Letter

Geometric Programming for Nonlinear Satellite Buffer Networks With Time Delays under L_1-Gain Performance

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Dear Editor,

This letter concerns the parameter tuning problem for nonlinear satellite buffer networks with communication delays, aiming to optimize their stability properties under L_1-gain. We first model the satellite buffer networks by a nonlinear time-delay positive system and propose a novel characterization under which the nonlinear system is asymptotically stable with a prescribed L_1-induced performance. Then, the problem of finding the minimum-cost parameters is reduced to geometric programming (GP), which can be resolved by convex optimization efficiently. The flexibility of GP allows simultaneous tuning of parameters in system state, input, and output matrices. A numerical example is presented to illustrate the efficiency of our proposed optimization framework.

Satellite networks are critical for providing communication services to remote rural areas, disaster response teams, and military operations [1]. Besides, the emerging concept of integrated sensing and communication (ISC) offers advantages over traditional satellite networks by improving data collection, transmission, and analysis while reducing complexity, power consumption, and cost [2]. To reduce the communication delay and loss rate of satellite networks sensing data, buffer management is necessary to store and forward data packets between different components, such as satellites, ground stations, and user terminals.

Before their employment in satellite networks, buffer networks play a critical role in managing data flows to prevent congestion in computer networks and IoT before their use in satellite networks [3]. The optimal design of buffer networks has been a hot research topic for several decades, with various studies proposing different models and algorithms for minimizing network congestion and maximizing throughput by tuning their parameters, such as flow size and routing paths [4]. One of the earliest studies in this field was conducted in [5], who developed a model for analyzing the performance of buffer networks. Recent research has studied factors influencing network performance, such as packet loss, throughput, and delay. In [6], nonlinear optimal buffer allocation approaches were proposed for quality of service (QoS) assurance in mobile networks. As confirmed by simulations, the proposed methods minimize wasted data traffic and

improve resource optimization, leading to improved QoS.

The famous SpaceX's Starlink project aims to provide high-speed Internet access to remote areas by significantly increasing the number of network satellites. It remains a daunting task to optimize the satellite network at such a massive scale. In this letter, we introduce GP for optimizing the parameters of satellite buffer networks with communication delay and nonlinear characteristics. GP is a nonlinear optimization technique employing generalized posynomials as objective functions and constraints [7]. It has been successfully applied in diverse areas, including the chemical industry, network power control [8], and resource allocation [9]. GP-based algorithms are efficient and robust in optimizing resource allocation for large-scale networks.

This letter introduces a convex optimization framework, specifically GP, to tune the system parameters for nonlinear satellite buffer networks. The main contributions are twofold: 1) A necessary and sufficient criterion is first derived to characterize the L_1-gain performance for buffer networks modeled as nonlinear time-delay positive systems; 2) Based on the obtained input-output performance criterion, we prove that the problem of searching the minimum-cost parameters can be reformulated into a GP one and solved by convex optimization efficiently.

Notations: $\mathbb{R}^{m \times n}$ represents the set of all $m \times n$ real matrices. Let $\mathbf{1}$ represent a column vector with all elements equal to one. $v \geq (>)0$ or $v \in \mathbb{R}^n_{0,+}(\mathbb{R}^n_+)$ represents a real vector v is a nonnegative (positive) vector whose elements are all nonnegative (positive). $A \geq (>)0$ or $A \in \mathbb{R}^{m \times n}_{0,+}(\mathbb{R}^{m \times n}_+)$ means a real matrix $A \in \mathbb{R}^{m \times n}$ is a nonnegative (positive) matrix. For two vectors v_1 and v_2, $v_1 \geq (>)v_2$ means $v_1 - v_2$ is a nonnegative (positive) vectors. Let $z \in \mathbb{R}^n$, the 1-norm of vector z is defined as $\|z\|_1 \triangleq \sum_{i=1}^n |z_i|$. The L_1-norm is defined as $\|z\|_{L_1} \triangleq \int_0^\infty \|z(t)\|_1 \, dt$. Let $\text{diag}(A_1, \ldots, A_n)$ stand for a diagonal matrix that has block diagonals $A_1, \ldots, A_n$.

Problem statement: We consider an ISC network consisting of N satellites as shown in Fig. 1. A weighted and directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is employed to depict the communication topology among satellites, where $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represent the vertex set and edge set. We denoted each edge (communication link) as $e_\ell = (e_\ell(1), e_\ell(2))$, where $e_\ell(1)$ and $e_\ell(2)$ represent the transmitter and the receiver of the communication edge. The weight of edge (i, j) is denoted as w_{ij}, representing the transmit rate. The adjacent matrix $A_\mathcal{G}$ for the weighted and directed graph is given as: $[A_\mathcal{G}]_{ij} = w_{ji}$, if $(j, i) \in \mathcal{E}$ and $[A_\mathcal{G}]_{ij} = 0$ for otherwise. Define in-neighborhood set and out-neighborhood set of satellite i as $\mathcal{N}_i^{\text{in}} = \{j \in \mathcal{V} : e_{ij} \in \mathcal{E}\}$ and $\mathcal{N}_i^{\text{out}} = \{j \in \mathcal{V} : e_{ji} \in \mathcal{E}\}$. Furthermore, the network $\mathcal{G}$ has at least one origin with in-degree being 0, and at least one terminal with in-degree being 0. Let $\mathcal{V}_o$ and $\mathcal{V}_r$ denote the observation satellites (space telescopes) and terrestrial receivers, respectively. Other than
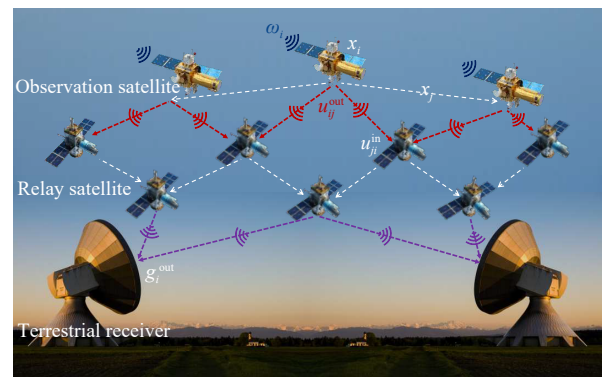

Fig. 1. Information flow in satellite buffer networks.

that, they are all relay satellites. The node i (satellite or receiver) in the dynamic buffer network satisfies

$$\frac{dx_i}{dt} = \begin{cases} -\sum\limits_{j\in\mathcal{N}_i^{\text{out}}} u_{ij}^{\text{out}} + \omega_i, & \text{if } i\in\mathcal{V}_o \\ \sum\limits_{j\in\mathcal{N}_i^{\text{in}}} u_{ji}^{\text{in}} - \sum\limits_{j\in\mathcal{N}_i^{\text{out}}} u_{ij}^{\text{out}}, & \text{if } i\notin\mathcal{V}_o\cup\mathcal{V}_r \\ \sum\limits_{j\in\mathcal{N}_i^{\text{in}}} u_{ji}^{\text{in}} - g_i^{\text{out}}, & \text{if } i\in\mathcal{V}_r \end{cases} \quad (1)$$

where $x_i(t)$ denotes the data amount of buffer node i, $u_{ij}(t)$ denotes the data flow from satellite i to j, ω_i and g_i^{out} represent the input data of the observation satellite and the removal of processed data at the terminal receiver. Furthermore, g_i^{out}, u_{ij}^{out}, and u_{ji}^{in} satisfy

$$g_i^{\text{out}}(t) = \phi_i f(x_i(t)), \quad u_{ij}^{\text{out}}(t) = \psi_i w_{ij} f(x_i(t))$$

$$u_{ji}^{\text{in}}(t) = \psi_j w_{ji} f(x_j(t-d))$$

where $\phi_i > 0$ $(i\in\mathcal{V}_d)$ and $\psi_i > 0$ $(i\in\mathcal{V}\backslash\mathcal{V}_d)$ are constants, representing the data processing capability and the transmission power, respectively. The function $f:\mathbb{R}^N\to\mathbb{R}^N$ is denoted as $f(x)=[f_1(x_1),\ldots,f_N(x_N)]^T$, which represents the data compression process. $d\ge 0$ denotes the transmission delay between satellites. $\phi_i = 0$ for all $i\notin\mathcal{V}_d$ and $\psi_i = 0$ for all $i\in\mathcal{V}_d$. The measurement output is defined as $z=[f^T(x),(\alpha u^{\text{in}})^T]^T$ with a weight constant $\alpha>0$ and the M-dimensional vector u^{in} is constructed by vertically stacking the flows u_{ji}^{in}. We denote that $u^{\text{in}} = \alpha R\Psi f(x(t-d))$ and $R\in\mathbb{R}^{M\times N}$ is defined as: $R_{\ell i} = w_{e_\ell}$, if $i = e_\ell(2)$, otherwise $R_{\ell i} = 0$, for all $i\in\{1,\ldots,N\}$ and $\ell\in\{1,\ldots,M\}$.

Define $\delta = [\Phi^T,\Psi^T]^T$, the nonlinear buffer network is written as

$$\Sigma_\delta: \begin{cases} dx(t)/dt = A(\delta)f(x(t)) + A_d(\delta)f(x(t-d)) + B(\delta)\omega(t) \\ z(t) = C(\delta)f(x(t)) + C_d(\delta)f(x(t-d)) + D(\delta)\omega(t) \end{cases}$$

where

$$A(\delta) = -\text{diag}(1_N^T A_\mathcal{G}\Psi) - \Phi, \quad A_d(\delta) = A_\mathcal{G}\Psi, \quad C(\delta) = [I_N, 0_{M\times N}]^T$$

$$B(\delta) = [I_{|\mathcal{V}_o|}, 0_{(N-|\mathcal{V}_o|)\times|\mathcal{V}_o|}]^T, \quad C_d(\delta) = [0_N,(\alpha R\Psi)^T]^T, \quad D(\delta) = 0$$

$$x(t) = [x_1(t),\ldots,x_N(t)]^T, \quad \omega(t) = [\omega_1(t),\ldots,\omega_{|\mathcal{V}_o|}(t)]^T$$

$$\Phi = \text{diag}(\phi_1,\phi_2,\ldots,\phi_N), \quad \Psi = \text{diag}(\psi_1,\psi_2,\ldots,\psi_N).$$

Definitions and lemmas:

Definition 1 (Monomial and posynomial): 1) Define positive variable $v=(v_1,\ldots,v_n)$. A real function h of v is called a monomial if $c>0$ and $a_1,\ldots,a_n\in\mathbb{R}$ exist such that $h(v) = cv_1^{a_1}v_2^{a_2}\cdots v_n^{a_n}$; 2) The function $z(v)$ is called a posynomial if $z(v)$ is the sum of finitely many monomials; 3) $z(v)$ is called a generalized posynomial if $z(v)$ is formulated in terms of posynomials by employing the operations of maximum, addition, multiplication, and positive power.

Assumption 1: The function $f(\cdot)$ satisfies the condition:

1) $f(\cdot)$ is locally Lipschitz continuous and satisfy $f(0)=0$. Furthermore, $f_i(\cdot)$ is strictly increasing for all $i=1,2,\ldots,n_x$.

2) When $x_i\to\infty$, $f_i(x)\to\infty$, for all $i=1,2,\ldots,n_x$.

Assumption 2: $A(\delta)$, $A_d(\delta)$, $B(\delta)$, $C(\delta)$, $C_d(\delta)$, $D(\delta)$ satisfy:

1) A matrix function $H(\delta) = \text{diag}(r_1(\delta),r_2(\delta),\cdots,r_{n_x}(\delta))$ exists such that each element of $\bar{A}(\delta) = A(\delta)+H(\delta)$ is either zero or a posynomial of δ, where diagonals $r_1(\delta),r_2(\delta),\ldots,r_{n_x}(\delta)$ are monomial.

2) Each element of $A_d(\delta)$, $B(\delta)$, $C(\delta)$, $C_d(\delta)$, and $D(\delta)$ is either zero or a posynomial of δ.

Assumption 3: There exist posynomials $s_1(\delta)$, $s_2(\delta),\ldots,s_l(\delta)$ such that the constraint set Δ satisfies

$$\Delta = \{\delta\in\mathbb{R}^{n_\delta}|\delta > 0, s_1(\delta)\le 1, s_2(\delta)\le 1,\ldots,s_l(\delta)\le 1\}.$$

Lemma 1 [10]: For any vector-valued function $f(\cdot)$ satisfying 1) of Assumption 1, system Σ_δ is positive if $A(\delta)$ is Metzler and $A_d(\delta)$, $B(\delta)$, $C(\delta)$, and $D(\delta)$ are nonnegative.

By Lemma 1, it follows that the system Σ_δ is positive if the Assumptions 1 and 2 hold. Then, the definition of L_1-gain is recalled.

Definition 2 (L_1-gain): For an asymptotically stable nonlinear positive system Σ_δ with zero initial conditions, the L_1-gain denotes the smallest $\gamma > 0$ such that $\|z\|_{L_1} \le \gamma\|\omega\|_{L_1}$ holds for all $\omega\in L_1$.

Main results: We first provide a characterization on the stability of system Σ_δ with L_1-gain.

Theorem 1 (L_1-gain of system Σ_δ): Suppose that Assumptions 1 and 2 hold, the following conditions are equivalent for system Σ_δ:

1) Positive system Σ_δ is asymptotically stable with L_1-gain less than γ;

2) There exists a positive vector $p\in\mathbb{R}_+^N$ such that

$$(A(\delta)+A_d(\delta))^T p + (C(\delta)+C_d(\delta))^T \mathbf{1} \prec 0 \quad (2)$$

$$B^T(\delta)p + D^T(\delta)\mathbf{1} \prec \gamma\mathbf{1}. \quad (3)$$

Proof (Sufficiency): In the following analysis of input-output performance, the notation "(δ)" may be omitted for simplicity. First, by (2) and [10, Theorem 1], it follows that system Σ_δ is asymptotically stable. Then, we introduce the following Lyapunov functional to investigate the input-output gain performance:

$$\varphi(x(t)) = x^T(t)p + \int_t^{t+d} f^T(x(\tau-d))A_d^T p d\tau. \quad (4)$$

According to system Σ_δ, the derivative $\varphi(x(t))$ satisfies

$$\dot{\varphi}(x(t)) = (f^T(x(t))A^T + f^T(x(t-d))A_d^T + \omega^T(t)B^T)p$$
$$+ (f^T(x(t)) - f^T(x(t-d)))A_d^T p$$
$$= f^T(x(t))(A+A_d)^T p + \omega^T(t)B^T p. \quad (5)$$

Under zero initial conditions, it follows that:

$$\|z\|_{L_1} = \int_0^{+\infty} z^T(t)\mathbf{1}dt$$
$$= \int_0^{+\infty} \left(f^T(x(t))C^T + f^T(x(t-d))C_d^T + \omega^T(t)D^T\right)\mathbf{1}dt$$
$$= \left[\int_0^{+\infty} f^T(x(t))(C+C_d)^T dt + \int_0^{+\infty} \omega^T(t)D^T dt\right]\mathbf{1}. \quad (6)$$

Let functional $\mathcal{H}$ be presented as follows:

$$\mathcal{H}(x,\omega,z) = \int_0^t \dot{\varphi}(x(\tau))d\tau + \int_0^t (\|z(\tau)\|_1 - \gamma\|\omega(\tau)\|_1)d\tau. \quad (7)$$

Substituting (5) and (6) into (7), as t goes to ∞, it follows that:

$$\mathcal{H}(x,\omega,z) = \int_0^\infty (\|z(\tau)\|_1 - \gamma\|\omega(\tau)\|_1)d\tau + \int_0^\infty \dot{\varphi}(x(\tau))d\tau$$
$$= \int_0^{+\infty} f^T(x(\tau))\left[(C+C_d)^T\mathbf{1} + (A+A_d)^T p\right]d\tau$$
$$+ \int_0^{+\infty} \omega^T(\tau)\left(B^T p + D^T\mathbf{1} - \gamma\mathbf{1}\right)d\tau. \quad (8)$$

Considering the positive system Σ_δ, we have

$$\int_0^{+\infty} f^T(x(\tau))d\tau > 0, \quad \int_0^{+\infty} \omega^T(\tau)d\tau > 0.$$

According to (2) and (3), under zero initial conditions, one has that $\mathcal{H}(x,\omega,z) < 0$ stands true for all $t\in\mathbb{R}_{0,+}$ and $\omega(t)\in\mathbb{R}_{0,+}^{|\mathcal{V}_o|}$. Since

$$\int_0^\infty \dot{\varphi}(x(\tau))d\tau = \varphi(x(\infty)) - \varphi(x(0))$$
$$= x^T(\infty)p + \lim_{t\to\infty}\left(\int_t^{t+d} f^T(x(\tau-d))A_d^T p d\tau\right) \ge 0$$

it follows that:

$$\int_0^\infty (\|z(\tau)\|_1 - \gamma\|\omega(\tau)\|_1)d\tau < 0, \quad \forall\omega(\tau)\in\mathbb{R}_{0,+}$$
$$\Rightarrow \sup_{\|\omega\|_{L_1}\le 1} \|z\|_{L_1} < \gamma. \quad (9)$$

Thus, the sufficiency of conditions (2) and (3) is proved. ∎

Proof (Necessity): If system Σ_δ is asymptotically stable, the matrix $A+A_d$ is Hurwitz. Let $\omega(t)$ be an impulse vector function $\sigma(t)$ satisfying $\int_0^\infty \sigma(\tau)d\tau = e_i$, where e_i is the i-th base vector of $\mathbb{R}^{|\mathcal{V}_o|}$. Integrating the first equation of system Σ_δ leads to

$$x(\infty) - x(0) = (A+A_d)\int_0^\infty f(x(\tau))d\tau + Be_i. \quad (10)$$

Since $A+A_d$ is Hurwitz, $(A+A_d)^{-1}$ exists. Based on (9), one has

$\int_0^\infty f(x(\tau))d\tau = -(A + A_d)^{-1}Be_i$. By integrating the second equation of system Σ_δ, we have

$$\int_0^\infty z(\tau)d\tau = C\int_0^\infty f(x(\tau))d\tau + C_d\int_0^\infty f(x(\tau - d))d\tau + De_i$$
$$= (C + C_d)\int_0^\infty f(x(\tau))d\tau + De_i$$
$$= \left[(-(C + C_d)(A + A_d)^{-1})B + D\right]e_i. \qquad (11)$$

Since $\|e_i\|_1 = 1$ and the L_1-gain of system Σ_δ is less than γ. The right-hand side of (11) satisfies

$$\mathbf{1}^T[-(C + C_d)(A + A_d)^{-1}B + D]e_i < \gamma \qquad (12)$$

for all $i \in \{1, 2, \ldots, |\mathcal{V}_o|\}$. Equality (11) can be rewritten as $-B(A + A_d)^{-1}(C + C_d)\mathbf{1} + D\mathbf{1} \prec \gamma\mathbf{1}$. According to Lemma 1 of [11], the above inequality is equivalent to the condition that a positive vector $p \in \mathbb{R}_+^N$ exists such that (2) and (3) hold, and the necessity of Theorem 1 is proved. ∎

Theorem 2 (GP for system Σ_δ): Given that Assumptions 1–3 hold, the posynomial cost function $L(\delta)$ with L_1-gain performance less than γ can be optimized by resolving the following GP problem:

$$\min_{\delta \in \mathbb{R}_+^{n_\delta}, p \in \mathbb{R}_+^N} L(\delta) \qquad (13)$$

$$\text{s.t. } \mathcal{D}_{\delta,p}^{-1}\left[(\bar{A}^T(\delta) + A_d^T(\delta))p + (C^T(\delta) + C_d^T(\delta))\mathbf{1}\right] \prec \mathbf{1} \qquad (13a)$$

$$\gamma^{-1}\left[B^T(\delta)p + D^T(\delta)\mathbf{1}\right] \prec \mathbf{1} \qquad (13b)$$

$$s_i(\delta) \leq 1, \quad i = 1, 2, \ldots, l \qquad (13c)$$

where $\mathcal{D}_{\delta,p} = \text{diag}(r_1(\delta)p_1, r_2(\delta)p_2, \ldots, r_N(\delta)p_N)$, $p = [p_1, p_2, \ldots, p_N]^T \in \mathbb{R}_+^N$.

Proof: By Theorem 1, if the system Σ_δ is asymptotically stable and L_1-gain input-output performance is less than γ, it follows that:

$$(A^T(\delta) + A_d^T(\delta))p + (C^T(\delta) + C_d^T(\delta))\mathbf{1} \prec 0 \qquad (14)$$

$$B^T(\delta)p + D^T(\delta)\mathbf{1} \prec \gamma\mathbf{1}. \qquad (15)$$

It can be found that (15) is equivalent to (13b). Since $A(\delta) = \bar{A}(\delta) - H(\delta)$, we have that (14) yields

$$(\bar{A}^T(\delta) + A_d^T(\delta))p + (C^T(\delta) + C_d^T(\delta))\mathbf{1} \prec \mathcal{D}_{\delta,p}\mathbf{1} \qquad (16)$$

which is equivalent to (13a). ∎

Remark 1: The optimal parameter selection problem (13) in Theorem 2 is a standard GP one [7], which can be transformed into a convex optimization problem by the logarithmic variable transformation. Moreover, standard software platforms such as Python, MATLAB, and MOSEK provide packages to formulate and solve GP problems directly.

Numerical example: We introduce a buffer network example with three observation satellites (triangle nodes), fifteen relay satellites (circle nodes), and two receivers (square nodes) shown in Fig. 2. The nonlinear function f is given as $f_i(x_i) = \ln(x + 1)$, $i \in \mathcal{V}_o$, $f_i(x) = x^{1/2} + \sin x$, $i \in \mathcal{V}_r$, and $f_i(x) = \arctan x$, $i \notin \mathcal{V}_o \cup \mathcal{V}_r$. We evaluate the cost of tuning system Σ_δ by the following sum function:

$$L(\phi, \psi) = \sum_{i \in \mathcal{V}_r} \phi_i + \sum_{i \in \mathcal{V} \setminus \mathcal{V}_r} \psi_i$$

where ϕ_i and ψ_i denote the costly resources of data transmitting power and processing capacity. The measurement output rate is $\alpha = 1/10$. Considering the physical restrictions, the upper bounds of tuned parameters are given as $\phi_i \leq \bar{\phi}_i = 5$ {M/s} and $\psi_i \leq \bar{\psi}_i = 5$ {M/s}. Define $H(\delta) = \text{diag}(\mathbf{1}_N^T A_\mathcal{G} \Psi) + \Phi$ and we have $\bar{A}(\delta) = A(\delta) + H(\delta) = 0$. Suppose that the weights of edges stemming from a node are equal and sum to 1. We give $w_{ij} = 1/|\mathcal{N}_i^{\text{out}}|$ for each satellite i. First, the system minimum achievable L_1 norm is determined as $\gamma^* = 1.0692$. Then, given different γ within the range $[\gamma^*, 4\gamma^*]$, the optimal parameter selection problem for the buffer network is solved to get the optimal values of ϕ and ψ. The optimal values of cost L for
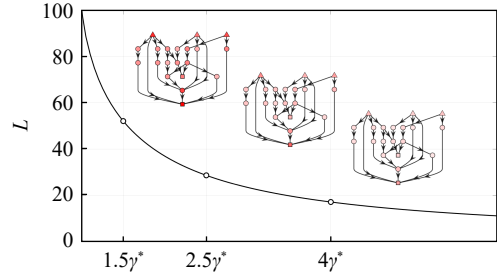

Fig. 2. The quantity of optimal source allocations under different γ. The redder the colors, the larger the optimized parameters ϕ_i and ψ_i.

different γ are shown in Fig. 2. For scenarios when $\gamma = 1.5\gamma^*$, $2\gamma^*$, and $4\gamma^*$, the optimal values of ϕ_i and ψ_i are given explicitly.

Conclusion: This letter has studied the parameter tuning problem for nonlinear satellite buffer networks. A fundamental performance characterization is first proposed to guarantee the asymptotic stability of nonlinear systems with L_1 induced performance. Then, a GP framework is proposed to find the minimum-cost parameters through convex optimization. Finally, the efficiency of the proposed method is demonstrated by a simulation example.

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