

# Dihedral Group Codes over Finite Fields

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## Abstract

Bazzi and Mitter [4] showed that binary dihedral group codes are asymptotically good. In this paper we prove that the dihedral group codes over any finite field with strong duality property are asymptotically good. If the characteristic of the field is even, self-dual dihedral group codes are asymptotically good. If the characteristic of the field is odd, maximal self-orthogonal dihedral group codes and LCD dihedral group codes are asymptotically good.

**Key words:** Dihedral group codes; finite fields; asymptotically good; self-dual codes; LCD codes.

## 1 Introduction

Let  $F$  be a finite field with cardinality  $|F| = q$ , where  $q$  is a power of a prime (just the characteristic  $\text{char } F$  of  $F$ ). Let  $n$  be a positive integer. Any nonempty subset  $C \subseteq F^n$  is called a code of length  $n$  over  $F$  in coding theory. The *Hamming weight*  $w(a)$  for  $a = (a_1, \dots, a_n) \in F^n$  is defined to be the number of the indexes  $i$  that  $a_i \neq 0$ , and the *Hamming distance*  $d(a, b) = w(a - b)$  for  $a, b \in F$ . And  $d(C) = \min\{d(c, c') \mid c \neq c' \in C\}$  is said to be the *minimum distance* of  $C$ , while  $\Delta(C) = \frac{d(C)}{n}$  is called the *relative minimum distance* of  $C$ . The rate of the code  $C$  is defined as  $R(C) = \frac{\log_q |C|}{n}$ . If  $C$  is a *linear code*, i.e., a linear subspace of  $F^n$ , then  $R(C) = \frac{\dim_F C}{n}$ . A class of codes is said to be *asymptotically good* if there is a code sequence  $C_1, C_2, \dots$  in the class such that

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the length  $n_i$  of  $C_i$  goes to infinity and both the rate  $R(C_i)$  and the relative minimum distance  $\Delta(C_i)$  are positively bounded from below.

Gilbert [11] and Varshamov [27] showed that, for linear codes whose relative minimum distances are at least  $\delta$ ,  $0 < \delta < 1 - q^{-1}$ , their rates attain the *GV-bound*  $g_q(\delta) = 1 - h_q(\delta)$  with high probability, where

$$h_q(\delta) = \delta \log_q(q-1) - \delta \log_q \delta - (1-\delta) \log_q(1-\delta), \quad 0 \leq \delta \leq 1 - q^{-1}, \quad (1.1)$$

is the *q-entropy* function. Note that  $h_q(\delta)$  is increasing and concave in the interval  $[0, 1 - q^{-1}]$ . More precisely, for linear codes of rate  $r$ , Pierce [24] proved that their relative minimum distances are asymptotically distributed at  $g_q^{-1}(r)$ , where  $g_q^{-1}(\cdot)$  is the inverse function of the GV-bound  $g_q(\cdot)$ . In particular, linear codes are asymptotically good. For codes of rate  $r$ , Barg and Forney [3] showed that their relative minimum distances are asymptotically distributed at  $g_q^{-1}(2r)$ .

Mathematical structures afforded by codes are useful for theory and practice. The euclidean inner product of  $F^n$  is defined as:

$$\langle a, b \rangle = \sum_{i=1}^n a_i b_i, \quad \forall a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in F^n. \quad (1.2)$$

And  $C^\perp = \{a \in F^n \mid \langle c, a \rangle = 0, \forall c \in C\}$  is the *orthogonal code* of  $C$ . If  $C \subseteq C^\perp$  ( $C = C^\perp$ , resp.), then  $C$  is said to be *self-orthogonal* (*self-dual*, resp.). Obviously,  $R(C) = \frac{1}{2}$  if  $C$  is self-dual. If  $C$  is self-orthogonal, but any code containing  $C$  properly is not self-orthogonal, then  $C$  is said to be *maximal self-orthogonal*. On the other hand,  $C$  is said to be a *linear complementary dual code*, or *LCD code* in short, if  $C \cap C^\perp = 0$ .

Let  $G$  be a finite group, and  $FG$  be the *group algebra* of  $G$  over the field  $F$ . Any left ideal of  $FG$  is called a *group code* of  $G$  over  $F$ , or an *FG-code* for short. Further, any  $FG$ -submodule of  $(FG)^2 = FG \oplus FG$  is called a *quasi-FG code of index 2*, or *2-quasi FG-code* in short. Quasi-FG codes of index  $m$  are defined similarly. If  $G$  is abelian (cyclic, resp.), quasi-FG codes are also called *quasi-abelian codes* (*quasi-cyclic codes*, resp.)

Let  $G$  be a cyclic group of order  $n$ . Then  $FG$ -codes are well-known as *cyclic codes* of length  $n$  over  $F$ , which are studied and applied extensively since the 1950's. Even so, it is still an open problem: whether or not the cyclic codes are asymptotically good? e.g., see [19]. In contrast, the quasi-cyclic codes of index 2 were proved asymptotically good, see [6, 7, 16]. Moreover, self-dual quasi-cyclic codes are asymptotically good, see [8, 18].

Now assume that  $G$  is a *dihedral group* of order  $2n$ , i.e.,  $G$  has a normal cyclic subgroup  $H = \langle u \rangle$  of order  $n$  generated by  $u$ , and an element  $v$  of order 2 such that  $vuv^{-1} = u^{-1}$ . Then  $FG$ -codes are called *dihedral group codes*, or *dihedral codes* in short. Bazzi and Mitter [4] proved that, if  $q = 2$ , the binary dihedral codes are asymptotically good. Their arguments are based on a result in [21, 25, 26], which estimates the number of the code words in a binary *balanced code* with weight bounded above, see Definition 2.1 and Lemma 2.2 below for details. Soon after, Martínez-Pérez and Willems [20] showed that the

binary doubly-even (hence must be self-dual) quasi-cyclic codes of index 2 are asymptotically good.

In [9] we generalized the result in [21, 25, 26] on estimating the number of the code words with bounded weight to any  $q$ -ary balanced codes, see Lemma 2.2 below for details; and showed that, like the linear codes, the relative minimum distances of the quasi-abelian codes of rate  $r$  are asymptotically distributed at  $g_q^{-1}(r)$ , where  $g_q^{-1}(\cdot)$  is the inverse function of the GV-bound  $g_q(\cdot)$ , see the outline around Eq.(1.1). In that paper we also said “... from it (means the generalization Lemma 2.2) quite a part of [4] can be extended to any  $q$ -ary case”.

For the case that  $\text{char } F = 2$ , Alahmadi, Özdemir and Solé [1] discovered an interesting fact: the self-dual double circulant codes over  $F$ , a family of self-dual quasi-cyclic codes of index 2, are in particular self-dual dihedral codes. Based on *Artin’s primitive root conjecture*, they proved that such codes are asymptotically good.

For odd  $p = \text{char } F$ , Borello and Willems [5] considered the *semidirect products* of the cyclic group of order  $p$  by suitable finite cyclic groups; with the help of the generalization Lemma 2.2, they proved the asymptotic goodness of such group codes.

In this paper we extend the asymptotic goodness of dihedral codes to any  $q$ -ary case. Specifically, we exhibit two kinds of random dihedral codes with strong duality property, and with nice asymptotic behavior as well. Suitably choosing a positive real number  $\delta$  and the code lengths  $2n_1, 2n_2, \dots$  going to infinity, we prove that the probability for the relative minimum distance of the random dihedral codes greater than  $\delta$  is convergent to 1. As consequences, we get the following asymptotic goodness.

**Theorem 1.1.** *Assume that  $0 < \delta < 1 - q^{-1}$  and  $0 < h_q(\delta) < \frac{1}{4}$ . If  $\text{char } F = 2$ , then there are self-dual dihedral group codes  $C_1, C_2, \dots$  over  $F$  with length of  $C_i$  going to infinity such that  $\Delta(C_i) > \delta$  for all  $i = 1, 2, \dots$ .*

For the case  $\text{char } F$  is odd, [28] had shown that there exist no self-dual dihedral codes.

**Theorem 1.2.** *Assume that  $0 < \delta < 1 - q^{-1}$  and  $0 < h_q(\delta) < \frac{1}{4}$ . If  $\text{char } F$  is odd, then:*

(1) *there are maximal self-orthogonal dihedral group codes  $C_1, C_2, \dots$  over  $F$  with length of  $C_i$  going to infinity such that  $\lim_{i \rightarrow \infty} R(C_i) = \frac{1}{2}$  and  $\Delta(C_i) > \delta$  for all  $i = 1, 2, \dots$ .*

(2) *there are LCD dihedral group codes  $C_1, C_2, \dots$  over  $F$  with length of  $C_i$  going to infinity such that  $R(C_i) = \frac{1}{2}$  and  $\Delta(C_i) > \delta$  for all  $i = 1, 2, \dots$ .*

If we ignore the action of the element of order 2 on the normal cyclic subgroup of order  $n$ , then the dihedral group codes are quasi-cyclic codes of index 2 (the converse is not true in general). So we have consequences:

**Corollary 1.3.** (1) *If char  $F=2$ , then the self-dual quasi-cyclic codes of index 2 over  $F$  are asymptotically good.*

(2) *If char  $F$  is odd, then the maximal self-orthogonal quasi-cyclic codes of index 2 and the LCD quasi-cyclic codes of index 2 are both asymptotically good.*

In the next section we sketch preliminaries. In §3 we explore the properties of the dihedral group algebras over  $F$ . In §4 we construct precisely our dihedral group codes of length  $2n$  with rate  $\frac{1}{2} - \frac{1}{2n}$  or  $\frac{1}{2}$ ; the two kinds of dihedral group codes may have different behavior. In §5 and §6 we exhibit the random properties of the two kinds of dihedral group codes constructed in §4. The two theorems listed above will be proved in §7.

## 2 Preliminaries

In this paper  $F$  is always a finite field with  $|F| = q$  which is a power of a prime, where  $|S|$  denotes the cardinality of any set  $S$ . And  $n > 1$  is an integer.

For any index set  $I = \{i_1, \dots, i_d\}$ ,  $F^I = \{(a_{i_1}, \dots, a_{i_d}) \mid a_{i_j} \in F\}$  is a vector space over  $F$  of dimension  $d$ . As usual,  $F^n = F^I$  with  $I = \{1, 2, \dots, n\}$ . For  $a \in F^n$ , the fraction  $\frac{w(a)}{n}$  is called the *relative weight* of  $a$ . Let  $\delta$  be a real number such that  $0 < \delta < 1 - q^{-1}$ . For any code  $C \subseteq F^n$ , we denote

$$C^{\leq \delta} = \{c \mid c \in C, \frac{w(c)}{n} \leq \delta\}.$$

**Definition 2.1.** Let  $C \subseteq F^n = F^I$  where  $I = \{1, 2, \dots, n\}$ . If there are subsets  $I_1, \dots, I_s$  (with repetition allowed) of the index set  $I$  and integers  $k$  and  $t$  such that every cardinality  $|I_j| = k$  and the following two hold:

- (1) for any  $i \in I$ , the number of such subscripts  $j$  that  $i \in I_j$  is equal to  $t$ ;
- (2) for any  $j = 1, \dots, s$ , the projection  $\rho_j : F^I \rightarrow F^{I_j}$  maps  $C$  bijectively onto  $F^{I_j}$ ;

then, following [4] and [25], we say that  $C$  is a *balanced code* over  $F$  of length  $n$  and *information length*  $k$ , and  $I_1, \dots, I_s$  form a *balanced system of information index sets* of  $C$ .

Note that the phrase “balanced codes” might be used for different concepts in literature, e.g., in [15]. And, in notation of the theory of block designs, the above definition is equivalent to saying that “there is a  $1-(n, k, t)$  design whose blocks are information sets” (thanks are given to the reviewers for showing the concise version).

The following result was proved in [21], [25] and [26] for binary case, and in [9] for the present version.

**Lemma 2.2.** *Let  $C$  be a balanced code over  $F$  of length  $n$  and information length  $k$ . Assume that  $0 < \delta < 1 - q^{-1}$ . Then  $|C^{\leq \delta}| \leq q^{kh_q(\delta)}$ .*

If  $C$  is a linear code, then  $w(C) = \min\{w(c) \mid 0 \neq c \in C\}$  is called the *minimum weight* of  $C$ , and  $w(C) = d(C)$ . So  $\Delta(C) = \frac{w(C)}{n}$ , and it is also called the *relative minimum weight* of  $C$ . And the rate  $R(C) = \frac{\dim_F C}{n}$ .

Let  $G$  be a finite group,  $FG = \{\sum_{x \in G} a_x x \mid a_x \in F\}$ , which is an  $F$ -vector space with a multiplication induced by the multiplication of the group  $G$ . So  $FG$  is an  $F$ -algebra, called the *group algebra* of  $G$  over  $F$ . Any  $\sum_{x \in G} a_x x \in FG$  is viewed as a sequence  $(a_x)_{x \in G}$  of  $F$  indexed by  $G$ . Any left ideal  $C$  of  $FG$  is called a *group code* of  $G$  over  $F$ . We also say that  $C$  is an  $FG$ -code for short. If  $e \in FG$  is an *idempotent*, i.e.,  $e^2 = e$ , then  $FGe$  is a left ideal and  $FG = FGe \oplus FGe'$ , where  $e' = 1 - e$  is also an idempotent and  $ee' = e'e = 0$ . Further, if the idempotent  $e$  is central, then  $FG = FGe \oplus FGe'$  with both  $FGe$  and  $FGe'$  being 2-sided ideals. If the greatest common divisor  $\gcd(|G|, q) = 1$ , then any ideals and any left ideals can be constructed by idempotents in this way; and  $e$  is called a *primitive* idempotent once  $FGe$  is a minimal left ideal (i.e., any left ideal contained in  $FGe$  is either 0 or  $FGe$  itself). Please see [14, Chapter 5, §3], or see [13, §4.3] for cyclic codes.

**Remark 2.3.** Any group code  $C$  of the group algebra  $FG$  is a balanced code, see [4, Lemma 2.2]. In fact, it can be proved in a similar way that any transitive permutation codes are balanced codes (a linear code is called a transitive permutation code if there is a group permuting the bits of the code transitively and the code is invariant under the group action, see [10]).

Mapping  $x$  to  $x^{-1}$  is an anti-automorphism of the group  $G$ , where  $x^{-1}$  denotes the inverse of  $x$ . We have an anti-automorphism of the algebra  $FG$ :

$$FG \longrightarrow FG, \quad \sum_{x \in G} a_x x \longmapsto \sum_{x \in G} a_x x^{-1}. \quad (2.1)$$

We denote  $\sum_{x \in G} a_x x^{-1} = \overline{\sum_{x \in G} a_x x}$ , and call Eq.(2.1) the “bar” map of  $FG$  for convenience. So,  $\overline{\overline{a}} = a$ ,  $\overline{ab} = \overline{b} \overline{a}$ , for  $a, b \in FG$ . It is an automorphism of  $FG$  once  $G$  is abelian. The following is a linear form of  $FG$ :

$$\sigma : FG \longrightarrow F, \quad \sum_{x \in G} a_x x \longmapsto a_{1_G} \quad (1_G \text{ is the identity of } G).$$

For  $a, b \in FG$ , we use the notation  $\langle a, b \rangle$  to denote the euclidean inner product of  $a$  and  $b$ , which are viewed as sequences  $(a_x)_{x \in G}$  and  $(b_x)_{x \in G}$  of length  $n$  over  $F$ ; see Eq.(1.2).

**Lemma 2.4.** (1)  $\sigma(ab) = \sigma(ba)$ ,  $\forall a, b \in FG$ .

(2)  $\langle a, b \rangle = \sigma(\overline{ab}) = \sigma(\overline{a} \overline{b})$ ,  $\forall a, b \in FG$ .

(3)  $\langle da, b \rangle = \langle a, \overline{d} b \rangle$ ,  $\forall a, b, d \in FG$ .

(4) If  $C$  is an  $FG$ -code, then so is  $C^\perp$ .

(5) For  $FG$ -codes  $C$  and  $D$ ,  $\langle C, D \rangle = 0$  if and only if  $C\overline{D} = 0$ .

*Proof.* The (1), (2) is verified directly. The (3) follows from (2). And (4) is checked by (3). For (5), the sufficiency follows from (2) directly. Conversely, if  $c\bar{d} \neq 0$  for  $c \in C$  and  $d \in D$ , write  $c\bar{d} = \sum_{x \in G} b_x x$  with a coefficient  $b_{x_0} \neq 0$ ; then  $x_0^{-1}c \in C$  and  $\langle x_0^{-1}c, d \rangle = \sigma(x_0^{-1}c\bar{d}) = b_{x_0} \neq 0$ .  $\square$

Assume that  $H$  is a cyclic group of order  $n$ . Then  $FH$ -codes are cyclic codes, and can be described by monic factors of the polynomial  $X^n - 1$ . In the following, we further assume that  $\gcd(n, q) = 1$ . Then monic factors of  $X^n - 1$  are determined by their zeros. As noted above,  $FH$ -codes are determined by idempotents. So each ideal  $FHe$  with  $e^2 = e \neq 0$  corresponds to a monic factor  $g(X) \mid X^n - 1$  such that  $FHe \cong F[X]/\langle g(X) \rangle$ . If the ideal  $FHe$  is simple, i.e.,  $e$  is a primitive idempotent, then  $g(X)$  is irreducible and  $FHe$  is a field over  $F$  with extension degree  $\dim_F FHe = \deg g(X)$ . Thus  $FH$  has finitely many primitive idempotents  $e_0, e_1, \dots, e_s$  such that  $1 = e_0 + e_1 + \dots + e_s$  and  $e_i e_j = 0$  for  $0 \leq i \neq j \leq s$ , where  $e_0 = \frac{1}{n} \sum_{x \in H} x$  and  $\dim_F FHe_0 = 1$ . And the automorphism “bar” in Eq.(2.1) permutes the primitive idempotents.

For any ring (with identity)  $R$ , by  $R^\times$  we denote the multiplicative group consisting of the units (invertible elements) of  $R$ . By  $\mathbb{Z}_n$  we denote the integer residue ring modulo  $n$ , hence  $\mathbb{Z}_n^\times$  is the multiplicative group consisting of the reduced residue classes. Then  $q \in \mathbb{Z}_n^\times$  (since  $\gcd(n, q) = 1$ ). In the multiplicative group  $\mathbb{Z}_n^\times$ ,  $\text{ord}_{\mathbb{Z}_n^\times}(q)$  denotes the order of  $q$ , and  $\langle q \rangle_{\mathbb{Z}_n^\times}$  denotes the cyclic subgroup generated by  $q$ . The following facts are well-known.

**Lemma 2.5.** *Let  $H$  be a cyclic group of odd order  $n$  with  $\gcd(n, q) = 1$ . Let  $e_0, e_1, \dots, e_s$  be all primitive idempotents of  $FH$ , where  $e_0 = \frac{1}{n} \sum_{x \in H} x$ . Let  $\lambda(n) = \min \{ \dim_F(FHe_1), \dots, \dim_F(FHe_s) \}$ .*

- (1) ([4, Lemma 2.5])  $\lambda(n) = \min \{ \text{ord}_{\mathbb{Z}_p^\times}(q) \mid p \text{ is a prime divisor of } n \}$ .
- (2) ([2, Theorem 6])  $\bar{e}_j \neq e_j$  for any  $j > 0$  if and only if  $\text{ord}_{\mathbb{Z}_n^\times}(q)$  is odd.
- (3) ([17, Theorem 1])  $\bar{e}_j = e_j$  for any  $j > 0$  if and only if  $-1 \in \langle q \rangle_{\mathbb{Z}_n^\times}$ .

We need some number-theoretic results. Let  $t > q$  be an integer, and  $\pi(t)$  be the number of the primes less or equal to  $t$ . By Gauss’ Lemma,  $\lim_{t \rightarrow \infty} \frac{\pi(t)}{t/\ln t} = 1$ .

**Lemma 2.6.** *Set  $\mathcal{G}_t = \{ \text{prime } p \mid q < p \leq t, \text{ord}_{\mathbb{Z}_p^\times}(q) \geq (\log_q t)^2 \}$ . Then the natural density  $\lim_{t \rightarrow \infty} \frac{|\mathcal{G}_t|}{\pi(t)} = 1$ .*

*Proof.* It was proved in [4, Lemma 2.6] for the binary case. For the general case, the proof is similar. Set  $\bar{\mathcal{G}}_t = \{ \text{prime } p \mid q < p \leq t, \text{ord}_{\mathbb{Z}_p^\times}(q) < (\log_q t)^2 \}$ . If  $r < (\log_q t)^2$  and  $p_1, \dots, p_k \in \bar{\mathcal{G}}_t$  satisfy that  $\text{ord}_{\mathbb{Z}_{p_i}^\times}(q) = r$ ,  $i = 1, \dots, k$ , then  $q^r - 1 = p_1 \cdots p_k s$ , hence  $k \leq \log_q(q^r - 1) < r < (\log_q t)^2$ . So  $|\bar{\mathcal{G}}_t| < (\log_q t)^4$ , and

$$\lim_{t \rightarrow \infty} \frac{|\bar{\mathcal{G}}_t|}{\pi(t)} < \lim_{t \rightarrow \infty} \frac{(\ln t / \ln q)^4}{t / \ln t} = 0. \quad \square$$

The following result was proved in [12] (for Dirichlet density) and in [23] (for natural density).

**Lemma 2.7** ([12], [23]). Let  $\mathcal{O}_t = \{\text{prime } p \mid q < p \leq t, \text{ord}_{\mathbb{Z}_p^\times}(q) \text{ is odd}\}$ . Then the natural density  $\lim_{t \rightarrow \infty} \frac{|\mathcal{O}_t|}{\pi(t)}$  is a positive fraction less than 1 (the exact value depends on the exponent of the prime power  $q$ , see [23, Theorem 1]).

With the above three lemmas and their notation, we conclude:

**Corollary 2.8.** (1) There is a sequence  $n_1, n_2, \dots$  of positive odd integers coprime to  $q$  such that  $\text{ord}_{\mathbb{Z}_{n_i}^\times}(q)$  are odd for all  $i = 1, 2, \dots$  and  $\lim_{i \rightarrow \infty} \frac{\log_q n_i}{\lambda(n_i)} = 0$ .

(2) There is a sequence  $n_1, n_2, \dots$  of positive odd integers coprime to  $q$  such that  $-1 \in \langle q \rangle_{\mathbb{Z}_{n_i}^\times}$  for all  $i = 1, 2, \dots$  and  $\lim_{i \rightarrow \infty} \frac{\log_q n_i}{\lambda(n_i)} = 0$ .

*Proof.* (1). The natural density

$$\lim_{t \rightarrow \infty} \frac{|\mathcal{O}_t \cap \mathcal{G}_t|}{\pi(t)} = \lim_{t \rightarrow \infty} \left( \frac{|\mathcal{O}_t|}{\pi(t)} + \frac{|\mathcal{G}_t|}{\pi(t)} - \frac{|\mathcal{O}_t \cup \mathcal{G}_t|}{\pi(t)} \right) = \lim_{t \rightarrow \infty} \frac{|\mathcal{O}_t|}{\pi(t)} > 0.$$

(2). Note that, if  $n$  is a prime, then  $\mathbb{Z}_n^\times$  is cyclic and has  $-1$  as the unique element of order 2. Hence,  $-1 \in \langle q \rangle_{\mathbb{Z}_n^\times}$  if and only if  $\text{ord}_{\mathbb{Z}_n^\times}(q)$  is even. Let  $\overline{\mathcal{O}}_t = \{\text{prime } p \mid q < p \leq t, \text{ord}_{\mathbb{Z}_p^\times}(q) \text{ is even}\}$ . By Lemma 2.7, the natural density  $\lim_{t \rightarrow \infty} \frac{|\overline{\mathcal{O}}_t|}{\pi(t)}$  is positive. So

$$\lim_{t \rightarrow \infty} \frac{|\overline{\mathcal{O}}_t \cap \mathcal{G}_t|}{\pi(t)} = \lim_{t \rightarrow \infty} \left( \frac{|\overline{\mathcal{O}}_t|}{\pi(t)} + \frac{|\mathcal{G}_t|}{\pi(t)} - \frac{|\overline{\mathcal{O}}_t \cup \mathcal{G}_t|}{\pi(t)} \right) = \lim_{t \rightarrow \infty} \frac{|\overline{\mathcal{O}}_t|}{\pi(t)} > 0. \quad \square$$

**Lemma 2.9.** Let  $q \geq 2$  and  $k_1 \leq k_2 \leq \dots \leq k_m$  be positive integers. If  $k_1 \geq \log_q m$ , then  $(q^{k_1} - 1)(q^{k_2} - 1) \dots (q^{k_m} - 1) \geq q^{k_1 + k_2 + \dots + k_m - 2}$ .

*Proof.* We have

$$\frac{(q^{k_1} - 1)(q^{k_2} - 1) \dots (q^{k_m} - 1)}{q^{k_1} q^{k_2} \dots q^{k_m}} = \left(1 - \frac{1}{q^{k_1}}\right) \left(1 - \frac{1}{q^{k_2}}\right) \dots \left(1 - \frac{1}{q^{k_m}}\right) \geq \left(1 - \frac{1}{q^{k_1}}\right)^m.$$

Note that the sequence  $(1 - t^{-1})^t$  for  $t = 2, 3, \dots$  is increasing and  $(1 - \frac{1}{2})^2 \geq \frac{1}{q^2}$ . Since  $m \leq q^{k_1}$ , we get that  $(1 - \frac{1}{q^{k_1}})^m \geq (1 - \frac{1}{q^{k_1}})^{q^{k_1}} \geq \frac{1}{q^2}$ .  $\square$

### 3 Dihedral group algebras

**Remark 3.1.** In the following we always assume that:

- $F$  is a finite field of cardinality  $q$ .
- $n > 1$  is an odd integer and  $\gcd(n, q) = 1$ .
- $G = \langle u, v \mid u^n = 1 = v^2, vuv^{-1} = u^{-1} \rangle$  is the dihedral group of order  $2n$ .  
 $H = \langle u \rangle \leq G$  is the cyclic subgroup generated by  $u$  of order  $n$ ;  
 $vH = \{v, vu, \dots, vu^{n-1}\} = Hv$  is the coset of  $H$  other than  $H$ ;  
Hence  $G = H \cup vH$ .

- $FG = \{ \sum_{x \in G} a_x x \mid a_x \in F \}$  is the group algebra of  $G$  over  $F$ .

**Lemma 3.2.**  *$FH$  is a commutative ring,  $FG = FH \oplus vFH$ ,  $vFH = FvH = FHv$ , and the following hold.*

(1) *Let  $e_0 = \frac{1}{n} \sum_{x \in H} x$ ,  $e_1, \dots, e_s$  be all primitive idempotents of  $FH$ . Then  $FH = FHe_0 \oplus FHe_1 \oplus \dots \oplus FHe_s$  is a direct sum of simple ideals  $FHe_j$ 's which are field extensions over  $F$ . In particular,  $FHe_0 = Fe_0$  is the trivial ideal with  $\dim_F FHe_0 = 1$ .*

(2)  *$H$  is normal in  $G$ , and  $v$  induces the automorphism “bar” on  $FH$ , i.e., in notation of Eq.(2.1),  $vav^{-1} = \bar{a}$ , for all  $a \in FH$ .*

(3) *The idempotent  $e_0$  is central in  $FG$  and the ideal  $FGe_0$  is of dimension 2. Set  $\hat{e}_0 = e_0 + ve_0$ ; then  $\hat{e}_0$  is central in  $FG$ ,  $\hat{e}_0 \bar{\hat{e}}_0 = \hat{e}_0^2 = 2\hat{e}_0$  and  $FG\hat{e}_0 = \{ a \sum_{x \in G} x \mid a \in F \} = F\hat{e}_0$  is an ideal of dimension 1 contained in  $FGe_0$ .*

*Proof.* (1) is well-known, see [13, §4.3]. The others can be checked straightforwardly.  $\square$

By  $M_2(F)$  we denote the  $2 \times 2$  matrix algebra over  $F$ .

**Lemma 3.3.** *Let  $e$  be a primitive idempotents of  $FH$  other than  $e_0$ . Then  $FHe$  is a field extension over  $F$ , and one of the following holds:*

(1) *If  $\bar{e} \neq e$ , then  $e + \bar{e}$  is a primitive central idempotent of  $FG$ , and the ideal  $FG(e + \bar{e}) = FHe \oplus FH\bar{e} \oplus vFHe \oplus vFH\bar{e} \cong M_2(\tilde{F})$ , where  $\tilde{F} = FHe$ .*

(2) *If  $\bar{e} = e$ , then  $e$  is a primitive central idempotent of  $FG$ , the extension degree  $\dim_F FHe$  is even,  $FHe$  has a subfield  $\tilde{F}$  with  $\dim_{\tilde{F}} FHe = 2$ , and the ideal  $FGe = FHe \oplus FHev \cong M_2(\tilde{F})$ .*

*Proof.* They are somewhat known, e.g., (2) is proved in [4] for binary case. We show a proof for (1), (2) by constructing specific isomorphisms (3.1), (3.4) for later quotation.

We have seen in Lemma 3.2(1) that  $FHe$  is a field.

(1). Since  $e\bar{e} = 0$ ,  $e + \bar{e}$  is an idempotent. By Lemma 3.2(2),  $v(e + \bar{e}) = \bar{e}v + ev = (e + \bar{e})v$ , i.e.,  $e + \bar{e}$  is central in  $FG$ . So

$$FG(e + \bar{e}) = (FH \oplus vFH)(e + \bar{e}) = FHe \oplus FH\bar{e} \oplus vFHe \oplus vFH\bar{e}$$

is an ideal of  $FG$ . Note that  $a = ae$  for  $a \in \tilde{F} = FHe$ . Define a map:

$$\begin{aligned} M_2(\tilde{F}) &\longrightarrow FHe \oplus vFHe \oplus vFH\bar{e} \oplus FH\bar{e}, \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &\longmapsto a_{11}e + va_{21}e + v\bar{a}_{12}\bar{e} + \bar{a}_{22}\bar{e}, \end{aligned} \quad (3.1)$$

which is obviously a linear isomorphism. Note that  $e\bar{e} = 0$ ,  $ev = v\bar{e}$  and  $ve = \bar{e}v$ , see Lemma 3.2(2). For any two elements of  $FG(e + \bar{e})$ :

$$a_{11}e + va_{21}e + v\bar{a}_{12}\bar{e} + \bar{a}_{22}\bar{e}, \quad b_{11}e + vb_{21}e + v\bar{b}_{12}\bar{e} + \bar{b}_{22}\bar{e},$$



where  $a_{ij}, b_{ij} \in \tilde{F}$  for  $1 \leq i, j \leq 2$ ,

$$\begin{aligned} & (a_{11}e + va_{21}e + v\overline{a_{12}}\bar{e} + \overline{a_{22}}\bar{e})(b_{11}e + vb_{21}e + v\overline{b_{12}}\bar{e} + \overline{b_{22}}\bar{e}) \\ = & (a_{11}b_{11} + a_{12}b_{21})e + v(a_{21}b_{11} + a_{22}b_{21})e \\ & + v\overline{(a_{11}b_{12} + a_{12}b_{22})}\bar{e} + \overline{(a_{21}b_{12} + a_{22}b_{22})}\bar{e}. \end{aligned}$$

Thus Eq.(3.1) is an algebra isomorphism.

(2). In this case,  $ve = \bar{e}v = ev$ , hence  $e$  is central in  $FG$ . Denote  $K = FHe$  which is a field with identity  $e$ . Note that  $n > 1$  is odd, see Remark 3.1. The map  $a \mapsto \bar{a}$  for  $a \in K$  is an automorphism of the field  $K$  of order 2. By Galois Theory,

$$\tilde{F} := \{a \mid a \in K, \bar{a} = a\} \subseteq K \text{ is a subfield and } |K : \tilde{F}| = 2.$$

Since  $FH = \sum_{i=0}^{n-1} Fu^i$ ,  $K = FHe = \sum_{i=0}^{n-1} F(ue)^i$ ; i.e.,  $K$  is generated as an  $F$ -algebra by  $ue$ . Then  $K$  is generated as an  $\tilde{F}$ -algebra also by  $ue$  (since  $\tilde{F} \supseteq F$ ), and the minimal polynomial  $\varphi_{ue}(X)$  over  $\tilde{F}$  of  $ue$  has degree 2. Let  $\varphi_{ue}(X) = X^2 + gX + h \in \tilde{F}[X]$ . Then  $K = \tilde{F}e + \tilde{F}(ue)$ , and  $(ue)^2 + g(ue) + h = 0$ . Hence  $(\overline{ue})^2 + g(\overline{ue}) + h = \overline{(ue)^2 + g(ue) + h} = 0$ . So  $ue$  and  $\overline{ue}$  are two roots ( $\overline{ue} \neq ue$  since  $ue \notin \tilde{F}$ ) of  $\varphi_{ue}(X)$ , hence  $(ue)(\overline{ue}) = h$ . In  $K$  we have  $\overline{ue} = v(ue)v^{-1} = vuv^{-1}e = u^{-1}e = (ue)^{-1}$ . Thus  $h = (ue)(\overline{ue}) = 1$ , and  $\varphi_{ue}(X) = X^2 + gX + 1$ . Note that, since  $\varphi_{ue}(X)$  is irreducible,  $g$  and 2 are not both zero. We set

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \eta = \begin{pmatrix} -g & 1 \\ -1 & 0 \end{pmatrix}, \quad \nu = \begin{pmatrix} -1 & 0 \\ -g & 1 \end{pmatrix}. \quad (3.2)$$

Then the characteristic polynomial of  $\eta$  is  $\varphi_\eta(X) = X^2 + gX + 1 = \varphi_{ue}(X)$ . Mapping  $e \mapsto \varepsilon$  and  $ue \mapsto \eta$ , we get a field isomorphism

$$K = \tilde{F}e + \tilde{F}(ue) \cong \tilde{F}[X]/\langle \varphi_\eta(X) \rangle \cong \tilde{F}\varepsilon + \tilde{F}\eta \subseteq M_2(\tilde{F}). \quad (3.3)$$

Comparing the  $K$ -dimension, we get that

$$M_2(\tilde{F}) = (\tilde{F}\varepsilon + \tilde{F}\eta) + (\tilde{F}\varepsilon + \tilde{F}\eta)\nu.$$

On the other hand,

$$FGe = FHe + FHev = K + K(ve) = \tilde{F}e + \tilde{F}ue + \tilde{F}ve + \tilde{F}uve.$$

Since

$$\nu^2 = \varepsilon, \quad \nu\eta\nu^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -g \end{pmatrix} = \eta^{-1},$$

by mapping  $ve \mapsto \nu$ ,  $uve \mapsto \eta\nu$ , we extend the isomorphism Eq.(3.3) to the following isomorphism (where  $a, b, c, d \in \tilde{F}$ ):

$$\begin{aligned} FGe & \longrightarrow M_2(\tilde{F}), \\ ae + bue + cve + duve & \longmapsto a\varepsilon + b\eta + c\nu + d\eta\nu, \end{aligned} \quad (3.4)$$

which completes the proof.  $\square$

Combining Lemma 3.2 and Lemma 3.3, we obtain the following theorem.

**Theorem 3.4.** *The dihedral group algebra  $FG$  is a direct sum of ideals  $A_t$ :*

$$FG = A_0 \oplus A_1 \oplus \cdots \oplus A_m,$$

where  $A_0 = FG e_0$  is described in Lemma 3.2(3) and, for  $t = 1, \dots, m$ , the ideal  $A_t \cong M_2(F_t)$  with  $F_t$  being a field extension over  $F$  and  $\dim_F F_t = k_t$ , hence

$$k_1 + \cdots + k_m = \frac{n-1}{2}. \quad (3.5)$$

For the identity  $1_{A_t}$  of  $A_t$ , which is a central idempotent of  $FG$ , one of the following two holds:

(1) The identity  $1_{A_t} = e + \bar{e}$  for a primitive idempotent  $e$  of  $FH$  with  $e \neq \bar{e}$ , and  $k_t = \dim_F(FHe)$ .

(2) The identity  $1_{A_t} = e$  is a primitive idempotent of  $FH$  with  $e = \bar{e}$ , and  $k_t = \frac{1}{2} \dim_F(FHe)$ .

**Corollary 3.5.** *For  $t = 1, \dots, m$ , we have  $2k_t \geq \lambda(n)$ .*

*Proof.* Recall from Lemma 2.5 that

$$\lambda(n) = \min \{ \dim_F FHe \mid e \text{ is a primitive idempotent of } FH \text{ other than } e_0 \}.$$

By Theorem 3.4, if  $\bar{e} \neq e$  then  $k_t = \dim_F FHe$ ; otherwise,  $k_t = \frac{1}{2} \dim_F FHe$ . That is,  $2k_t \geq \lambda(n)$ .  $\square$

We collect in the following lemma the properties of  $2 \times 2$  matrix algebras which we need to study the dihedral group codes.

**Lemma 3.6.** *Let  $M = M_2(F)$ ,  $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , hence  $Z(M) = F\varepsilon \cong F$  is the center of  $M$ . Then  $M$  has a subalgebra  $E$  which is a field extension over  $F$  with  $\dim_F E = 2$ , and the following hold:*

(1) *If  $c \in M$  has  $\text{rank}(c) = 1$ , then  $Mc = Ec$  is a simple left ideal of  $M$ .*

(2) *Let  $L = E\varepsilon_{11} = M\varepsilon_{11}$ , where  $\varepsilon_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ . Then, for  $0 \neq c \in L$  and  $a, b \in E^\times$ ,  $ac = cb$  if and only if  $a = b \in (F\varepsilon)^\times$ .*

(3) *For  $\beta \in E^\times$ ,  $L\beta$  is a simple left ideal of  $M$ . And, when  $\beta$  runs over  $E^\times$ , the  $L\beta$  runs over all the simple left ideals of  $M$ , each of them appears exactly  $q - 1$  times.*

*Proof.* The finite field  $F$  has an extension of degree 2, in other words, there is an irreducible polynomial  $\varphi(X)$  of degree 2 over  $F$ . Let  $\eta \in M$  be a matrix with characteristic polynomial  $\varphi(X)$ . Then

$$E = F\varepsilon + F\eta = (F\varepsilon) + (F\varepsilon)\eta \cong F[X]/\langle \varphi(X) \rangle$$

is a field extension over  $F$  of degree 2.

(1). Obviously,  $Ec \subseteq Mc$  and  $Mc$  is a left ideal of  $M$  with  $\dim_F Mc = 2$ . Since  $E$  is a field and  $Ec \neq 0$ , hence  $\dim_E Ec = 1$  and  $\dim_F Ec = 2$ . So  $Ec = Mc$ . Since any left  $M$ -submodule contained in  $Mc$  is also a left  $E$ -submodule and  $\dim_E Mc = 1$ ,  $Mc = Ec$  is a simple left ideal.

(2). The sufficiency is obvious. We prove the necessity. First assume that  $c = \varepsilon_{11}$ ; i.e.,  $a\varepsilon_{11} = \varepsilon_{11}b$ . Write  $\eta = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ , then  $g_{12} \neq 0 \neq g_{21}$ , otherwise the characteristic polynomial of  $\eta$  is  $(X - g_{11})(X - g_{22})$  which is reducible. Write  $a = a_1\varepsilon + a_2\eta$  and  $b = b_1\varepsilon + b_2\eta$ , where  $a_i, b_i \in F$ . Then

$$(a_1\varepsilon + a_2\eta)\varepsilon_{11} = \varepsilon_{11}(b_1\varepsilon + b_2\eta),$$

i.e.,

$$(a_1 - b_1)\varepsilon_{11} + a_2\eta\varepsilon_{11} - b_2\varepsilon_{11}\eta = 0;$$

in matrix version,

$$\begin{pmatrix} a_1 - b_1 + a_2g_{11} - b_2g_{11} & -b_2g_{12} \\ a_2g_{21} & 0 \end{pmatrix} = 0.$$

So  $a_2g_{21} = -b_2g_{12} = 0$ . Since  $g_{12} \neq 0 \neq g_{21}$ , we obtain that  $a_2 = b_2 = 0$  and  $a_1 = b_1$ ; i.e.,  $a = b \in (F\varepsilon)^\times$ .

Next, assume that  $0 \neq c \in L$  and  $ac = cb$ . Since  $L = E\varepsilon_{11}$ , there is a  $d \in E^\times$  such that  $c = d\varepsilon_{11}$ . So  $ad\varepsilon_{11} = d\varepsilon_{11}b$ . Note that  $d^{-1} \in E$  commutes with  $a$ . Left multiplying by  $d^{-1}$ , we get  $a\varepsilon_{11} = \varepsilon_{11}b$ . Thus  $a = b \in (F\varepsilon)^\times$ .

(3). Because  $\beta$  is invertible, the map  $L \rightarrow L\beta$ ,  $c \mapsto c\beta$ , is an isomorphism of left  $M$ -modules. Hence  $L\beta$  is a simple left ideal of  $M$ .

Next, for  $\beta, \beta' \in E^\times$ ,  $L\beta = L\beta'$  if and only if  $L = L\beta'\beta^{-1}$ . Denote  $b = \beta'\beta^{-1} \in E$ . Note that  $L = E\varepsilon_{11}$  and  $Lb = E\varepsilon_{11}b$ . Hence  $L = Lb$  if and only if there is an  $a \in E$  such that  $a\varepsilon_{11} = \varepsilon_{11}b$ . By the above (2),  $a\varepsilon_{11} = \varepsilon_{11}b$  if and only if  $a = b \in (F\varepsilon)^\times$ . We get that

- For  $\beta, \beta' \in E^\times$ ,  $L\beta = L\beta'$  if and only if  $\beta'\beta^{-1} \in (F\varepsilon)^\times$ .

Thus, when  $\beta$  runs over  $E^\times$ , we obtain altogether  $\frac{q^2-1}{q-1} = q+1$  distinct simple left ideals  $L\beta$  of  $M$ , each of them appears  $q-1$  times. On the other hand, any simple left ideal of  $M$  consists of the zero matrix and  $q^2-1$  matrices of rank 1. Furthermore, the intersection of any two distinct simple left ideals of  $M$  is 0. The number of the matrices of rank 2 is equal to  $(q^2-1)(q^2-q) = q^4 - q^3 - q^2 + q$ . Hence the number of the matrices of rank 1 is equal to

$$q^4 - 1 - (q^4 - q^3 - q^2 + q) = q^3 + q^2 - q - 1 = (q+1)(q^2-1).$$

So the number of the simple left ideals of  $M$  is:  $(q+1)(q^2-1)/(q^2-1) = q+1$ . In other words, when  $\beta$  runs over  $E^\times$ , we obtain all  $q+1$  simple left ideals  $L\beta$  of  $M$ , each of them appears  $q-1$  times.  $\square$

## 4 Dihedral group codes

By Theorem 3.4 and Lemma 3.6, from now on we fix the following notation.

**Remark 4.1.**  $FG = A_0 \oplus A_1 \oplus \cdots \oplus A_m$ , where the ideal  $A_0 = FGe_0$  and ideals  $A_t \cong M_2(F_t)$ ,  $t = 1, \dots, m$ . For  $t = 1, \dots, m$ , we always assume:

- (1)  $Z_t = Z(A_t)$  which is corresponding to the center  $Z(M_2(F_t))$ , so  $Z_t \cong F_t$  is a field and  $\dim_F Z_t = k_t$ ;
- (2) fix a field  $K_t \subseteq A_t$  which is, by notation of Lemma 3.6, corresponding to the field  $E$  contained in  $M_2(F_t)$  of dimension 2 over  $F_t$ , in particular,  $\dim_F K_t = 2k_t$ ;
- (3)  $C_t$  is the simple left ideal of  $A_t$  corresponding to  $M_2(F_t) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ .

And set:

- (4)  $A = A_1 \oplus \cdots \oplus A_m$ , so  $\dim_F A = 4k_1 + \cdots + 4k_m = 2(n-1)$ ;
- (5)  $Z = Z_1 \oplus \cdots \oplus Z_m$ , so  $\dim_F Z = k_1 + \cdots + k_m = \frac{n-1}{2}$ ;
- (6)  $K = K_1 \oplus \cdots \oplus K_m$ , so  $\dim_F K = 2k_1 + \cdots + 2k_m = n-1$ ;
- (7)  $C = C_1 \oplus \cdots \oplus C_m$ , so  $\dim_F C = 2k_1 + \cdots + 2k_m = n-1$ ;
- (8)  $\widehat{C} = C_0 \oplus C_1 \oplus \cdots \oplus C_m$  where  $C_0 = F\widehat{e}_0$  as described in Lemma 3.2(3), so  $\dim_F \widehat{C} = 1 + 2k_1 + \cdots + 2k_m = n$ .

Then  $C$  and  $\widehat{C}$  are dihedral group codes of rate  $\frac{1}{2} - \frac{1}{2n}$  and  $\frac{1}{2}$ , respectively. The multiplicative group of  $K$ :  $K^\times = K_1^\times \times \cdots \times K_m^\times$ , is not a subgroup of the multiplicative group  $(FG)^\times$ . Let

$$K^* = \{e_0\} \times K^\times = \{e_0\} \times K_1^\times \times \cdots \times K_m^\times, \quad (4.1)$$

where  $\{e_0\}$  is the identity subgroup of  $A_0^\times$ . Then  $K^*$  is a subgroup of  $(FG)^\times$ . Note that, if something within  $A$ , e.g., the code  $C$ , is considered, then the actions of  $K^*$  and  $K^\times$  are the same because  $e_0C = Ce_0 = 0$ .

By Theorem 3.4, for any  $j = 0, 1, \dots, m$ ,  $\overline{1_{A_j}} = 1_{A_j}$ ,

$$\overline{A_j} = \overline{FG \cdot 1_{A_j}} = 1_{A_j} \cdot FG = FG \cdot 1_{A_j} = A_j, \quad j = 0, 1, \dots, m.$$

For any  $0 \leq j \neq j' \leq m$ ,  $A_j \overline{A_{j'}} = A_j A_{j'} = 0$ . So, by Lemma 2.4(5),

$$\langle A_j, A_{j'} \rangle = 0, \quad \forall 0 \leq j \neq j' \leq m. \quad (4.2)$$

**Lemma 4.2.** *We keep the notation of Remark 4.1. Let  $1 \leq t \leq m$ .*

(1) *If  $1_{A_t} = e + \bar{e}$  for a primitive idempotent  $e$  of  $FH$  with  $e \neq \bar{e}$ , then  $C_t \bar{C}_t = 0$  hence  $\langle C_t, C_t \rangle = 0$ .*

(2) *Assume that  $1_{A_t} = e$  for a primitive idempotent  $e$  of  $FH$  with  $e = \bar{e}$ .*

(i) *If  $\text{char } F = 2$ , then  $C_t \bar{C}_t = 0$  hence  $\langle C_t, C_t \rangle = 0$ .*

(ii) *If  $\text{char } F$  is odd, then  $C_t \bar{C}_t \neq 0$  hence  $\langle C_t, C_t \rangle \neq 0$ .*

*Proof.* (1). By Lemma 3.3 and its isomorphism Eq.(3.1),  $e$  is corresponding to  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ , so  $C_t = A_t e$ . Then  $C_t \bar{C}_t = A_t e \bar{e} A_t = 0$ , since  $e \bar{e} = 0$ .

(2). By Eq.(3.2),  $\varepsilon - \nu = \begin{pmatrix} 2 & 0 \\ g & 0 \end{pmatrix}$ , whose first column  $\begin{pmatrix} 2 \\ g \end{pmatrix} \neq 0$ , see the note before Eq.(3.2). So  $M_2(F_t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = M_2(F_t) \begin{pmatrix} 2 & 0 \\ g & 0 \end{pmatrix}$ . By the isomorphism Eq.(3.4),  $e$  and  $ve$  are corresponding to  $\varepsilon$  and  $\nu$  respectively. So  $C_t = A_t(e - ve)$ . By the definition of the “bar” map in Eq.(2.1),  $\bar{v} = v$ . So

$$C_t \bar{C}_t = A_t(e - ve)(e - ve)A_t = A_t(2e - 2ve)A_t.$$

If  $\text{char } F = 2$ , then  $2e - 2ve = 0$ , hence  $C_t \bar{C}_t = 0$ . Thus (i) holds. If  $\text{char } F$  is odd, then  $2e \neq 0$ . Note that  $2e \in FH$ ,  $2ve \in vFH$ ; by Lemma 3.2,  $2e - 2ve \neq 0$ . Hence  $C_t \bar{C}_t \neq 0$ .  $\square$

**Theorem 4.3.** *If  $\text{char } F = 2$ , then for any  $\beta \in K^*$ ,  $\widehat{C}\beta$  is a self-dual dihedral group code.*

*Proof.* First we show that  $\widehat{C}$  is self-dual. For any  $c = c_0 + c_1 + \cdots + c_m$  and  $c' = c'_0 + c'_1 + \cdots + c'_m$ , where  $c_j, c'_j \in C_j$ ,  $j = 0, 1, \dots, m$ , by Eq.(4.2),

$$\langle c_0 + c_1 + \cdots + c_m, c'_0 + c'_1 + \cdots + c'_m \rangle = \langle c_0, c'_0 \rangle + \langle c_1, c'_1 \rangle + \cdots + \langle c_m, c'_m \rangle.$$

By Lemma 2.4(2) and Lemma 3.2(3),  $\langle \widehat{e}_0, \widehat{e}_0 \rangle = \sigma(\widehat{e}_0 \widehat{e}_0) = \sigma(2\widehat{e}_0) = \frac{2}{n} = 0$ , hence  $\langle c_0, c'_0 \rangle = 0$ . By Lemma 4.2,  $\langle c_t, c'_t \rangle = 0$  for  $1 \leq t \leq m$ . That is,  $\langle c, c' \rangle = 0$ . So  $\widehat{C}$  is self-orthogonal. Further, the rate  $R(\widehat{C}) = \frac{1}{2}$ . Thus  $\widehat{C}$  is self-dual.

For the general case, since  $\overline{\widehat{C}\widehat{C}} = 0$  (see Lemma 2.4(5)), we have

$$\overline{\widehat{C}\beta} \cdot \widehat{C}\beta = \overline{\beta\widehat{C}}\widehat{C}\beta = 0, \quad \text{hence } \langle \widehat{C}\beta, \widehat{C}\beta \rangle = 0.$$

And  $R(\widehat{C}\beta) = \frac{1}{2}$ . We obtain that  $\widehat{C}\beta$  is self-dual.  $\square$

The word “module” in this paper means a left module, except for other declarations. Note that  $\widehat{C}$  is an  $FG$ -module.

**Lemma 4.4.** *If  $D$  is an  $FG$ -submodule of  $\widehat{C}$ , then*

$$D = (D \cap C_0) \oplus (D \cap C_1) \oplus \cdots \oplus (D \cap C_m),$$

*and each  $D \cap C_j$  is either 0 or  $C_j$ , for  $j = 0, 1, \dots, m$ .*

*Proof.* The identity  $1_{FG} = e_0 + 1_{A_1} + \cdots + 1_{A_m}$  is a sum of central idempotents, and  $e_0 1_{A_t} = 0$ ,  $1_{A_t} 1_{A_{t'}} = 0$  for  $1 \leq t \neq t' \leq m$ . For any  $d \in D$  we have

$$d = (e_0 + 1_{A_1} + \cdots + 1_{A_m})d = e_0 d + 1_{A_1} d + \cdots + 1_{A_m} d \in (D \cap C_0) \oplus \cdots \oplus (D \cap C_m).$$

So the equality of the lemma holds. Since the  $FG$ -module  $C_j$  is simple,  $D \cap C_j$  is either 0 or  $C_j$ .  $\square$

**Theorem 4.5.** *Assume that  $\text{char } F$  is odd, and  $\beta \in K^*$ .*

(1) *If  $\text{ord}_{\mathbb{Z}_n^\times}(q)$  is odd, then  $C\beta$  is a maximal self-orthogonal code of rate  $\frac{1}{2} - \frac{1}{2n}$ .*

(2) *If  $-1 \in \langle q \rangle_{\mathbb{Z}_n^\times}$ , then  $\widehat{C}\beta$  is an LCD code of rate  $\frac{1}{2}$ .*

*Proof.* (1). By Lemma 2.5(2) and Lemma 4.2(1),  $C_t \overline{C}_t = 0$  for  $t = 1, \dots, m$ . By the same argument as in the proof of Theorem 4.3,  $C\beta$  is a self-orthogonal code of rate  $\frac{1}{2} - \frac{1}{2n}$ . But this time  $\langle \widehat{e}_0, \widehat{e}_0 \rangle = \frac{2}{n} \neq 0$ ,  $C_0 = F\widehat{e}_0$  is not self-orthogonal, hence  $C$  is maximal self-orthogonal.

(2). Write  $\beta = e_0 + \beta_1 + \cdots + \beta_m$ , where  $\beta_t \in K_t^\times$  for  $t = 1, \dots, m$ . Then

$$\widehat{C}\beta = C_0 \oplus C_1 \beta_1 \oplus \cdots \oplus C_m \beta_m.$$

As shown above,  $C_0$  is not self-orthogonal. For  $1 \leq t \leq m$ , by Lemma 2.5(3) and Lemma 4.2(2),  $C_t \overline{C}_t \neq 0$  (i.e.  $\overline{C}_t C_t \neq 0$ ); hence

$$\overline{C}_t \beta_t \cdot C_t \beta_t = \overline{\beta}_t \overline{C}_t C_t \beta \neq 0,$$

i.e.,  $C_t \beta_t$  is not self-orthogonal. Denote  $D = (\widehat{C}\beta) \cap (\widehat{C}\beta)^\perp$ . By Lemma 4.4,  $D = (D \cap C_0) \oplus \bigoplus_{t=1}^m (D \cap C_t \beta_t)$ . But,  $D \cap C_0$ ,  $D \cap C_t \beta_t$  must be self-orthogonal, hence  $D \cap C_0 \neq C_0$ ,  $D \cap C_t \beta_t \neq C_t \beta_t$ . By Lemma 4.4,  $D \cap C_0 = 0$ ,  $D \cap C_t \beta_t = 0$ . Then  $D = 0$ , and  $\widehat{C}\beta$  is an LCD code.  $\square$

## 5 Random dihedral codes of rate $\frac{1}{2} - \frac{1}{2n}$

Keep the assumptions in Remark 3.1 and Remark 4.1. For  $k_t$  in Remark 4.1(1), we further assume that  $k_1 \leq k_2 \leq \cdots \leq k_m$ . By Corollary 3.5,  $2k_1 \geq \lambda(n)$ . By Lemma 2.5(1) and Lemma 2.6, we can further assume  $\lambda(n) > \log_q n$ . Hence, in the following we always assume that

$$\frac{1}{2} \log_q n < \frac{1}{2} \lambda(n) \leq k_1 \leq k_2 \leq \cdots \leq k_m. \quad (5.1)$$

From now on, let  $\delta$  be a real number satisfying that

$$0 < \delta < 1 - q^{-1}, \quad \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{\lambda(n)} > 0. \quad (5.2)$$

Note that, if  $h_q(\delta) < \frac{1}{4}$ , by Lemma 2.5(1) and Lemma 2.6, there are infinitely many odd integers  $n > 1$  coprime to  $q$  such that  $\frac{1}{4} - h_q(\delta) - \frac{\log_q n}{\lambda(n)}$  are positively bounded from below.

For any left ideal  $L$  of  $FG$  and any  $(\alpha, \beta) \in K^* \times K^*$ ,  $\alpha$  is a unit of  $FG$ , see Eq.(4.1); so  $(FG)\alpha = FG = \alpha(FG)$ , hence  $\alpha L\beta = \alpha \cdot FG \cdot L\beta = FG \cdot L\beta = L\beta$  is a left ideal.

**Definition 5.1.** Consider  $K^* \times K^*$  as a probability space with equal probability for every sample. Let  $(\alpha, \beta) \in K^* \times K^*$ . We have the following:

- (1)  $C_{\alpha, \beta} := \alpha C\beta = C\beta$  is a random  $FG$ -code with rate  $R(C_{\alpha, \beta}) = \frac{1}{2} - \frac{1}{2n}$ .
- (2)  $\Delta(C_{\alpha, \beta}) = \frac{w(\alpha C\beta)}{2n}$  is a random variable.
- (3) For  $c \in C$ , define a 0-1 variable:  $X_c = \begin{cases} 1, & 0 < \frac{w(\alpha c\beta)}{2n} \leq \delta; \\ 0, & \text{otherwise.} \end{cases}$
- (4) Let  $X = \sum_{c \in C} X_c$ , which stands for the number of the non-zero code-words  $\alpha c\beta$  whose relative weights are at most  $\delta$ .

By  $\Pr(\Delta(C_{\alpha, \beta}) \leq \delta)$  we denote the probability that  $\Delta(C_{\alpha, \beta}) \leq \delta$ , and by  $E(X)$  we denote the expectation of the random variable  $X$ . Then

$$\Pr(\Delta(C_{\alpha, \beta}) \leq \delta) = \Pr(X \geq 1).$$

By Markov's inequality (c.f. [22, Theorem 3.1]), for the non-negative integer variable  $X$  we have  $\Pr(X \geq 1) \leq E(X)$ . So

$$\Pr(\Delta(C_{\alpha, \beta}) \leq \delta) \leq E(X). \quad (5.3)$$

If  $c = 0$  then  $X_0 = 0$  obviously. By the linearity of expectations,

$$E(X) = \sum_{c \in C} E(X_c) = \sum_{0 \neq c \in C} E(X_c). \quad (5.4)$$

Since  $X_c$  is a 0-1 variable,

$$E(X_c) = \Pr(X_c = 1) = \Pr\left(0 < \frac{w(\alpha c\beta)}{2n} \leq \delta\right). \quad (5.5)$$

We estimate  $E(X_c)$  for  $0 \neq c \in C$ . Set  $C_t^+ = C_t \setminus \{0\}$ ,  $t = 1, \dots, m$ . For the non-zero  $c \in C$ , there is a subset  $\omega = \{t_1, \dots, t_r\} \subseteq \{1, 2, \dots, m\}$  such that

$$c = c_{t_1} + c_{t_2} + \dots + c_{t_r}, \quad c_{t_j} \in C_{t_j}^+ = C_{t_j} \setminus \{0\}, \quad j = 1, \dots, r. \quad (5.6)$$

Then  $Ac = A_{t_1}c_{t_1} \oplus \dots \oplus A_{t_r}c_{t_r}$ . Since  $A_{t_j}c_{t_j} \neq 0$  is a submodule of  $C_{t_j}$  and  $C_{t_j}$  is simple, we have  $A_{t_j}c_{t_j} = C_{t_j}$ . So,

$$Ac = C_{t_1} \oplus \dots \oplus C_{t_r}, \quad (5.7)$$

and  $\dim_F(Ac) = 2k_{t_1} + \dots + 2k_{t_r}$  is even. Denote

$$\ell_c = \frac{\dim_F(Ac)}{2} = k_{t_1} + \dots + k_{t_r}, \quad (5.8)$$

then  $k_1 \leq \ell_c \leq \frac{n-1}{2}$  (cf. Eq.(5.1) and Remark 4.1(5)).

**Lemma 5.2.** *Let the notation be as above. Then  $E(X_c) < q^{-3\ell_c+4\ell_ch_q(\delta)+4}$ .*

*Proof.* Let  $\tilde{\omega} = \{1, 2, \dots, m\} \setminus \omega = \{1, 2, \dots, m\} \setminus \{t_1, t_2, \dots, t_r\}$ . Let

$$\begin{aligned} A_\omega &= A_{t_1} \oplus \dots \oplus A_{t_r}, & A_{\tilde{\omega}} &= \bigoplus_{t \in \tilde{\omega}} A_t, & \text{hence } A &= A_\omega \oplus A_{\tilde{\omega}}; \\ K_\omega^\times &= K_{t_1}^\times \times \dots \times K_{t_r}^\times, & K_{\tilde{\omega}}^\times &= \prod_{t \in \tilde{\omega}} K_t^\times, & \text{hence } K^\times &= K_\omega^\times \times K_{\tilde{\omega}}^\times; \\ Z_\omega^\times &= Z_{t_1}^\times \times \dots \times Z_{t_r}^\times. \end{aligned}$$

For  $(\alpha, \beta), (\alpha', \beta') \in K^* \times K^*$ , by Eq.(4.1), we can write  $\alpha = e_0 + \alpha_\omega + \alpha_{\tilde{\omega}}$  with  $\alpha_\omega \in K_\omega^\times$  and  $\alpha_{\tilde{\omega}} \in K_{\tilde{\omega}}^\times$ ; since  $e_0c = 0 = ce_0$ ,

$$\alpha c \beta = (\alpha_\omega + \alpha_{\tilde{\omega}})c(\beta_\omega + \beta_{\tilde{\omega}}) = \alpha_\omega c \beta_\omega, \quad \text{and} \quad \alpha' c \beta' = \alpha'_\omega c \beta'_\omega.$$

By Lemma 3.6(2),  $\alpha c \beta = \alpha' c \beta'$  if and only if  $\alpha'_\omega^{-1} \alpha_\omega = \beta'_\omega \beta_\omega^{-1} \in Z_\omega^\times$ , if and only if there are  $z_\omega \in Z_\omega^\times$  and  $(\alpha'_{\tilde{\omega}}, \beta'_{\tilde{\omega}}) \in K_{\tilde{\omega}}^\times \times K_{\tilde{\omega}}^\times$  such that

$$\alpha' = \alpha_\omega z_\omega^{-1} + \alpha'_{\tilde{\omega}}, \quad \beta' = \beta_\omega z_\omega + \beta'_{\tilde{\omega}}.$$

So, for  $d \in K^* c K^*$ , there are exactly  $|Z_\omega^\times| \cdot |K_{\tilde{\omega}}^\times|^2$  pairs  $(\alpha, \beta)$  in  $K^* \times K^*$  such that  $\alpha c \beta = d$ . Since

$$K^* c K^* = K^\times c K^\times = K_{t_1}^\times c_{t_1} K_{t_1}^\times \times \dots \times K_{t_r}^\times c_{t_r} K_{t_r}^\times \subseteq A_{t_1} \oplus \dots \oplus A_{t_r} = A_\omega,$$

and  $A_\omega$  is an ideal in  $FG$  of dimension  $4\ell_c$  over  $F$ , we get

$$|(K^* c K^*)^{\leq \delta}| \leq |(A_\omega)^{\leq \delta}| \leq q^{4\ell_ch_q(\delta)},$$

where the last inequality follows from Lemma 2.2. Thus, there are at most  $|Z_\omega^\times| \cdot |K_{\tilde{\omega}}^\times|^2 \cdot q^{4\ell_ch_q(\delta)}$  pairs  $(\alpha, \beta) \in K^* \times K^*$  such that  $0 < \frac{w(\alpha c \beta)}{2n} \leq \delta$ . By Eq.(5.5), we obtain that

$$E(X_c) \leq \frac{|Z_\omega^\times| \cdot |K_{\tilde{\omega}}^\times|^2 \cdot q^{4\ell_ch_q(\delta)}}{|K^* \times K^*|} = \frac{|Z_\omega^\times| \cdot q^{4\ell_ch_q(\delta)}}{|K_{\tilde{\omega}}^\times|^2}.$$

We estimate the cardinalities  $|Z_\omega^\times|$  and  $|K_{\tilde{\omega}}^\times|$  as follows:

$$\begin{aligned} |Z_\omega^\times| &= \prod_{j=1}^r (q^{k_{t_j}} - 1) < q^{k_{t_1}} \dots q^{k_{t_r}} = q^{k_{t_1} + \dots + k_{t_r}} = q^{\ell_c}, \\ |K_{\tilde{\omega}}^\times| &= \prod_{j=1}^r (q^{2k_{t_j}} - 1) > q^{2(k_{t_1} + \dots + k_{t_r}) - 2} = q^{2\ell_c - 2}; \end{aligned}$$

where the second inequality follows from Lemma 2.9 and Eq.(5.1). Then

$$E(X_c) \leq \frac{q^{\ell_c} \cdot q^{4\ell_ch_q(\delta)}}{(q^{2\ell_c - 2})^2} = q^{-3\ell_c + 4\ell_ch_q(\delta) + 4}. \quad \square$$

By Lemma 4.4, the following  $\Omega$  is the set of all  $A$ -submodules of  $C$ :

$$\Omega = \{C_{t_1} \oplus \dots \oplus C_{t_r} \mid \{t_1, \dots, t_r\} \subseteq \{1, 2, \dots, m\}\}. \quad (5.9)$$



**Lemma 5.3.** For  $D = C_{t_1} \oplus \cdots \oplus C_{t_r} \in \Omega$ , let  $D^+ = C_{t_1}^+ \oplus \cdots \oplus C_{t_r}^+$  where  $C_{t_j}^+ = C_{t_j} \setminus \{0\}$  as before. For  $k_1 \leq \ell \leq \frac{n-1}{2}$ , let  $\Omega_\ell = \{D \in \Omega \mid \dim_F D = 2\ell\}$  (it is possible that  $\Omega_\ell = \emptyset$ ). Then

- (1)  $\Omega = \bigcup_{\ell=k_1}^{(n-1)/2} \Omega_\ell$ , and  $C \setminus \{0\} = \bigcup_{\ell=k_1}^{(n-1)/2} \bigcup_{D \in \Omega_\ell} D^+$ .
- (2)  $|\Omega_\ell| \leq n^{\ell/k_1}$ .

*Proof.* (1) is proved directly.

If  $C_{t_1} \oplus \cdots \oplus C_{t_r} \in \Omega_\ell$ , then  $k_{t_1} + \cdots + k_{t_r} = \ell$ ; in particular,  $r \leq \ell/k_1$ . Thus

$$|\Omega_\ell| \leq \sum_{j=1}^{\ell/k_1} \binom{m}{j} \leq \sum_{j=1}^{\ell/k_1} \binom{n}{j} \leq n^{\ell/k_1}. \quad \square$$

**Theorem 5.4.**  $E(X) < q^{-4k_1 \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{2k_1} \right) + 4}$ .

*Proof.* By Eq.(5.4) and Lemma 5.3(1), we have

$$E(X) = \sum_{0 \neq c \in C} E(X_c) = \sum_{\ell=k_1}^{(n-1)/2} \sum_{D \in \Omega_\ell} \sum_{c \in D^+} E(X_c).$$

For  $D \in \Omega_\ell$  and  $c \in D^+$ ,  $\ell_c = \frac{1}{2} \dim_F D = \ell$ , see Eq.(5.7) and Eq.(5.8). By Lemma 5.2 and Lemma 5.3(2),

$$\begin{aligned} \sum_{D \in \Omega_\ell} \sum_{c \in D^+} E(X_c) &< \sum_{D \in \Omega_\ell} \sum_{c \in D^+} q^{-3\ell + 4\ell h_q(\delta) + 4} \\ &< \sum_{D \in \Omega_\ell} q^{2\ell} \cdot q^{-3\ell + 4\ell h_q(\delta) + 4} \leq n^{\frac{\ell}{k_1}} q^{-\ell + 4\ell h_q(\delta) + 4} \\ &= q^{-4\ell \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{4k_1} \right) + 4} \leq q^{-4k_1 \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{4k_1} \right) + 4}. \end{aligned}$$

The last inequality holds since  $\ell \geq k_1$  (by Eq.(5.8)) and  $\frac{1}{4} - h_q(\delta) - \frac{\log_q n}{4k_1} > 0$  (by Eq.(5.2)). Further,  $\frac{n-1}{2} - k_1 + 1 \leq n = q^{\log_q n}$ . So

$$\begin{aligned} E(X) &< \sum_{\ell=k_1}^{(n-1)/2} q^{-4k_1 \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{4k_1} \right) + 4} \\ &= \left( \frac{n-1}{2} - k_1 + 1 \right) q^{-4k_1 \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{4k_1} \right) + 4} \\ &\leq q^{\log_q n} q^{-4k_1 \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{4k_1} \right) + 4}. \end{aligned}$$

That is,  $E(X) < q^{-4k_1 \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{2k_1} \right) + 4}$ .  $\square$

**Theorem 5.5.**  $\Pr(\Delta(C_{\alpha, \beta}) \leq \delta) < q^{-2\lambda(n) \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{\lambda(n)} \right) + 4}$ .

*Proof.* Combining Theorem 5.4 with Eq.(5.3), we get

$$\Pr(\Delta(C_{\alpha, \beta}) \leq \delta) < q^{-4k_1 \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{2k_1} \right) + 4}.$$

By Corollary 3.5,  $2k_1 \geq \lambda(n)$ ; and by Eq.(5.2),

$$\frac{1}{4} - h_q(\delta) - \frac{\log_q n}{2k_1} > \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{\lambda(n)} > 0.$$

we get that

$$q^{-4k_1\left(\frac{1}{4}-h_q(\delta)-\frac{\log_q n}{2k_1}\right)+4} \leq q^{-2\lambda(n)\left(\frac{1}{4}-h_q(\delta)-\frac{\log_q n}{\lambda(n)}\right)+4}. \quad \square$$

## 6 Random dihedral codes of rate $\frac{1}{2}$

Keep the notation in §5. In particular, Eq.(5.1), Eq.(5.2) hold and  $K^* \times K^*$  is considered as a probability space with equal probability for each sample. We start from  $\widehat{C} = C_0 \oplus C$  where  $C_0 = F\widehat{e}_0$ , see Remark 4.1(8). Then

$$\widehat{C}_{\alpha,\beta} = \alpha\widehat{C}\beta, \quad (\alpha, \beta) \in K^* \times K^*, \quad (6.1)$$

is a random code with  $R(\widehat{C}_{\alpha,\beta}) = \frac{1}{2}$ . Define

$$\widehat{X}_c = \begin{cases} 1, & 0 < \frac{w(\alpha c \beta)}{2n} \leq \delta; \\ 0, & \text{otherwise;} \end{cases} \quad c \in \widehat{C}; \quad \text{and} \quad \widehat{X} = \sum_{c \in \widehat{C}} \widehat{X}_c.$$

We still have

$$\Pr(\Delta(\widehat{C}_{\alpha,\beta}) \leq \delta) = \Pr(\widehat{X} \geq 1) \leq E(\widehat{X}). \quad (6.2)$$

Recall that  $\Omega = \bigcup_{\ell=k_1}^{(n-1)/2} \Omega_\ell$  is the set of all non-zero submodules of  $C$ , see Eq.(5.9) and Lemma 5.3. It is easy to check the following.

**Lemma 6.1.** *Denote  $C_0 \oplus \Omega = \{C_0 \oplus D \mid D \in \Omega\}$ . For  $D' = C_0 \oplus D$  with  $D \in \Omega$ , let  $D'^+ = C_0^+ \oplus D^+$  ( $D^+$  is defined in Lemma 5.3). Then*

$$\widehat{C} \setminus \{0\} = C_0^+ \cup \left( \bigcup_{D \in \Omega} D^+ \right) \cup \left( \bigcup_{D' \in C_0 \oplus \Omega} D'^+ \right).$$

We already have the estimation of  $\sum_{D \in \Omega} \sum_{c \in D^+} E(\widehat{X}_c)$ , see Theorem 5.4.

For  $0 \neq c \in C_0$  and  $(\alpha, \beta) \in K^* \times K^*$ , it is trivial that  $\alpha c \beta = c$  and  $\frac{w(\alpha c \beta)}{2n} = 1$ . So  $E(\widehat{X}_c) = 0$ . Hence

$$\sum_{c \in C_0^+} E(\widehat{X}_c) = 0. \quad (6.3)$$

**Lemma 6.2.**  $\sum_{D' \in (C_0 \oplus \Omega)} \sum_{c \in D'^+} E(\widehat{X}_c) < q^2 q^{-4k_1\left(\frac{1}{4}-h_q(\delta)-\frac{\log_q n}{2k_1}\right)+4}$ .

*Proof.* For  $k_1 \leq \ell \leq \frac{n-1}{2}$ , let  $D' = C_0 \oplus D$  with  $D \in \Omega_\ell$ , and let  $c' \in D'^+$ . Similarly to the proof of Lemma 5.2, we assume that  $\omega = \{t_1, \dots, t_r\} \subseteq \{1, \dots, m\}$  such that

$$c' = c_0 + c_{t_1} + \dots + c_{t_r}, \quad c_0 \in C_0^+, \quad c_{t_j} \in C_{t_j}^+, \quad j = 1, \dots, r;$$

and construct

$$A'_\omega = C_0 \oplus A_{t_1} \oplus \cdots \oplus A_{t_r}, \quad K'_\omega{}^\times = \{e_0\} \times K_{t_1}^\times \times \cdots \times K_{t_r}^\times.$$

It is the same as in the proof of Lemma 5.2, except that  $\dim_F(A'_\omega) = 4\ell + 1$ , hence

$$|(K^* c' K^*)^{\leq \delta}| \leq |(A'_\omega)^{\leq \delta}| \leq q^{(4\ell+1)h_q(\delta)}.$$

We obtain

$$\mathbb{E}(\widehat{X}_c) < q^{-3\ell+4\ell h_q(\delta)+h_q(\delta)+4} < q^{-3\ell+4\ell h_q(\delta)+5}.$$

Because  $|D'^+| < |D'| = q^{2\ell+1}$ ,

$$\sum_{c \in D'^+} \mathbb{E}(\widehat{X}_c) < q^{2\ell+1} q^{-3\ell+4\ell h_q(\delta)+5} = q^{-\ell+4\ell h_q(\delta)+6}.$$

Then, similarly to Theorem 5.4, we obtain

$$\begin{aligned} \sum_{D' \in (C_0 \oplus \Omega)} \sum_{c \in D'^+} \mathbb{E}(\widehat{X}_c) &= \sum_{\ell=k_1}^{(n-1)/2} \sum_{D' \in (C_0 \oplus \Omega_\ell)} \sum_{c \in D'^+} \mathbb{E}(\widehat{X}_c) \\ &< q^{-4k_1 \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{2k_1} \right) + 6}. \end{aligned}$$

That is,

$$\sum_{D' \in (C_0 \oplus \Omega)} \sum_{c \in D'^+} \mathbb{E}(\widehat{X}_c) < q^2 q^{-4k_1 \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{2k_1} \right) + 4}. \quad \square$$

**Theorem 6.3.**  $\mathbb{E}(\widehat{X}) < (1 + q^2) q^{-4k_1 \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{2k_1} \right) + 4}$ .

*Proof.* By Lemma 6.1, Eq.(6.3), Theorem 5.4 and Lemma 6.2,

$$\begin{aligned} \mathbb{E}(\widehat{X}) &= \sum_{0 \neq c \in \widehat{C}} \mathbb{E}(\widehat{X}_c) \\ &= \sum_{c \in C_0^+} \mathbb{E}(\widehat{X}_c) + \sum_{D \in \Omega} \sum_{c \in D^+} \mathbb{E}(\widehat{X}_c) + \sum_{D' \in (C_0 \oplus \Omega)} \sum_{c \in D'^+} \mathbb{E}(\widehat{X}_c) \\ &< 0 + q^{-4k_1 \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{2k_1} \right) + 4} + q^2 q^{-4k_1 \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{2k_1} \right) + 4} \\ &= (1 + q^2) q^{-4k_1 \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{2k_1} \right) + 4}. \end{aligned}$$

We are done. □

Similarly to Theorem 5.5, we obtain:

**Theorem 6.4.**  $\Pr(\Delta(\widehat{C}_{\alpha, \beta}) \leq \delta) < (1 + q^2) q^{-2\lambda(n) \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n}{\lambda(n)} \right) + 4}$ .

## 7 Proofs of the main theorems

For a sequence  $n_1, n_2, \dots$  of odd positive integers  $n_i$  coprime to  $q$  with  $n_i \rightarrow \infty$ , we have a sequence  $G^{(1)}, G^{(2)}, \dots$  of dihedral groups  $G^{(i)}$  of order  $2n_i$ , and have random  $FG^{(i)}$ -codes:

- $C_{\alpha,\beta}^{(i)}$  of rate  $\frac{1}{2} - \frac{1}{2n_i}$ , defined in Definition 5.1;
- $\widehat{C}_{\alpha,\beta}^{(i)}$  of rate  $\frac{1}{2}$ , defined in Eq.(6.1);

hence we have two sequences of random dihedral codes:

$$C_{\alpha,\beta}^{(1)}, C_{\alpha,\beta}^{(2)}, C_{\alpha,\beta}^{(3)}, \dots; \quad (7.1)$$

$$\widehat{C}_{\alpha,\beta}^{(1)}, \widehat{C}_{\alpha,\beta}^{(2)}, \widehat{C}_{\alpha,\beta}^{(3)}, \dots. \quad (7.2)$$

**Theorem 7.1.** *Assume that  $0 < \delta < 1 - q^{-1}$  and  $0 < h_q(\delta) < \frac{1}{4}$ . Assume that  $\text{char } F = 2$ . Then there is a sequence  $n_1, n_2, \dots$  of odd integers  $n_i$  coprime to  $q$  with  $n_i \rightarrow \infty$  such that*

- (1) *The sequence in Eq.(7.2) consists of self-dual dihedral codes;*
- (2)  $\lim_{i \rightarrow \infty} \Pr(\Delta(\widehat{C}_{\alpha,\beta}^{(i)}) > \delta) = 1$ .

*Proof.* By Lemma 2.6, there is a series  $n_1, n_2, \dots$  of odd integers coprime to  $q$  such that  $\lim_{i \rightarrow \infty} \frac{\log_q n_i}{\lambda(n_i)} = 0$ . Then (1) follows from Theorem 4.3. And, since  $\frac{1}{4} - h_q(\delta) - \frac{\log_q n_i}{\lambda(n_i)} > 0$  and  $\lambda(n_i) \rightarrow \infty$ , by Theorem 6.4,

$$\lim_{i \rightarrow \infty} \Pr(\Delta(\widehat{C}_{\alpha,\beta}^{(i)}) \leq \delta) < \lim_{i \rightarrow \infty} (1 + q^2)q^{-2\lambda(n_i)(\frac{1}{4} - h_q(\delta) - \frac{\log_q n_i}{\lambda(n_i)}) + 4} = 0.$$

That is, (2) holds.  $\square$

Theorem 1.1 is obviously a consequence of Theorem 7.1. On the other hand, Theorem 1.2 is a consequence of the following theorem.

**Theorem 7.2.** *Assume that  $0 < \delta < 1 - q^{-1}$  and  $0 < h_q(\delta) < \frac{1}{4}$ . Assume that  $\text{char } F$  is odd.*

(1) *There is a sequence  $n_1, n_2, \dots$  of odd integers  $n_i$  coprime to  $q$  with  $n_i \rightarrow \infty$  such that Eq.(7.1) is a sequence of maximal self-orthogonal dihedral codes of rate  $\frac{1}{2} - \frac{1}{2n_i}$  and  $\lim_{i \rightarrow \infty} \Pr(\Delta(C_{\alpha,\beta}^{(i)}) > \delta) = 1$ .*

(2) *There is a sequence  $n_1, n_2, \dots$  of odd integers  $n_i$  coprime to  $q$  with  $n_i \rightarrow \infty$  such that Eq.(7.2) is a sequence of LCD dihedral codes of rate  $\frac{1}{2}$  and  $\lim_{i \rightarrow \infty} \Pr(\Delta(\widehat{C}_{\alpha,\beta}^{(i)}) > \delta) = 1$ .*

*Proof.* (1). By Corollary 2.8(1), there is a sequence  $n_1, n_2, \dots$  of odd integers  $n_i$  coprime to  $q$  such that  $\text{ord}_{\mathbb{Z}_{n_i}^\times}(q)$  are all odd and  $\lim_{i \rightarrow \infty} \frac{\log_q n_i}{\lambda(n_i)} = 0$ . By Theorem 4.5(1), Eq.(7.1) is a sequence of maximal self-orthogonal dihedral codes of

rate  $\frac{1}{2} - \frac{1}{2n_i}$ . By Theorem 5.5,

$$\lim_{i \rightarrow \infty} \Pr(\Delta(C_{\alpha, \beta}^{(i)}) \leq \delta) < \lim_{i \rightarrow \infty} q^{-2\lambda(n_i) \left( \frac{1}{4} - h_q(\delta) - \frac{\log_q n_i}{\lambda(n_i)} \right) + 4} = 0.$$

(2). The proof is similar to the above by citing Corollary 2.8(2), Theorem 4.5(2) and Theorem 6.4.  $\square$

## 8 Conclusion

We decompose the group algebra of a finite dihedral group of order  $2n$  with  $n$  being odd over any finite field  $F$  into an orthogonal direct sum of a special component of dimension 2 and some  $2 \times 2$  matrix algebras. With the structure we find two kinds of random dihedral group codes. The random dihedral codes are self-dual, or maximal self-orthogonal, or LCD under different conditions. And the random dihedral codes have nice asymptotic behavior so that, suitably choosing a positive real number  $\delta$  and the code lengths  $2n_1, 2n_2, \dots$  going to infinity, we proved that the probability for the relative minimum distance of the random dihedral codes greater than  $\delta$  is convergent to 1.

As consequences, if  $\text{char } F = 2$ , then self-dual dihedral codes are asymptotically good. In the case that  $\text{char } F$  is odd, there exist asymptotically good maximal self-orthogonal dihedral codes of rate tending to  $\frac{1}{2}$ ; and, LCD dihedral codes of rate  $\frac{1}{2}$  are asymptotically good.

In the case that  $\text{char } F = 2$ , both the two kinds of random dihedral codes we discussed are not LCD. Though the LCD dihedral codes exist in that case, for the moment we have no good idea to study their asymptotic behavior, and we guess that they probably have no good asymptotic property.

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