Outage Minimization via Power Adaptation and Allocation for Truncated Hybrid ARQ

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Abstract

In this work, we analyze hybrid ARQ (HARQ) protocols over the independent block fading channel. We assume that the transmitter is unaware of the channel state information (CSI) but has a knowledge about the channel statistics. We consider two scenarios with respect to the feedback received by the transmitter: *i)* "conventional", one-bit feedback about the decoding success/failure (ACK/NACK), and *ii)* the multi-bit feedback scheme when, on top of ACK/NACK, the receiver provides additional information about the state of the decoder to the transmitter. In both cases, the feedback is used to allocate (in the case of one-bit feedback) or adapt (in the case of multi-bit feedback) the power across the HARQ transmission attempts. The objective in both cases is the minimization of the outage probability under long-term average and peak power constraints. We cast the problems into the dynamic programming (DP) framework and solve them for Nakagami-m fading channels. A simplified solution for the high signal-to-noise ratio (SNR) regime is presented using a geometric programming (GP) approach. The obtained results quantify the advantage of the multi-bit feedback over the conventional approach, and show that the power optimization can provide significant gains over conventional power-constant HARQ transmissions even in the presence of peak-power constraints.

Index Terms

Chase Combining, Dynamic Programming, Geometric Programming, HARQ, Incremental Redundancy, Outage Probability, Nakagami-m Fading.

I. Introduction

TO GUARANTEE reliable data transmissions over unreliable channels, two fundamental techniques are commonly used: forward error correction (FEC) and automatic repeat request (ARQ) [1]. In FEC schemes, error correcting codes are used to combat transmission errors. In ARQ schemes, error detecting codes are used and retransmission is requested every time a negative acknowledgment (NACK) is sent to the transmitter via the feedback channel. An HARQ scheme combines ARQ and FEC, and provides better performances compared to each scheme alone [2]. In typical HARQ protocols, a retransmission request is repeated until the codeword is received without errors—in which case a positive acknowledgment (ACK) is sent on the feedback channel—or a maximum number of transmissions is reached; this particular case is called truncated HARQ [3], [4]. HARQ schemes can be classified into two categories: the Chase combining HARQ (CC-HARQ) [5], where all retransmitted packets are identical, and the incremental redunda [6] where each retransmission carries a different piece of the "mother code" that generates the complete coded version of the message.

In this paper, we design transmission schemes and power assignment strategies which minimize the outage probability subject to both peak power and long-term average power constraints for IR-HARQ and CC-HARQ protocols in block fading channels. We analyze both cases when one bit ACK/NACK or multi-bits feedback is available at the transmitter. The multi-bit feedback scenario covers the case when the transmitter may obtain the CSI from the receiver through the feedback channel, but—due, e.g., to

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long communication/processing delays—the CSI is fully outdated (i.e., independent of the CSI in the subsequent transmissions).

To improve HARQ's performance, many power policies have been proposed in the litterature. In [7], a power adaptation was proposed to increase the throughput in the case of a discretized CSI. An asymptotically optimal power control algorithm that attains the diversity limit in long-term static channels has been presented in [8]. The case of long-term static channels was also studied in [9], where the authors determined the optimal power assignment strategy to minimize the total average transmission power subject to outage probability constraints. The optimization of power efficiency with a packet error rate (PER) constraint was solved as a GP problem in [10] for the case of space-time coded HARQ, and in [11] for CC-HARQ over independent Rayleigh block fading channels. In [12], the authors derived an optimal power allocation scheme which minimizes the packet drop probability under a total average transmit power constraint for IR-HARQ with two transmissions. A suboptimal feedback and power adaptation rule was proposed for multiple-input multiple-output (MIMO) IR-HARQ block fading channels in [13], achieving the optimal outage diversity.

The objective of this paper is to assess the value of the multi-bit feedback for power assignment schemes in HARQ, and the main contributions of this work are the following:

1) We show how to use the well-known DP methods [14] to find the optimal power adaptation policies for truncated HARQ in order to minimize the outage probability under constraints on peak and long-term average power. The method can be applied for both CC-HARQ and IR-HARQ and for any channel with a continuous cumulative distribution function. Unlike [13], where the proposed power strategies are sub-optimal in terms of outage performance, our power policies are optimal in terms of outage.
2) We show how to optimize the power-allocation policy for IR-HARQ and CC-HARQ over Nakagami-m fading channels. The optimal solutions are given in parametrized closed form for an arbitrary number of transmissions. We note that only two transmissions were allowed in [12]; in [11] and [12] only Rayleigh block fading channels were considered.
3) We present a simplified allocation policy for the high SNR regime obtained using the geometric programming (GP) framework.
4) We provide numerical results for practically interesting wireless channel models, comparing the outage probability with various power adaptation/allocation methods.

The rest of the paper is organized as follows. In Sec. II, we introduce the adopted system model and, in Sec. III, we define the optimization problem. We show the optimization method for power adaptation policies in Sec. IV and the power allocation is treated in Sec. V. The optimal allocation for the high SNR regime is discussed in Sec. VI. We provide numerical examples that illustrate the advantages obtained using the optimal power policies in Sec. VII. Conclusions are drawn in Sec. VIII.

II. System Model

We consider a block-fading model where the channel between the transmitter and the receiver is varying (fading) randomly from one transmission to another but stays invariant during each of the transmissions, thus the signal received on the k^{th} ARQ transmission round is given by

$$\mathbf{y}_k = \sqrt{\gamma_k \cdot P_k(\mathsf{CSI}_{k-1})} \cdot \mathbf{x}_k + \mathbf{z}_k, \quad k = 1, ..., K \tag{1}$$

where $\mathbf{z}_k$ is a zero-mean, unit-variance Gaussian noise, $\mathbf{x}_k$ is the unit-variance transmitted signal, $P_k(\mathsf{CSI}_{k-1}) \geq 0$ is the transmit power and is a function of the previous realization of the channel $\mathsf{CSI}_{k-1} = [\gamma_1, \gamma_2, ..., \gamma_{k-1}]$, where $\sqrt{\gamma_k}$ is the instantaneous channel gain. Then, γ_k has the meaning of an instantaneous nominal SNR (i.e., which considers unitary power transmission) which is assumed to be perfectly known at the receiver but unknown to the transmitter. Thus, the transmitter cannot adjust the communication rate in the k^{th} transmission based on γ_k.

To recover from decoding errors, the coded versions of a data packet are transmitted at most K times. On top of the conventional one-bit signaling between the transmitter and the receiver (ACK/NACK messages), we also allow the receiver to send the CSI collected during unsuccessful transmission attempts back to the transmitter (entirely defined through SNR realizations γ_k) via the feedback channel (which is assumed error-free). The transmitter should be able to *adapt* the transmit power during the k^{th} transmission attempt using the knowledge of $\gamma_1, \ldots, \gamma_{k-1}$. Thus, we will talk about *power adaptation* when the CSI are used to adjust the power used in each transmission. On the other hand, the *power allocation* covers the case when only the "conventional" one-bit feedback (ACK/NACK) is available. In this case, the transmitter responds to the reception of a NACK message by retransmitting the packet with a power that depends only on the transmission idex $k = 1, \cdots, K$.

We assume that γ_k can be modelled as independent and identically distributed (i.i.d.) random variables with $\overline{\gamma}_k = \mathbb{E}_{\gamma_k}[\gamma_k]$, where $\mathbb{E}_{\gamma}[\cdot]$ denotes the mathematical expectation calculated with respect to γ. The independence of γ_k can be justified by the practical scenario where the successive transmissions are not sent in adjacent time instants and, being sufficiently well separated, the realizations of the channel become—to all practical extent—independent [15].

Most of the derivations will be done in abstraction of the particular fading distribution, but in numerical examples, we consider the popular Nakagami-m fading profile. Hence, the channel normalized SNR γ_k follows a gamma distribution with a probability density function (PDF) $p_\gamma(x)$ given by

$$p_\gamma(x) = \frac{m^m}{\Gamma(m)\overline{\gamma}^m} x^{m-1} \mathrm{e}^{-mx/\overline{\gamma}}, \qquad x > 0,$$

and the cumulative density function (CDF) $F_\gamma(x)$ is given by

$$F_\gamma(x) = 1 - \frac{\Gamma(m, mx/\overline{\gamma})}{\Gamma(m)}, \tag{2}$$

where $\Gamma(x)$ and $\Gamma(s,x)$ denote respectively the gamma function and the upper incomplete gamma function.

We assume that the decoding is successful if the average accumulated mutual information at the receiver is larger than the overall transmission rate for IR-HARQ. In the case of CC-HARQ, the decoding is successful if the accumulated SNR is larger than an SNR threshold. Thus, the decoding fails after k transmissions with the probability

$$\begin{aligned} f_k &= \begin{cases} \Pr\left\{\sum_{l=1}^{k} \log\left(1+\gamma_l \cdot P_l(\mathsf{CSI}_{l-1})\right) < R\right\}, & \text{for IR-HARQ} \\ \Pr\left\{\log\left(1+\sum_{l=1}^{k}\gamma_l \cdot P_l(\mathsf{CSI}_{l-1})\right) < R\right\}, & \text{for CC-HARQ} \end{cases} \\ &= \Pr\left\{I_k < i_{\text{th}}\right\} \qquad (3) \\ &= F_{I_k}(i_{\text{th}}) = \int_0^{i_{\text{th}}} p_{I_k}(x)\mathrm{d}x, \qquad (4) \end{aligned}$$

where

$$\begin{cases} I_k = \sum_{l=1}^{k} C_l, \quad i_{\text{th}} = R & \text{for IR-HARQ} \\ I_k = \sum_{l=1}^{k} \sigma_l, \quad i_{\text{th}} = \gamma_{\text{th}} = 2^R - 1 & \text{for CC-HARQ} \end{cases}, \tag{5}$$

and $\sigma_l = \gamma_l \cdot P_l(\mathsf{CSI}_{l-1})$, $C_l = \log\left(1 + \gamma_l \cdot P_l(\mathsf{CSI}_{l-1})\right)$ and $p_{I_k}(x)$ is the PDF of I_k.

With this notation, the scenarios we consider are defined as follows:

- Constant power (CO) HARQ, where $P_k(\mathsf{CSI}_{k-1}) \equiv \bar{P}$, i.e., the power is the same throughout retransmissions.
- Power Allocation (AL), where the CSI feedback is ignored (or, simply not available) and the power varies solely as a function of the transmission's index, i.e., $P_k(\mathsf{CSI}_{k-1}) \equiv \hat{P}_k \cdot \mathbb{I}(I_{k-1} \leq i_{\text{th}})$ where

$\mathbb{I}(x)=1$ if x is true, and 0 otherwise, since the k^{th} transmission is necessary only if the previous $(k-1)$ transmissions were unsuccessful. Finding the scalars $\hat{P}_k$ is a problem of power allocation.

- Power Adaptation (AD), where the power is modified in each transmission attempt using the CSI provided over the feedback channel. From (3), the decoding error event in the k^{th} transmission depends uniquely on I_{k-1} and γ_k (which is unknown, and cannot be predicted from the previous CSI $\gamma_1,\ldots,$ γ_{k-1} due to the independence assumption). Consequently, I_{k-1} (which is a scalar representation of the vector CSI_{k-1}) is the only parameter eventually required to adapt the power $P_k(\mathsf{CSI}_{k-1})$ via a scalar function

$$P_k(\mathsf{CSI}_{k-1}) \equiv \tilde{P}_k(I_{k-1})\cdot\mathbb{I}(I_{k-1}\leq i_{\text{th}}),\quad k=1,...,K \tag{6}$$

where $I_0 \triangleq 0$. Finding the function $\tilde{P}_k(I_{k-1})$ is a problem of power adaptation.

For simplicity, we assume that the transmitter has a perfect knowledge of I_{k-1}, that is, we ignore all eventual transmission and discretization errors. This assumption lets us know the maximum gain that can be achieved using information about the decoder's state contained in I_{k-1}.

III. Optimization Problem

According to the reward-renewal theorem [16], the long-term average consumed power is the ratio between the average transmit power between two consecutive renewals (sending a new data packet) $\mathbb{E}_{\mathsf{CSI}_K}\Big[\mathcal{P}(\mathsf{CSI}_K)\Big]$ and the expected number of transmissions $\mathbb{E}_{\mathsf{CSI}_K}\Big[\mathcal{T}(\mathsf{CSI}_K)\Big]$ needed to deliver the packet with up to K transmission attempts [7], [15]:

$$\overline{P} \triangleq \frac{\mathbb{E}_{\mathsf{CSI}_K}\Big[\mathcal{P}(\mathsf{CSI}_K)\Big]}{\mathbb{E}_{\mathsf{CSI}_K}\Big[\mathcal{T}(\mathsf{CSI}_K)\Big]} = \frac{\sum_{k=1}^{K}\mathbb{E}_{\mathsf{CSI}_{k-1}}\Big[P_k(\mathsf{CSI}_{k-1})\Big]}{\sum_{k=0}^{K-1} f_k}, \tag{7}$$

where f_k is the probability of a decoding failure after k transmission attempts given by (3), and $\mathbb{E}_{\mathsf{CSI}_{k-1}}\Big[P_k(\mathsf{CSI}_{k-1})\Big]$ is the expected transmit power during the k^{th} transmission attempt, obtained by considering all the events yielding the k^{th} transmission, i.e., the event $I_{k-1}<i_{\text{th}}$.

In this work, we aim at minimizing the outage probability f_K with respect to the power policy $\{P_k(\mathsf{CSI}_{k-1})\}_{k=1}^{K}$ for a given long-term average power $\overline{P}_{\text{max}}$, peak allowed power P_{max} and a transmission rate R. Taking (7) into consideration, the optimization problem can be formulated as follows:

$$\min_{P_1,P_2(\mathsf{CSI}_1),\ldots,P_K(\mathsf{CSI}_{K-1})} f_K,\qquad \text{s.t.}\quad \begin{cases}\overline{P}\leq\overline{P}_{\text{max}}\\ 0\leq P_k(\mathsf{CSI}_{k-1})\leq P_{\text{max}},\ 1\leq k\leq K\end{cases}. \tag{8}$$

The problem (8) requires an optimization over the scalar P_1 and the functions $P_k(\mathsf{CSI}_{k-1})$, so to solve it we will discretize the functions using N equidistant points. Then we define the Lagrangian function $L:\mathbf{R}_+^{N(K-1)+1}\times\mathbf{R}\rightarrow\mathbf{R}$ associated with the problem (8) as

$$L\left(P_1,P_2(\mathsf{CSI}_1),...,P_K(\mathsf{CSI}_{K-1}),\lambda\right)=f_K+\lambda\cdot\left(\sum_{k=1}^{K}\mathbb{E}_{\mathsf{CSI}_{k-1}}\Big[P_k(\mathsf{CSI}_{k-1})\Big]-\overline{P}_{\text{max}}\sum_{k=0}^{K-1}f_k\right), \tag{9}$$

where we left implicit all power constraints $0\leq P_k(\mathsf{CSI}_{k-1})\leq P_{\text{max}}$. Without any loss of generality, we consider $\overline{P}_{\text{max}}=1$ in what follows.

IV. OUTAGE-OPTIMAL POWER ADAPTATION

For power adaptation (6), the expected transmit power during the k^{th} transmission is given by:

$$\mathbb{E}_{\mathsf{CSI}_{k-1}}\Big[P_k(\mathsf{CSI}_{k-1})\Big] = \mathbb{E}_{I_{k-1}}\Big[\tilde{P}_k(I_{k-1})\Big] = \int_0^{i_{\text{th}}} \tilde{P}_k(x) p_{I_{k-1}}(x) \mathrm{d}x. \tag{10}$$

Thus the Lagrangian function L defined in (9) can be written as:

$$L\left(\tilde{P}_1, \tilde{P}_2(I_1), \ldots, \tilde{P}_K(I_{K-1}), \lambda\right) = f_K + \lambda \cdot \left(\sum_{k=1}^{K} \mathbb{E}_{I_{k-1}}\Big[\tilde{P}_k(I_{k-1})\Big] - \sum_{k=0}^{K-1} f_k \right). \tag{11}$$

To solve the primal problem in (8), it is difficult to use the Karush–Kuhn–Tucker (KKT) conditions on the Lagrangian function (11) since it requires to solve analytically a system of an infinite number of equations where, in addition, closed form expressions of f_k (for $2 \leq k \leq K$) are unknown. To overcome these difficulties, we can solve the dual problem.

In the general case, the dual problem provides a solution which is a lower bound to the solution of (8); the difference between the lower bound and the true optimum is called the "duality gap". However, according to a result in [17, Theorem 1], optimization problems with expectations over possibly non-convex functions of random variables in both objective and constraint functions have a zero duality gap, given that the PDF of the random variable of interest has no points of strictly positive probability (i.e., its CDF is continuous). For this reason, we express the outage probability and the average transmit power (10) as a function of the channel normalized SNR γ_k, $k = 1, \ldots, K$, with a continuous CDF, indeed verifying the above requirement:

$$f_K = \Pr\left\{I_K < i_{\text{th}}\right\} = \mathbb{E}_{\gamma_1,\gamma_2,\ldots\gamma_K}[\mathbb{I}(I_K < i_{\text{th}})], \tag{12}$$

and

$$\mathbb{E}_{\mathsf{CSI}_K}[P] = \tilde{P}_1 + \sum_{k=2}^{K} \mathbb{E}_{\gamma_1,\gamma_2,\ldots\gamma_{k-1}}\Big[\tilde{P}_k(I_{k-1})\Big] = \tilde{P}_1 + \mathbb{E}_{\gamma_1,\gamma_2,\ldots\gamma_{K-1}}\left[\sum_{k=2}^{K} \tilde{P}_k(I_{k-1})\right]. \tag{13}$$

According to [17, Theorem 1], the objective and constraint functions must be expectations over (possibly non-convex) functions of random variables. However, $\tilde{P}_1$ is independent of any random variable. So we introduce a sub-optimization problem for each value of $\tilde{P}_1 > 0$

$$\hat{f}_K(\tilde{P}_1) \triangleq \min_{\tilde{P}_2(I_1),\ldots,\tilde{P}_K(I_{K-1})} f_K, \quad \text{s.t.} \begin{cases} \sum_{k=2}^{K} \mathbb{E}_{I_{k-1}}\Big[\tilde{P}_k(I_{k-1})\Big] - \sum_{k=2}^{K-1} f_k \leq 1 + f_1 - \tilde{P}_1 \\ 0 \leq \tilde{P}_k\left(I_{k-1}\right) \leq P_{\text{max}}, \quad \text{for} \quad 1 \leq k \leq K \end{cases}. \tag{14}$$

The optimal solution of (8) is then given by

$$\min_{\tilde{P}_1} \hat{f}_K(\tilde{P}_1). \tag{15}$$

Defining the Lagrange dual function $d : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ as

$$d(\tilde{P}_1, \lambda) \triangleq \min_{\tilde{P}_2(I_1),\ldots,\tilde{P}_K(I_{K-1})} L\Big(\tilde{P}_1, \tilde{P}_2(I_1), \ldots, \tilde{P}_K(I_{K-1}), \lambda\Big), \tag{16}$$

the dual optimization problem is thus given by

$$D(\tilde{P}_1) = \min_{\lambda\geq 0} d(\tilde{P}_1, \lambda). \tag{17}$$

Note that, since the problem (14) and its dual (17) have a zero duality gap, we can guarantee that $D(\tilde{P}_1) \equiv \hat{f}_K(\tilde{P}_1)$ for $\tilde{P}_1 > 0$.

Finally, (16) can be rewritten in a recursive form characteristic of dynamic programming optimization (DP):

$$d(\tilde{P}_1, \lambda) = J_1(I_0)$$

$$J_1(I_0) = \Big\{ -\lambda \cdot \mathbb{E}_{\gamma_1}[\mathbb{I}(I_1 < i_{\text{th}})] + \lambda \cdot \tilde{P}_1 + \mathbb{E}_{\gamma_1}[J_2(I_1)] \Big\} \tag{18}$$

$$J_2(I_1) = \min_{\tilde{P}_2(I_1)} \Big\{ -\lambda \cdot \mathbb{E}_{\gamma_2}[\mathbb{I}(I_2 < i_{\text{th}})] + \lambda \cdot \tilde{P}_2(I_1) + \mathbb{E}_{\gamma_2}[J_3(I_2)] \Big\} \tag{19}$$

$$\vdots$$

$$J_k(I_{k-1}) = \min_{\tilde{P}_k(I_{k-1})} \Big\{ -\lambda \cdot \mathbb{E}_{\gamma_k}[\mathbb{I}(I_k < i_{\text{th}})] + \lambda \cdot \tilde{P}_k(I_{k-1}) + \mathbb{E}_{\gamma_k}[J_{k+1}(I_k)] \Big\} \tag{20}$$

$$\vdots$$

$$J_K(I_{K-1}) = \min_{\tilde{P}_K(I_{K-1})} \Big\{ \lambda \cdot \tilde{P}_K(I_{K-1}) + \mathbb{E}_{\gamma_K}[\mathbb{I}(I_K < i_{\text{th}})] \Big\}, \tag{21}$$

where I_k is a function of I_{k-1}, $\tilde{P}_k(I_{k-1})$, and γ_k in the form

$$I_k = \begin{cases} I_{k-1} + \log\Big(1 + \gamma_k \tilde{P}_k(I_{k-1})\Big), & \text{for IR-HARQ} \\ I_{k-1} + \gamma_k \tilde{P}_k(I_{k-1}), & \text{for CC-HARQ} \end{cases}. \tag{22}$$

For a given I_k—noting that $I_k \in [0, i_{\text{th}}]$ should be discretized over N points—we can optimize the value of the function $\tilde{P}_k(I_{k-1})$ provided that the function $J_{k+1}(I_k)$ is known. Thus, the global optimization of the possibly non-convex problem in (14) over the set of N^{K-1} values is reduced to a series of $(K-1) \cdot N$ one-dimensional optimizations thanks to the DP formulation equations in (18)-(21).

A. Radio silence

In this context of IR-HARQ transmissions, we have

$$\mathbb{E}_{\gamma_k}\Big[\mathbb{I}(I_k < i_{\text{th}})\Big] = F_{\gamma_k}\left(\frac{2^{R-I_{k-1}} - 1}{\tilde{P}_k(I_{k-1})}\right). \tag{23}$$

The condition to guarantee a minimum in the last DP step is that the derivative of the function under minimization in (21) equals zero, i.e.,

$$\begin{aligned} u(\tilde{P}_K) &\triangleq \lambda - \frac{2^{R-I_{K-1}} - 1}{\tilde{P}_K^2} \cdot p_{\gamma_K}\left(\frac{2^{R-I_{K-1}} - 1}{\tilde{P}_K}\right) \\ &= \lambda - \frac{1}{2^{R-I_{K-1}} - 1} \cdot q\left(\frac{2^{R-I_{K-1}} - 1}{\tilde{P}_K}\right) = 0, \end{aligned} \tag{24}$$

where it is easy to show that $q(x) \triangleq x^2 p_{\gamma_K}(x)$ satisfies $q(x) > 0$, $q(0) = 0$, and $q(\infty) = 0$, and hence, $q(x)$ has a maximum $q_{\text{max}} = \max_x q(x)$. Since the derivative $u(0) = \lambda$ and $u(\tilde{P}_K)$ must be locally non-increasing around $\tilde{P}_K = 0$ (i.e., $u'(0) \leq 0$), the solution of (24) does not exist if $\lambda \cdot (2^{R-I_{K-1}} - 1) > q_{\text{max}}$; meaning that the minimum is obtained by setting $\tilde{P}_K = 0$ and yielding $J_K(I_{K-1}) = 1$. When $\lambda \cdot (2^{R-I_{K-1}} - 1) < q_{\text{max}}$,

$u(\tilde{P}_K)$ has at least two zeros[1] and the optimal solution corresponds to the point where the second derivative is positive.

In Fig. 1, we show the adaptation policy $\tilde{P}_k(x)$ in a case of IR-HARQ with $K = 4$. As we see, the optimal solution requires a "radio silence", that is, knowing in the k^{th} transmission that the accumulated mutual information at the receiver is below a threshold $i_{0,k-1}$, the transmitter decides to stay silent (zero transmit power) until the maximum number of transmissions is attained. This "silence time" guarantees that the power is "saved" when the transmitter does not have a "reasonable hope" of successfully terminating the transmission.

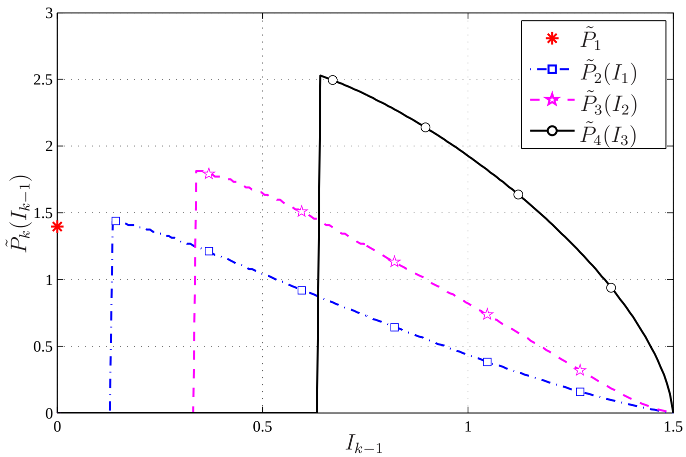


Fig. 1. Optimized adaptation policies $\tilde{P}_k(I_{k-1})$ for the case of IR-HARQ in Nakagami-m fading model, when $m = 2$, $K = 4$, $R = 1.5$ and $\overline{\gamma} = -4$ dB. The "radio silence" means that if the accumulated mutual information after the k^{th} transmission is less than a specific threshold $i_{0,k}$ (i.e., if $I_k < i_{0,k}$ where $i_{0,1} \approx 0.12$, $i_{0,2} \approx 0.33$, $i_{0,3} \approx 0.63$ in this example), the HARQ process decides to use zero power for all reminder transmissions, i.e., $P_l(I_{l-1}) = 0,\ k < l \leq K$.

B. *Outage calculation*

To calculate the outage probability, we use (4) where the CDF of I_k is $F_{I_k}(x)$ and taking into consideration that $\tilde{P}_k(x) = 0$ for $x \in [0, i_{0,k-1}]$, we obtain

$$F_{I_k}(x) = \Pr\left\{I_{k-1} + \log(1 + \gamma_k \cdot \tilde{P}_k(I_{k-1})) < x\right\}$$
$$= \begin{cases} F_{I_{k-1}}(x), & \text{if } x < i_{0,k-1} \\ F_{I_{k-1}}(i_{0,k-1}) + \displaystyle\int_{i_{0,k-1}}^{x} F_{\gamma}\left(\frac{2^{x-y} - 1}{\tilde{P}_k(y)}\right) \cdot p_{I_{k-1}}(y)\,\mathrm{d}y, & \text{if } x > i_{0,k-1} \end{cases}, \tag{25}$$

which depends on the PDF $p_{I_{k-1}}(y)$ of I_{k-1}.

The differentiation of (25) yields a recursive relationship for the PDF

$$p_{I_k}(x) = \begin{cases} p_{I_{k-1}}(x), & \text{if } x < i_{0,k-1} \\ \int_{i_{0,k-1}}^{x} \frac{\log(2)2^{x-y}}{\tilde{P}_k(y)} p_{I_{k-1}}(y) \cdot p_{\gamma}\left(\frac{2^{x-y}-1}{\tilde{P}_k(y)}\right) \mathrm{d}y & \text{if } x > i_{0,k-1} \end{cases}, \tag{26}$$

[1]In the case of a Rayleigh fading channel, there are exactly two zeros: one corresponds to the local maximum, and the other one to the local minimum.

where

$$p_{I_1}(x) = \frac{\log(2)2^x}{\tilde{P}_1} \cdot p_\gamma\left(\frac{2^x - 1}{\tilde{P}_1}\right). \tag{27}$$

Considering $\mathbb{E}_{\gamma_k}[\mathbb{I}(I_k < i_{\text{th}}] = F_{\gamma_k}\left(\frac{\gamma_{\text{th}} - I_{k-1}}{\tilde{P}_k(I_{k-1})}\right)$, the same analysis as (25) and (26) can be done in the case of CC-HARQ. Thus,

$$F_{I_k}(x) = \Pr\left\{I_{k-1} + \gamma_k \cdot \tilde{P}_k(I_{k-1}) < x\right\}$$
$$= \begin{cases} F_{I_{k-1}}(x), & \text{if } \; x < i_{0,k-1} \\ F_{I_{k-1}}(i_{0,k-1}) + \int_{i_{0,k-1}}^{x} F_\gamma\left(\frac{x-y}{\tilde{P}_k(y)}\right) \cdot p_{I_{k-1}}(y)\, \mathrm{d}y, & \text{if } \; x > i_{0,k-1} \end{cases}, \tag{28}$$

which again depends on the PDF $p_{I_{k-1}}(y)$ of I_{k-1}.

Differentiation of (28) yields the recursive relationship for the pdf

$$p_{I_k}(x) = \begin{cases} p_{I_{k-1}}(x) & \text{if} \quad x < i_{0,k-1} \\ \int_{i_{0,k-1}}^{x} \frac{1}{\tilde{P}_k(y)} p_{I_{k-1}}(y) p_\gamma\left(\frac{x-y}{\tilde{P}_k(y)}\right) \mathrm{d}y, & \text{if} \quad x > i_{0,k-1} \end{cases}, \tag{29}$$

where

$$p_{I_1}(x) = \frac{1}{P_1} p_\gamma\left(\frac{x}{P_1}\right). \tag{30}$$

V. OUTAGE OPTIMAL POWER ALLOCATION

In this section, we consider the problem of optimal power allocation $\Big($i.e., $P_k(\mathcal{CSI}_{k-1}) \equiv \hat{P}_k \cdot \mathbb{I}(I_{k-1} \leq i_{\text{th}})\Big)$. The expected power consumed in the k^{th} transmission attempt is given by

$$\mathbb{E}_{\mathsf{CSI}_k}[P] = \mathbb{E}_{I_{k-1}}[\hat{P}_k \cdot \mathbb{I}(I_{k-1} \leq i_{\text{th}})] = \hat{P}_k \cdot f_{k-1}, \tag{31}$$

and the long-term average power (7) by

$$\overline{P} = \frac{\sum_{k=1}^{K} \hat{P}_k \cdot f_{k-1}}{\sum_{k=0}^{K-1} f_k}. \tag{32}$$

Thus, the Lagrangian function L defined in (9) can be expressed as

$$L(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_K, \lambda) = f_K + \lambda \cdot \left(\sum_{k=1}^{K} \hat{P}_k f_{k-1} - \sum_{k=0}^{K} f_k\right). \tag{33}$$

To cast (33) into the DP formulation, we have to find the "states" S_k such that: (i) f_k may be calculated from S_k, and (ii) state S_{k+1} may be obtained from S_k and $\hat{P}_{k+1}$. Because the closed form expressions of f_k are unknown, we use an accurate approximation to express f_k in terms of $\{\hat{P}_l\}_{l=1}^{k}$; as shown in the Appendix A, we can obtain the following relationship:

$$f_k \approx \frac{h_k}{\hat{P}_k^{m_k}} \cdot f_{k-1}, \quad \text{for} \quad 1 \leq k \leq K, \tag{34}$$

where the parameter h_k is independent from $\hat{P}_k$ and m_k is the parameter of the Nakagami-m channel at the k^{th} transmission. Considering $f_0 = 1$, the optimization problem in (33) can be reformulated recursively as follows

$$\begin{aligned} L(\hat{P}_1, \hat{P}_2, ..., \hat{P}_K, \lambda) &= J_1(f_0) \\ J_1(f_0) &= \lambda \cdot (\hat{P}_1 - 1) \cdot f_0 + J_2(f_1) \quad (35) \\ J_2(f_1) &= \lambda \cdot (\hat{P}_2 - 1) \cdot f_1 + J_3(f_2) \quad (36) \\ &\vdots \\ J_{K-1}(f_{K-1}) &= \lambda \cdot (\hat{P}_K - 1) \cdot f_{K-1} + f_K. \quad (37) \end{aligned}$$

From the KKT necessary conditions, and starting with (37), we find a unique (therefore, the optimal) solution as

$$\hat{P}_k = \begin{cases} \min\left\{ \left(\frac{m_k \cdot g_{k+1} \cdot h_k}{\lambda} \right)^{\frac{1}{m_k+1}}, P_{\text{max}} \right\}, & \text{for } 1 \leq k \leq K-1 \\ \min\left\{ \left(\frac{m_K \cdot h_K}{\lambda} \right)^{\frac{1}{m_K+1}}, P_{\text{max}} \right\}, & \text{for } k = K \end{cases}, \quad (38)$$

where

$$g_k \triangleq \begin{cases} \lambda \cdot (\hat{P}_k - 1) + \frac{g_{k+1} \cdot h_k}{\hat{P}_k^{m_k}}, & \text{for } 1 \leq k \leq K-1 \\ \lambda \cdot (\hat{P}_K - 1) + \frac{h_K}{\hat{P}_K^{m_K}}, & \text{for } k = K \end{cases}. \quad (39)$$

VI. Approximate solutions

We target the high SNR regime, where typically $\lim_{\overline{\gamma} \to \infty} f_k = 0$ for $k \in \{1, ..., K\}$. Thus, the long term average power (defined in (32)) can be expressed in the high SNR regime as:

$$\overline{P} \approx \sum_{k=1}^{K} \hat{P}_k \cdot f_{k-1}. \quad (40)$$

We note that [11] and [12] define the long term average power as in (40) even if it is only a valid approximation in high SNR regime. Thus, the optimization problem (8) can be rewritten in the case of power allocation as:

$$\min_{\hat{P}_1, \hat{P}_2, ..., \hat{P}_K} f_K, \qquad \text{s.t.} \ \sum_{k=1}^{K} \hat{P}_k \cdot f_{k-1} \leq 1, \quad (41)$$

where we assume that $P_{\text{max}} = \infty$ and $\overline{P}_{\text{max}} = 1$.

As shown in Appendix A, for Nakagami-m fading channel, a unified approximation of the outage probability f_K and the expected transmit power for both IR-HARQ and CC-HARQ can be written as

$$f_K \approx A_K \cdot \prod_{k=1}^{K} \hat{P}_k^{-m}, \quad (42)$$

and

$$\hat{P}_k \cdot f_{k-1} \approx A_{k-1} \cdot \hat{P}_k \prod_{l=1}^{k-1} \hat{P}_l^{-m}. \quad (43)$$

where $A_0 = 1$, A_k is defined in (61) and (52) for the case of IR-HARQ and CC-HARQ respectively.

Thus, the optimization problems (41) can be written in the standard primal form of geometric programming problems [18], [19], [20]:

$$\min_{\hat{P}_1,\hat{P}_2,\ldots,\hat{P}_K} \left\{ A_K \cdot \prod_{k=1}^{K} \hat{P}_k^{-m} \right\}, \quad \text{s.t.} \ \sum_{k=1}^{K} A_{k-1} \cdot \hat{P}_k \prod_{l=1}^{k-1} \hat{P}_l^{-m} \leq 1, \tag{44}$$

As shown in Appendix B, the optimal solution of (44) is given by

$$f_K = (\lambda(\boldsymbol{\delta}^*))^{\lambda(\boldsymbol{\delta}^*)} \cdot A_K \cdot \prod_{k=2}^{K+1} \left(\frac{A_{k-2}}{\delta_k^*} \right)^{\delta_k^*}, \tag{45}$$

where $\boldsymbol{\delta}^* = [\delta_1^*, \ldots, \delta_{K+1}^*]$ and

$$\delta_k^* = \begin{cases} 1, & \text{for } k = 1 \\ m \cdot (m+1)^{(K+1-k)}, & \text{for } k \in \{2, ..., K+1\} \end{cases}, \tag{46}$$

and thus $\lambda(\boldsymbol{\delta}^*) = (m+1)^K - 1$, cf. (66). Therefore, the optimal power policy corresponding to the optimization problem (41) is given by (69)

$$P_k = \begin{cases} \dfrac{\delta_2^*}{\lambda(\boldsymbol{\delta}^*) \cdot A_0}, & \text{if } k = 1 \\ \dfrac{\delta_{i+1}^*}{\lambda(\boldsymbol{\delta}^*) \cdot A_{i-1} \cdot \prod_{j=1}^{i-1} \left(P_j^*\right)^{-m}}, & \text{if } k \in \{2, .., K\} \end{cases}. \tag{47}$$

On the other hand, the diversity is defined as [21]

$$\mathcal{D} = - \lim_{\overline{\gamma} \to \infty} \frac{\log(f_K)}{\log(\overline{\gamma})}. \tag{48}$$

In the A_k's expressions, the exponent of $\overline{\gamma}$ is equal to $-mk$. Thus, according to (45) the diversity $\mathcal{D}$ for power allocation is given by

$$\begin{aligned} \mathcal{D} &= Km\delta_1 + \sum_{k=2}^{K+1} (k-2) m \delta_k \\ &= Km\delta_1 + 0m\delta_2 + 1m\delta_3 + ... + (K-1)m\delta_{K+1} \\ &= Km + m^2[(m+1)^{K-2} + 2(m+1)^{K-3} + ... + (K-2)(m+1) + (K-1)] \\ &= Km + m^2 \cdot K \cdot \left(1 + (m+1) + ... + (m+1)^{K-2}\right) - 1 - 2 \cdot (m+2) - ... - (K-1) \cdot (m+1)^{K-2} \\ &= (m+1)^K - 1 \end{aligned} \tag{49}$$

We note that the same diversity value was obtained in [13] where, in addition, it was proven that both multi-bit feedback (i.e., adaptation) and single-bit feedback (i.e., allocation) have the same diversity gain for an infinite constellation size; this is confirmed by our results. On the other hand, the diversity of constant power HARQ is given by $\mathcal{D} = Km$.

VII. NUMERICAL EXAMPLES

For the case of $m = 2$, Fig. 2a and Fig. 2b present the optimized outage probability in the case of IR-HARQ and CC-HARQ, respectively. We show both cases when $P_{\text{max}} = 5$ and $P_{\text{max}} = \infty$. We also plot the outage probability of constant-power transmissions (CO). We can see, that for high SNR, the optimized results outperform CO HARQ, which is due to the increases diversity of both power adaptation and allocation strategies. On the other hand, the results for CO HARQ can outperform AL HARQ because the latter is based on the approximations, which loose their validity for low SNR. For example, for $\overline{\gamma} < -2$ dB in Fig. 2a and for $\overline{\gamma} < 0$ dB in Fig. 2b.

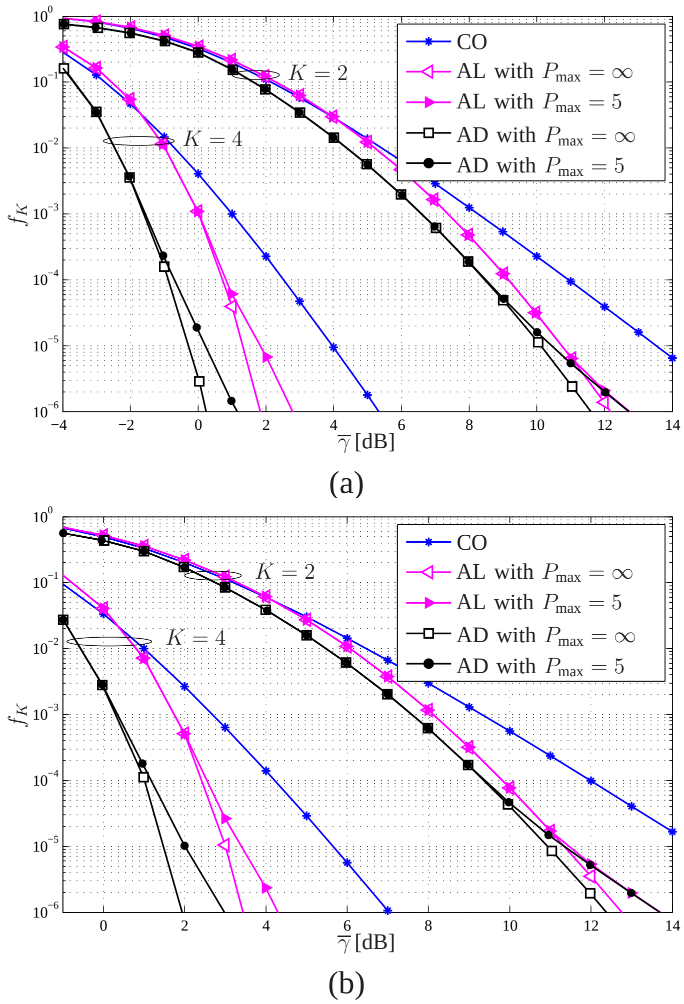


Fig. 2. Optimized outage probability when using the optimized power adaptation (AD) and allocation (AL) compared with the outage probability of constant-power transmission (CO) (i.e., $P_k = \overline{P} = 1$, $\forall k$) in the case of (a) IR-HARQ, and (b) CC-HARQ. $K = 2, 4$ and Nakagami-m fading with $m = 2$; $R = 1.5$. Unconstrained peak (i.e., $P_{\max} = \infty$) and constrained peak (i.e., $P_{\max} = 5$) cases are shown.

When $P_{\max} = 5$ instead of $P_{\max} = \infty$, the gain of the optimized outage compared with CO starts to decrease after a specific value $\overline{\gamma}_0$ of the average SNR $\overline{\gamma}$. For example, in the case of allocation with $K = 4$, $\overline{\gamma}_0 \approx 0$ dB for IR-HARQ and $\overline{\gamma}_0 \approx 2$ dB for CC-HARQ. This is justified by the fact that the constraint $P_{\max} = 5$ becomes active for $\overline{\gamma} \geq \overline{\gamma}_0$, as can be seen in Fig. 3. Moreover, when the maximum power constraints are active, the diversity of the adaptation/allocation schemes is the same as the diversity of the constant power transmission.

Fig. 4 illustrates the gain of the power adaptation over allocation strategies, where it is clear that the gain is not only a function of the maximum number of transmissions K, but is also depends on the channel parameter m. In particular, for $K = 2$ we obtain the gain of approximately 0.1 dB, 0.2 dB, and 0.5 dB for $m = 1$, $m = 2$, and $m = 3$, respectively. For $K = 4$, the respective gains increase and also grow with m; they are approximately given by 0.5 dB, 1.5 dB, and 1.8 dB. In Fig. 5, we compare the optimized solutions obtained using DP and GP for CC-HARQ and IR-HARQ. For high SNR, and as expected, the solutions of GP converge to the optimized solution obtained with DP.

VIII. Conclusion

In this paper, we analyzed the impact of multi-bit feedback on the performance of HARQ protocols in terms of outage probability. We analyzed HARQ with Chase combining and Incremental Redundancy in

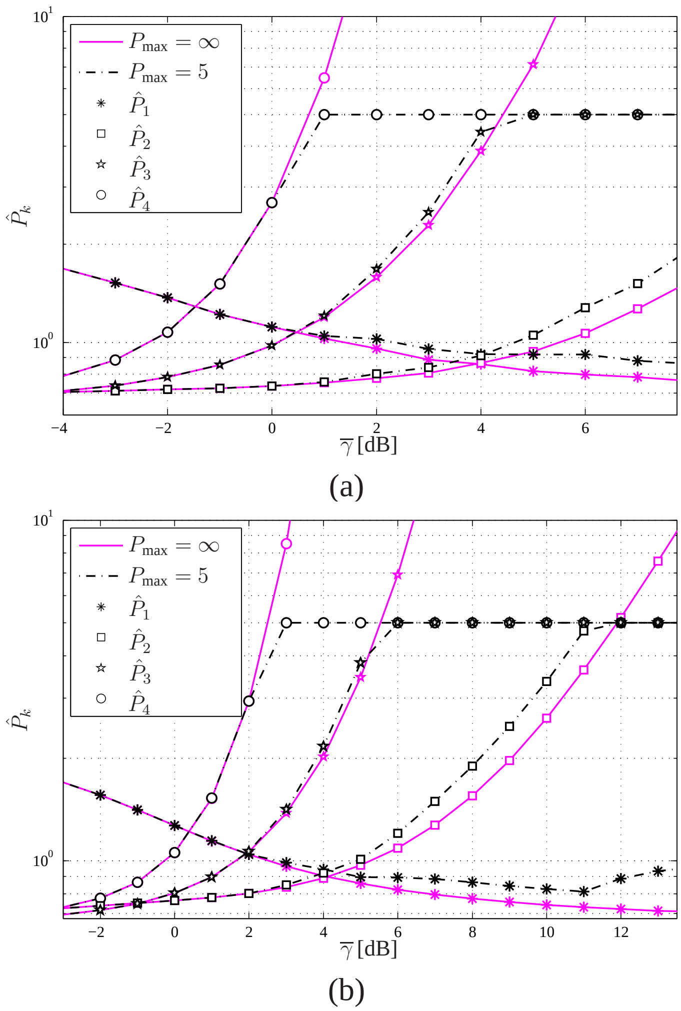


Fig. 3. Optimized allocation policies $\hat{P}_k$ as a function of $\overline{\gamma}$ in the case of (a) IR-HARQ and (b) CC-HARQ. $K = 4$, $m = 2$, $R = 1.5$. Both cases of unbounded (i.e., $P_{\max} = \infty$) and bounded peak power (i.e., $P_{\max} = 5$) are shown for comparison.

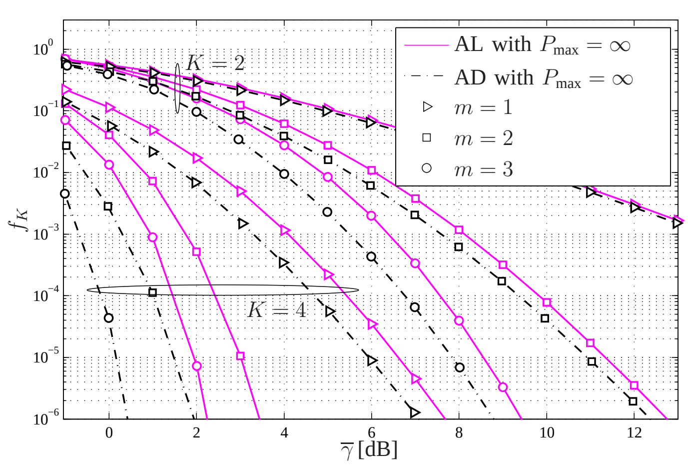
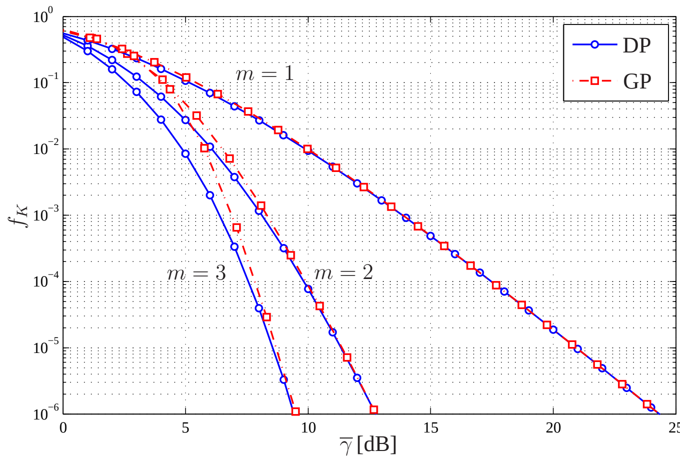
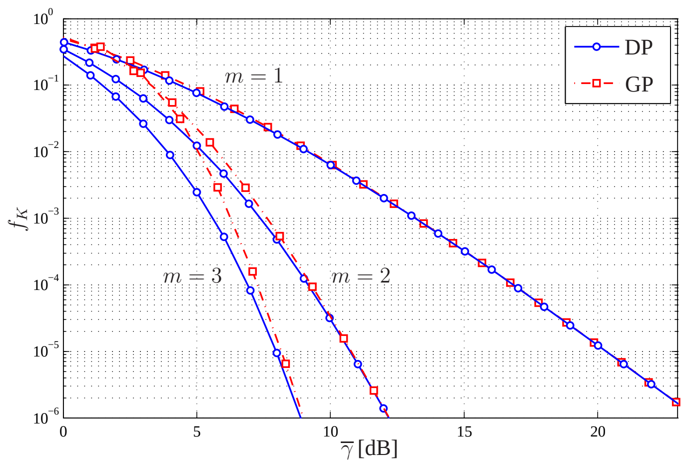


Fig. 4. Optimized outage probability when using the optimized power adaptation (AD) and allocation (AL) policies in the case of CC-HARQ when $K = 2, 4$ and all variables correspond to Nakagami-m fading with $m = 1, 2, 3$, $R = 1.5$, and $P_{\max} = \infty$.

Nakagami-m block fading channels. We show that an optimized power allocation/adaptation strategy



(a)



(b)

Fig. 5. Outage probability when HARQ process use the optimized allocation power policies find by dynamic programming (DP) and geometric programming (GP) in the case of (a) CC-HARQ and (b) IR-HARQ. $K = 2$ and all variables correspond to Nakagami-m fading with $m = 1, 2, 3$ and $R = 1.5$.

throughout the transmissions leads to notable gains over the power-constant HARQ. Adding multi-bit feedback improves the performance and the achievable gains increase with the allowed number of transmissions as well as with the parameter m of the Nakagami-m distribution.

Appendix A

We aim to determine the expression of h_k and A_K (required in (34) and (42) respectively) for the case of IR-HARQ and CC-HARQ. For that, we will derive a simple and accurate approximation of f_k defined in (3). Clearly, calculating the outage probability in (3) for the power allocation scheme requires the derivation of the CDF of the sum of k independent random variables: $C_k = \log_2\left(1 + \gamma_k \hat{P}_k\right)$ in the case of IR-HARQ and $\sigma_k = \gamma_k \hat{P}_k$ in the case of CC-HARQ.

A. CC-HARQ

We use a simple and accurate method to evaluate the outage probability at the output of maximum ratio combining receivers in arbitrarily fading channels introduced in [22]. The approximation is based on the so-called saddle-point approximation (SPA) [23], [24]. For the special case of Nakagami-m fading channels, the outage probability can be approximated by [22]

$$f_K \approx \left(\frac{\exp(1)\cdot\gamma_{\text{th}}}{\tilde{m}_K}\right)^{\tilde{m}_K} \cdot \frac{1}{\sqrt{2\pi\tilde{m}_K}} \cdot \prod_{k=1}^{K}\left(\frac{m_k}{\overline{\gamma}_k \hat{P}_k}\right)^{m_k}, \tag{50}$$

$$= A_K^{\text{CC}} \cdot \prod_{k=1}^{K} \frac{1}{\hat{P}_k^{m_k}}, \tag{51}$$

where $\tilde{m}_K = \sum_{k=1}^{K} m_k$ and

$$A_K^{\text{CC}} = \left(\frac{\exp(1)\cdot\gamma_{\text{th}}}{\tilde{m}_K}\right)^{\tilde{m}_K} \cdot \frac{1}{\sqrt{2\pi\tilde{m}_K}} \cdot \prod_{k=1}^{K}\left(\frac{m_k}{\overline{\gamma}_k}\right)^{m_k}. \tag{52}$$

In this case, we can easily show that h_k, required in (34), is given by

$$h_k = \begin{cases} \left(\frac{\exp(1)\gamma_{\text{th}} m_k}{\overline{\gamma}_k \tilde{m}_k}\right)^{m_k} \cdot \left(1-\frac{m_k}{\tilde{m}_k}\right)^{\tilde{m}_{k-1}+0.5}, & \text{for } 2 \le k \le K \\ \frac{1}{\sqrt{2\pi m_1}}\left(\frac{\exp(1)\cdot\gamma_{\text{th}}}{\overline{\gamma}_1}\right)^{m_1}, & \text{for } k=1 \end{cases}. \tag{53}$$

B. IR-HARQ

In [25] and [26], the authors develop a simple and powerful way of characterizing the performance of diversity schemes via limiting analysis of outage probabilities in Rayleigh fading channels. We use a similar analysis to approximate the outage probability in the case of IR-HARQ for Nakagami-m fading channels. The key idea is the following:

Theorem 1: [25, Theorem. 1] [26, Lemma. 1]
Let Z and W be two independent random variables. If their CDF verify

$$\lim_{\overline{\gamma}\to\infty} \overline{\gamma}^{n_1} \cdot F_Z(t) = a \cdot q(t), \tag{54}$$

$$\lim_{\overline{\gamma}\to\infty} \overline{\gamma}^{n_2} \cdot F_W(t) = b \cdot g(t), \tag{55}$$

where n_1, n_2, a and b are constants, $g(t)$ and $q(t)$ are monotonically increasing functions, and the derivative of $q(t)$ (denoted as $q'(t)$) is integrable, then the CDF of the sum $Y = Z + W$ satisfies

$$\lim_{\overline{\gamma}\to\infty} \overline{\gamma}^{n_1+n_2} F_Y(t) = ab \cdot \int_0^t g(x)\cdot q'(t-x)\mathrm{d}x. \tag{56}$$

Since γ_k follows a gamma distribution, it is easy to show that

$$\lim_{\overline{\gamma}\to\infty} \overline{\gamma}^{m_k} F_{C_k}(t) = \frac{m_k^{m_k}}{\hat{P}_k^{m_k}\Gamma(m_k+1)} \cdot \left(2^t-1\right)^{m_k}. \tag{57}$$

Using Theorem 1 with $t = R$ recursively, we have an approximation of the outage probability as

$$f_K \approx \hat{f}_K = g_K(R) \cdot \prod_{k=1}^{K} \frac{m_k^{m_k}}{\hat{P}_k^{m_k}\overline{\gamma}^{m_k}\Gamma(m_k+1)}, \tag{58}$$

$$= A_K^{\text{IR}} \cdot \prod_{k=1}^{K} \frac{1}{\hat{P}_k^{m_k}}, \tag{59}$$

where

$$g_k(t) = \int_0^t g_{k-1}(x) q_k'(t-x) \mathrm{d}x, \tag{60}$$

with $g_0 = 1$ and $q_k(t) = (2^t - 1)^{m_k}$. Thus, A_K^{IR} is given by

$$A_K^{\text{IR}} = g_K(R) \cdot \prod_{k=1}^{K} \frac{m_k^{m_k}}{\overline{\gamma}^{m_k} \Gamma(m_k + 1)}, \tag{61}$$

where $g_k(R)$ can be calculated numericaly. We show the accuracy of the approximation in Fig. 6.

Since $g_k(t)$ are independent of the transmitted power and/or SNR, the expression of h_k, required in (34), is thus given by

$$h_k = \begin{cases} \dfrac{g_k(R)}{g_{k-1}(R)} \cdot \dfrac{m_k^{m_k}}{\overline{\gamma}^{m_k} \Gamma(m_k+1)}, & 2 \le k \le K \\ \left(\dfrac{m_1(2^R - 1)}{\overline{\gamma}} \right)^{m_1} \cdot \dfrac{1}{\Gamma(m_1+1)}, & k = 1 \end{cases}. \tag{62}$$

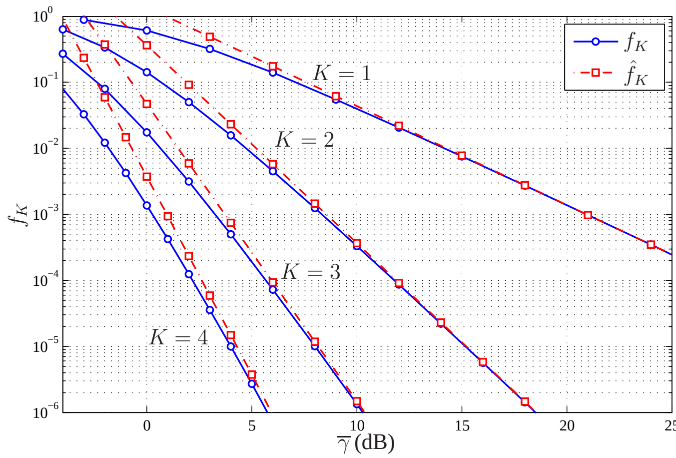


Fig. 6. Exact outage probability f_K compared with the SPA approximation $\hat{f}_K$ (58) in the case of IR-HARQ for Nakagami-m when $K = 1, 2, 3, 4$, $R = 1$ and $m = 1.5$.

Appendix B

In this appendix, we show how we solve the optimization problem (44) written as a GP problem [18] , [19] , [20] in the standard primal form

$$\min_{\boldsymbol{x}} \left\{ g_0(\boldsymbol{x}) = A_K \cdot \prod_{k=1}^{K} x_k^{-m} \right\} \quad \text{s.t. } g_1(\boldsymbol{x}) = \sum_{k=1}^{K} A_{k-1} \cdot x_k \prod_{l=1}^{k-1} x_l^{-m} \le 1, \tag{63}$$

where $\boldsymbol{x} = [x_1, x_2, \dots, x_K]$. The dual problem corresponding to the primal problem (63) is defined as

$$\max_{\boldsymbol{\delta}} v(\boldsymbol{\delta}) = (\lambda(\boldsymbol{\delta}))^{\lambda(\boldsymbol{\delta})} \left(\frac{A_K}{\delta_1} \right)^{\delta_1} \prod_{i=2}^{K+1} \left(\frac{A_{i-2}}{\delta_i} \right)^{\delta_i}, \tag{64}$$

$$\text{s.t.} \quad \begin{cases} \delta_1 = 1, & \\ \delta_i \ge 0, & \forall i \in \{1, .., K+1\} \\ -m \cdot \delta_1 + \delta_j - m \cdot \sum_{i=j+1}^{K+1} \delta_i = 0, & \forall j \in \{2, .., K+1\} \end{cases}, \tag{65}$$

where $\boldsymbol{\delta} = [\delta_1, \ldots, \delta_{K+1}]$ and

$$\lambda(\boldsymbol{\delta}) = \sum_{i=2}^{K+1} \delta_i. \quad (66)$$

This is a GP with a zero degree of difficulty [19], which implies that the unique solution $\boldsymbol{\delta}^*$ of the dual constraints (65) is also the solution of (64) . Because the dual constraints are linear, $\boldsymbol{\delta}^*$ can be determined easily by solving (65) as

$$\delta_i^* = \begin{cases} 1, & \text{if } i = 1 \\ m \cdot (m+1)^{(K+1-i)}, & \text{if } i \in \{2, ..., K+1\} \end{cases}. \quad (67)$$

Defining $\boldsymbol{x}^*$ as the argument which maximizes (63), the optimal solution of (63) is given by [18, pp. 114-116]

$$g_1(\boldsymbol{x}^*) = v(\boldsymbol{\delta}^*) = (\lambda(\boldsymbol{\delta}^*))^{\lambda(\boldsymbol{\delta}^*)} \cdot A_K \cdot \prod_{i=2}^{K+1} \left(\frac{A_{i-2}}{\delta_i^*} \right)^{\delta_i^*}, \quad (68)$$

if and only if

$$x_i^* = \begin{cases} \dfrac{\delta_2^*}{\lambda(\boldsymbol{\delta}^*) \cdot A_0}, & \text{if } i = 1 \\ \dfrac{\delta_{i+1}^*}{\lambda(\boldsymbol{\delta}^*) \cdot A_{i-1} \cdot \prod_{j=1}^{i-1} \left(x_j^*\right)^{-m}}, & \text{if } i \in \{2, .., K\} \end{cases}, \quad (69)$$

where

$$\lambda(\boldsymbol{\delta}^*) = (m+1)^K - 1. \quad (70)$$

Basically, (69) is the closed-form expression of the optimal power policy.

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