Turbo Decoding on the Binary Erasure Channel: Finite-Length Analysis and Turbo Stopping Sets

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Abstract

This paper is devoted to the finite-length analysis of turbo decoding over the binary erasure channel (BEC). The performance of iterative belief-propagation (BP) decoding of low-density parity-check (LDPC) codes over the BEC can be characterized in terms of *stopping sets*. We describe turbo decoding on the BEC which is simpler than turbo decoding on other channels. We then adapt the concept of stopping sets to turbo decoding and state an exact condition for decoding failure. Apply turbo decoding until the transmitted codeword has been recovered, or the decoder fails to progress further. Then the set of erased positions that will remain when the decoder stops is equal to the unique maximum-size *turbo stopping set* which is also a subset of the set of erased positions. Furthermore, we present some improvements of the basic turbo decoding algorithm on the BEC. The proposed improved turbo decoding algorithm has substantially better error performance as illustrated by the given simulation results. Finally, we give an expression for the turbo stopping set size enumerating function under the uniform interleaver assumption, and an efficient enumeration algorithm of small-size turbo stopping sets for a particular interleaver. The solution is based on the algorithm proposed by Garello *et al.* in 2001 to compute an exhaustive list of *all* low-weight codewords in a turbo code.

Index Terms

Binary erasure channel, improved decoding, stopping set, turbo decoding, uniform interleaver, weight spectrum.

I. INTRODUCTION

Low-density parity-check (LDPC) codes as opposed to turbo codes have been studied extensively on the binary erasure channel (BEC). In [1], an iterative decoding algorithm for LDPC codes over the BEC was proposed, and it was shown that this scheme approaches channel capacity arbitrarily close. Although carefully optimized irregular LDPC codes with iterative decoding can achieve channel capacity on the BEC as the code length tends to infinity,

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there is still some performance loss compared to maximum-likelihood (ML) decoding of a given fixed code of finite length. Recently, some progress has been made towards efficient ML or near ML decoding of LDPC codes over the BEC [2]–[4]. The Tanner graph representation of an LDPC code is a bipartite graph with left and right nodes. The left nodes correspond to codeword bits. The right nodes correspond to parity-check constraints. It is known that iterative belief-propagation (BP) decoding fails if and only if the set of erased bit-positions contains a *stopping set* [5]. A stopping set is a subset of the bit-positions such that the corresponding left nodes in the Tanner graph have the property that all neighboring nodes are connected to the set at least twice.

In this work we consider turbo decoding over the BEC. Turbo codes have gained considerable attention since their introduction by Berrou *et al.* [6] in 1993 due to their near-capacity performance and low decoding complexity. Here we consider the conventional turbo code which is the parallel concatenation of two identical recursive systematic convolutional encoders separated by a pseudo-random interleaver. To accurately describe turbo decoding on the BEC we introduce the concept of a *turbo stopping set*, and we identify an exact condition for decoding failure. Assume that we transmit codewords of a turbo code over the BEC. Apply turbo decoding until either the codeword has been recovered, or the decoder fails to progress further. Then the set of erased positions that will remain when the decoder stops is equal to the unique maximum-size turbo stopping set which is also a subset of the set of erased positions. We also consider improved turbo decoding on the BEC. The algorithm applies turbo decoding until the transmitted codeword is recovered, or the decoder fails to progress further. Then, an unknown (systematic) bit-position is identified and its value is guessed, after which turbo decoding is applied again. Thus, the algorithm is based on guessing bit-values in erased bit-positions when turbo decoding does not progress further and has the same *structure* as the algorithms in [4].

Recently, several algorithms have been introduced to compute the first few terms of the weight distribution of both parallel and serial turbo codes. Both exact algorithms (e.g., [7]-[9]) and approximate algorithms (e.g., [10]-[12]) have been presented. In this work we also show that basically all *trellis-based* algorithms can be adapted to find the first few terms of the turbo stopping set size enumerating function. In particular, we have considered in detail how to adapt the algorithm by Garello *et al.* introduced in [8] and the improved algorithm in [7]. Also, an expression for the (average) turbo stopping set size enumerating function under the uniform interleaver assumption is presented.

Using linear programming (LP) to decode binary linear codes has recently been considered by Feldman *et al.* [13]. See also the seminal papers [14], [15] where LP decoding of turbo-like codes is considered. In particular, repeat-accumulate (RA) codes are considered. A description of LP decoding of arbitrary concatenated codes is given in [16, Ch. 6]. The obvious polytope for LP decoding is the convex hull of all codewords, in which case LP decoding is equivalent to ML decoding. However, the convex hull has a description complexity that is exponential in the codeword length for a general binary linear code. Thus, Feldman *et al.* [13] proposed a relaxed polytope which contains all valid codewords as vertices, but also additional non-codeword vertices. The vertices of the relaxed polytope are basically what the authors called *pseudo-codewords* in [13]. One desirable property of the LP decoder is the ML certificate property, i.e., when the LP decoder outputs a codeword, it is guaranteed to be the ML

codeword. Experimental results with LDPC codes show that the performance of the relaxed LP decoder is better than with the iterative min-sum algorithm, but slightly worse than with iterative BP decoding.

Recently, some understanding of the performance of iterative BP decoding of finite-length LDPC codes over general memoryless channels have been developed. Finite graph covers of the Tanner graph and the codes defined by these covers play an essential role in the analysis [17]. The low complexity of iterative BP decoding is due to the fact that the algorithm operates *locally* on the Tanner graph of the code. This property is also the main weakness of iterative BP decoding, since the decoder cannot distinguish if it operates on the original Tanner graph or on any of the finite covers. Hence, codewords in the code defined by a finite cover of the Tanner graph will influence on iterative decoding. These codewords are basically what Vontobel and Koetter referred to as pseudo-codewords in [17]. It turns out that the set of pseudo-codewords of all finite covers of the Tanner graph is equal to the set of points where all entries are rational numbers from the relaxed polytope of LP decoding of LDPC codes as introduced by Feldman et al. in [13]. A similar connection between the relaxed polytope of LP decoding of turbo codes, as described in [16, Ch. 6], and the pseudo-codewords of all finite covers of the turbo code factor graph [18] was established in [19]. Furthermore, Rosnes also showed in [19] that there is a many-to-one correspondence between these pseudo-codewords and turbo stopping sets in the following sense. The support set of any pseudo-codeword, i.e., the set of non-zero coordinates, is a turbo stopping set, and for any turbo stopping set there is a pseudocodeword with support set equal to the turbo stopping set. A similar connection also holds for pseudo-codewords of finite graph covers of Tanner graphs and stopping sets [17], [20].

For LDPC codes it has been observed that stopping sets, to some degree, also reflect the performance of iterative decoding for other channels than the BEC [17], [21]. It is therefore our hope that the notion of turbo stopping sets also can provide some useful insight into turbo decoding on the additive white Gaussian noise (AWGN) channel or on other memoryless channels. In [19], Rosnes presented some simulation results to indicate that this may be the case.

This paper is organized as follows. In Section II we define some basic notation and describe simplified turbo decoding on the BEC. Section III introduces the concept of a turbo stopping set. We further give some of the basic properties and show that turbo stopping sets characterize exactly the performance of turbo decoding on the BEC. An improved turbo decoding algorithm on the BEC is introduced in Section IV, and its superiority compared to conventional turbo decoding is illustrated by simulation examples. Finally, in Section V, we consider enumeration of small-size turbo stopping sets for a particular interleaver and under the uniform interleaver assumption. Conclusions and a discussion of future work are given in Section VI.

II. PRELIMINARIES

In this section we introduce the channel, define some basic notation, and describe simplified turbo decoding on the BEC.

A. The BEC

The BEC model was introduced by Elias [22] in 1955. The channel has recently been used for modeling information transmission over the Internet. The BEC is a two-input, three-output discrete memoryless channel. Each input bit is erased with probability ϵ , or received correctly with probability $1 - \epsilon$.

B. Some Definitions and Basic Notation

Let $C = C(K, C_a, C_b, \pi)$ denote a parallel concatenated convolutional code (PCCC), or turbo code, with information length K, constituent encoders C_a and C_b of rate R = k/n, and interleaver π . In this work we consider dual termination [23], which implies that the length of the interleaver $I = K + 2\nu$ where ν is the constraint length. We assume here that I is a multiple of k. Let N_a and N_b denote the lengths of the constituent codes. The length of the turbo code is denoted by N. For an unpunctured turbo code, $N_a = N_b = I/R$ and N = (2/R - 1)I. In general, the values of N_a , N_b , and N depend on the puncturing pattern P and the termination scheme.

We will now define some useful mappings, but we advise the reader that the formal definitions below will be easier to understand after taking a look at Fig. 1.

Define two mappings $\mu_a : \mathcal{Z}_N \to \mathcal{Z}_{N_a} \cup \{*\}$ and $\mu_b : \mathcal{Z}_N \to \mathcal{Z}_{N_b} \cup \{*\}$ where $\mathcal{Z}_N = \{0, 1, \dots, N-1\}$ for a positive integer N. The mapping μ_a gives the index in the first constituent codeword of the turbo codeword index if such a relation exists, or * if not. Similarly, the mapping μ_b gives the index in the second constituent codeword of the turbo codeword index if such a relation exists, or * if not. Note that for turbo codeword indices that correspond to systematic bits, the interleaver is used to get the correct constituent codeword index for the second encoder.

Define two mappings $\psi_a : Z_{N_a} \to Z_I \cup \{*\}$ and $\psi_b : Z_{N_b} \to Z_I \cup \{*\}$. The mapping ψ_a gives the systematic sequence index of the first constituent codeword index if such a relation exists, or * if not. Similarly, the mapping ψ_b gives the interleaved systematic sequence index of the second constituent codeword index if such a relation exists, or * if not.

Example 1: Consider a turbo code composed of two identical nominal rate-1/2 constituent convolutional codes. The interleaver length I = 6 and parity bits from the two constituent encoders are punctured alternatively to create a nominal rate-1/2 turbo code. The lengths of the first and second constituent codes are $N_a = N_b = 9$. The interleaver π is defined by $\{3, 5, 1, 4, 0, 2\}$ (i.e., $0 \rightarrow 3, 1 \rightarrow 5$, and so on). The ordering of bits in the turbo codeword is

$$I_0 P_0^a I_1 P_1^b I_2 P_2^a I_3 P_3^b I_4 P_4^a I_5 P_5^b$$

where I_i and P_i^a denote the *i*th systematic and parity bit from the first constituent code, respectively, and where P_i^b denotes the *i*th parity bit from the second constituent code. The mappings μ_a , μ_b , ψ_a , and ψ_b are depicted graphically in Fig. 1.

C. Turbo Decoding on the BEC

The aim of turbo decoding on the BEC is to find a set of paths through each constituent code trellis that is consistent with the received sequence. The decoding starts with a set of all paths and iteratively eliminates



Fig. 1. Graphical representation of the mappings μ_a , μ_b , ψ_a , and ψ_b for the turbo code from Example 1.

those that are inconsistent. This iterative process continues until either there is only one possible path left in each constituent trellis (successful decoding), or there is no change from one iteration step to the next. We will describe the basic algorithm, but remark that the BEC version of turbo decoding allows more efficient implementations. For simplicity, we omit the details.

Let \mathcal{T}_{info}^x denote an *information bit-oriented trellis* [24]–[26] for constituent code C_x , x = a, b. The trellis \mathcal{T}_{info}^x is time-variant, but periodic, and has two edges out of each vertex. For details, see Section V-A.1. The number of trellis depths in \mathcal{T}_{info}^x is I + 1, i.e., there is one trellis section for each information bit. Below, \bar{x} denotes the complement of x, x = a, b.

Like turbo decoding for an AWGN channel, the constituent decoders work with a forward and a backward pass through the constituent code trellis \mathcal{T}_{info}^x , and during these passes the state metric $\alpha_x^{(i)}(v, j)$ (resp. $\beta_x^{(i)}(v, j)$) for vertex v at trellis depth j, at the *i*th iteration, is updated. For the BEC, however, the state metrics are boolean and define forward (resp. backward) paths that are consistent with the current estimate of the transmitted sequence $\hat{\mathbf{c}} = (\hat{c}_0, \dots, \hat{c}_{N-1})$. The decoding for constituent code C_x is performed as follows.

Initially, we set $\alpha_x^{(1)}(0,0) = \beta_x^{(1)}(0,I) = \beta_x^{(0)}(0,I) = \mathbf{true}; \ \beta_x^{(0)}(v,j) = \mathbf{true}$ for all vertices v at trellis depth j in \mathcal{T}_{info}^x , $j = 1, \ldots, I-1$; and $\alpha_x^{(1)}(v,0) = \beta_x^{(1)}(v,I) = \beta_x^{(0)}(v,I) = \mathbf{false}$ for every non-zero vertex v at trellis depths 0 and I, respectively, in \mathcal{T}_{info}^x . Finally, set the estimate of the transmitted sequence $\hat{\mathbf{c}}$ equal to the received sequence.

Then the forward pass calculates, for j = 1, ..., I and for each vertex v' at trellis depth j in \mathcal{T}_{info}^x , $\zeta_j(v')$ and subsequently $\alpha_x^{(i)}(v', j) = (\beta_x^{(i-1)}(v', j) \text{ AND } \zeta_j(v'))$. Here $\zeta_j(v')$ is a boolean variable which is true if there exists, in the (j-1)th trellis section of \mathcal{T}_{info}^x , an edge consistent with $\hat{\mathbf{c}}$, with right vertex v', and left vertex v such that $\alpha_x^{(i)}(v, j-1) = \mathbf{true}$. Initially, prior to the forward pass, we set $\alpha_x^{(i)}(v, 0) = \alpha_x^{(1)}(v, 0)$ for all vertices v at trellis depth 0 in \mathcal{T}_{info}^x .

In a similar way the backward pass calculates $\beta_x^{(i)}(v, j)$ for all vertices v at trellis depth j in \mathcal{T}_{info}^x , $j = 0, \dots, I-1$. Finally, the estimate $\hat{\mathbf{c}}$ of the transmitted sequence is updated so that only information values consistent with legal The codeword is said to be *recovered* if $\hat{c}_j \neq \star$, where \star denotes an erasure, for all j such that $\mu_a(j) \neq \star$ and $\psi_a(\mu_a(j)) \neq \star$, j = 0, ..., N - 1. The decoder is said to *fail to progress further* if, for x = a, b, the state metrics $\alpha_x^{(l)}(v, j)$ and $\beta_x^{(l)}(v, j)$, for some positive integer l > 1, are equal to the state metrics $\alpha_x^{(l-1)}(v, j)$ and $\beta_x^{(l-1)}(v, j)$, respectively, for all vertices v at trellis depth j in \mathcal{T}_{info}^x , j = 1, ..., I - 1.

III. TURBO STOPPING SETS

In this section we will introduce the concept of a turbo stopping set. A turbo stopping set is the equivalent of an LDPC stopping set when turbo decoding and *not* iterative BP decoding is performed.

Definition 1: Let C denote a given PCCC with interleaver π . A set $S = S(\pi) \subseteq \{0, ..., N-1\}$ is a turbo stopping set if and only if there exist two linear subcodes $\bar{C}_a \subseteq C_a \subseteq \{0,1\}^{N_a}$ and $\bar{C}_b \subseteq C_b \subseteq \{0,1\}^{N_b}$ of dimension > 0 with support sets $\chi(\bar{C}_a)$ and $\chi(\bar{C}_b)$, respectively, such that

$$\chi(\bar{C}_a) = \mu_a(\mathcal{S}) \setminus \{*\}$$
$$\chi(\bar{C}_b) = \mu_b(\mathcal{S}) \setminus \{*\}$$
$$\pi(\psi_a(\chi(\bar{C}_a)) \setminus \{*\}) = \psi_b(\chi(\bar{C}_b)) \setminus \{*\}.$$
(1)

The *size* of a turbo stopping set S is its cardinality.

The lemmas below state some of the properties of a turbo stopping set.

Lemma 1: Let C denote a given PCCC with interleaver π . The support set of any non-zero codeword from C is a turbo stopping set of size equal to the Hamming weight of the given codeword. Thus, the minimum turbo stopping set size is upper-bounded by the minimum Hamming weight.

Proof: Denote the turbo codeword by **c** and the corresponding first and second constituent codewords by \mathbf{c}_a and \mathbf{c}_b , respectively. Furthermore, let $\bar{C}_a = \{\mathbf{c}_a, \mathbf{0}_{N_a}\}$ and $\bar{C}_b = \{\mathbf{c}_b, \mathbf{0}_{N_b}\}$ where $\mathbf{0}_{N_x}$ denotes an all-zero sequence of length N_x , x = a, b. The result follows immediately from Definition 1, since $\chi(\bar{C}_a) = \mu_a(\chi(\mathbf{c})) \setminus \{*\}$, $\chi(\bar{C}_b) = \mu_b(\chi(\mathbf{c})) \setminus \{*\}$, and $\pi(\psi_a(\mu_a(\chi(\mathbf{c})) \setminus \{*\}) \setminus \{*\}) = \psi_b(\mu_b(\chi(\mathbf{c})) \setminus \{*\}) \setminus \{*\}$.

Lemma 2: Let C denote a given PCCC with interleaver π , and let $S = S(\pi)$ denote a turbo stopping set. If \bar{C}_a and \bar{C}_b can both be decomposed into *direct sums* of linear subcodes of dimension 1 with disjoint support sets, then S is the support set of a turbo codeword of Hamming weight |S|. The converse is also true.

Proof: Assume that \bar{C}_a and \bar{C}_b can both be decomposed into direct sums of linear subcodes of dimension 1 with disjoint support sets. In more detail,

$$\bar{C}_a = \{\mathbf{c}_a^{(0)}, \mathbf{0}_{N_a}\} + \dots + \{\mathbf{c}_a^{(p)}, \mathbf{0}_{N_a}\} \text{ and } \bar{C}_b = \{\mathbf{c}_b^{(0)}, \mathbf{0}_{N_b}\} + \dots + \{\mathbf{c}_b^{(q)}, \mathbf{0}_{N_b}\}$$

where p and q are non-negative integers. The codewords $\mathbf{c}_a = \sum_{i=0}^p \mathbf{c}_a^{(i)}$ and $\mathbf{c}_b = \sum_{i=0}^q \mathbf{c}_b^{(i)}$ have support sets $\chi(\mathbf{c}_a) = \chi(\bar{C}_a)$ and $\chi(\mathbf{c}_b) = \chi(\bar{C}_b)$, respectively, since the support sets of the direct sum subcodes are disjoint. Since the two sets $\pi(\psi_a(\mu_a(S) \setminus \{*\}) \setminus \{*\}) = \pi(\psi_a(\chi(\mathbf{c}_a)) \setminus \{*\})$ and $\psi_b(\mu_b(S) \setminus \{*\}) \setminus \{*\} = \psi_b(\chi(\mathbf{c}_b)) \setminus \{*\}$ are equal (from Definition 1), there exists a turbo codeword \mathbf{c} with first and second constituent codewords \mathbf{c}_a and \mathbf{c}_b , respectively. Thus, the turbo stopping set is the support set of a turbo codeword of Hamming weight $|\mathcal{S}|$, since $\chi(\mathbf{c}) = \mathcal{S}$.

We prove the converse using the following argument. The turbo stopping set S is the support set of a turbo codeword **c**. The corresponding first and second constituent codewords are denoted by \mathbf{c}_a and \mathbf{c}_b , respectively. Then, there exist subcodes $\bar{C}_a = {\mathbf{c}_a, \mathbf{0}_{N_a}} \subset C_a$ and $\bar{C}_b = {\mathbf{c}_b, \mathbf{0}_{N_b}} \subset C_b$ both of dimension 1 which satisfy the constraints in (1) with $S = \chi(\mathbf{c})$, and the result follows. (The codewords \mathbf{c}_a and \mathbf{c}_b may or may not be further decomposed.)

Construction 1: Let $T_a = (V_a, E_a)$ and $T_b = (V_b, E_b)$ denote two arbitrarily chosen Tanner graphs for the first and second constituent codes, respectively. The node set V_x can be partitioned into three disjoint subsets V_x^s , V_x^p , and V_x^c , corresponding to systematic bits, parity bits, and parity-check equations, respectively, where x = a, b. There is an edge $e \in E_x$ connecting a node $v_x^v \in V_x^v = V_x^s \cup V_x^p$ to a node $v_x^c \in V_x^c$ if and only if the first (x = a)or second (x = b) constituent codeword bit represented by v_x^v is checked by the parity-check equation represented by v_x^c . Next, define

$$E'_{b} = \{(v^{s}_{a,j}, v^{c}_{b}) : v^{c}_{b} \in V^{c}_{b}, (v^{s}_{b,\pi(j)}, v^{c}_{b}) \in E_{b}, j = 0, \dots, I-1\} \cup \{(v^{p}_{b}, v^{c}_{b}) : v^{p}_{b} \in V^{p}_{b}, v^{c}_{b} \in V^{c}_{b}, (v^{p}_{b}, v^{c}_{b}) \in E_{b}\}.$$

Then, the graph $T = (V_a \cup V_b^p \cup V_b^c, E_a \cup E_b')$ is a Tanner graph for the turbo code C.

We remark that for a given binary linear code there exist in general several full-rank parity-check matrices, and thus several (distinct) Tanner graphs. The minimum stopping set size is in general a function of the Tanner graph [27]–[29]. Note that Construction 1 gives a specific class of Tanner graphs for a turbo code that is a proper subset of the class of all Tanner graphs of the given turbo code. We will consider this specific class of Tanner graphs below.

Lemma 3: Let C denote a given PCCC with interleaver π . For any turbo stopping set $S = S(\pi)$ there is an LDPC stopping set of cardinality |S| in any Tanner graph T for C within the class of Tanner graphs from Construction 1.

Proof: We use the notation introduced in Construction 1. The set $\{v_{x,j}^v : j \in \mu_x(S) \setminus \{*\}\}$, where $v_{x,j}^v$ denotes the *j*th node in V_x^v , is an LDPC stopping set in any Tanner graph T_x of C_x , since $\mu_x(S) \setminus \{*\}$ is the support set $\chi(\bar{C}_x)$ (from Definition 1) of some subcode \bar{C}_x of C_x of dimension > 0. We have here used the fact that the variables nodes corresponding to the support set of a non-zero codeword constitute an LDPC stopping set. Furthermore, the set

$$\left(\left\{v_{a,j}^{v}: j \in \mu_{a}(\mathcal{S}) \setminus \{*\}\right\} \cup \left\{v_{b,j}^{v}: j \in \mu_{b}(\mathcal{S}) \setminus \{*\}\right\}\right) \setminus V_{b}^{s}$$

$$(2)$$

is an LDPC stopping set in the Tanner graph T of the turbo code C due to the last condition in Definition 1. The cardinality of the LDPC stopping set in (2) is |S|.

Note that the converse is not necessarily true (i.e., for an LDPC stopping set in any Tanner graph T for C within the class of Tanner graphs from Construction 1, there is not necessarily a turbo stopping set). Thus, iterative BP decoding using a Tanner graph within the class of Tanner graphs of a turbo code from Construction 1 is inferior to turbo decoding on the BEC. The following theorem states an exact condition for decoding failure. Theorem 1: Let C denote a given PCCC with interleaver π that we use to transmit information over the BEC. The received vectors are decoded using turbo decoding until either the codeword has been recovered, or the decoder fails to progress further. Then the set of erased positions that will remain when the decoder stops is equal to the unique maximum-size turbo stopping set which is also a subset of \mathcal{E} , where \mathcal{E} denotes the subset of erased positions.

Proof: The proof given here is inspired by the proof given by Di *et al.* in [5, Lemma 1.1] in the context of an LDPC stopping set. Let S be a turbo stopping set contained in \mathcal{E} . The claim is that the basic turbo decoder cannot determine the bits corresponding to the positions in the turbo stopping set S. Assume that all other bits are known. Turbo decoding starts by activating the first constituent decoder. For the first constituent decoder, the forward-backward algorithm will determine $|\bar{C}_a| > 1$ possible paths through the trellis. The support set of these possible paths is equal to $\mu_a(S) \setminus \{*\}$, and they are all equally likely. Consequently, no additional codeword bits can be determined. Thus, the extrinsic probability distributions for systematic bits in positions in $\psi_a(\mu_a(S) \setminus \{*\}) \setminus \{*\}$ are uniform. For the second constituent decoder, the forward-backward algorithm will determine $|C_b| > 1$ possible paths through the trellis. The support set of these possible paths is equal to $\mu_b(S) \setminus \{*\}$ and, since the *a priori* probability distributions for systematic bits in positions in $\pi(\psi_a(\mu_a(S) \setminus \{*\}) \setminus \{*\})$ are uniform, and the two sets $\pi(\psi_a(\mu_a(S) \setminus \{*\}) \setminus \{*\})$ and $\psi_b(\mu_b(S) \setminus \{*\}) \setminus \{*\}$ are equal (see Definition 1), all paths are equally likely. Consequently, no additional codeword bits can be determined. Thus, the extrinsic probability distributions for systematic bits in positions in $\psi_b(\mu_b(S) \setminus \{*\}) \setminus \{*\}$ are uniform. One iteration of the basic turbo decoding algorithm has been performed and no additional codeword bits have been determined. Since there is no new information available to the first constituent decoder, no additional bits will be determined in the next round of turbo decoding either. It follows that the decoder cannot determine the bits corresponding to the positions in the unique maximum-size turbo stopping set which is also a subset of \mathcal{E} . Note that there is a unique maximum-size turbo stopping set which is also a subset of \mathcal{E} , since the union of two turbo stopping sets is also a turbo stopping set. Conversely, if the decoder terminates at a set S, then there will exist subcodes $\bar{C}_a \subseteq C_a$ and $\bar{C}_b \subseteq C_b$ of dimension > 0 with support sets $\mu_a(S) \setminus \{*\}$ and $\mu_b(S) \setminus \{*\}$, respectively. Since the turbo decoder terminates, the two sets $\pi(\psi_a(\mu_a(S) \setminus \{*\}) \setminus \{*\})$ and $\psi_b(\mu_b(S) \setminus \{*\}) \setminus \{*\}$ are equal. From Definition 1, it follows that S is a turbo stopping set and, since no erased bit-positions contained in a turbo stopping set can be determined by the turbo decoder, it must be the maximum-size turbo stopping set which is also a subset of \mathcal{E} .

A. A (155, 64, 18) Turbo Code

In [30], a particularly nice (3, 5)-regular LDPC code of length 155, dimension 64, and minimum Hamming distance 20 was constructed. The underlying Tanner graph has girth 8 which makes the code an excellent candidate for iterative decoding. This is the reason behind the selected code parameters.

The turbo code is obtained by puncturing of a nominal rate-1/3 turbo code with nominal rate-1/2, constraint length $\nu = 4$ constituent codes defined by the parity-check matrix $\mathbf{H}(D) = (1 + D + D^2 + D^4 + 1 + D^3 + D^4)$. The last polynomial which is irreducible and primitive has been chosen as the parity polynomial making the constituent encoders recursive. The information block size is 64, the interleaver length is 72 due to dual termination [23], and

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the interleaver is a dithered relative prime (DRP) interleaver [10], [31]. The puncturing pattern is designed using the algorithm in [32]. The minimum distance of 18 of the code has been computed using the algorithm in [7]. For this code there exists a turbo stopping set of size 17. The turbo stopping set is depicted in Fig. 2. In Fig. 2, the three upper rows of nodes correspond to the first constituent encoder, while the three remaining rows of nodes correspond to the second constituent encoder. The nodes in row number 5(i-1)+1 give the bit-position in the turbo codeword of the corresponding systematic (i = 1) and parity (i = 2) bits. The nodes in row number 3(i - 1) + 2 and i + 2 correspond to parity and systematic bits, respectively, from constituent encoder i, i = 1, 2. The blue and dark green nodes correspond to systematic bits (can *not* be assigned freely). The red and yellow nodes correspond to erased bits and punctured (parity) bits, respectively. The light green nodes correspond to parity bits. The arrows in between the upper three rows and the remaining three rows correspond to interleaving of erased information bits. The remaining part of the interleaver is of no concern for the following discussion. In fact, possible choices for \overline{C}_a and \overline{C}_b are the linear subcodes spanned by

$$\mathbf{c}_{a}^{(1)} = (0\diamond, \dots, 00, 1\diamond, 10, 00, 11, 01, 1\diamond, 00, \dots, 00, 00, 0\diamond, 0\diamond, 00, 00, 0\diamond, \dots, 0\diamond)$$
$$\mathbf{c}_{a}^{(2)} = (0\diamond, \dots, 00, 0\diamond, 00, 00, 00, 0\diamond, 00, \dots, 00, 11, 0\diamond, 0\diamond, 10, 11, 0\diamond, \dots, 0\diamond)$$
(3)

and

respectively. The symbol \diamond indicates that the bit-position has been punctured. From Fig. 2 we get

$$S = \{10, 11, 15, 16, 18, 19, 69, 70, 73, 75, 76, 115, 116, 117, 123, 124, 126\}$$

$$\mu_a(S) = \{10, 11, 15, 16, 18, 19, 69, 70, 73, 75, 76, *\}$$

$$\mu_b(S) = \{9, 11, 13, 14, 15, 17, 32, 34, 38, 39, 101, 104, 105, *\}$$

$$\psi_a(\mu_a(S) \setminus \{*\}) = \{8, 9, 11, 13, 43, 46, 47, *\}$$

$$\psi_b(\mu_b(S) \setminus \{*\}) = \{7, 10, 11, 25, 61, 64, 65, *\}.$$
(5)

Furthermore, it holds that $\chi(\bar{C}_a) = \mu_a(S) \setminus \{*\}, \chi(\bar{C}_b) = \mu_b(S) \setminus \{*\}$, and $\pi(\psi_a(\chi(\bar{C}_a)) \setminus \{*\}) = \psi_b(\chi(\bar{C}_b)) \setminus \{*\}$, which shows that S is a turbo stopping set.



Fig. 2. Turbo stopping set of size 17 for the example (155, 64, 18) turbo code. The direct sum of the two upper (lower) sets of binary vectors (extended to length I with zeros) is the systematic part of the subcode \bar{C}_a (\bar{C}_b).

The turbo stopping set depicted in Fig. 2 is given as an example to illustrate the concept of a turbo stopping set. Actually, it is possible to find an exhaustive list of all turbo stopping sets of size less than some threshold using a modification of the algorithm in [7]. The details of the algorithm are outlined in Section V. For this code there are 2 turbo stopping sets of size 16 and 13 of size 17.

B. A (201, 64, 21) Turbo Code

From Fig. 2 it is apparent that some of the punctured bits can be reinserted without increasing the size of the depicted turbo stopping set. In fact, 27 out of the 31 punctured parity bits from the first constituent code can be reinserted. For the second constituent code, 19 out of the 30 punctured parity bits can be reinserted. The minimum distance of 21 of the resulting turbo code has been determined by the algorithm in [7]. Note that this is a *constructed* example, since it is possible to design a puncturing pattern of weight 15 that will give a length 201 and dimension 64 turbo code of minimum distance 25 using the same mother code and the algorithm in [32]. The minimum distance of the unpunctured mother code is 27.

C. Remarks

The code considered in Section III-A is *not* a rare example in the sense of having turbo stopping sets of size smaller than the minimum distance. We have found several examples of excellent turbo codes with this property. For further examples see Section V-E.

IV. IMPROVED TURBO DECODING

In [4], Pishro-Nik and Fekri consider improved iterative BP decoding of LDPC codes on the BEC. When standard iterative BP decoding fails, the improved algorithm chooses one of the unknown variables nodes and guesses its value. The decoding continues until either the transmitted codeword is recovered, the decoder does not progress

further, or the decoder reaches an *inconsistency*. The decoder is said to reach an inconsistency if all the variables nodes connected to a check node are known, but the check node is not satisfied. If the decoder has reached an inconsistency, then the value of the guessed variable node is changed and the decoding is repeated. This time the decoder will not reach an inconsistency, but the decoding can stop again. More sophisticated algorithms for improved iterative BP decoding of LDPC codes on the BEC were recently proposed by Ravisankar and Fekri in [2]. These algorithms improve upon the algorithms in [4]. Furthermore, the algorithms in [2] use the Tanner graph representation of the LDPC code actively to identify *equivalent* bit-positions when iterative BP decoding has stopped in a stopping set. In this context, two bit-positions are equivalent if and only if the knowledge of one of the bits implies the knowledge of the other bit after a series of message-passing in the subgraph of the Tanner graph composed of all unknown variable nodes and the neighboring check nodes of degree 2.

Improved turbo decoding, the pseudo-code of which is given below, has the same *structure* as the algorithm in [4]. In more detail, it is based on guessing unknown systematic bit-positions and then decode until either the transmitted codeword is recovered, the decoder does not progress further, or the decoder reaches some kind of inconsistency. In the context of turbo decoding, however, the unknown bit-position to guess can be chosen more efficiently than in the context of improved iterative BP decoding of LDPC codes, since the forward and backward passes on the constituent trellises give information on legal paths. Bit-position selection is considered in more detail in Section IV-A.

In the pseudo-code below $\gamma_{x,j}^{(l)}$, x = a, b, denotes the number of vertices of \mathcal{T}_{info}^x at trellis depth j that are legal after the *l*th iteration.

step 1 (Initialization):

- 1) Choose the maximal number of iterations l_{max} .
- 2) Initialize J and T with zero.

step 2 (Original turbo decoding; bit-position selection):

- Run turbo decoding as described in Section II-C, with a maximum of l_{max} iterations, until either the codeword is recovered, or the decoder fails to progress further. Let l_J ≤ l_{max} denote the number of iterations carried out and increment T with l_J.
- 2) If $T = l_{\text{max}}$, or the codeword is recovered, make a decision based on \hat{c} , and terminate the algorithm.
- 3) Compute $\gamma_{x,0}^{(T)}, \ldots, \gamma_{x,I}^{(T)}$ for x = a, b and select a bit-position v_T to guess based on these values. See Section IV-A below for details.
- 4) Initialize a list \mathcal{L} with the *ordered* sets $\{(v_T, 0)\}$ and $\{(v_T, 1)\}$.
- 5) Initialize \mathcal{M}_0 and \mathcal{M}_1 with $\hat{\mathbf{c}}$.

step 3 (Performing additional iterations; bit-position selection):

- 1) Increment J.
- 2) $L = \{(L_1^{(0)}, L_2^{(0)}), \dots, (L_1^{(|L|-1)}, L_2^{(|L|-1)})\}$ is chosen and removed from \mathcal{L} .
- 3) Initialize $\hat{\mathbf{c}}$ with $\mathcal{M}_{L_2^{(0)},\dots,L_2^{(|L|-1)}}$ and set \hat{c}_l equal to $L_2^{(j)}$ where $\psi_a(\mu_a(l)) = L_1^{(j)}$ and j = |L| 1.

- 4) Run turbo decoding as described in Section II-C with a maximum of l_{max} − T iterations, but without assigning values to ĉ in the initialization step of the algorithm, until the decoder fails to progress further, or there is a trellis depth j in which all vertices in T^x_{info} have false forward or backward state metrics for some x, x = a, b. In this case the decoder is said to have reached an *inconsistency*. Let l_J ≤ l_{max} − T denote the number of iterations carried out and increment T with l_J.
- 5) If $T = l_{\text{max}}$, or the decoder has not reached an inconsistency and $\hat{c}_j \neq \star$ for all j such that $\mu_a(j) \neq \star$ and $\psi_a(\mu_a(j)) \neq \star$, $j = 0, \ldots, N-1$, make a decision based on \hat{c} , and terminate the algorithm.
- 6) Delete $\mathcal{M}_{L_2^{(0)},...,L_2^{(|L|-1)}}$.
- 7) If the decoder has not reached an inconsistency, perform the following.
 - a) Compute $\gamma_{x,0}^{(T)}, \ldots, \gamma_{x,I}^{(T)}$ for x = a, b and select a bit-position v_T to guess based on these values. See Section IV-A below for details.
 - b) Add the two elements $L_0 = L \cup \{(v_T, 0)\}$ and $L_1 = L \cup \{(v_T, 1)\}$ to the list \mathcal{L} , and initialize $\mathcal{M}_{L_{0,2}^{(0)},...,L_{0,2}^{(|L_0|-1)}}$ and $\mathcal{M}_{L_{1,2}^{(0)},...,L_{1,2}^{(|L_1|-1)}}$ with $\hat{\mathbf{c}}$.

step 4 (Repeating): Repeat step 3.

The list \mathcal{L} in improved turbo decoding can be implemented as a *last-in first-out* queue or as a *first-in first-out* queue. The first-in first-out implementation requires more memory than the last-in first-out implementation. The efficiency of an improved decoding algorithm with the above structure is very dependent on the selection of bit-positions to guess.

A. Bit-Position Selection for Improved Turbo Decoding

When turbo decoding fails to progress further, the unknown bit-positions constitute a turbo stopping set. Thus, guessing a bit-position in $\chi(\bar{C}_x)$ will *free* at least one additional bit. Some of the positions in $\chi(\bar{C}_x)$ can be determined from the numbers $\gamma_{x,j}^{(l)}$ of legal vertices at trellis depth j in \mathcal{T}_{info}^x (l is the iteration number). In particular, the *j*th systematic bit is unknown if $\gamma_{a,j}^{(l)} = 1$ and $\gamma_{a,j+1}^{(l)} = 2$, or $\gamma_{a,j+1}^{(l)} = 1$ and $\gamma_{a,j}^{(l)} = 2$. When selecting a bit-position to guess we would also like to free as many unknown positions as possible. We propose the following bit-position selection algorithm.

- 1) For x = a, b do the following.
 - a) Let l_x be the number of indices j with the property that $\gamma_{x,j}^{(T)} = 1$ and $\gamma_{x,j+1}^{(T)} = 2$.
 - b) Let $w_{x,f}$ be the largest non-negative integer such that there exists an index f_x with the property that $\gamma_{x,f_x-1}^{(T)} = 1$ and $\gamma_{x,f_x}^{(T)} = \cdots = \gamma_{x,f_x+w_{x,f}}^{(T)} = 2.$
 - c) Let $w_{x,r}$ be the largest non-negative integer such that there exists an index r_x with the property that $\gamma_{x,r_x+1}^{(T)} = 1$ and $\gamma_{x,r_x}^{(T)} = \cdots = \gamma_{x,r_x-w_{x,r}}^{(T)} = 2.$
 - d) Let $w_x = \max(w_{x,f}, w_{x,r})$.
- 2) If $l_a > l_b$, or $l_a = l_b$ and $w_a \ge w_b$, then set x = a. Otherwise, set x = b.

If w_{x,f} ≥ w_{x,r}, then v_T is set equal to f_x − 1 if x = a and π⁻¹(f_x − 1) if x = b. Otherwise, v_T is set equal to r_x if x = a and π⁻¹(r_x) if x = b.

B. Remarks

We remark that several variations of the above bit-position selection algorithm are possible. In particular, one could use both the number of legal edges in each trellis section, or equivalently the edge entropy, and the number of legal vertices for each trellis depth, or equivalently the vertex entropy, in combination with the interleaver to improve the algorithm as described below.

Let v_T denote a chosen systematic bit-position within a vertex entropy transition from level j at time t to level j+1 at time t+1 (a forward transition from level j at time t), or from level j at time t to level j+1 at time t-1(a backward transition from level j at time t) for one of the constituent trellises. The effective length of v_T is the number of undetermined systematic bit-positions that will be determined by the forward-backward algorithm on the considered constituent trellis before any interleaving, if v_T is guessed. To simplify notation we assume that v_T is chosen based on the first constituent trellis. The effective length of v_T is a positive integer smaller than or equal to the number of undetermined systematic bit-positions within the range $[t, t + w_s - 1]$ for a forward transition, or within the range $[t-w_s, t-1]$ for a backward transition. The positive integer w_s is the smallest integer such that the vertex entropy at time $t + w_s$, for a forward transition, or at time $t - w_s$, for a backward transition, is different from j+1. A general upper bound on the effective length of v_T is w_s . When j=0, we can use the edge entropy to find the exact value of the effective length of v_T . Let w_e denote the smallest positive integer such that the edge entropy for the $(t+w_e)$ th trellis section, i.e., for the transition from time $t+w_e$ to time $t+w_e+1$, for a forward transition, or for the $(t - 1 - w_e)$ th trellis section, for a backward transition, is 2. A general upper bound on the effective length of v_T in this case is $\min(w_s, w_e)$. The exact value is the number of undetermined systematic bit-positions within the range $[t, t + \min(w_s, w_e) - 1]$, for a forward transition, or within the range $[t - \min(w_s, w_e), t - 1]$, for a backward transition. The selection of bit-positions can be improved even further by actively using the interleaver. If we choose a bit-position v_T with the property that both v_T and $\pi(v_T)$ are within vertex entropy transitions, then the performance will be improved.

Finally, we remark that the simple version described in Section IV-A provides good results as indicated in Section IV-D below.

C. Some Properties of Improved Turbo Decoding

In this subsection we establish some basic results of improved turbo decoding as described above. The following lemma is simple, but important, since the bit-position selection algorithm in Section IV-A is based on this result.

Lemma 4: Apply improved turbo decoding as described above using the bit-position selection algorithm in Section IV-A. Then, the channel value corresponding to the selected bit-position v_T is an erasure.

Proof: The bit-position selection algorithm in Section IV-A selects only (systematic) bit-positions $v_T \in \{0, \ldots, I-1\}$ with the property that $\gamma_{x,\pi_x(v_T)}^{(T)} \neq \gamma_{x,\pi_x(v_T)+1}^{(T)}$ for x = a or b, where $\pi_x(v_T) = v_T$ for x = a

and $\pi_x(v_T) = \pi(v_T)$ for x = b. If the channel value is *not* erased, then we know the v_T th information bit with probability 1. Since the v_T th information bit is known with probability 1, and both constituent trellises are information bit-oriented, there is only a single legal edge out of each legal vertex at trellis depth $\pi_x(v_T)$ for x = a or b. Consequently, the number of legal trellis vertices at trellis depth $\pi_x(v_T) + 1$ is equal to the number of legal trellis vertices at trellis depth $\pi_x(v_T)$.

Lemma 5: Let C denote a given PCCC with interleaver π . Let $S = S(\pi)$ denote a turbo stopping set, and erase all bit-positions in S. Then, choose any bit-position $j \in S$, and do the following.

- 1) Fix the bit-value in bit-position j to 0 and perform turbo decoding until either the decoder fails to progress further, or the decoder reaches an inconsistency. If the decoder does not reach an inconsistency, denote the set of erased positions that remain when the decoder stops by $S_i^{(0)}$.
- 2) Fix the bit-value in bit-position j to 1 and perform turbo decoding until either the decoder fails to progress further, or the decoder reaches an inconsistency. If the decoder does not reach an inconsistency, denote the set of erased positions that remain when the decoder stops by $S_i^{(1)}$.

If the decoder does *not* reach an inconsistency in either of the two cases above, then the two sets $S_j^{(0)}$ and $S_j^{(1)}$ are equal.

Proof: Fix the bit-value in bit-position j to c where c = 0 or 1. Then the number of possible paths in the first constituent code is immediately reduced by a factor of 2, since the subcode \bar{C}_a corresponding to S is linear. Let this reduced set of legal paths in the first constituent code be denoted by $P_{a,j}^{(c)}$. Furthermore, the forward-backward algorithm for the first constituent code will determine additional bit-positions (which are previously unknown) contained within a set $S_{a,j}^{(c)}$. For a given bit-position $i \in S_{a,j}^{(c)}$, all paths in $P_{a,j}^{(c)}$ will have the same bit-value of \tilde{c} (depending on the value of c) in this bit-position. Since the subcode \bar{C}_a is linear, all paths in $P_{a,j}^{(\bar{c})}$, where \bar{c} denotes the complement of c, will also have a fixed bit-value of \bar{c} in bit-position i. Thus, it holds that $i \in S_{a,j}^{(\bar{c})}$, from which it follows that $S_{a,j}^{(c)} = S_{a,j}^{(\bar{c})}$, since c and i both are arbitrarily chosen. For the second constituent code, several bit-positions are fixed due to extrinsic information from the first constituent code. However, we can apply the same type of arguments as above to show that the sequence of additional bit-positions determined by the forward-backward algorithm in the second constituent code is independent of c. The result follows by applying these arguments in an iterative fashion until there is no further progress.

Lemma 6: Apply improved turbo decoding as described above with the bit-position selection algorithm in Section IV-A. For any two elements

$$L = \{ (L_1^{(0)}, L_2^{(0)}), \dots, (L_1^{(|L|-1)}, L_2^{(|L|-1)}) \} \in \mathcal{L} \text{ and } \tilde{L} = \{ (\tilde{L}_1^{(0)}, \tilde{L}_2^{(0)}), \dots, (\tilde{L}_1^{(|\tilde{L}|-1)}, \tilde{L}_2^{(|\tilde{L}|-1)}) \} \in \mathcal{L} \}$$

with the property that $|L| = |\tilde{L}|$, it holds that $L_1^{(i)} = \tilde{L}_1^{(i)}$ for all i, i = 0, ..., |L| - 1, i.e., the actual bit-values in the guessed bit-positions do not influence on which bit-positions are selected next by the bit-position selection algorithm in Section IV-A, as long as no inconsistency is reached.

Proof: The result follows directly from Lemma 5.

We remark that due to Lemma 6 we can reduce the number of times we need to run the bit-position selection algorithm from Section IV-A when performing improved turbo decoding.

Theorem 2: Improved turbo decoding is ML decoding on the BEC when $l_{\max} \rightarrow \infty$.

Proof: The proof is two-fold. First we prove that if the algorithm terminates, then we have an ML decoder. Secondly, we prove that the algorithm will always terminate.

- 1) It follows from the pseudo-code above (step 2, item 2), and step 3, item 5)) that if the algorithm terminates, then the decoder has not reached an inconsistency and ĉ_j ≠ * for all j such that µ_a(j) ≠ * and ψ_a(µ_a(j)) ≠ *, j = 0,..., N 1, since l_{max} → ∞. The original turbo decoding algorithm will not introduce bit errors and neither will the improved turbo decoding algorithm due to items 5) and 7) in step 3. In more detail, the algorithm will not terminate if the decoder has reached an inconsistency, and no further guessing is performed if this is the case. Thus, if the improved turbo decoding algorithm terminates, then transmitted codeword is recovered, or there exists a different turbo codeword c' with support set χ(c') ⊆ E where E denotes the subset of remaining erased bit-positions. In the latter case the transmitted codeword is not recovered. An ML decoder will not be able to determine the transmitted codeword are equally likely to have been transmitted. Thus, improved turbo decoding.
- 2) As the algorithm proceeds, erased bit-positions are guessed. Each time an element L is removed from the list L, turbo decoding is performed. If the decoder does not reach an inconsistency and ĉ_j = ★ for some j such that μ_a(j) ≠ * and ψ_a(μ_a(j)) ≠ *, j = 0,..., N − 1, then a *new* erased bit-position is guessed (see the bit-position selection algorithm in Section IV-A and Lemma 4 for details). The decoder will always terminate, since there is a finite number of bit-positions to guess.

D. Numerical Results

Here we present some simulation results of improved turbo decoding on the BEC. The simulated frame error rate (FER) is presented in Fig. 3 for the (155, 64, 18) turbo code introduced in Section III-A. We have used the bit-position selection algorithm described in Section IV-A in the simulations. The truncated union bound (TUB) in Fig. 3 is computed from the first 5 non-zero terms of the code's weight distribution. The near ML decoding curve is obtained using improved turbo decoding with a large number for l_{max} . In Table I we have tabulated, for different values of the channel erasure probability ϵ , the empirical value of l_{max} such that improved turbo decoding is near ML decoding when the fraction of ML-decodable frame errors observed in the simulation is ≤ 0.05 . The corresponding estimated values of the expected number of iterations E[T] are tabulated in the third row of the table. The gap between the TUB and the near ML performance curve at moderate-to-high values of ϵ is due to the fact that only a limited number of codewords are taken into account in the summation of the union bound. The two remaining curves show the FER for two different values of l_{max} . Observe that when l_{max} is increased, the performance improves. In Table II estimated values of E[T]

of improved turbo decoding are tabulated for different values of ϵ and l_{max} . From Table II we observe that the difference in the estimated values of E[T] for $l_{\text{max}} = 170$ and $l_{\text{max}} = 10$ decreases when ϵ decreases. In particular, for $\epsilon = 0.40$, there is almost no difference in the expected number of iterations. However, as can be seen from Fig. 3, there is a large difference in performance. The numbers in Tables I and II are based on more than 1000 observed frame errors for $\epsilon = 0.40, 0.45, 0.50, 0.55$, and more than 100 frame errors for $\epsilon = 0.35$. Similar performance improvements have been observed for the (3600, 1194, 49) turbo code from [10].



Fig. 3. FER on the BEC of the (155, 64, 18) turbo code from Section III-A.

V. TURBO STOPPING SET ENUMERATION

A. Convolutional Codes and Trellises

An edge-labeled directed graph is a triple (V, E, A), consisting of a set V of vertices, a finite set A called the alphabet, and a set E of ordered triples (v, a, v'), with $v, v' \in V$ and $a \in A$ called edges. The edge (v, a, v') begins at v, ends at v', and has label a.

Let C denote a linear (n, k, ν) convolutional code over some finite field F_q of q elements, where ν is the constraint length or the code degree. In this work the convolutional code symbols are taken from the binary field $F_2 = GF(2)$. A convolutional code can be defined by an $(n - k) \times n$ polynomial parity-check matrix $\mathbf{H}(D)$. We

TABLE I

Estimated expected number of iterations E[T] and empirical l_{\max} such that improved turbo decoding is near ML

ϵ	0.55	0.50	0.45	0.40	0.35
$l_{\rm max}$	2600	2400	2000	1100	400^{a}
E[T]	402	49	4.4	1.46	1.10

decoding for the (155, 64, 18) turbo code from Section III-A

^aThe estimate is less reliable, since only 6 ML-decodable frame errors have been observed compared to about 50 for $\epsilon > 0.35$.

TABLE II

Estimated expected number of iterations E[T] of improved turbo decoding for different values of ϵ and l_{\max} for the

$l_{\max}\downarrow,\epsilon\rightarrow$	0.55	0.50	0.45	0.40	0.35
10	8.3	5.1	2.42	1.42	1.10
170	74	19.5	3.57	1.45	1.10

(155, 64, 18) turbo code from Section III-A

assume in general a *canonical* parity-check matrix [33]. The maximum degree of the polynomials in the *i*th row is the *i*th row degree, denoted by ν_i .

Let \mathbf{H}^{L} be the matrix $\mathbf{H}(0)$, and let \mathbf{H}^{H} be the matrix $\operatorname{diag}(D^{\nu_{1}}, \ldots, D^{\nu_{n-k}})\mathbf{H}(D^{-1})$ with D = 0. When a matrix is given an integer interval subscript we mean the submatrix consisting of the columns with indices in the interval. The columns in a matrix are indexed from left to right with positive integers.

1) Trellis Representation of Convolutional Codes: The minimal trellis of C can be constructed from a paritycheck matrix of the code as outlined in [34]. The minimal trellis can be regarded (after an initial transient) as the infinite composition of a basic building block which is called the *trellis module*. The trellis module $T = (V, E, F_q)$ of C is an edge-labeled directed graph with the property that the vertex set V can be partitioned as

$$V = V_0 \cup V_1 \cup \dots \cup V_n \tag{6}$$

such that every edge in E begins at a vertex in V_i and ends at a vertex in V_{i+1} , for some i, i = 0, ..., n-1. The *depth* of the trellis module is n. The ordered index set $\mathcal{I} = \{0, 1, ..., n\}$ induced by the partition in (6) is called the *time axis* for T. The partition in (6) also induces a partition $E = E_0 \cup E_1 \cup \cdots \cup E_{n-1}$ of the edge set E where E_i is the subset of edges that begin at a vertex in V_i and end at a vertex in V_{i+1} .

Define $b_0 = 0$ and $b_i = \operatorname{rank}[\mathbf{H}_{n-i+1,n}^L]$, i = 1, ..., n, and $f_0 = 0$ and $f_i = \operatorname{rank}[\mathbf{H}_{1,i}^H]$, i = 1, ..., n. The vertex set V_i is a vector space of dimension $\dim(V_i) = \nu - n + k + f_i + b_{n-i} \le \nu + n - k$, from which it follows that $\dim(V_0) = \dim(V_n) = \nu$. The edge set E_i is also a vector space of dimension $\dim(E_i) = \nu - n + k + f_i + b_{n-i-1} + 1$.

Let \mathcal{I}_{info} be the subset of $\mathcal{I} \setminus \{n\}$ consisting of *all* integers *i* with the property that $b_{n-i} = b_{n-i-1}$. Furthermore, we assume without loss of generality that $b_n = b_{n-1}$ which implies that $0 \in \mathcal{I}_{info}$. Let $n_i = j + 1$ where *j* is

the largest non-negative integer $\leq n - i - 1$ such that $b_{n-i-1} \neq \cdots \neq b_{n-i-j-1}$ for every $i \in \mathcal{I}_{info}$. As argued, for instance in [26], there are n - k time instances $i \in \mathcal{I} \setminus \mathcal{I}_{info}$ in which there is only a single edge out of each vertex in V_i . By *sectionalization* the depth of the trellis module T can be reduced to k. This reduced trellis module is called an information bit-oriented trellis module and is denoted by $T_{info} = (V_{info}, E_{info}, \bigcup_{i \in \mathcal{I}_{info}} F_q^{n_i})$ where $V_{info} = \bigcup_{i \in \mathcal{I}_{info} \cup \{n\}} V_i$ and $E_{info} = \bigcup_{i \in \mathcal{I}_{info}} E'_i$. The edge set E'_i is the set of paths that begin at a vertex in V_i and end at a vertex in V_{i+n_i} . The label of an edge in E'_i is the label sequence along the defining path which is a q-ary sequence of length n_i . The edge set E'_i is a vector space of the same dimension as E_i .

In the trellis module T_{info} there are q edges out of each vertex in $V_{info} \setminus V_n$. Thus, we can assign a q-ary input label to each edge $e \in E_{info}$, and the trellis module T_{info} can used for encoding.

2) Trellis Representation of Subcodes of Convolutional Codes: A trellis representing subcodes of C can be written (after an initial transient) as the infinite composition of an *extended* trellis module $\overline{T} = (\overline{V}, \overline{E}, F_2)$ where $\overline{V} = \overline{V}_0 \cup \cdots \cup \overline{V}_n$ and $\overline{E} = \overline{E}_0 \cup \cdots \cup \overline{E}_{n-1}$ are partitions of \overline{V} and \overline{E} , respectively. Each vertex of \overline{V}_i corresponds to a subspace of V_i , and each edge of \overline{E}_i corresponds to a subspace of E_i . The number of distinct k-dimensional subspaces of an n-dimensional vector space over F_q , $k = 1, \ldots, n$, denoted by S(k, n, q), is [35, p. 444]

$$S(k,n,q) = \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)\cdots(q - 1)} = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

from which it follows that

$$|\bar{V}_i| = 1 + \sum_{j=1}^{\dim(V_i)} \left[\frac{\dim(V_i)}{j} \right]_q$$
 and $|\bar{E}_i| = 1 + \sum_{j=1}^{\dim(E_i)} \left[\frac{\dim(E_i)}{j} \right]_q$.

The left (resp. right) vertex of an edge e is denoted by $v^L(e)$ (resp. $v^R(e)$). The label of an edge e is denoted by c(e). Note that the edge e could either be an edge in the trellis module T or in the extended trellis module \overline{T} .

The connections in the extended trellis module \overline{T} are established as follows. An edge $\overline{e} \in \overline{E}_i$ corresponds to a subspace of E_i of dimension $d(\overline{e})$ and basis $\{e_0(\overline{e}), \ldots, e_{d(\overline{e})-1}(\overline{e})\}$. The vertex in \overline{V}_i that corresponds to the vector space spanned by $\{v^L(e_0(\overline{e})), \ldots, v^L(e_{d(\overline{e})-1}(\overline{e}))\}$ is connected to the vertex in \overline{V}_{i+1} that corresponds to the vector space spanned by $\{v^R(e_0(\overline{e})), \ldots, v^R(e_{d(\overline{e})-1}(\overline{e}))\}$ by the edge \overline{e} . The binary label $c(\overline{e})$ of \overline{e} is 1 if at least one of the q-ary labels $c(e_j(\overline{e}))$ of $e_j(\overline{e}) \in E_i$, $j = 0, \ldots, d(\overline{e}) - 1$, is non-zero. Otherwise, it is 0. In the case of parallel edges in \overline{T} with the same label, we keep only one.

Note that an information bit-oriented extended trellis module \overline{T}_{info} can be obtained by sectionalization as described above. A trellis $\overline{\mathcal{T}}_{info}$ constructed as the infinite composition of the trellis module \overline{T}_{info} has paths that are in oneto-many correspondence with subcodes of C. The label sequence of a path in $\overline{\mathcal{T}}_{info}$ is a binary sequence where the set of 1-positions is equal to the support set of the subcodes represented by the given path. Since distinct subcodes could have equal support sets, there could be paths in $\overline{\mathcal{T}}_{info}$ that have equal label sequences.

Input labels can be assigned to the edges in the trellis module \bar{T}_{info} , and thus \bar{T}_{info} can be used for *encoding*, but the encoding is *not* one-to-one, since there could be more than one path in \bar{T}_{info} with the same input (and output) label sequence.

TABLE III

Vertex and edge complexities of the trellis modules $T_{
m info}$ and $ar{T}_{
m info}$ for (n,1,
u) binary convolutional codes

ν	2	3	4	5	6
$\mu(T_{\rm info})$	4	8	16	32	64
$\phi(T_{\rm info})$	8	16	32	64	128
$\mu(\bar{T}_{info})$	5	16	67	374	2825
$\phi(\bar{T}_{info})$	16	67	374	2825	29212

Example 2: Consider the (2, 1, 2) binary convolutional code defined by the parity-check matrix $\mathbf{H}(D) = (1 + D^2 1 + D + D^2)$. The trellis module T_{info} and the extended trellis module \bar{T}_{info} are both depicted in Fig. 4. Note that the trellis module \bar{T}_{info} is *non-linear*.



Fig. 4. (a) Basic trellis module T_{info} , and (b) extended trellis module \bar{T}_{info} of the convolutional code from Example 2.

The numbers of vertices and edges of an (extended) information bit-oriented trellis module T_{info} divided by k are called the vertex and edge complexities of T_{info} , and are given, respectively, by

$$\mu(T_{\text{info}}) = \frac{1}{k} \sum_{i \in \mathcal{I}_{\text{info}}} |V_i| \text{ and } \phi(T_{\text{info}}) = \frac{1}{k} \sum_{i \in \mathcal{I}_{\text{info}}} |E_i|$$

The vertex and edge complexities of T_{info} and \overline{T}_{info} are tabulated in Table III for different values of ν for $(n, 1, \nu)$ binary convolutional codes.

B. The Uniform Interleaver

Consider an (n, k, ν) systematic convolutional code which is terminated to the all-zero vertex at trellis depth $I \ge \lceil \nu/k \rceil k$ in \mathcal{T}_{info} where I is assumed to be a multiple of k. The resulting linear block code C has length

 $\delta = (I/k)n$ and dimension $I - \nu$. When $\nu = 0$, the convolutional code is actually a linear block code in which case we choose I = k.

Partition all the subcodes of C of dimension $d, d = 1, ..., I - \nu$, into equivalence classes based on their support sets. In particular, all subcodes within a specific subcode class are required to have the same support set, but the subcodes may have different dimensions.

We define the *subcode input-redundancy support size enumerating function* (SIRSEF) of C. The SIRSEF has the form

$$A^{C}(W,Z) = \sum_{w=1}^{I} \sum_{z=0}^{\delta-I} a^{C}_{w,z} W^{w} Z^{z}$$
(7)

where $a_{w,z}^C$ is the number of subcode classes of *C* of *input* support set size *w* and *parity* support set size *z*. When analyzing the performance it is useful to group the terms in the SIRSEF according to input support set size. The *conditional* SIRSEF

$$A_{w}^{C}(Z) = \sum_{z=0}^{\delta - I} a_{w,z}^{C} Z^{z}$$
(8)

enumerates subcode classes of different parity support set sizes associated with a particular input support set size. The conditional SIRSEF $A_w^C(Z)$ and the SIRSEF $A^C(W, Z)$ are related to one another through the following pair of expressions

$$A^{C}(W,Z) = \sum_{w=1}^{I} W^{w} A^{C}_{w}(Z) \text{ and } A^{C}_{w}(Z) = \frac{1}{w!} \cdot \frac{\partial^{w} A^{C}(W,Z)}{\partial W^{w}} \Big|_{W=0}.$$
(9)

Let $A_w^{C_x}(Z)$, x = a, b, denote the conditional SIRSEF for the constituent code C_x of a given PCCC C with interleaver length I. We assume that the interleaver is uniform. With a uniform interleaver, the $\binom{I}{w}$ possible sequences of length I and weight w occur with equal probability at the output of the interleaver when the interleaver is fed with a length-I and weight-w sequence. Let $S^{C}(W, Z)$ denote the *input-redundancy turbo stopping set size enumerating* function (IRTSSEF) of C. The IRTSSEF has the form

$$S^{\mathcal{C}}(W,Z) = \sum_{w=1}^{I} \sum_{z=0}^{2(\delta-I)} s^{\mathcal{C}}_{w,z} W^{w} Z^{z}$$
(10)

where $s_{w,z}^{\mathcal{C}}$ is the number of turbo stopping sets of \mathcal{C} of *input* size w and *parity* size z. The conditional IRTSSEF

$$S_{w}^{\mathcal{C}}(Z) = \sum_{z=0}^{2(\delta-I)} s_{w,z}^{\mathcal{C}} Z^{z}$$
(11)

enumerates turbo stopping sets of different parity sizes associated with a particular input size. The conditional IRTSSEF $S_w^{\mathcal{C}}(Z)$ and the IRTSSEF $S^{\mathcal{C}}(W, Z)$ are related to one another through the following pair of expressions

$$S^{\mathcal{C}}(W,Z) = \sum_{w=1}^{I} W^{w} S^{\mathcal{C}}_{w}(Z) \text{ and } S^{\mathcal{C}}_{w}(Z) = \frac{1}{w!} \cdot \frac{\partial^{w} S^{\mathcal{C}}(W,Z)}{\partial W^{w}} \Big|_{W=0}.$$
 (12)

A PCCC with a uniform interleaver has a uniform probability of matching a given support set in $A_w^{C_a}(Z)$ with any given support set in $A_w^{C_b}(Z)$. Thus, it follows that the conditional IRTSSEF for C is

$$S_{w}^{\mathcal{C}}(Z) = \frac{A_{w}^{C_{a}}(Z)A_{w}^{C_{b}}(Z)}{\binom{I}{w}}.$$
(13)

The turbo stopping set size enumerating function (TSSEF) for C is

$$S^{\mathcal{C}}(X) = \sum_{i=1}^{2\delta - I} s_i^{\mathcal{C}} X^i, \text{ where } s_i^{\mathcal{C}} = \sum_{w=1}^{\min(i,I)} s_{w,i-w}^{\mathcal{C}}$$
(14)

and $s_i^{\mathcal{C}}$ is the number of turbo stopping sets of size *i*. If only 1-dimensional subcodes are considered in the constituent conditional SIRSEFs $A_w^{C_a}(Z)$ and $A_w^{C_b}(Z)$, then we get the weight enumerating function (WEF) for \mathcal{C} .

1) The (7,4) Hamming Code: We consider the (7,4) Hamming code in its cyclic form in which case $\nu = 0$, I = k = 4, and $\delta = n = 7$. The SIRSEF of the Hamming code is

$$W(3Z^{2} + Z^{3}) + W^{2}(3Z + 3Z^{2} + 6Z^{3}) + W^{3}(1 + 3Z + 12Z^{2} + 4Z^{3}) + W^{4}(3Z + 3Z^{2} + Z^{3}).$$

The IRTSSEF for the PCCC \mathcal{C} with a uniform interleaver is

$$W(2.25Z^{4} + 1.5Z^{5} + 0.25Z^{6}) + W^{2}(1.5Z^{2} + 3Z^{3} + 7.5Z^{4} + 6Z^{5} + 6Z^{6}) + W^{3}(0.25 + 1.5Z + 8.25Z^{2} + 20Z^{3} + 42Z^{4} + 24Z^{5} + 4Z^{6}) + W^{4}(9Z^{2} + 18Z^{3} + 15Z^{4} + 6Z^{5} + Z^{6})$$
(15)

from which the TSSEF can be calculated. The result is

$$0.25X^3 + 3X^4 + 13.5X^5 + 38X^6 + 66.25X^7 + 45X^8 + 10X^9 + X^{10}$$

Note that the WEF for C is

$$1 + 0.25X^3 + 3X^4 + 7.5X^5 + 3X^6 + 0.25X^7 + X^{10}$$

We can check the result in (15) by computing the IRTSSEFs of the PCCCs constructed using all the 4! = 24 possible interleavers. The results are tabulated in Table IV. Only two types of IRTSSEF are possible. It is easy to verify that the average over all possible interleavers is equal to the expression in (15). Note that the WEF is dependent on the interleaver while the *non-codeword* TSSEF is the same for *all* interleavers.

2) Convolutional Codes as Constituent Codes: Let $T^{C_x}(W, Z, \Gamma, \Sigma)$, x = a, b, enumerate all subcode classes of C_x constructed from trellis paths in constituent trellis $\overline{\mathcal{T}}_{info}^x$ leaving the all-zero vertex at trellis depth zero, and remerging into the all-zero vertex at or before trellis depth I, with possible remerging into the all-zero vertex at other depths in between, subject to the constraint that, after remerging, the paths leave the all-zero vertex at the same trellis depth. In general,

$$T^{C_x}(W, Z, \Gamma, \Sigma) = \sum_{w=1}^{I} \sum_{z=0}^{\delta-I} \sum_{\gamma=2}^{I} \sum_{\sigma=1}^{\lfloor \gamma/2 \rfloor} t^{C_x}_{w, z, \gamma, \sigma} W^w Z^z \Gamma^{\gamma} \Sigma^{\sigma}$$
(16)

where $t_{w,z,\gamma,\sigma}^{C_x}$ is the number of subcode classes of C_x of input support set size w and parity support set size z constructed from trellis paths of length γ , and with σ remergings with the all-zero vertex. Notice that each subcode class in $T^{C_x}(W, Z, \Gamma, \Sigma)$ of input support set size w and parity support set size z constructed from trellis paths of length γ , and with σ remergings with the all-zero vertex, gives rise to $\binom{I-\gamma+\sigma}{\sigma}$ subcode classes with the same input and parity support set sizes. Thus, the conditional SIRSEF $A_w^{C_x}(Z)$ can be written as

$$A_w^{C_x}(Z) = \sum_{z=0}^{\delta-I} \left[\sum_{\gamma=2}^{I} \sum_{\sigma=1}^{\lfloor \gamma/2 \rfloor} \binom{I-\gamma+\sigma}{\sigma} t_{w,z,\gamma,\sigma}^{C_x} \right] Z^z.$$
(17)

TABLE IV

IRTSSEFS, TSSEFS, AND WEFS OF THE PCCCS CONSTRUCTED USING ALL THE 4! = 24 possible interleavers and with the (7, 4)

π	$S^{\mathcal{C}}(W,Z), S^{\mathcal{C}}(X), \text{ and } WEF(X)$
3210	
3201	
3120	
3102	
3012	
3021	
2310	$S^{\mathcal{C}}(W,Z) = 1 + W(2Z^4 + 2Z^5)$
2301	$+ W^2(Z^2 + 4Z^3 + 7Z^4 + 6Z^5 + 6Z^6)$
2130	+ $W^3(2Z + 8Z^2 + 20Z^3 + 42Z^4 + 24Z^5 + 4Z^6)$
2031	$+ W^4(9Z^2 + 18Z^3 + 15Z^4 + 6Z^5 + Z^6)$
1230	$S^{\mathcal{C}}(X) = 1 + 3X^4 + 14X^5 + 38X^6 + 66X^7 + 45X^8 + 10X^9 + X^{10}$
1320	WEF(X) = $1 + 3X^4 + 8X^5 + 3X^6 + X^{10}$
1302	
1032	
0231	
0132	
0312	
0321	
2103	$S^{\mathcal{C}}(W,Z) = 1 + W(3Z^4 + Z^6) + W^2(3Z^2 + 9Z^4 + 6Z^5 + 6Z^6)$
2013	$+ W^{3}(1+9Z^{2}+20Z^{3}+42Z^{4}+24Z^{5}+4Z^{6})$
1203	+ $W^4(9Z^2 + 18Z^3 + 15Z^4 + 6Z^5 + Z^6)$
1023	$S^{\mathcal{C}}(X) = 1 + X^3 + 3X^4 + 12X^5 + 38X^6 + 67X^7 + 45X^8 + 10X^9 + X^{10}$
0213	WEF(X) = $1 + X^3 + 3X^4 + 6X^5 + 3X^6 + X^7 + X^{10}$
0123	

HAMMING CODE AS CONSTITUENT CODES.

Finding a closed-form expression for the conditional SIRSEF (as we did in Section V-B.1 when the constituent codes were Hamming codes) for a given (large) value of the interleaver length I is difficult. For this reason, we will use an algorithmic approach to compute the most significant terms of the conditional SIRSEF in (17). One approach is to use the algorithm to be described in Section V-C with only one constituent code. See Section V-C below for details.

C. Enumeration of Small-Size Turbo Stopping Sets for a Particular Interleaver

Let T_{info}^x and \bar{T}_{info}^x denote the information bit-oriented and the extended information bit-oriented trellis modules of constituent code C_x , x = a, b. We assume that T_{info}^x and \bar{T}_{info}^x both have input and output labels assigned to the edges. The trellises constructed from the trellis modules T_{info}^x and \bar{T}_{info}^x are denoted by \mathcal{T}_{info}^x , respectively.

Let S_{C_x} denote the subset of $\{0,1\}^{N_x}$ of label sequences of paths that begin and end at the all-zero vertex at trellis depths 0 and I of the trellis $\overline{\mathcal{T}}_{info}^x$, x = a, b. Finally, let $S_{\mathcal{C}} = S_{\mathcal{C}}(K, S_{C_a}, S_{C_b}, \pi)$ denote the PCCC with

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information length K, constituent encoders S_{C_a} and S_{C_b} , and interleaver π . Note that the support sets of words of S_C are in one-to-one correspondence with *all* turbo stopping sets of C. Thus, the minimum Hamming weight of S_C is equal to the minimum turbo stopping set size of C. In general, the set of turbo stopping sets can be obtained by *turbo encoding* using the constituent encoders S_{C_a} and S_{C_b} . Note that there could exist several paths in $\overline{\mathcal{T}}_{info}^x$ with the same input label sequence from which it follows that the complexity of turbo encoding could be more than linear in K.

Let $\Pi(S_{\mathcal{C}}, \tau)$ be the problem of finding all words of $S_{\mathcal{C}}$ of Hamming weight $\leq \tau$. This problem is equivalent of finding an exhaustive list of all turbo stopping sets of size $\leq \tau$. To simplify notation we assume below that the I - K redundant systematic bits appear at the end of the input block.

A constraint set F is a set $\{(p_i, u_{p_i}) : u_{p_i} \in \{0, 1\} \forall p_i \in \Gamma_p\}$, where $\Gamma_p \subseteq \{0, \dots, K-1\}$ is a set of distinct positions. For any constraint set F, let $U^{(F)}$ be the set of length-K vectors $\{\mathbf{u} = (u_0, \dots, u_{K-1}) : u_j = u \text{ if } (j, u) \in F, u_j \in \{0, 1\} \text{ if } j \notin \Gamma_p\}$. Let the length l = l(F) be the number of constraints. We will start with a constraint set F of length l of the form $\{(0, u_0), (1, u_1), \dots, (l-1, u_{l-1})\}$, i.e., it applies consecutively to the first l positions. When the turbo interleaver π acts on F, we obtain a new constraint set $\pi F = \{(\pi(p_i), u_{p_i})\}$, where in general the constrained positions are scattered over the input block.

Let $\mathcal{S}_{\mathcal{C}}^{(F)}$ be the subset of $\mathcal{S}_{\mathcal{C}}$ that is obtained by encoding the input vectors in $U^{(F)}$, let w(F) be the minimum Hamming weight of $\mathcal{S}_{\mathcal{C}}^{(F)}$, and let w'(F) be any lower bound for w(F). The pseudo-code of the algorithm to solve $\Pi(\mathcal{S}_{\mathcal{C}},\tau)$ is given below. Note that the algorithm has the same *structure* as the algorithm proposed by Garello *et al.* in [8] to solve $\Pi(\mathcal{C},\tau)$. We will refer to this algorithm as the GPB algorithm. However, there are some differences that we will discuss below.

Let $\mathcal{S}_{C_a}^{(F)}$ be the subset of words generated by the constituent encoder \mathcal{S}_{C_a} when the input vectors are contained in $U^{(F)}$, and let $w_a(F)$ be the minimum Hamming weight of $\mathcal{S}_{C_a}^{(F)}$. Select any vector from $U^{(F)}$ as an input sequence of \mathcal{S}_{C_a} . After l(F) time units the encoder has reached a subset $\{\sigma_{a,0}^{(F)}, \ldots, \sigma_{a,\rho(F)-1}^{(F)}\}$ of cardinality $\rho(F)$ of the set

of trellis vertices of $\overline{\mathcal{T}}_{info}^{a}$ at trellis depth l(F). The trellis path of $\overline{\mathcal{T}}_{info}^{a}$ from the all-zero vertex at trellis depth 0 to vertex $\sigma_{a,i}^{(F)}$ at trellis depth l(F) of minimum Hamming weight is denoted by $\mathbf{c}_{a,i}^{(F)}$. Let $w(\mathbf{c}_{a,i}^{(F)})$ be the Hamming weight of $\mathbf{c}_{a,i}^{(F)}$, and let $w(\sigma_{a,i}^{(F)}, l(F))$ be the minimum Hamming weight of any path from vertex $\sigma_{a,i}^{(F)}$ at trellis depth l(F) to the all-zero vertex at trellis depth *I*. In general, it holds that

$$w_{a}(F) = \min_{i=0,\dots,\rho(F)-1} \left(w(\mathbf{c}_{a,i}^{(F)}) + w(\sigma_{a,i}^{(F)}, l(F)) \right).$$
(18)

The weights $w(\sigma_{a,i}^{(F)}, l(F))$ can be computed in a preprocessing stage using the Viterbi algorithm. Actually, the weights $w(\sigma_{a,i}^{(F)}, l(F))$ depend only on the vertex $\sigma_{a,i}^{(F)}$ if l(F) is not too close to K and $\overline{\mathcal{T}}_{info}^{a}$ is *non-catastrophic*. This reduces the memory requirements. However, the weights $w(\mathbf{c}_{a,i}^{(F)})$ have to be computed during the course of the algorithm. In general, the weights $w(\mathbf{c}_{a,i}^{(F)})$ can be computed by a *constrained* Viterbi algorithm. Note that in the original GPB algorithm there is no need to apply a constrained Viterbi algorithm here, since $\rho(F) = 1$, and there is a unique path in \mathcal{T}_{info}^{a} from the all-zero vertex at trellis depth 0 to the vertex $\sigma_{a,0}^{(F)}$ at trellis depth l(F).

Similarly, let $S_{C_b}^{(\pi F)}$ be the subset of words generated by the constituent encoder S_{C_b} when the input vectors are contained in $U^{(\pi F)}$. Also, let $w_b(\pi F)$ be the minimum Hamming *parity* weight of $S_{C_b}^{(\pi F)}$. We have

$$w_{\text{bound}}(F) = w_a(F) + w_b(\pi F) \le w(F).$$
⁽¹⁹⁾

Note that $w_{\text{bound}}(F)$ is a lower bound on w(F), since the sequence of input bits giving the minimum-weight path in the second constituent encoder trellis $\overline{\mathcal{T}}_{\text{info}}^b$ is not necessarily an interleaved version of the sequence of input bits giving the minimum-weight path in the first constituent encoder trellis $\overline{\mathcal{T}}_{\text{info}}^a$. The value of $w_b(\pi F)$ can be determined by the use of a constrained Viterbi algorithm. Since the positions of πF are in general not consecutive, the complexity of calculating the value of $w_b(\pi F)$ by a constrained Viterbi algorithm is larger than the complexity of calculating the weights $w(\mathbf{c}_{a,i}^{(F)})$, $i = 0, \dots, \rho(F) - 1$, needed in (18).

In [7] we outlined several improvements to the basic GPB algorithm for solving $\Pi(\mathcal{C}, \tau)$. All of the improvements described in the context of solving $\Pi(\mathcal{C}, \tau)$ can be applied when solving $\Pi(\mathcal{S}_{\mathcal{C}}, \tau)$.

From Table III we observe that the edge complexity of \bar{T}_{info} is large compared to the edge complexity of T_{info} even for $\nu = 4$. To reduce complexity we propose to remove some of the edges from \bar{T}_{info}^x . For instance, all edges in \bar{T}_{info}^x that correspond to edge subspaces of T_{info}^x of dimension $\geq \alpha$, for some integer $\alpha \geq 2$, can be removed.

D. Remarks

We remark that in principle every *trellis-based* turbo code weight spectrum computation or estimation algorithm can be adapted in a straightforward manner to find small-size turbo stopping sets. The only requirement is that the basic trellis module is substituted with the extended trellis module introduced above. We have considered the impulse methods by Berrou and Vaton [12], Vila-Casado and Garello [11], and Crozier *et al.* [10] with promising results.

E. Numerical Results

We have applied the algorithm from Section V-C with the improvements from [7] on a few example codes. The example codes are constructed without considering turbo stopping sets. Consider the (828, 270, 36) turbo code constructed by Crozier *et al.* in [10]. The code has an optimized minimum distance, is based on a DRP interleaver, and is constructed from nominal rate-1/2, $\nu = 3$ constituent codes defined by the parity-check matrix $\mathbf{H}(D) = (1 + D + D^3 \ 1 + D^2 + D^3)$. The last polynomial which is irreducible and primitive has been chosen as the parity polynomial making the constituent encoders recursive. For this code there are 59 (0), 58 (58), and 283 (283) stopping sets (codewords) of size (weight) 33, 36, and 37, respectively. The minimum turbo stopping set size is smaller than the code's minimum distance. This is typically what happens for both short and moderate-length distance-optimized DRP interleavers. With the uniform interleaver, however, there are

 $0.08538(0.08538), 1.245 \cdot 10^{-7}(1.245 \cdot 10^{-7}), 0.01958(0.01958), 0.66860(0.66860), 1.91184(1.90691), 0.01958(0.01958), 0.66860(0.66860), 0.91184(0.90691))$

2.48949 (1.83803), 7.22456 (6.12582), 8.24370 (6.66759), 7.72356 (4.50883)

stopping sets (codewords) of size (weight) 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, and 23, respectively. Thus, small-size turbo stopping sets is not a problem with the uniform interleaver for these parameters. As another example consider an interleaver length of 1200. In [10], Crozier *et al.* constructed an impressive DRP interleaver of length 1200 with a dither length of 8. The corresponding turbo code has length 3600, dimension 1194, and minimum distance 49. For this code we have found turbo stopping sets of size 47. Results for a range of interleaver lengths are given in Fig. 5. The turbo codes are constructed from the same nominal rate-1/2 constituent codes as the (828, 270, 36) code above. For each interleaver length between 32 and 320, in which the dither length is a divisor, we have found the best (in terms of turbo code minimum distance) DRP interleaver with a dither length of 4. Also, for the same interleavers we have found the minimum turbo stopping set size. These results are plotted in Fig. 5. Note that for several interleaver lengths the minimum turbo stopping set size is smaller than the minimum distance.

VI. CONCLUSION AND FUTURE WORK

In this work we have considered the finite-length analysis of turbo decoding on the BEC. In the same way as iterative BP decoding of LDPC codes is simpler on the BEC than on other channels, turbo decoding can also be simplified on this channel. Based on this simplified turbo decoding algorithm we have introduced turbo stopping sets by adapting the concept of stopping sets from the theory of iterative BP decoding of LDPC codes. These turbo stopping sets characterize turbo decoding on the BEC, and an exact condition for decoding failure has been established as follows. Apply turbo decoding until the transmitted codeword has been recovered, or the decoder fails to progress further. Then the set of erased positions that will remain when the decoder stops is equal to the unique maximum-size turbo stopping set which is also a subset of the set of erased positions. Furthermore, we have presented some improvements of the basic turbo decoding algorithm on the BEC.



Fig. 5. Minimum distance and minimum turbo stopping set size as a function of interleaver length for distance-optimized DRP interleavers with a dither length of 4.

decoding algorithm has substantially better error performance as illustrated by simulation examples. In the second part of the paper an expression for the turbo stopping set size enumerating function under the uniform interleaver assumption was derived. Also, an efficient enumeration algorithm of small-size turbo stopping sets for a particular interleaver was given. The solution is based on the algorithm proposed by Garello *et al.* in 2001 to compute an exhaustive list of *all* low-weight codewords in a turbo code. In fact, it turns out that every trellis-based weight spectrum computation or estimation algorithm for turbo codes can be adapted to the case of finding small-size turbo stopping sets. In particular, the impulse methods by Berrou and Vaton, Vila-Casado and Garello, and Crozier *et al.* can be adapted in a straightforward manner.

One interesting topic for future work is the design of interleavers in which one considers both low-weight codewords and small-size turbo stopping sets. Trellis-based interleaver design algorithms can in a similar manner be adapted to this problem using the extended trellis module.

Finally, we remark that the findings in this paper can be adapted in a fairly straightforward manner to other turbo-like codes, e.g., RA codes, serial concatenated convolutional codes, and product codes.

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