptotic diversity estimates coincide with the well-known asymptotic DMT characterization as summarized in the following theorem.

*Theorem 2:* Assuming full-rank transmit spatial correlation, the optimal DMT, for the uncorrelated and correlated cases, is given by the asymptotic diversity estimate, that is:

$$\lim_{\eta \rightarrow \infty} \hat{d}^{uncorr}(r, \eta) = \lim_{\eta \rightarrow \infty} \hat{d}^{corr}(r, \eta) = d_{asym}, \quad (13)$$

where  $d_{asym} = -\lim_{\eta \rightarrow \infty} \frac{\log_2 P_{out}}{\log_2 \eta}$ .

*Proof:* The proof is presented in Appendix II. ■

Theorem 2 states that our asymptotic diversity estimate is exactly the high-SNR DMT. Therefore, it can be seen as a generalization of the DMT for the spatially correlated and uncorrelated channels. Note that Lemma 2 and Theorem 2 agree with the results in [6, Corollary 1]. On the other hand, Lemma 2 and Theorem 2 confirm a recently established result concerning the asymptotic diversity [8]. However, our result is broader, since it allows understanding the impact of spatial correlation at finite-SNR which is not discussed in [8]. More importantly, the framework presented here provides some guidelines on designing space-time codes at practical SNR values. As an example, the following corollary defines the maximum achievable diversity gain by any full diversity -based space-time code.

*Corollary 2 (Maximum diversity):* The maximum diversity gain is the same for both correlated and uncorrelated spatial fading channels and is given by:

$$\hat{d}^{max}(\eta) = \lim_{r \rightarrow 0} \hat{d}^{uncorr}(r, \eta) = \lim_{r \rightarrow 0} \hat{d}^{corr}(r, \eta) = N_t N_r \left[ 1 - \frac{g\eta}{(1 + g\eta) \ln(1 + g\eta)} \right]. \quad (14)$$

*Proof:* The proof is presented in [7]. ■

Corollary 2 agrees with [6, Theorem 6] even though our diversity estimates are different from those in [6]. Corollary

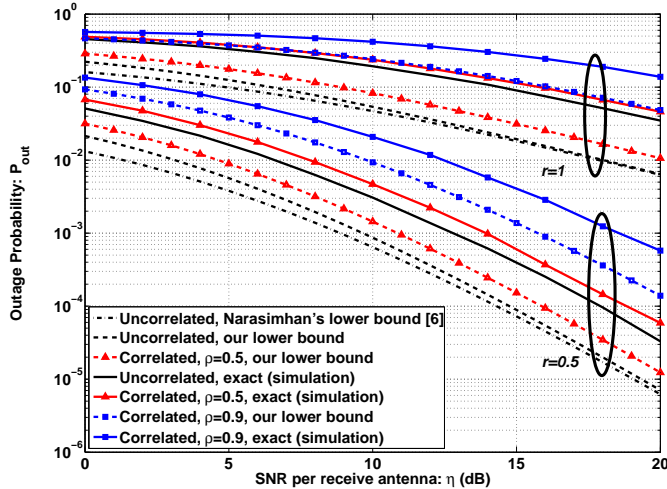


Fig. 1. Comparison of lower bounds on  $P_{out}$  and the exact simulation in the uncorrelated ( $\rho = 0$ ), and correlated ( $\rho = 0.5, 0.9$ ) spatial fadings, for  $N_t = N_r = 2$ .

2 also indicates that the estimated maximum diversity gain is unaffected by the spatial correlation. This has been previously pointed out for a high-SNR regime in [9]. Corollary 2 is however stronger, since it holds for all values of  $\eta$ , and in particular for  $\eta = +\infty$ . It is to be reminded that in establishing this last result, we have assumed that  $\mathbf{R}_t$  has full rank. However, should  $\mathbf{R}_t$  be rank-deficient, then  $\mathbf{H}$  would also be rank-deficient and it may be expected that the maximum diversity gain may be lower than that given by (14). Moreover, as suggested by (5), all the results given here apply when the receive spatial correlation is considered separately.

## VI. NUMERICAL RESULTS

In this section, simulation results for an MEA system with  $N_t = N_r = 2$  are presented. The transmit correlation matrix  $\mathbf{R}_t$  is chosen according to a single coefficient spatial correlation model [9], [11], i.e., the entry of  $\mathbf{R}_t$  at the  $i^{th}$  row and the  $j^{th}$  column is  $(R_t)_{i,j} = \rho^{(i-j)^2}$ . The lower bound on  $P_{out}$  given by Theorem 1 is plotted in Fig.1 together with the exact  $P_{out}$ , given by simulation, for  $r = 0.5$  and  $r = 1$ . Figure 1 also shows the lower bound found by Narasimhan in [6] for the uncorrelated case. As was proven in section III, our bound is tighter especially at low SNR values. More importantly, our lower bounds follow the same shape as the exact curves and the gap between the exact  $P_{out}$  and the lower bound is independent of the correlation coefficients, regardless of the SNR values. For comparison in the transmit spatial correlation case, we have plotted in Fig. 2 the exact  $P_{out}$ , our lower bound given by (9) and the lower bound in [6], for  $r = 0.5$  and  $r = 1$ . All curves in Fig. 2 have been obtained using the same transmit spatial correlation matrix in [6]. As can be seen in Fig. 2, our lower bound is again slightly tighter than Narasimhan's lower bound at low SNR. Beyond SNR=30 dB, the lower bound curves are exactly the same. The exact diversity gain, obtained by Monte-Carlo

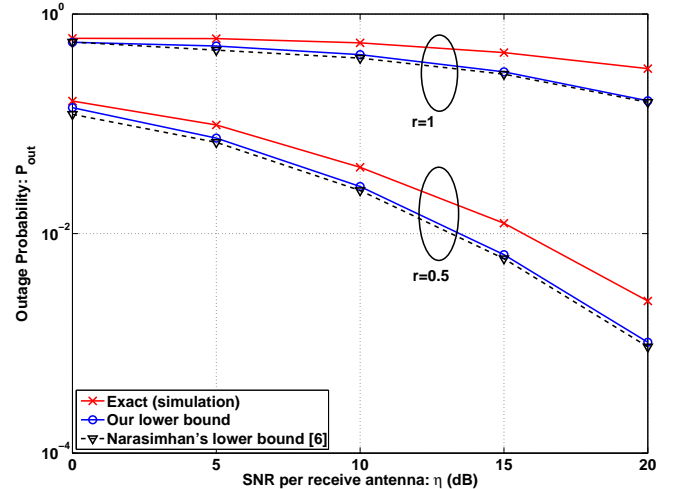


Fig. 2. Comparison of our lower bound and Narasimhan's lower bound, for the transmit spatial correlation given by (49) in [6] and for  $N_t = N_r = 2$ .

simulations using (4) and an estimated diversity gain computed using Corollary 1 are plotted in Fig.3 for an SNR=15 dB. Figure 3 indicates that the DMT estimate is a good fit to the exact simulation tradeoff curve. Therefore, the estimated diversity can be used to obtain an insight on the DMT over spatially correlated and uncorrelated channels while avoiding time consuming simulations. Interestingly, it can be noticed that with a correlation coefficient  $\rho = 0.5$ , the diversity gain is only slightly degraded and one may expect to achieve a quasi-equal uncorrelated diversity gain as shown in Fig.3. However, the diversity is substantially degraded when  $\rho = 0.9$ . For example, as illustrated in Fig.4, an MEA system operating at  $r \geq 0.8$  and an SNR of 5 dB in a moderately correlated channel ( $\rho = 0.5$ ), achieves a better diversity gain than a system operating at the same  $r$  and an SNR of 10 dB in a highly correlated channel ( $\rho = 0.9$ ). These observations are confirmed with the exact diversity curves. In Fig. 5, we have plotted the relative diversity-estimate gain, defined as  $\frac{\hat{d}_{corr}}{\hat{d}_{uncorr}}$  for different SNR values. As predicted by Lemma 2, the relative diversity-estimate gain converges toward 1 as  $\eta \rightarrow \infty$  regardless of the multiplexing gain values. However, the convergence would be faster for small values of  $r$ . Finally, as predicted by Theorem 2, Fig. 6 illustrates the convergence of the uncorrelated diversity estimate to the asymptotic DMT as  $\eta \rightarrow \infty$ .

## VII. CONCLUSION

In this paper, we have addressed the finite-SNR diversity-multiplexing tradeoff over spatially correlated MEA channels. We first derived lower bounds on the outage probability for spatially correlated and uncorrelated MEA channels. Then, using these bounds, estimates of the corresponding DMT were determined. The diversity estimates provide an insight on the finite-SNR DMT of MEA systems. Furthermore, extensions to the asymptotic behavior of the diversity-estimate gain, as either the SNR goes to infinity or the multiplexing gain tends toward

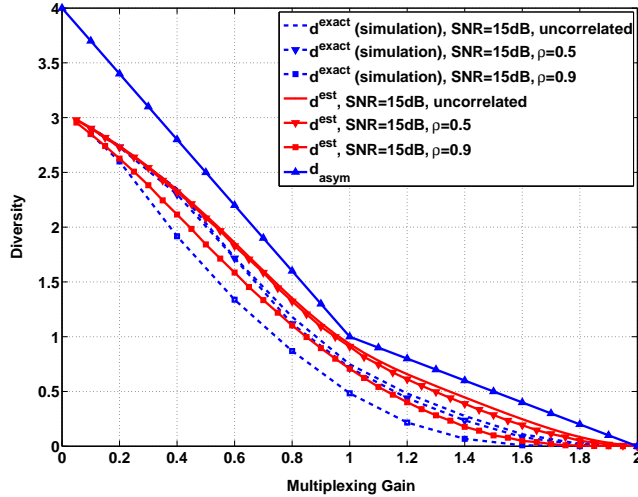


Fig. 3. The impact of spatial correlation on the diversity estimates for  $N_t = N_r = 2$  and SNR=15 dB.

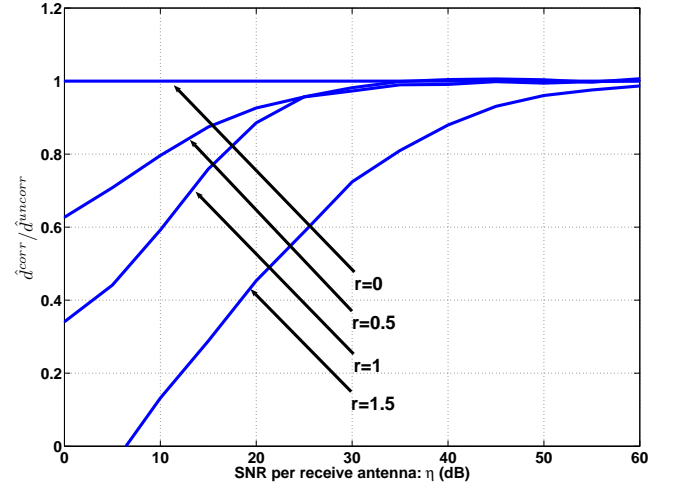


Fig. 5. Relative diversity-estimate gain versus SNR for different multiplexing gains  $r$ .

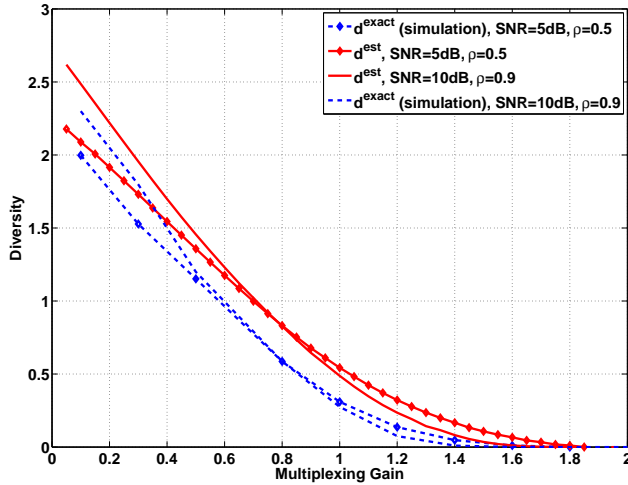


Fig. 4. Comparison of our diversity estimate and the exact (simulation) for  $\rho = 0.5$ , SNR = 5 dB and  $\rho = 0.9$ , SNR = 10 dB, when  $N_t = N_r = 2$ .

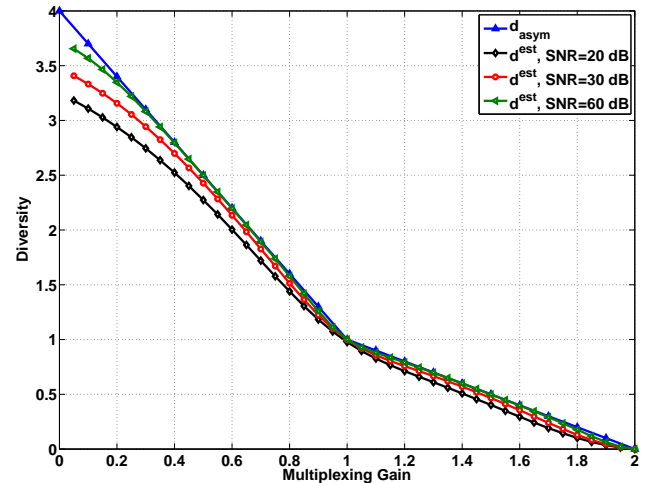


Fig. 6. Convergence of the uncorrelated diversity estimate  $\hat{d}^{uncorr}(r, \eta)$  to the asymptotic diversity  $d_{asym}$ , as  $SNR \rightarrow \infty$ .

zero have been derived. The asymptotic behavior provides some guidelines for the design of diversity-oriented space-time codes. More interestingly, this asymptotic analysis reveals that our framework includes well-known results about the asymptotic DMT for correlated and uncorrelated channels. Hence, the framework presented here can be seen as a generalization of the asymptotic DMT. Finally, it is worth mentioning that although we have focused on the transmit spatial correlation, we showed that all the results still hold when the receive spatial correlation is considered instead.

#### APPENDIX I PROOF OF LEMMA 2

First, note that as  $\eta \rightarrow \infty$ , we have:

$$\xi_l \approx N_t g^{b_l} \eta^{b_l - 1}. \quad (15)$$

Let us define  $J_l(\xi_l) = \frac{\xi_l^{N_t + N_r - 2l} e^{-\xi_l} / (N_t + N_r - 2l)!}{\Gamma_{inc}(\xi_l, N_r + N_t - 2l + 1)}$ , and  $K_l(\xi_l) = ((1 + g\eta)^{b_l} - b_l g \eta (1 + g\eta)^{b_l - 1} - 1)$ , for  $l = 1, \dots, t$ . To prove Lemma 2, it suffices to prove that:

$$\lim_{\eta \rightarrow \infty} J_l(\xi_l) K_l(\xi_l) = \lim_{\eta \rightarrow \infty} \frac{Q_l(\xi_l)}{P_l(\xi_l)} K_l(\xi_l). \quad (16)$$

Indeed, to prove (16) for each  $l = 1, \dots, t$ , we distinguish three cases:

- $b_l > 1$ : In this case,  $\xi_l \rightarrow \infty$ . Since  $\Gamma_{inc}(\xi_l, N_r + N_t - 2l + 1) \rightarrow 1$  as  $\xi_l \rightarrow \infty$ , then  $J_l(\xi_l) K_l(\xi_l) \rightarrow 0$ , and so does the term  $\frac{Q_l(\xi_l)}{P_l(\xi_l)} K_l(\xi_l)$ .
- $b_l < 1$ : In this case,  $\xi_l \rightarrow 0$ . Note that  $J_l(\xi_l) = \frac{f'(\xi_l)}{f(\xi_l)}$ , where  $f(\xi_l) = \Gamma_{inc}(\xi_l, N_r + N_t - 2l + 1)$ , and  $f'(\xi_l)$  denotes the derivative of  $f(\xi_l)$ . Using Taylor expansion

of  $f(\xi_l)$  and  $f'(\xi_l)$  around 0, we obtain:

$$J_l(\xi_l) \approx \frac{1/(N_t + N_r - 2l)! \xi_l^{N_t + N_r - 2l}}{1/(N_t + N_r - 2l + 1)! \xi_l^{N_t + N_r - 2l + 1}} = \frac{N_t + N_r - 2l + 1}{\xi_l}. \quad (17)$$

On the other hand,  $Q_l(\xi_l) = f_{\Delta_l}(\xi_l)$ . Since the  $n^{\text{th}}$  derivative of  $f_{\Delta_l}(x)$  can also be expressed by  $f_{\Delta_l}^{(n)}(x) = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} (-jv)^n e^{-jvx} \Psi_{\Delta_l}(jv) d(jv)$ ,  $n \geq 0$ , then it can be shown, using the Residue Theorem, that  $f_{\Delta_l}^{(n)}(0) = 0$  for  $n \leq N_t + N_r - 2l - 1$ . This is because  $N_t + N_r - 2l + 1$  is the degree of  $\Psi_{\Delta_l}(jv)$ 's denominator. Observing again that  $Q_l(\xi_l)$  is the derivative of  $P_l(\xi_l)$  and using Taylor expansion of  $Q_l(\xi_l)$  and  $P_l(\xi_l)$  to the  $(N_t + N_r - 2l)^{\text{th}}$  term, we also find that  $\frac{Q_l(\xi_l)}{P_l(\xi_l)} \approx \frac{N_t + N_r - 2l + 1}{\xi_l}$ .

- $b_l = 1$ : In this case,  $\xi_l = gN_t$ . Since  $J_l(\xi_l)$  and  $\frac{Q_l(\xi_l)}{P_l(\xi_l)}$  are finite, and  $K_l(\xi_l) = 0$ , (16) still holds.

## APPENDIX II PROOF OF THEOREM 2

To prove Theorem 2, it is sufficient to prove that  $\lim_{\eta \rightarrow \infty} \hat{d}^{uncorr}(r, \eta) = d_{asym}$ , since the other equality follows from Lemma 2. In Appendix I, it was shown that, if  $b_l \geq 1$ ,  $l = 1, \dots, t$ , then  $J_l(\xi_l)K_l(\xi_l) \rightarrow 0$  as  $\eta \rightarrow \infty$ . Thus, only the case  $b_l < 1$  is of interest. Moreover, using (15), (17) and the fact that  $K_l(\xi_l)$  can be approximated by  $K_l(\xi_l) \approx g^{b_l}(1 - b_l)\eta^{b_l}$ , we obtain that  $J_l(\xi_l)K_l(\xi_l) \approx \frac{N_t + N_r - 2l + 1}{N_t}(1 - b_l)\eta$ . Hence,  $\hat{d}^{uncorr}(r, \eta)$  given by (10) can be written as:

$$\begin{aligned} \hat{d}^{uncorr}(r, \infty) &= \sum_{l=1, b_l < 1}^t (N_t + N_r - 2l + 1)(1 - b_l) \\ &= \sum_{l=1}^t (N_t + N_r - 2l + 1)(1 - b_l)^+ \\ &= \sum_{l=1}^t (N_t + N_r - 2l + 1)\alpha_l, \end{aligned} \quad (18)$$

where  $\alpha_l = (1 - b_l)^+$  and  $(x)^+ = \max(x, 0)$  for each real number  $x$ . Next, we show that the  $\alpha_l$ 's that satisfy (18) are exactly the coefficients leading to the asymptotic DMT given in [3]. First, recall that  $b = (b_1, \dots, b_t) \in \mathcal{A} = \{(b_1, \dots, b_t) \in \mathcal{R}^{t+} \mid \sum_{l=1}^t b_l = r\}$ , and  $b$  maximizes the lower bound (8).

$$\max_{b \in \mathcal{A}} \prod_{l=1}^t \Gamma_{inc}(\xi_l, N_r + N_t - 2l + 1). \quad (19)$$

As  $\eta \rightarrow \infty$ ,  $\Gamma_{inc}(\xi_l, N_r + N_t - 2l + 1)$  is independent of  $b_l$ , for  $b_l \geq 1$ . This is because when  $b_l = 1$  then  $\xi_l = gN_t$ , and when  $b_l > 1$  then  $\xi_l \rightarrow \infty$  and  $\Gamma_{inc}(\xi_l, N_r + N_t - 2l + 1) \rightarrow 1$ . Indeed, if we let  $\kappa$  be the number of coefficients  $b_l < 1$ , the maximization problem (19) reduces to:

$$K \cdot \max_{b \in \mathcal{B}} \prod_{l=1}^{\kappa} \Gamma_{inc}(\xi_l, N_r + N_t - 2l + 1), \quad (20)$$

where  $K$  is a constant factor and  $\mathcal{B}$  is given by:

$$\mathcal{B} = \{(b_1, \dots, b_{\kappa}) \in \mathcal{R}^{\kappa+} \mid \sum_{l=1}^{\kappa} b_l \leq r\}.$$

Maximization (20) involves only  $b_l < 1$ , for which  $\xi_l \rightarrow 0$  as  $\eta \rightarrow \infty$ . Using (15) and the fact that around zero,  $\Gamma_{inc}(x, m)$  can be approximated by  $\Gamma_{inc}(x, m) \approx \frac{x^m}{m!}$ , we have:  $\Gamma_{inc}(\xi_l, N_r + N_t - 2l + 1) \approx C(g\eta)^{(b_l - 1)(N_r + N_t - 2l + 1)}$ , where  $C = \frac{g^{N_t}}{(N_t + N_r - 2l + 1)!}$  is a constant independent of  $b_l$ ,  $l = 1, \dots, t$ . Then, (20) is equivalent to:

$$\min_{b \in \mathcal{B}} \sum_{l=1}^{\kappa} (1 - b_l)(N_r + N_t - 2l + 1).$$

Thus, the asymptotic diversity estimate given by (18) can be expressed as:

$$\hat{d}^{uncorr}(r, \infty) = \min_{\alpha \in \mathcal{A}'} \sum_{l=1}^t (N_r + N_t - 2l + 1)\alpha_l,$$

where  $\alpha = (\alpha_1, \dots, \alpha_t)$ , and  $\mathcal{A}'$  is given by:

$$\mathcal{A}' = \{(\alpha_1, \dots, \alpha_t) \in \mathcal{R}^{t+} \mid \sum_{l=1}^t (1 - \alpha_l)^+ \leq r\},$$

which is exactly the asymptotic DMT  $d_{asym}$  established in [3]. This completes the proof of Theorem 2.

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