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Preface

This book is a revised and updated version, including a substantial portion of new material, of J. D. Cole's text *Perturbation Methods in Applied Mathematics*, Ginn–Blaisdell, 1968. We present the material at a level which assumes some familiarity with the basics of ordinary and partial differential equations. Some of the more advanced ideas are reviewed as needed; therefore this book can serve as a text in either an advanced undergraduate course or a graduate level course on the subject.

The applied mathematician, attempting to understand or solve a physical problem, very often uses a perturbation procedure. In doing this, he usually draws on a backlog of experience gained from the solution of similar examples rather than on some general theory of perturbations. The aim of this book is to survey these perturbation methods, especially in connection with differential equations, in order to illustrate certain general features common to many examples. The basic ideas, however, are also applicable to integral equations, integrodifferential equations, and even to difference equations.

In essence, a perturbation procedure consists of constructing the solution for a problem involving a small parameter ε, either in the differential equation or the boundary conditions or both, when the solution for the limiting case $\varepsilon = 0$ is known. The main mathematical tool used is asymptotic expansion with respect to a suitable asymptotic sequence of functions of ε.

In a regular perturbation problem a straightforward procedure leads to an approximate representation of the solution. The accuracy of this approximation does not depend on the value of the independent variable and gets better for smaller values of ε. We will not discuss this type of problem here as it is well covered in other texts. For example, the problem of calculating the perturbed eigenvalues and eigenfunctions of a self adjoint differential operator is a regular perturbation problem discussed in most texts on differential equations.

Rather, this book concentrates on singular perturbation problems which are very common in physical applications and which require special techniques. Such singular perturbation problems may be divided into two broad categories: layer-type problems and cumulative perturbation problems.

In a layer-type problem the small parameter multiplies a term in the differential equation which becomes large in a thin layer near a boundary (e.g., a boundary-layer) or in the interior (e.g., a shock-layer). Often, but not always, this is the highest derivative in the differential equation and the $\varepsilon = 0$ approximation is therefore governed by a lower order equation which cannot satisfy all the initial or boundary conditions prescribed. In a cumulative perturbation problem the small parameter multiplies a term which never becomes large. However, its cumulative effect becomes important for large values of the independent variable. In some applications both categories occur simultaneously and require the combined use of the two principal techniques we study in this book.

This book is written very much from the point of view of the applied mathematician; much less attention is paid to mathematical rigor than to rooting out the underlying ideas, using all means at our disposal. In particular, physical reasoning is often used as an aid to understanding a problem and to formulating the appropriate approximation procedure.

The first chapter contains some background on asymptotic expansions. The more advanced techniques in asymptotics such as the methods of steepest descents and stationary phase are not covered as there are excellent modern texts including these techniques which, strictly speaking, are not perturbation techniques. In addition, we introduce in this chapter the basic ideas of limit process expansions, matching asymptotic expansions, and general asymptotic expansions.

Chapter 2 gives a deeper exposition of limit process expansions through a sequence of examples for ordinary differential equations. Chapter 3 is devoted to cumulative perturbation problems using the so-called multiple variable expansion procedure. Applications to nonlinear oscillations, flight mechanics and orbital mechanics are discussed in detail followed by a survey of other techniques which can be used for this class of problems.

In Chapter 4 we apply the procedures of the preceding chapters to partial differential equations, presenting numerous physical examples. Finally, the last chapter deals with a typical use of asymptotic expansions, the construction of approximate equations; simplified models such as linearized and transonic aerodynamics, and shallow water theory are derived from more exact equations by means of asymptotic expansions. In this way the full meaning of laws of similitude becomes evident.

The basic ideas used in this book are, as is usual in scientific work, the ideas of many people. In writing the text, no particular attempt has been made to cite the original authors or to have a complete list of references and bibliography. Rather, we have tried to present the "state of the art" in a systematic manner starting from elementary applications and progressing gradually to areas of current research.

For a deeper treatment of the fundamental ideas of layer-type expansions and related problems the reader is referred to the forthcoming book by P. A. Lagerstrom and J. Boa of Caltech.

To a great extent perturbation methods were pioneered by workers in fluid mechanics and these traditional areas are given full coverage. Applications in celestial mechanics, nonlinear oscillations, mathematical biology, wave propagation, and other areas have also been successfully explored since the publication of J. D. Cole's 1968 text. Examples from these more recent areas of application are also covered.

We believe that this book contains a unified account of perturbation theory as it is understood and widely used today.

Fall 1980

J. Kevorkian
J. D. Cole

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