# Ordinal sum of two binary operations being a t-norm on bounded lattice<sup>\*</sup>

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### Abstract

The ordinal sum of t-norms on a bounded lattice has been used to construct other t-norms. However, an ordinal sum of binary operations (not necessarily t-norms) defined on the fixed subintervals of a bounded lattice may not be a t-norm. Some necessary and sufficient conditions are presented in this paper for ensuring that an ordinal sum on a bounded lattice of two binary operations is, in fact, a t-norm. In particular, the results presented here provide an answer to an open problem put forward by Ertuğrul and Yeşilyurt [Ordinal sums of triangular norms on bounded lattices, Inf. Sci., 517 (2020) 198-216].

Keywords: Incomparability; lattice; ordinal sum; triangular norm.

#### 1. Introduction

Triangular norms (t-norms) were systematically investigated by Schweizer and Sklar [1, 2, 3] in the framework of probabilistic metric spaces aiming at an extension of the triangle inequality. As an extension of the logical connective conjunction in classical two-valued logic, t-norms have been used widely in many different areas, such as in decision making [4, 5, 6, 7, 8, 9], statistics [10], fuzzy set theory [11, 12, 13]. Clifford introduced the notion of ordinal sum on the unit interval [0, 1], providing a method to produce new t-norms from given ones since the unit interval together with a t-norm forms a semigroup [14]. As a result, a continuous t-norm can be represented as an ordinal sum of the product t-norm and Łukasiewicz's t-norm [15, 16, 17]. Afterward, t-norms were generalized to more general structures, including posets and bounded lattices, and their characteristics were extensively investigated [18, 19, 20, 21]. Saminger [22] extended the ordinal sum of t-norms on the unit interval to the ordinal sum of t-norms on subintervals of a bounded lattice. However, Saminger's definition of ordinal sum of t-norms on a bounded lattice does not always generate a t-norm. From this point of view, the constraints were provided in [23, 24] to ensure that the ordinal sum of t-norms defined on the subintervals of a bounded lattice generates a t-norm. Then, some researchers presented some construction methods for t-norms on a bounded lattice to modify Saminger's ordinal sum and considered the ordinal sum problem for a particular class of lattices. In particular, in the papers [25, 26, 27, 28, 29, 30, 31, 32, 33, 34], Saminger's ordinal sum method with one special summand was modified to guarantee that it is a t-norm on a bounded lattice. El-Zekey [35] studied the ordinal sum of t-norms on bounded lattices written as a lattice-based sum of lattices. Dvořák and Holčapek [30, 31] introduced a new ordinal sum construction of t-norms on bounded lattices based on interior and closure operators. Ouyang et al. [36] proposed an alternative definition of ordinal sum of countably many t-norms on subintervals of a complete lattice and proved that it is a t-norm.

Recently, Ertuğrul and Yeşilyurt [37] have extended the results related to ordinal sum with one summand either based on some arbitrary or fixed subinterval and dealt with the ordinal sum operations

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with more summands such that the ordinal sum on a bounded lattice of arbitrary t-norms yields again a t-norm. In particular, they have shown that it is possible to give a construction method to obtain a t-norm on a bounded lattice L derived from two t-norms  $T_1$  and  $T_2$  on the subintervals [a, 1] and [0, a], respectively, for  $a \in L \setminus \{0, 1\}$  without any additional requirement. In the same paper, they have also proposed an open problem: if we take an associative, commutative, and monotone binary operation instead of at least one of t-norms on the subintervals of L, will the same method work? If not, what kind of modification is required? Although there are many results on t-norms on bounded lattices, their structure is still unclear. Hence, the proposal in [37] requires further study of t-norms on bounded lattices to obtain as many of their new classes as possible. In this paper, motivated by the above-mentioned suggestion, we first demonstrate that the ordinal sum method with two summands introduced in [37] may not work on a bounded lattice L when taking  $T_2$  on the subinterval [0, a] as an associative, commutative, and increasing binary operation, not necessarily a t-norm. Then, we look for necessary and sufficient conditions to ensure that such an ordinal sum method is increasing. Moreover, considering  $T_1$  and  $T_2$  as two t-subnorms on the subintervals [a, 1] and [0, a], respectively, we provide some necessary and sufficient conditions for the ordinal sum of  $T_1$  and  $T_2$  being a t-norm on a bounded lattice L. In this way, we give a complete answer to the above open problem.

The remainder of this paper is organized as follows. In Section 2, we briefly recall some basic notions and results related to lattices and t-norms on a bounded lattice. In Section 3, we first review the related ordinal sum methods on a bounded lattice in the sense of Saminger [22], and Ertuğrul and Yeşilyurt [37]. Then, we are interested in two open problems proposed in [37]. According to their proposals, Section 4 is devoted to investigating whether the ordinal sum method with two summands on a bounded lattice in the sense of Ertuğrul and Yeşilyurt [37] is increasing. Here, we consider any binary operation instead of at least one of two summands being a t-norm on the fixed subinterval. Some necessary and sufficient conditions are presented in Section 5 for guaranteeing that the ordinal sum method on a bounded lattice of two t-subnorms is, in fact, a t-norm. We end with some concluding remarks and future works in Section 6.

## 2. Preliminaries

In this section, we recall some basic notions and results related to lattices and t-norms on a bounded lattice.

A *lattice* is a nonempty set L equipped with a partial order  $\leq$  such that each two elements  $x, y \in L$  have a greatest lower bound, called *meet* or *infimum*, denoted by  $x \wedge y$ , as well as a smallest upper bound, called *join* or *supremum*, denoted by  $x \vee y$  [38]. For  $x, y \in L$ , the symbol x < y means that  $x \leq y$  and  $x \neq y$ . If  $x \leq y$  or y < x, then we say that x and y are comparable. Otherwise, we say that x and y are *incomparable*, in this case, we use the notation x || y. The set of all elements of L that are incomparable with a is denoted by  $I_a$ , i.e.,  $I_a = \{x \in L : x || a\}$ .

A bounded lattice is a lattice  $(L, \leq)$  which has the top and the bottom elements, which are denoted by 1 and 0, respectively; that is, two elements  $1, 0 \in L$  exist such that  $0 \leq x \leq 1$  for all  $x \in L$ .

**Definition 1** ([38]). Let  $(L, \leq, 0, 1)$  be a bounded lattice and  $a, b \in L$  with  $a \leq b$ . The subinterval [a, b] is defined by

$$[a,b] = \{x \in L : a \le x \le b\}.$$

Other subintervals such as [a, b), (a, b] and (a, b) can be defined similarly. Obviously,  $([a, b], \leq)$  is a bounded lattice with the top element b and the bottom element a.

Let  $(L, \leq, 0, 1)$  be a bounded lattice, [a, b] be a subinterval of L,  $T_1$  and  $T_2$  be two binary operations on  $[a, b]^2$ . If there holds  $T_1(x, y) \leq T_2(x, y)$  for all  $(x, y) \in [a, b]^2$ , then we say that  $T_1$  is less than or equal to  $T_2$  or, equivalently, that  $T_2$  is greater than or equal to  $T_1$ , and written as  $T_1 \leq T_2$ .

**Definition 2** ([18, 19, 36]). Let  $(L, \leq, 0, 1)$  be a bounded lattice and [a, b] be a subinterval of L. A binary operation  $T : [a, b]^2 \longrightarrow [a, b]$  is said to be a *t*-norm on [a, b] if, for any  $x, y, z \in [a, b]$ , the following conditions are fulfilled:

- (T<sub>1</sub>) (commutativity) T(x, y) = T(y, x);
- (T<sub>2</sub>) (associativity) T(T(x,y),z) = T(x,T(y,z));
- (T<sub>3</sub>) (increasingness) If  $x \le y$ , then  $T(x, z) \le T(y, z)$ ;
- (T<sub>4</sub>) (neutrality) T(b, x) = x.

Notice that b is a neutral element for a t-norm  $T : [a, b]^2 \longrightarrow [a, b]$  while a is a zero element for T, i.e., T(a, x) = a for all  $x \in [a, b]$ .

**Example 1.** The following binary operations are examples of t-norms on the subinterval [a, b] of a bounded lattice L. The meet (or infimum) t-norm  $T_M$  on [a, b] and the drastict product t-norm  $T_D$  on [a, b] are defined by

$$T_M: [a,b]^2 \longrightarrow [a,b], \ T_M(x,y) = x \wedge y,$$

and

$$T_D: [a,b]^2 \longrightarrow [a,b], \ T_D(x,y) = \begin{cases} x \wedge y, & b \in \{x,y\}, \\ a, & \text{otherwise.} \end{cases}$$

We observe that  $T_M$  and  $T_D$ , respectively, are the greatest and the least smallest t-norms on the subinterval [a, b].

## 3. Ordinal sums of t-norms on bounded lattices

T-norms have been extensively studied on bounded lattices similarly to their counterparts on the unit interval. Saminger proposed an ordinal sum of t-norms defined on some fixed subinterval of a bounded lattice [22]. We can present Saminger's method as follows:

**Theorem 1** ([22]). Let  $(L, \leq, 0, 1)$  be a bounded lattice and  $a \in L \setminus \{0, 1\}$ . If  $T_1 : [a, 1]^2 \longrightarrow [a, 1]$  and  $T_2 : [0, a]^2 \longrightarrow [0, a]$  are two t-norms on the subintervals [a, 1] and [0, a] of L, respectively, then, the binary operation  $T^{(S)} : L^2 \longrightarrow L$  defined by the following formula (3.1) is an ordinal sum of t-norms  $T_1$  and  $T_2$  on L.

$$T^{(S)}(x,y) = \begin{cases} T_1(x,y), & (x,y) \in [a,1]^2, \\ T_2(x,y), & (x,y) \in [0,a]^2, \\ x \wedge y, & otherwise. \end{cases}$$
(3.1)

According to Saminger [22], however, the above ordinal sum of t-norms on the fixed subinterval is not always a t-norm. She introduced some conditions which the bounded lattice L needs to fulfill such that the ordinal sum method given by the formula (3.1) produces a t-norm on L.

**Theorem 2** ([22]). Let  $(L, \leq, 0, 1)$  be a bounded lattice,  $a \in L \setminus \{0, 1\}$ ,  $T_1 : [a, 1]^2 \longrightarrow [a, 1]$  and  $T_2 : [0, a]^2 \longrightarrow [0, a]$  be two t-norms on the subintervals [a, 1] and [0, a] of L, respectively. Then the following statements are equivalent:

- I) The ordinal sum  $T: L^2 \longrightarrow L$  of  $T_1$  and  $T_1$  defined by the formula (3.1) is a t-norm on L.
- II) For all  $x \in L$ , it holds that
  - II-1) if x is incomparable with a, then it is incomparable to all element in [a, 1),
  - II-2) if x is incomparable with a, then it is incomparable to all elements in (0, a].

Several researchers characterized when Saminger's ordinal sum of t-norms always yield a t-norm on a bounded lattice [23, 24], while other researchers attempted to modify Saminger's ordinal sum or considered the ordinal sum problem for a particular class of lattices [25, 26, 27, 33, 34]. In recent times, Ertuğrul and Yeşilyurt have introduced an ordinal sum method to produce a t-norm on a bounded lattice L by using the t-norms defined on the indicated subintervals of L [37]. Their ordinal sum method is presented in the following Theorem 3.

**Theorem 3** ([37]). Let  $(L, \leq, 0, 1)$  be a bounded lattice and  $a \in L \setminus \{0, 1\}$ . If  $T_1 : [a, 1]^2 \longrightarrow [a, 1]$  and  $T_2 : [0, a]^2 \longrightarrow [0, a]$  are two t-norms on the subintervals [a, 1] and [0, a] of L, respectively, then, the binary operation  $T : L^2 \longrightarrow L$  defined by the following formula (3.2) is a t-norm on L.

$$T(x,y) = \begin{cases} T_1(x,y), & (x,y) \in [a,1)^2, \\ T_2(x,y), & (x,y) \in [0,a)^2, \\ x \wedge y, & (x,y) \in [0,a) \times [a,1) \\ & \cup ([a,1) \times [0,a)) \\ & \cup (L \times \{1\}) \\ & \cup (\{1\} \times L), \\ T_2(x \wedge a, y \wedge a), & otherwise. \end{cases}$$
(3.2)

By this, the following Question 1 arises quite naturally, which has been proposed as an open problem by Ertuğrul and Yeşilyurt [37].

Question 1 ([37]). Given a bounded lattice  $(L, \leq, 0, 1)$  and  $a \in L \setminus \{0, 1\}$ , if we take in Theorem 3 an associative, commutative and monotone binary operation instead of at least one of the t-norms  $T_1$  and  $T_2$  defined on the subintervals [a, 1] and [0, a] of L, respectively, does the function T given by the formula (3.2) need to yield a t-norm on L?

We introduce the following Proposition 1 to help us answer this question.

**Proposition 1.** Let  $(L, \leq, 0, 1)$  be a bounded lattice,  $a \in L \setminus \{0, 1\}, T_1 : [a, 1]^2 \longrightarrow [a, 1]$  and  $T_2 : [0, a]^2 \longrightarrow [a, 1]$ [0, a] be two binary operations on the subintervals [a, 1] and [0, a] of L, respectively. If the function  $T: L^2 \longrightarrow L$  defined by the formula (3.2) is increasing, then  $T_1(x_1, y_1) \leq x_1 \wedge y_1$  and  $T_2(x_2, y_2) \leq x_2 \wedge y_2$ for any  $(x_1, y_1) \in [a, 1)^2$  and  $(x_2, y_2) \in [0, a)^2$  hold.

*Proof.* From the increasingness and the definition of T, it follows that for any  $(x_1, y_1) \in [a, 1)^2$ ,  $T_1(x_1, y_1) =$  $T(x_1, y_1) \leq T(x_1, 1) = x_1$  and  $T_1(x_1, y_1) = T(x_1, y_1) \leq T(1, y_1) = y_1$ . Hence, we have  $T_1(x_1, y_1) \leq T(x_1, y_1$  $x_1 \wedge y_1$  for any  $(x_1, y_1) \in [a, 1)^2$ . 

Similarly, it is shown that  $T_2(x_2, y_2) \le x_2 \land y_2$  for any  $(x_2, y_2) \in [0, a)^2$ .

By using Proposition 1, we provide the following Example 2 answering negatively to Question 1. We first take in Theorem 3 an associative, commutative, and monotone binary operation instead of at least one of the t-norms on the subintervals of a bounded lattice L. Then, we show that the function T given by the formula (3.2) is not a t-norm on L.

**Example 2.** Let L = [0,1] and  $a = \frac{1}{2}$ . Define  $T_1 : [\frac{1}{2},1]^2 \longrightarrow [\frac{1}{2},1]$  and  $T_2 : [0,\frac{1}{2}]^2 \longrightarrow [0,\frac{1}{2}]$  by  $T_1(x,y) = x \wedge y$  and  $T_2(x,y) = \frac{1}{2}$  for all  $x, y \in [0,\frac{1}{2}]$ , respectively. It is easy to observe that  $T_1$  is a t-norm on  $[\frac{1}{2}, 1]$ , and  $T_2$  is an associative, commutative, and monotone binary operation on  $[0, \frac{1}{2}]$ . It follows from Proposition 1 that  $T: [0,1]^2 \longrightarrow [0,1]$  given by the formula (3.2) is not increasing, i.e.,  $\tilde{T}$  is not a t-norm. Because, if we take  $x = \frac{1}{4} \le y = \frac{2}{3}$  and  $z = \frac{1}{4}$ , we obtain  $T(x,z) = T(\frac{1}{4},\frac{1}{4}) = T_2(\frac{1}{4},\frac{1}{4}) = \frac{1}{2} > \frac{1}{4} = \frac{2}{3} \land \frac{1}{4} = \frac{1}{2} \land \frac{1}{4} = \frac{1}{4} \land \frac{1}{4} \land \frac{1}{4} = \frac{1}{4} \land \frac{1}{4}$  $T(\frac{2}{3}, \frac{1}{4}) = T(y, z).$ 

In the paper [37], it has also been posed another open problem introduced in Question 2.

Question 2 ([37]). If the function T given by the formula (3.2) does not need to be a t-norm on a bounded lattice L when considering an associative, commutative and monotone binary operation instead of at least one of the t-norms on the subintervals of L in Theorem 3, what kind of modification is required?

We are motivated in this paper from Example 2 that gives a negative answer to Question 1. We aim to present a sufficient and necessary condition for ensuring that the ordinal sum of two families of binary operations on the subintervals of a bounded lattice L defined by the formula (3.2) is always a t-norm on L. In particular, this paper, including Question 2, obtains some types of characterizations for the ordinal sum defined by the formula (3.2) being an increasing binary operation in Section 3 and a t-norm in Section 4. These characterizations exactly solve Question 2 and show that the ordinal sum defined by the formula (3.2) is closely related to the boundary values of the binary operation on  $((\{a\} \cup I_a) \times \{a\}) \cup ((\{a\} \cup I_a) \times \{a\}) \text{ for } a \in L \setminus \{0, 1\}.$ 

### 4. Ordinal sum operation with two summands being increasing

In this section, we concentrate on conditions that the operation  $T: L^2 \longrightarrow L$  defined by the formula (3.2) is increasing with respect to both variables on a bounded lattice L.

The following Theorem 4 provides a partial answer to Question 2. To put a finer point on it, we present a sufficient and necessary condition for the ordinal sum  $T: L^2 \longrightarrow L$  of two commutative operations  $T_1$ and  $T_2$  defined by the formula (3.2) being an increasing operation on a bounded lattice L.

**Lemma 1.** Let  $(L, \leq, 0, 1)$  be a bounded lattice and  $a \in L \setminus \{0, 1\}$ . Then, there holds  $x \wedge a < a$  for any  $x \in I_a$ .

The result is proved straightforwardly.

**Theorem 4.** Let  $(L, \leq, 0, 1)$  be a bounded lattice and  $a \in L \setminus \{0, 1\}$ . Assume that  $T_1 : [a, 1]^2 \longrightarrow [a, 1]$ and  $T_2: [0,a]^2 \longrightarrow [0,a]$  are two commutative binary operations on the subintervals [a,1] and [0,a] of L, respectively, where  $T_1(x,y) \leq x \wedge y$  for all  $x, y \in [a,1]$  and  $T_2(x,y) \leq x \wedge y$  for all  $x, y \in [0,a]$ . Then the following statements are equivalent:

I) The binary operation  $T: L^2 \longrightarrow L$  defined by the formula (3.2) is increasing with respect to both variables on L.

- II) The following hold:
  - II-1)  $T_1$  and  $T_2$  are increasing with respect to both variables on the subintervals [a, 1) and [0, a) of L, respectively.
  - II-2) { $x \in L : x \in I_a \text{ and } T_2(x \wedge a, a) < x \wedge a$ } =  $\emptyset$ .

*III)* The following hold:

- III-1)  $T_1$  and  $T_2$  are increasing with respect to both variables on the subintervals [a, 1) and [0, a) of L, respectively.
- III-2)  $I_a = \emptyset$  or  $T_2(z \land a, a) = z \land a$  for all  $z \in I_a$ .

*Proof.* (I) $\Longrightarrow$ (II).

II-1) It is obtained straightforwardly.

II-2) On the contrary, suppose that  $\{x \in L : x \in I_a \text{ and } T_2(x \wedge a, a) < x \wedge a\} \neq \emptyset$ . This implies that there exists an element  $y \in I_a$  such that  $T_2(y \wedge a, a) < y \wedge a$ . By Lemma 1, it is obvious that  $a \wedge y \in [0, a)$ . Then, we have

$$T(y,a) = T_2(y \land a, a) < y \land a = (y \land a) \land a = T(y \land a, a).$$

$$(4.1)$$

By the increasingness of T, for  $a \land y \leq y$ , we have  $T(a \land y, a) \leq T(y, a)$ . This is a contradiction with the formula (4.1). Hence, it holds that  $\{x \in L : x \in I_a \text{ and } T_2(x \land a, a) < x \land a\} = \emptyset$ .

 $(II) \Longrightarrow (III).$ 

III-1) It is obtained straightforwardly.

III-2) Since  $\{x \in L : x \in I_a \text{ and } T_2(x \land a, a) < x \land a\} = \emptyset$ , then  $I_a = \emptyset$  or  $T_2(z \land a, a) \not\leq z \land a$  for any  $z \in I_a$ . If  $I_a = \emptyset$ , the proof is completed. Suppose that  $I_a \neq \emptyset$ . Then,  $T_2(z \land a, a) \not\leq z \land a$ . From the fact that the binary operation  $T_2 : [0, a]^2 \longrightarrow [0, a], T_2(x, y) \leq x \land y$  is assumed, it follows  $T_2(a \land z, a) \leq a \land z$ . Then, we obtain  $T_2(z \land a, a) = z \land a$ .

 $(III) \Longrightarrow (I).$ 

From the assumption of the binary operation  $T_2: [0,a]^2 \longrightarrow [0,a], T_2(x,y) \le x \land y$  and the definition of T, the commutativity of T and the fact that 1 and 0 are the neutral and zero elements of T, respectively, are evident. We demonstrate the increasingness of T. The proof is split into all remaining possible cases.

Claim 1.  $T(x, y) \leq x \wedge y$  for any  $x, y \in L$ .

Considering the commutativity of T, it is sufficient to check only that  $T(x, y) \leq x$ .

1-1) If x = 1, it is clear that  $T(1, y) = y \le 1$ .

1-2) If  $x \in [0, a]$ , then by the assumption that  $T_2: [0, a]^2 \longrightarrow [0, a], T_2(x, y) \le x \land y$ , it is verified that

$$T(x,y) = \begin{cases} T_2(x,y), & y \in [0,a), \\ x \land y, & y \in [a,1), \\ x, & y = 1, \\ T_2(x,y \land a), & y \in I_a, \end{cases}$$
$$\leq \begin{cases} x \land y, & y \in [0,a), \\ x, & y \in [a,1), \\ x, & y = 1, \\ x \land y \land a, & y \in I_a, \end{cases}$$
$$< y.$$

1-3) If  $x \in [a,1)$ , then by the assumptions that  $T_2: [0,a]^2 \longrightarrow [0,a], T_2(x,y) \leq x \wedge y$  and  $T_1:$ 

 $[a,1]^2 \longrightarrow [a,1], T_1(x,y) \le x \land y$ , it is verified that

$$T(x,y) = \begin{cases} x \land y, & y \in [0,a), \\ T_1(x,y), & y \in [a,1), \\ x, & y = 1, \\ T_2(a,y \land a), & y \in I_a, \end{cases}$$
$$\leq \begin{cases} x \land y, & y \in [0,a), \\ x \land y, & y \in [a,1), \\ x, & y = 1, \\ y \land a, & y \in I_a, \end{cases}$$
$$\leq y.$$

1-4) If  $x \in I_a$ , then by the assumption that  $T_2: [0,a]^2 \longrightarrow [0,a], T_2(x,y) \le x \land y$ , it is verified that

$$T(x,y) = \begin{cases} x, & y = 1, \\ T_2(x \land a, y \land a), & \text{otherwise,} \end{cases}$$
$$\leq \begin{cases} x, & y = 1, \\ x \land a \land y, & \text{otherwise,} \end{cases}$$
$$\leq y.$$

Claim 2.  $T(x, z) \leq T(y, z)$  for all  $x, y, z \in L$  with  $x \leq y$ . 2-1) If x = 1, then y = 1. This implies that  $T(x, z) = z \le z = T(y, z)$ . 2-2) If z = 1, then  $T(x, z) = x \le y = T(y, z)$ . 2-3) If y = 1, then by applying Claim 1,  $T(x, z) \le x \land z \le z = T(y, z)$ . 2-4) If  $x, y \in [0, a)$ , it is verified that

$$T(x,z) = \begin{cases} T_2(x,z), & z \in [0,a), \\ x \wedge z, & z \in [a,1), \\ T_2(x,z \wedge a), & z \in I_a, \end{cases}$$

and

$$T(y,z) = \begin{cases} T_2(y,z), & z \in [0,a), \\ y \land z, & z \in [a,1), \\ T_2(y,z \land a), & z \in I_a. \end{cases}$$

These, together with (III-1) and the fact that  $z \wedge a < a$  for  $z \in I_a$ , imply that  $T(x, z) \leq T(y, z)$ . 2-5). If  $x, y \in [a, 1)$ , it is verified that

$$T(x,z) = \begin{cases} x \land z, & z \in [0,a), \\ T_1(x,z), & z \in [a,1), \\ T_2(a,z \land a), & z \in I_a, \end{cases}$$

and

$$T(y,z) = \begin{cases} y \land z, & z \in [0,a), \\ T_1(y,z), & z \in [a,1), \\ T_2(a,z \land a), & z \in I_a. \end{cases}$$

These, together with (III-1) and the fact that  $z \wedge a < a$  for  $z \in I_a$ , imply that  $T(x, z) \leq T(y, z)$ . 2-6) If  $x, y \in I_a$ , then  $x \wedge a < a$  and  $y \wedge a < a$ . By (III-2), it is verified that

2-6) If 
$$x, y \in I_a$$
, then  $x \wedge a < a$  and  $y \wedge a < a$ . By (III-2), it is verified that

$$T(x, z) = \begin{cases} T_2(x \land a, z \land a), & z \in \{z_1 \in L \setminus \{1\} : z_1 \land a < a\}, \\ x \land a, & z \in \{z_1 \in L \setminus \{1\} : z_1 \land a = a\}, \end{cases}$$

and

$$T(y, z) = \begin{cases} T_2(y \land a, z \land a), & z \in \{z_1 \in L \setminus \{1\} : z_1 \land a < a\}, \\ y \land a, & z \in \{z_1 \in L \setminus \{1\} : z_1 \land a = a\}. \end{cases}$$

These, together with (III-1), imply that  $T(x, z) \leq T(y, z)$ .

2-7) If  $x \in [0, a)$  and  $y \in I_a$ , then  $y \wedge a < a$ . By (III-2), it is verified that

$$T(x,z) = \begin{cases} T_2(x,z), & z \in [0,a), \\ x, & z \in [a,1), \\ T_2(x,z \wedge a), & z \in I_a, \end{cases}$$

and

$$T(y, z) = \begin{cases} T_2(y \land a, z \land a), & z \in \{z_1 \in L \setminus \{1\} : z_1 \land a < a\}, \\ y \land a, & z \in \{z_1 \in L \setminus \{1\} : z_1 \land a = a\}, \end{cases}$$

These, together with (III-1) and the fact that  $z \wedge a < a$  for  $z \in I_a$ , imply that  $T(x, z) \leq T(y, z)$ .

2-8) If  $x \in I_a$  and  $y \in [a, 1)$ , then  $x \wedge a < a$ . By (III-2) and the assumption that  $T_2 : [0, a]^2 \longrightarrow [0, a]$ ,  $T_2(x,y) \leq x \wedge y$ , it is verified that

$$T(x, z) = \begin{cases} T_2(x \land a, z \land a), & z \in \{z_1 \in L \setminus \{1\} : z_1 \land a < a\}, \\ x \land a, & z \in \{z_1 \in L \setminus \{1\} : z_1 \land a = a\}, \end{cases}$$
$$\leq \begin{cases} x \land z \land a, & z \in \{z_1 \in L \setminus \{1\} : z_1 \land a < a\}, \\ x \land a, & z \in \{z_1 \in L \setminus \{1\} : z_1 \land a < a\}, \end{cases}$$

and

$$T(y,z) = \begin{cases} y \land z, & z \in [0,a), \\ T_1(y,z), & z \in [a,1), \\ T_2(a,z \land a), & z \in I_a, \end{cases}$$
$$\geq \begin{cases} y \land z, & z \in [0,a), \\ a, & z \in [a,1), \\ z \land a, & z \in I_a. \end{cases}$$

These, together with (III-1) and the fact that  $z \wedge a < a$  for  $z \in I_a$ , imply that  $T(x, z) \leq T(y, z)$ . 2-9) If  $x \in [0, a)$  and  $y \in [a, 1)$ , then by (III-2) and the assumption that  $T_2 : [0, a]^2 \longrightarrow [0, a]$ ,  $T_2(x,y) \leq x \wedge y$ , it is verified that

$$T(x,z) = \begin{cases} T_2(x,z), & z \in [0,a), \\ x, & z \in [a,1), \\ T_2(x,z \wedge a), & z \in I_a, \end{cases}$$
$$\leq \begin{cases} x \wedge z, & z \in [0,a), \\ a, & z \in [a,1), \\ x \wedge z \wedge a, & z \in I_a, \end{cases}$$

and

$$T(y,z) = \begin{cases} z, & z \in [0,a), \\ T_1(y,z), & z \in [a,1), \\ T_2(a,z \wedge a), & z \in I_a, \end{cases}$$
$$\geq \begin{cases} z, & z \in [0,a), \\ a, & z \in [a,1), \\ z \wedge a, & z \in I_a. \end{cases}$$

These imply that  $T(x, z) \leq T(y, z)$ .

Thereofere, we obtain that the binary operation  $T: L^2 \longrightarrow L$  defined by the formula (3.2) is increasing with respect to both variables on L.  By Theorem 4, the increasingness of the binary operation  $T: L^2 \longrightarrow L$  defined by the formula (3.2) is equivalent to the increasingness of the binary operations  $T_1$  on [a, 1) and  $T_2$  on [0, a) (excluding the right endpoint). This means that the ordinal sum of two commutative binary operations  $T_1$  and  $T_2$  given by the formula (3.2) satisfies the increasingness, commutativity, and neutrality properties, where  $T_1: [a, 1]^2 \longrightarrow [a, 1], T_1(x, y) \leq x \wedge y$  and  $T_2: [0, a]^2 \longrightarrow [0, a], T_2(x, y) \leq x \wedge y$  under the assumptions (III-1) and (III-2). This ordinal sum operation is not interested in the values of  $T_1$  and  $T_2$  on the boundary  $(\{a\} \times ([0, a] \setminus \{z \wedge a : z \in I_a\})) \cup (([0, a] \setminus \{z \wedge a : z \in I_a\}) \times \{a\})$  and  $(\{1\} \times [a, 1]) \cup ([a, 1] \times \{1\})$ , respectively.

## 5. Ordinal sum operation with two summands being a t-norm

Theorem 4 deals with an ordinal sum of two commutative binary operations on the fixed subintervals being increasing on a bounded lattice L. In this section, we extend this result to ordinal sums with two summands yielding a t-norm on a bounded lattice L. To be more precise, we focus on an ordinal sum of t-norms on L built from two binary operations defined on the subintervals [0, a] and [a, 1] for  $a \in L \setminus \{0, 1\}$ .

Recently, El-Zekey has extended the concept of t-subnorm on the unit interval introduced by Klement et al. ([9]) to lattices [35]. In the following, we define t-subnorms on bounded lattices, which are used in the sequel.

**Definition 3** ([35]). Let  $(L, \leq, 0, 1)$  be a bounded lattice and [a, b] be a subinterval of L. A binary operation  $F : [a, b]^2 \longrightarrow [a, b]$  is said to be a *t*-subnorm on [a, b] if it is commutative, associative, increasing in both arguments and it satisfies the range condition  $F(x, y) \leq x \wedge y$  for all  $x, y \in L$ .

Every t-norm is a t-subnorm. Nevertheless, the contrary of this argument does not need to true. For example, given the trivial t-subnorm defined by F(x, y) = 0 for all  $x, y \in L$ , it is not a t-norm since there is no neutral element of F. It should be pointed out that, like any t-norm, any t-subnorm F on a bounded lattice L has the zero element 0, i.e., F(0, x) = 0 for all  $x \in L$ .

We now investigate which types of binary operations defined on the fixed subinterval of L are appropriate candidates for the ordinal sum with two summands given by the formula (3.2), guaranteeing that it yields a t-norm on a bounded lattice L. The following Theorem 5 provides a sufficient and necessary condition for the ordinal sum  $T: L^2 \longrightarrow L$  of two t-subnorms on subintervals of L given by the formula (3.2) being a t-norm on a bounded lattice L. In this way, we provide a complete answer to Question 2.

**Theorem 5.** Let  $(L, \leq, 0, 1)$  be a bounded lattice,  $a \in L \setminus \{0, 1\}$ ,  $T_1 : [a, 1]^2 \longrightarrow [a, 1]$  and  $T_2 : [0, a]^2 \longrightarrow [0, a]$  be two t-subnorms on the subintervals [a, 1] and [0, a] of L, respectively. The following statements are equivalent:

- I) The binary operation  $T: L^2 \longrightarrow L$  defined by the formula (3.2) is a t-norm on L.
- II)  $\{x \in L : x \in I_a \text{ and } T_2(x \wedge a, a) < x \wedge a\} = \emptyset.$
- III)  $I_a = \emptyset$  or  $T_2(z \wedge a, a) = z \wedge a$  for all  $z \in I_a$ .

Proof. (I)  $\Longrightarrow$  (II)  $\Longrightarrow$  (III).

It follows from Theorem 4.

$$(III) \Longrightarrow (I).$$

Let  $I_a = \emptyset$  or  $T_2(z \land a, a) = z \land a$  for all  $z \in I_a$ . It follows from Theorem 4 that the commutativity and increasingness of T hold. We demonstrate the associativity of T. Taking into account of the commutativity of T, the proof is split into all remaining possible cases.

Claim: T(x, T(y, z)) = T(T(x, y), z) for any  $x, y, z \in L$ .

1) If one of the elements x, y and z is equal to 1, the equality holds.

2) Let  $x, y \in [0, a)$ .

2-1) If  $z \in [0, a)$ ,

$$\begin{split} &T(x,T(y,z))\\ =&T(x,T_2(y,z))=T_2(x,T_2(y,z))\\ =&T_2(T_2(x,y),z)=T(T_2(x,y),z)=T(T(x,y),z). \end{split}$$

2-2) If  $z \in [a, 1)$ ,

$$T(x, T(y, z)) = T(x, y) = T_2(x, y) = T(T_2(x, y), z) = T(T(x, y), z).$$

2-3) If  $z \in I_a$ ,

$$T(x, T(y, z))$$
  
=T(x, T<sub>2</sub>(y, z \land a)) = T<sub>2</sub>(x, T<sub>2</sub>(y, z \land a))  
=T<sub>2</sub>(T<sub>2</sub>(x, y), z \land a) = T(T<sub>2</sub>(x, y), z) = T(T(x, y), z).

3) Let  $x \in [0, a)$  and  $y \in [a, 1)$ . 3-1) If  $z \in [0, a)$ ,

$$T(x, T(y, z)) = T(x, z) = T(x, z) = T(T(x, y), z)$$

3-2) If  $z \in [a, 1)$ ,

$$T(x, T(y, z)) = T(x, T_1(y, z)) = x = T(x, z) = T(T(x, y), z).$$

3-3) If  $z \in I_a$ ,

$$T(x, T(y, z))$$
  
=T(x, T<sub>2</sub>(a, z \lambda a)) = T(x, z \lambda a)  
=T<sub>2</sub>(x, z \lambda a) (by (III))

and

 $T(T(x,y),z) = T(x,z) = T_2(x,z \wedge a),$ 

which imply that T(x, T(y, z)) = T(T(x, y), z). 4) Let  $x \in [0, a)$  and  $y \in I_a$ . 4-1) If  $z \in [0, a)$ ,

$$T(x, T(y, z)) = T(x, T_2(y \land a, z)) = T_2(x, T_2(y \land a, z)) = T_2(T_2(x, y \land a), z) = T(T_2(x, y \land a), z) = T(T(x, y), z).$$

4-2) If  $z \in [a, 1)$ ,

$$T(x, T(y, z))$$
  
=T(x, T<sub>2</sub>(y \wedge a, a)) = T(x, y \wedge a)  
=T<sub>2</sub>(x, y \wedge a) (by (III))

and

$$T(T(x,y),z) = T(T_2(x,y \land a),z) = T_2(x,y \land a)$$

which imply that T(x, T(y, z)) = T(T(x, y), z). 4-3) If  $z \in I_a$ ,

$$T(x, T(y, z)) = T(x, T_2(y \land a, z \land a))$$
  
=  $T_2(x, T_2(y \land a, z \land a)) = T_2(T_2(x, y \land a), z \land a)$   
=  $T(T_2(x, y \land a), z) = T(T(x, y), z).$ 

5) Let  $x, y \in [a, 1)$  and  $z \in I_a$ .

$$T(x, T(y, z)) = T(x, T_2(a, z \land a)) = T(x, z \land a) = z \land a \text{ (by (III))}$$

and

$$T(T(x,y),z) = T(T_1(x,y),z) = T_2(a,z \wedge a) = z \wedge a$$
 (by (III)),

which imply that T(x, T(y, z)) = T(T(x, y), z). 6) Let  $x \in [a, 1)$  and  $y \in I_a$ . 6-1) If  $z \in [0, a)$ ,

$$T(x,T(y,z)) = T(x,T_2(y \land a,z)) = T_2(y \land a,z)$$

and

$$T(T(x,y),z) = T(T_2(a,y \wedge a),z) = T_2(y \wedge a,z)$$
 (by (III)),

which imply that T(x, T(y, z)) = T(T(x, y), z).

6-2) If  $z \in [a, 1)$ ,

$$T(x, T(y, z)) = T(x, T_2(y \land a, a)) = T(x, y \land a) = y \land a \text{ (by (III))},$$

and

$$T(T(x,y),z) = T(T_2(a,y \wedge a),z) = T_2(y \wedge a,z) = y \wedge a \quad (by (III)),$$

which imply that T(x, T(y, z)) = T(T(x, y), z).

6-3) If  $z \in I_a$ ,

$$T(x, T(y, z)) = T(x, T_2(y \land a, z \land a)) = T_2(y \land a, z \land a)$$

and

$$T(T(x, y), z)$$
  
= $T(T_2(a, y \land a), z) = T_2(y \land a, z)$   
= $T_2(y \land a, z \land a)$  (by (III)),

which imply that T(x, T(y, z)) = T(T(x, y), z).

7) Let  $x, y, z \in I_a$ .

$$T(x, T(y, z))$$
  
=T(x, T<sub>2</sub>(y \wedge a, z \wedge a)) = T<sub>2</sub>(x \wedge a, T<sub>2</sub>(y \wedge a, z \wedge a))  
=T<sub>2</sub>(T<sub>2</sub>(x \wedge a, y \wedge a), z \wedge a) = T(T<sub>2</sub>(x \wedge a, y \wedge a), z)  
=T(T(x, y), z).

Therefore, we obtain that the binary operation  $T: L^2 \longrightarrow L$  defined by the formula (3.2) is a t-norm on L.

We should note that Theorem 5 exactly answers Question 2. We can also observe from Theorem 5 that the ordinal sum  $T: L^2 \longrightarrow L$  of two binary operations  $T_1: [a, 1]^2 \longrightarrow [a, 1]$  and  $T_2: [0, a]^2 \longrightarrow [0, a]$  defined by the formula (3.2) being a t-norm on a bounded lattice L is closely interested in the value of the binary operation  $T_2$  on  $(\{a\} \cup I_a) \times \{a\}$ , while the binary operation  $T_1$  is inefficiency. Such an ordinal sum T of two t-subnorms  $T_1$  and  $T_2$  generates a t-norm on a bounded lattice L under the assumption (III), whatever the values of  $T_2$  and  $T_1$  are on the boundary  $(\{a\} \times ([0, a] \setminus \{z \land a : z \in I_a\})) \cup (([0, a] \setminus \{z \land$ 

Take in Theorem 3 the binary operation  $T_2 : [0, a]^2 \longrightarrow [0, a]$  defined by  $T_2(x, y) = x \wedge y$  for all  $x, y \in [0, a]$ . Then the ordinal sum operation with two summands given by the formula (3.2) reduces to the ordinal sum operation with one summand introduced in [33], which is given in the following Corollary 1.

**Corollary 1** ([33]). Let  $(L, \leq, 0, 1)$  be a bounded lattice and  $a \in L \setminus \{0, 1\}$ . If  $T_1 : [a, 1]^2 \longrightarrow [a, 1]$  is a *t*-norm on [a, 1], then the binary operation  $T^{(1)} : L^2 \longrightarrow L$  defined by the formula (5.1) is a *t*-norm on L.

$$T^{(1)}(x,y) = \begin{cases} x \wedge y, & x = 1 \text{ or } y = 1, \\ T_1(x,y), & (x,y) \in [a,1)^2, \\ x \wedge y \wedge a, & otherwise. \end{cases}$$
(5.1)

Take in Theorem 3 the binary operation  $T_2: [0, a]^2 \longrightarrow [0, a]$  defined by  $T_2(x, y) = x \wedge y$  for  $a \in \{x, y\}$ and  $T_2(x, y) = 0$  for  $x, y \in [0, a]^2$ . Then the ordinal sum operation with two summands given by the formula (3.2) reduces to the ordinal sum operation with one summand introduced in [26], which is given in the following Corollary 2.

**Corollary 2** ([26]). Let  $(L, \leq, 0, 1)$  be a bounded lattice and  $a \in L \setminus \{0, 1\}$ . If  $T_1 : [a, 1]^2 \longrightarrow [a, 1]$  is a *t*-norm on [a, 1], then the binary operation  $T^{(2)} : L^2 \longrightarrow L$  defined by the formula (5.2) is a *t*-norm on L.

$$T^{(2)}(x,y) = \begin{cases} x \wedge y, & x = 1 \text{ or } y = 1, \\ 0, & (x,y) \in [0,a)^2 \cup ([0,a) \times I_a) \\ & \cup (I_a \times [0,a)) \cup (I_a \times I_a), \\ T_1(x,y), & (x,y) \in [a,1)^2, \\ x \wedge y \wedge a, & otherwise. \end{cases}$$
(5.2)

In Theorem 5, for the ordinal sum operation  $T: L^2 \longrightarrow L$  of two t-subnorms  $T_1: [a, 1]^2 \longrightarrow [a, 1]$  and  $T_2: [0, a]^2 \longrightarrow [0, a]$ , which is given by the formula (3.2), generating a t-norm on a bounded lattice L, the condition (III) cannot be omitted, in general. In the following, we provide an example of a bounded lattice L, and a t-subnorm  $T_2: [0, a]^2 \longrightarrow [0, a]$  violating the condition (III) on which the ordinal sum T defined by the formula (3.2) is not a t-norm on L.

**Example 3.** Given the bounded lattice  $L_1$  with Hasse diagram shown in Fig. 1, it is clear that  $I_a = \{c\}$ , i.e.,  $I_a \neq \emptyset$ . If we consider the binary operations  $T_1 : [a, 1]^2 \longrightarrow [a, 1]$  and  $T_2 : [0, a]^2 \longrightarrow [0, a]$  defined by  $T_1(x, y) = a$  for all  $x, y \in [a, 1]$  and  $T_2(x, y) = 0$  for all  $x, y \in [0, a]$ , then both  $T_1$  and  $T_2$  are t-subnorms on [a, 1] and [0, a], respectively. Notice that from  $T_2(c \land a, a) = 0$  and  $I_a \neq \emptyset$ , the condition (III) is violated. By applying the construction approach in the formula (3.2), we obtain the ordinal sum operation  $T : L_1 \times L_1 \longrightarrow L_1$  given in Table 1. It is easily seen that T is not increasing on  $L_1$  since T(b, a) = b > 0 = T(c, a) for  $b \leq c$ . Therefore, T is not a t-norm on  $L_1$ .

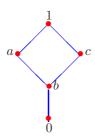


Figure 1: Hasse diagram of the lattice  $L_1$  in Example 3

Table 1	1:	Th	e ordinal	sum	T	$_{\mathrm{in}}$	Example $3$	

T	0	b	a	c	1
0	0	0	0	0	0
b	0	0	b	0	b
a	0	b	a	0	a
c	0	0	0	0	c
1	0	b	a	С	1

In the following example, we consider a bounded lattice  $L_2$ , and two t-subnorms  $T_1$  and  $T_2$  on the subintervals [a, 1] and [0, a] of L, respectively, which satisfy the condition (III) in Theorem 5. Taking into account of Theorem 5, we observe that the ordinal sum operation  $T: L_2 \times L_2 \longrightarrow L_2$  with two summands given by the formula (3.2) is a t-norm on  $L_2$ .

**Example 4.** Consider the bounded lattice  $L_2$  with Hasse diagram shown in Fig. 2. If we define the binary operations  $T_1 : [a, 1]^2 \longrightarrow [a, 1]$  and  $T_2 : [0, a]^2 \longrightarrow [0, a]$  by  $T_1(x, y) = a$  for all  $x, y \in [a, 1]$  and  $T_2(x, y) = 0$  for all  $x, y \in [0, a]$ , then neither  $T_1$  nor  $T_2$  is a t-norm on [a, 1] and [0, a], respectively.  $T_1$  and  $T_2$  are t-subnorms on [a, 1] and [0, a], respectively, as well as  $T_2(a, b \land a) = b \land a = 0$ . That is, the condition (III) is satisfied. By using the method in the fromula (3.2), we define the ordinal sum operation  $T : L_2 \times L_2 \longrightarrow L_2$  as in Table 2. In view of Theorem 5, we observe that T is a t-norm on  $L_2$ .

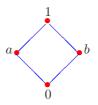


Figure 2: Hasse diagram of the lattice L in Example 4

Table 2: The ordinal sum T in Example 4

T	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	0	b
1	0	a	b	1

#### 6. Concluding Remarks

This paper continues to study the ordinal sum operation with two summands on a bounded lattice. Ertuğrul and Yeşilyurt have recently introduced an ordinal sum construction for generating t-norms on the bounded lattice L derived from two t-norms  $T_1$  and  $T_2$  on the indicated subintervals [a, 1] and [0, a]of L, respectively, for  $a \in L \setminus \{0,1\}$  [37]. Furthermore, they have put forward an open problem as follows: if we take an associative, commutative, and monotone binary operation instead of at least one of the t-norms on the subintervals of L, will the same method work? By providing extra necessary and sufficient conditions to ensure that an ordinal sum with two summands is a t-norm on a bounded lattice L, we present a complete solution for their proposal. We first give an example to show that the ordinal sum T defined in Theorem 3 may not be a t-norm on L if we take an associative, commutative, and monotone binary operation instead of at least one of the t-norms  $T_1$  and  $T_2$ . Moreover, in Theorem 4, given two commutative operations  $T_1$  and  $T_2$  on the subintervals [a, 1] and [0, a] of L, respectively, we provide a sufficient and necessary condition for the ordinal sum operation T of  $T_1$  and  $T_2$  being increasing on L. Thanks to this theorem, we give a partial answer to the mentioned problem. Then, in Theorem 4, we introduce a sufficient and necessary condition for ensuring that the ordinal sum operation T of two t-subnorms  $T_1$  and  $T_2$  on the subintervals [a, 1] and [0, a] of L produces a t-norm on L. Through this theorem, we answer exactly the above problem. For future work, it is interesting to consider how to define the ordinal sum construction of t-subnorms on subintervals of a bounded lattice being a t-norm without any additional constraint.

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