On one-bit quantized ZF precoding for the multiuser massive MIMO downlink

A Kant Saxena, Inbar Fijalkow, A Lee Swindlehurst

On One-Bit Quantized ZF Precoding for the Multiuser Massive MIMO Downlink

Amodh Kant Saxena[1], Inbar Fijalkow[2], A. Lee Swindlehurst[1]

[1] Center for Pervasive Communications and Computing, University of California Irvine, Irvine, CA 92697, USA, {aksaxena, swindle}@uci.edu

[2] ETIS, UMR 8051 / ENSEA, Université Cergy-Pontoise, CNRS, F-95000 Cergy, France, inbar.fijalkow@ensea.fr

Abstract - We study low complexity precoding for a downlink massive MIMO multiuser system assuming a base station that employs one-bit digital-to-analog converters (DACs) in order to mitigate power usage. The use of one-bit DACs is equivalent to constraining the transmit signal to be drawn from a QPSK alphabet. While the precoding problem can be formulated using a standard maximum likelihood (ML) encoder, the implementation cost is prohibitive for massive numbers of antennas, even if a sphere encoding approach is used. Instead, we study the performance of a one-bit quantized zero-forcing precoder, and we show that it asymptotically provides the desired downlink vector with low complexity. Simulations show that the quantized ZF precoder can actually outperform the ML encoder for low to moderate signal-to-noise ratios.

I. Introduction

Massive MIMO involves the use of many (perhaps hundreds) of antennas at the base station (BS), and can potentially provide large increases in capacity via spatial multiplexing [1]. Prior work on Mulitple Input Multiple Output (MIMO) downlink precoding has assumed that each RF chain is equipped with a high-resolution digital-to-analog converter (DAC). However, the power consumption of the DACs grows exponentially with the number of quantization bits [2], and also grows with increases in bandwidth and sampling rate. For massive MIMO configurations with many antennas and DACs, the resulting cost and power consumption will be prohibitive, and alternative approaches are needed.

In this paper we study a scenario in which simple one-bit DACs are implemented at the BS. One-bit DACs do not require highly linear amplifiers, and hence they can be implemented with very low cost and power consumption [3], [4]. One-bit analog-to-digital converters (ADCs) have been studied in prior work for uplink communication scenarios, with the focus primarily on channel estimation and information theoretic capacity analyses [5]-[12]. While there has been previous research focused on downlink precoding for massive MIMO (see e.g., [13]-[15]) very little has been reported on the impact of low-resolution DACs on transmit processing [16].

The effect of one-bit DACs on the downlink is equivalent to constraining the transmit signals to be drawn from the Quadrature Phase Shift Keying (QPSK) constellation points: $\pm 1 \pm j$. The problem can thus be posed as choosing a finite-alphabet signal such that when passed through the channel, results in the desired symbol at the individual users. This problem is equivalent to maximum likelihood (ML) decoding, or more appropriately in this context, ML encoding. However, an ML encoding approach is not feasible when massive numbers of transmit antennas are involved, since a different QPSK symbol has to be chosen for each antenna. This is true even if one chooses to use the more computationally efficient sphere (en)decoder [17].

Instead, in this paper, we choose to focus on a more computationally efficient approach that simply performs a one-bit quantization of the zero-forcing (ZF) precoder. We show that, with an accurate channel estimate and assuming an asymptotically large number of antennas, this simple approach yields received signals that will be correctly detected as the desired symbols at each user. Simulations show that this approach even outperforms the ML encoder at low to moderate signal-to-noise ratios (SNRs) due to the fact that it essentially relaxes the requirement for the precoding to produce received signals as close as possible to the desired constellation points, which is unnecessary if, for example, the user terminals also employ one-bit ADCs and the desired symbols are thus QPSK. In such cases, all that is needed for a correct detection is that the received signal lies in the correct quadrant.

The paper is organized as follows. Section II covers the system model and lays the foundation for our analysis. Section III introduces the proposed algorithm and Section IV provides a proof that the algorithm will yield accurate symbol detections when the number of antennas is asymptotically large. Finally Section V provides simulation results to support the conclusions of the paper.

II. System Model and Precoding Algorithms

An M-antenna BS desires to send a vector $\mathbf{s}$ of K symbols to K single-antenna users in a cell (one symbol to each user), as depicted in Fig. 1. The symbol vector $\mathbf{s}$ is input to a precoder P in order to generate the M-long vector $\mathbf{x}$ that will be transmitted over the propagation channel. Due to the assumption of one-bit DACs at the BS, our aim is to design a precoder that generates a suitable $\mathbf{x}$ whose elements are constrained to come from a QPSK alphabet $\mathcal{S} = \{-1-j, 1-j, -1+j, 1+j\}$. We will further assume that the elements of the desired symbol vector $\mathbf{s}$ are QPSK as well.

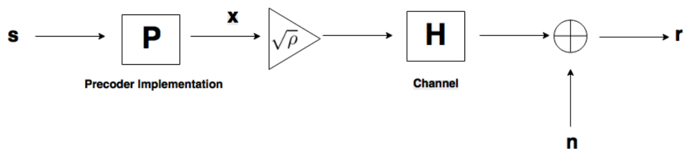

Fig. 1: System Model

The downlink transmission model can be written as

$$\mathbf{r} = \sqrt{\rho}\mathbf{H}\mathbf{x} + \mathbf{n} \ , \tag{1}$$

where $\mathbf{H}$ is the $K \times M$ channel matrix and $\mathbf{n}$ represents additive noise. $\rho = \frac{P_t}{M}$, P_t being the total transmission power at the BS. In order to focus on the impact of the precoding, we assume that $\mathbf{H}$ is perfectly known at the transmitter, although we recognize that in practice only a noisy estimate will be available. We further assume that $M \gg K$.

A. ML Precoding

Ideally, in the absence of noise, one might attempt to design the precoder P such that $\mathbf{Hx} \simeq \mathbf{s}$, so that a simple threshold detector applied to $\mathbf{r}$ would yield a minimal probability of error. In such a case, $\mathbf{x}$ could be of the form $\mathbf{H}^{\dagger}\mathbf{s} + \mathbf{u}$, where $\mathbf{H}^{\dagger}$ denotes the pseudo-inverse of the channel and $\mathbf{u}$ belongs to the nullspace of $\mathbf{H}$. Due to the finite alphabet constraint imposed by the one-bit DACs, one would have to find such an $\mathbf{x}$ with QPSK entries, which may not be possible.

Assuming Gaussian noise, one could claim that the best precoder P should solve the following ML optimization problem:

$$\mathbf{x} = \arg \min_{\mathbf{v} \in \mathcal{S}^M} ||\mathbf{s} - \mathbf{Hv}||^2 \ , \tag{2}$$

which is equivalent to ML detection, $\mathcal{S}^M$ being the set of all possible M long vectors composed of elements of $\mathcal{S}$. However, ML detection has a complexity of $O(4^M)$, which is too large for massive MIMO scenarios where $M \gg K$. The Sphere Decoding algorithm [18] provides a more computationally efficient solution to (2), although it must be noticed that we consider the unusual case of a matrix $\mathbf{H}$ with many more columns than rows. In such cases, one should transform (2) to

$$\mathbf{x} = \arg \min_{\mathbf{v} \in \mathcal{S}^M} ||\mathbf{D}(\mathbf{z} - \mathbf{v})||^2 \ , \tag{3}$$

where $\mathbf{D}$ is the upper triangular matrix obtained by the Cholesky factorization of $\mathbf{G} = \mathbf{H}^H\mathbf{H} + \alpha\mathbf{I}_M$ ($\mathbf{I}_M$ being an Identity matrix of dimension M), $\mathbf{z} = \mathbf{G}^{-1}\mathbf{H}^H\mathbf{s}$ and α is a small regularization parameter as explained in [17]. Though less complex than ML detection, the generalized sphere encoder still has a complexity exponential in $M - K$, which is prohibitive when $M - K$ is very large. Thus, in the sequel we consider a low complexity solution suited for large values of M.

B. One-Bit ZF Precoding

In addition to reducing computational complexity, an additional motivation for considering an alternative to ML detection or sphere decoding is the observation that, when the desired signals at the user terminals are drawn from a QPSK alphabet, it is less important that $\mathbf{s} \simeq \mathbf{Hx}$; instead, all that is necessary is that the ith element of $\mathbf{Hx}$ lies in the same quadrant as s_i, the ith element of $\mathbf{s}$. The user will then decode the received signal as the desired symbol.

Consider the unconstrained ZF precoder $\mathbf{H}^H(\mathbf{HH}^H)^{-1}\mathbf{s}$, and let $\mathbf{h}_i$ represent the ith column of $\mathbf{H}$. For massive MIMO scenarios, in the limit as $M \to \infty$, we have $\mathbf{HH}^H \propto \mathbf{I}_K$. Thus the ZF precoder signal transmitted from antenna i is asymptotically proportional to $\mathbf{h}_i^H\mathbf{s}$, which is the complex number that best rotates the vector $\mathbf{h}_i$ in the direction of $\mathbf{s}$. Since it is the direction and not the magnitude of the elements of $\mathbf{s}$ that is crucial for detection, we simply truncate the number to the QPSK constellation. The sum of a large number M of such contributions should then result in correct decisions at the user terminals. In the next section, we provide a mathematical analysis that validates this heuristic conclusion.

The one-bit ZF precoding algorithm is summarized below as Algorithm 1. Note that the precoding complexity is $O(MK)$, which will be orders of magnitude less than that of the sphere encoder for large M. We will see in simulations presented later that in some cases this simple approach can significantly outperform ML detection for finite alphabet $\mathbf{s}$. $\Re(.)$ and $\Im(.)$ denote real part and imaginary part of the corresponding complex arguments respectively, for the algorithm description and the rest of the paper.

Algorithm 1: One-bit ZF precoding algorithm

Inputs: $\mathbf{s}, \mathbf{H}$

1) $\hat{\mathbf{x}} = \mathbf{H}^{\dagger}\mathbf{s}$, where $\mathbf{H}^{\dagger} = \mathbf{H}^H(\mathbf{HH}^H)^{-1}$

2) $\mathbf{x} = \mathbb{Q}(\hat{\mathbf{x}}) = \text{sign}(\Re(\hat{\mathbf{x}})) + j.\text{sign}(\Im(\hat{\mathbf{x}}))$

III. Proof of One-Bit ZF Precoder Efficiency

The analysis follows the algorithm and channel transmission step by step. Our aim is to show that asymptotically, $\mathbf{Hx}$ is proportional to $\mathbf{s}$. For this proof, we assume that the entries of $\mathbf{H}$ are $i.i.d$ random variables drawn from a circular symmetric gaussian distribution, $\mathbb{C}\mathcal{N}\left(0, \sigma^2\right)$. Also, we assume $\mathbf{s}$ is known at the BS.

a) Approximation of $\hat{\mathbf{x}}$:

$$\hat{\mathbf{x}} = \mathbf{H}^H(\mathbf{HH}^H)^{-1}\mathbf{s} \tag{4}$$

Since $M \gg 1$ and the entries of $\mathbf{H}$ are taken to be $i.i.d.$, by Law of Large Numbers [19],

$$\mathbf{HH}^H \simeq 2M\sigma^2\mathbf{I}_K \tag{5}$$

From (4) and (5), we have

$$\hat{\mathbf{x}} \simeq (\mathbf{H}^H\mathbf{s})\frac{1}{2M\sigma^2} \tag{6}$$

Therefore, we have the m^{th} element of $\hat{\mathbf{x}}$ as

$$\begin{aligned}\hat{x}_m &= \frac{1}{2M\sigma^2}\sum_{k=1}^{K} h_{km}^* s_k \\ &= \frac{1}{2M\sigma^2}(\sum_{k=1}^{K}(\Re(h_{km})\Re(s_k) + \Im(h_{km})\Im(s_k)) \\ &+ j\sum_{k=1}^{K}(\Re(h_{km})\Im(s_k) - \Im(h_{km})\Re(s_k)))\ ,\end{aligned} \tag{7}$$

where h_{km} is element k, m of $\mathbf{H}$.

b) Approximation of the quantizer output: We approximate any element of the output of the nonlinear function, $\mathbb{Q}$, by a logistic function [20]. The following asymptotically represents the characteristics of $\mathbb{Q}(.)$ as the parameter n becomes large:

$$(\mathbb{Q}(\hat{\mathbf{x}}))_m \approx \frac{e^{n\Re(\hat{x}_m)} - 1}{e^{n\Re(\hat{x}_m)} + 1} + j\frac{e^{n\Im(\hat{x}_m)} - 1}{e^{n\Im(\hat{x}_m)} + 1} \tag{8}$$

c) Approximation of the noiseless channel output: Next we denote $\tilde{\mathbf{s}}$ as the signal vector at the user terminals prior to detection/quantization. Here it is assumed that the transmit power, $P_t = 1$.

$$\tilde{\mathbf{s}} = \sqrt{\frac{1}{M}}\mathbf{H}\mathbb{Q}(\hat{\mathbf{x}}) \simeq \sqrt{\frac{1}{M}}\mathbf{H}\mathbb{Q}\left(\frac{\mathbf{H}^H\mathbf{s}}{2M\sigma^2}\right) \tag{9}$$

We write the l^{th} element of $\tilde{\mathbf{s}}$ separately in terms of the real and imaginary parts to obtain

$$\begin{aligned}&\Re(\tilde{s}_l) = \\ &\sqrt{\frac{1}{M}}\sum_{m=1}^{M}\Re(h_{lm})\frac{e^{\frac{n}{2M\sigma^2}(\sum_{k=1}^{K}(\Re(h_{km})\Re(s_k)+\Im(h_{km})\Im(s_k)))} - 1}{e^{\frac{n}{2M\sigma^2}(\sum_{k=1}^{K}(\Re(h_{km})\Re(s_k)+\Im(h_{km})\Im(s_k)))} + 1} \\ &- \Im(h_{lm})\frac{e^{\frac{n}{2M\sigma^2}(\sum_{k=1}^{K}(\Re(h_{km})\Im(s_k)-\Im(h_{km})\Re(s_k)))} - 1}{e^{\frac{n}{2M\sigma^2}(\sum_{k=1}^{K}(\Re(h_{km})\Im(s_k)-\Im(h_{km})\Re(s_k)))} + 1}\end{aligned} \tag{10}$$

The expression for the imaginary part is similar.

The algorithm will be shown to give the desired result for the first of the two terms of the real part of $\tilde{s}_l$. The extension to the second term and to the imaginary part is similar. Let X denote the real part of an entry of the channel matrix drawn from an $i.i.d.$ circular symmetric gaussian distribution:

$$X = \Re(h_{lm}) \sim \mathcal{N}(0, \sigma^2) \tag{11}$$

For a given $\mathbf{s}$, let Y be a random variable defined as

$$Y = e^{\frac{n}{2M\sigma^2}(\sum_{k=1}^{K}(\Re(h_{km})\Re(s_k)+\Im(h_{km})\Im(s_k)))} \tag{12}$$

and then define

$$Z = X\frac{Y-1}{Y+1} = g(X, Y) \tag{13}$$

We can easily see that any single term of the summation in (10) can be given by the random variable Z, with all other terms being independent and identically distributed to Z.

Let $\mu_X = E(X)$ and $\mu_Y = E(Y)$. The function $g(X, Y)$ is differentiable to the second order in the neighborhood around (μ_X, μ_Y). Using a second order Taylor expansion for $g(X, Y)$ about the point (μ_X, μ_Y), we have

$$\begin{aligned}Z &\simeq g(\mu_X, \mu_Y) + \frac{\partial g}{\partial x}(\mu_X, \mu_Y)(X - \mu_X) \\ &+ \frac{\partial g}{\partial y}(\mu_X, \mu_Y)(Y - \mu_Y) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(\mu_X, \mu_Y)(X - \mu_X)^2 \\ &+ \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(\mu_X, \mu_Y)(Y - \mu_Y)^2 \\ &+ \frac{\partial^2 g}{\partial x \partial y}(\mu_X, \mu_Y)(X - \mu_X)(Y - \mu_Y)\end{aligned} \tag{14}$$

Since $M \gg 1$, by the law of large numbers we can use $E(Z)$ to approximate the summation over M in (10). Using (14),

$$\begin{aligned}E(Z) &\simeq g(\mu_X, \mu_Y) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(\mu_X, \mu_Y)\sigma_X^2 \\ &+ \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(\mu_X, \mu_Y)\sigma_Y^2 + \frac{\partial^2 g}{\partial x \partial y}(\mu_X, \mu_Y)\sigma_{XY}\end{aligned} \tag{15}$$

By definition of X, $\mu_X = 0$, and

$$\begin{aligned}\mu_Y &= E(Y) \\ &= \prod_{k=1}^{K} E(e^{\frac{n}{2M\sigma^2}(X_{kR}\Re(s_k)+X_{kI}\Im(s_k))}) \\ &= e^{\frac{2Kt^2\sigma^2}{2}} = e^{\frac{n^2 K}{4M^2\sigma^2}}\end{aligned} \tag{16}$$

where $t = \frac{n\Re(s_k)}{2M\sigma^2}$, so that $t^2 = \frac{n^2}{4M^2\sigma^4}$ since $\Re(s_k) = \pm 1$ and $\Im(s_k) = \pm 1$. To get (16), we use the fact that $X_{kR}, X_{kI} \sim \mathcal{N}(0, \sigma^2)\ \forall k = 1, \ldots, K$ and are $i.i.d.$ due to the assumption on the distribution of the channel elements. Also, we have,

$$\begin{aligned}\sigma_{XY} &= E((X - \mu_X)(Y - \mu_Y)) \\ &= E(Xe^{\frac{n}{2M\sigma^2}(\sum_{k=1}^{K}(X_{kR}\Re(s_k)+X_{kI}\Im(s_k)))}) \\ &= E(Xe^{\frac{n}{2M\sigma^2}X_{lR}\Re(s_l)})\left(e^{\frac{n^2}{8M^2\sigma^2}}\right)^{2K-1}\end{aligned} \tag{17}$$

Since $X \sim \mathcal{N}(0, \sigma^2)$, and $X = X_{lR}$ using (10) and (11),

$$\begin{aligned}E(Xe^{\frac{n}{2M\sigma^2}X\Re(s_l)}) &= \int_{-\infty}^{\infty} xe^{\frac{n}{2M\sigma^2}x\Re(s_l)}\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{x^2}{2\sigma^2}}dx \\ &= \frac{n}{2M}\Re(s_l)e^{\frac{n^2}{8M^2\sigma^2}}\end{aligned} \tag{18}$$

Using (16) and (17), (15) becomes

$$E(Z) \simeq \Re(s_l)\frac{ne^{\frac{n^2 K}{4M^2\sigma^2}}}{M(e^{\frac{n^2 K}{4M^2\sigma^2}} + 1)^2} \tag{19}$$

From (10) and (19), we have

$$\Re(\tilde{s}_l) \simeq \Re(s_l)\sqrt{\frac{1}{M}}\frac{2n}{e^{\frac{n^2 K}{4M^2\sigma^2}}} \tag{20}$$

$$\Im(\tilde{s}_l) \simeq \Im(s_l)\sqrt{\frac{1}{M}}\frac{2n}{e^{\frac{n^2 K}{4M^2\sigma^2}}} \tag{21}$$

Thus, for values of $s_l \in \mathcal{S}$, $\tilde{s}_l$ is directly proportional to s_l, with a positive scaling factor, which will result in correct detections.

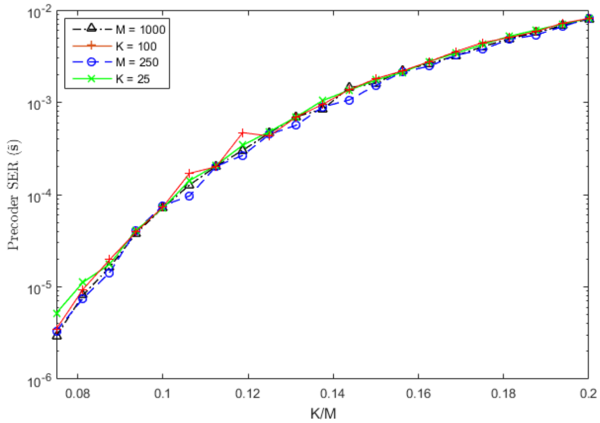

Fig. 2: One-bit ZF precoding SER versus $\frac{K}{M}$.

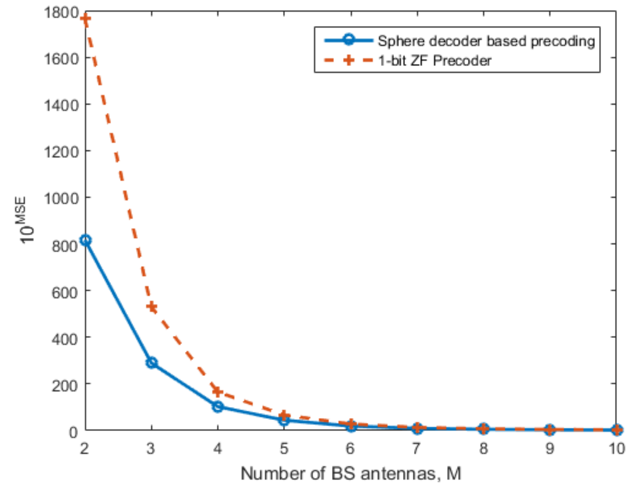

Fig. 3: $10^{MSE(\mathbf{s}-\mathbf{Hx})}$ versus M for $K = 2$.

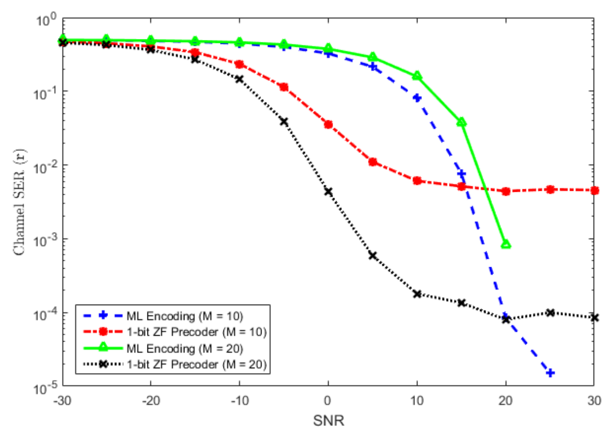

Fig. 4: SER versus SNR for ML encoder and one-bit ZF precoder for $K = 2$ and $M = 10, 20$.

IV. Simulation Results

In the first example, we study the precoder symbol error rate (SER), which denotes here the average rate at which the received symbols are incorrectly decoded without noise at the receivers. Fig. 2 plots the SER as a function of $\frac{K}{M}$ for four different cases: $M = 250, 1000, K = 25, 100$. Fig. 2 validates that the performance is a function of $\frac{K}{M}$ and SERs below 10^{-4} are achieved as long as $\frac{K}{M} \leq 0.1$, which is a typical loading for massive MIMO systems. Here, the elements of $\mathbf{x}$ and $\mathbf{s}$ are constrained to come from a QPSK alphabet.

In Fig. 3, the Mean Square Error (MSE) between the true symbol vector $\mathbf{s}$ and the channel output $\mathbf{Hx}$ is plotted for the one-bit ZF precoding and generalized sphere decoding algorithms. The elements of $\mathbf{x}$ and $\mathbf{s}$ are again constrained to come from a QPSK alphabet. In order to make the problem tractable for the sphere encoder, we only study MSE assuming $K = 2$ and for small values of M. In both cases, the MSE decreases rapidly with M, and the one-bit ZF precoder matches the ML encoder even for values of M less than 10.

In Fig. 4, we fix $K = 2$ and $M = 10, 20$ and plot the SER at the noisy channel output with respect to the transmit SNR. In this case, for simplicity we choose $\mathbf{x}$ and $\mathbf{s}$ to be BPSK: ± 1, and M to be small. It is interesting to note that although the one-bit ZF precoder has a high-SNR error floor, it significantly outperforms the ML encoder for low to moderate SNRs up to 20 dB. The one-bit ZF precoder helps in scaling up the symbol sent, at its reception at the receiver, which helps in enhancing its noise performance with respect to the ML encoder due to a larger received SNR. The mean scale factor is observed to increase with M, as seen from the simulations. For the case of $M = 10$, and BPSK inputs, a scaling of 1.69 is observed. In the case of $M = 20$, a scaling of 2.45 is observed. This is a promising result since it is at these lower SNR regimes where massive MIMO systems are targeted.

V. Conclusion

We have studied the use of a simple quantized zero-forcing precoder for massive MIMO downlink with one-bit DACs. The approach performs particularly well when the desired symbols to be received at the user terminals are QPSK, and thus need only lie in the quadrant associated with the desired signal in order to be correctly decoded. We provided an analysis to show that asymptotically, in the number of antennas M, the algorithm yields signals at the user terminals that are scaled versions of the desired symbols, and hence will be correctly detected. Simulations show that the algorithm outperforms ML encoder for low to moderate SNRs when the ratio K/M is smaller than 0.1. Also, The algorithm has a complexity of $O(MK)$, as opposed to complexities of $O(4^M)$ and $O(4^{M-K})$ for ML and sphere encoders respectively, which makes it computationally feasible to use.

Acknowledgment

The research was supported by the National Science Foundation under Grant ECCS-1547155, and by the Technische Universität München Institute for Advanced Study, funded by the German Excellence Initiative and the European Union Seventh Framework Programme under grant agreement No. 291763, and by the European Union under the Marie Curie COFUND Program, and by French CNRS.

REFERENCES

[1] E. Larsson, O. Edfors, F. Tufvesson, and T. Marzetta, “Massive MIMO for next generation wireless systems,” *IEEE Communication Magazine*, vol. 52, no. 2, pp. 186-195, Feb 2014

[2] R.H. Walden, “Analog-to-digital converter survey and analysis,” *IEEE Journal on Selected Areas in Communications*, vol. 17, no. 4, pp. 539-550, Apr 1999

[3] J. Mo, and R.W. Heath, “Capacity analysis of one-bit quantized MIMO systems with transmitter channel state information,” *IEEE Transactions on Signal Processing*, vol. 63, pp. 5498-5512, Oct 2015

[4] J. Singh, S. Ponnuru, and U. Madhow, “Multi-Gigabit communication: the ADC bottleneck,” in *IEEE International Conference on Ultra-Wideband*, Vancouver, Canada, 2009, pp. 22-27

[5] N. Liang, and W. Zhang, “A mixed-ADC receiver architecture for massive MIMO systems,” in *IEEE Information Theory Workshop*, Jerusalem, 2015 pp. 229-233

[6] J. Choi, and R.W. Heath, “Near maximum-likelihood detector with one-bit ADCs for multiuser massive MIMO systems,” in *IEEE 6th International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, Cancun, Mexico, 2015, pp. 397-400

[7] A. Mezghani, and J.A. Nossek, “Analysis of Rayleigh-fading channels with 1-bit quantized output,” in *IEEE International Symposium on Information Theory*, Toronto, Canada, 2008, pp. 260-264

[8] J.A. Nossek, and M.T. Ivrlac, “Capacity and coding for quantized MIMO systems,” in *Proceedings of the 2006 international conference on Wireless communications and mobile computing*, Vancouver, Canada, 2006, pp. 1387-1392

[9] C. Risi, D. Persson, and E.G. Larsson, “Massive MIMO with 1-bit ADC,” *arXiv preprint arXiv*:1404.7736 (2014)

[10] J. Choi, J. Mo, and R.W. Heath, “Near maximum-likelihood detector and channel estimator for uplink multiuser massive MIMO systems with one-bit ADCs,” *arXiv preprint arXiv*:1507.04452 (2015).

[11] J. Mo, P. Schniter, N.P. Gonzalez, and R.W. Heath, “Channel estimation in millimeter wave MIMO systems with one-bit quantization,” in *48th Asilomar Conference on Signals, Systems and Computers*, Pacific Grove, CA, 2014, pp. 957-961

[12] S. Jacobsson, G. Durisi, M. Coldrey, U. Gustavsson, and C. Studer, “One-bit massive MIMO: Channel estimation and high-order modulations,” in *IEEE International Conference on Communication Workshop*, London, UK, 2015, pp. 1304-1309

[13] L. Zhao, K. Zheng, H. Long, and H. Zhao, “Performance analysis for downlink massive MIMO system with ZF precoding,” *Transactions on Emerging Telecommunications Technologies*, vol. 25., no. 12, pp. 1219-1230, Dec 2014

[14] J. Hoydis, S. Brink, and M. Debbah, “Comparison of linear precoding schemes for downlink massive MIMO,” in *IEEE International Conference on Communications*, Ottawa, ON, 2012, pp. 2135-2139

[15] J. Hoydis, S. Brink, and M. Debbah, “Massive MIMO in the UL/DL of Cellular Networks: How Many Antennas Do We Need?” *IEEE J. Sel. Areas Commun.*, vol. 31, no. 2, pp. 160-171, Feb 2013

[16] A. Mezghani, R. Ghiat, and J.A. Nossek, “Transmit processing with low resolution D/A-converters,” in *16th IEEE International Conference on Electronics, Circuits, and Systems*, Yasmine Hammamet, Tunisia, 2009, pp. 683-686

[17] T. Cui and C. Tellambura, “An efficient generalized sphere decoder for rank-deficient MIMO systems,” *IEEE Commun. Lett.*, vol. 9, no. 5, pp. 423-425, May 2005

[18] U. Fincke and M. Pohst, “Improved methods for calculating vectors of short length in a lattice, including a complexity analysis,” *Mathematics of computation*, vol. 44, no. 170, pp. 463-471, Apr 1985

[19] J. Gubner, “More about discrete random variables,” in *Probability and random processes for electrical and computer engineers*, New York, Cambridge University Press, 2006, ch. 3, pp. 108-137

[20] S. Theodoridis and K. Koutroumbas, “Nonlinear Classifiers,” in *Pattern Recognition*, Academic Press, 2009, ch. 4, pp. 151-260