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► **To cite this version:**

Olivier Rioul, Max H. M. Costa. Almost There: Corner Points of Gaussian Interference Channels. 2015 IEEE Information Theory and Applications Workshop (ITA 2015), Feb 2015, San Diego, CA, United States. 10.1109/ITA.2015.7308963 . hal-02288425

**HAL Id: hal-02288425**

**<https://telecom-paris.hal.science/hal-02288425v1>**

Submitted on 10 Aug 2022

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# Almost There

## Corner Points of Gaussian Interference Channels

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**Abstract**—Almost-Gaussian (AG) and almost-lossless (AL) properties are used to derive almost trivial proofs of almost-achievable corner points of the capacity region of Gaussian interference channels. For the missing corner point, the proof is almost complete.

### I. INTRODUCTION

This paper is about the determination of the corner points of the capacity region of the two-user memoryless Gaussian interference channel. The channel is depicted in Fig. 1 in standard form [1]. Because the joint distribution of the Gaussian noises  $(Z_1, Z_2)$  at the decoder sides is not relevant for the communication problem, we find it notationally convenient to set  $Z_1 = Z_2 = Z$ . In standard form, we may assume unit noise powers  $N_1 = N_2 = 1$ , but we keep the notations  $N_1 = N_2 = N$  to stay with dimensionally homogeneous expressions—e.g., we write  $\log(1+P/N)$  instead of  $\log(1+P)$ .

Sender  $i = 1, 2$  produces a uniformly distributed  $M_i$ -ary message  $W_i$ , where<sup>1</sup>  $W_1 \perp W_2$ . Encoder  $i$  maps  $W_i$  to a random vector  $X_i \in \mathbb{R}^n$  of dimension  $n$  which satisfies the power constraint  $\|X_i\|^2 \leq nP_i$ . Decoder  $i$  maps the output  $Y_i$  to an  $M_i$ -ary decoded message  $\widehat{W}_i$ .

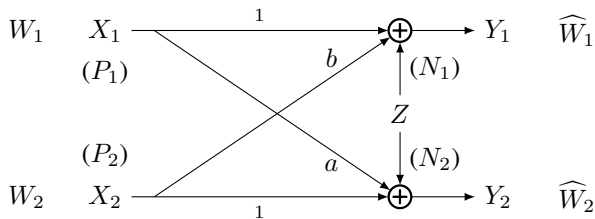


Fig. 1: Gaussian interference channel.

The capacity region of the channel is classically defined as the closure of the set of all achievable rate pairs  $(R_1, R_2)$ . We prefer to define it as the set of all *almost achievable* rate pairs, which are limit points of all sequences<sup>2</sup>  $(R_1, R_2)$  for which the corresponding sequence of encoding and decoding functions with  $M_i = e^{nR_i}$  are such that  $\mathbb{P}\{\widehat{W}_i \neq W_i\}$  ( $i = 1, 2$ ) tend to 0 as  $n \rightarrow +\infty$ . It is easily seen that the two definitions are equivalent. The capacity region is a subset of the rectangle  $R_1 \leq C_1, R_2 \leq C_2$ , where  $C_i = \frac{1}{2} \log(1 + P_i/N_i)$ . A typical shape is shown in Fig. 2, where the two *corner points*  $(C_1, C'_2)$  and  $(C'_1, C_2)$  are marked with circles.

<sup>1</sup>We use the notation  $\perp$  to denote independence.

<sup>2</sup>For notational convenience we have dropped the index  $n$  when writing  $R_1, R_2, W_1, W_2, X_1, X_2, Y_1, Y_2, Z_1, Z_2$ . However,  $P_1, P_2$  and  $N$  are constants.

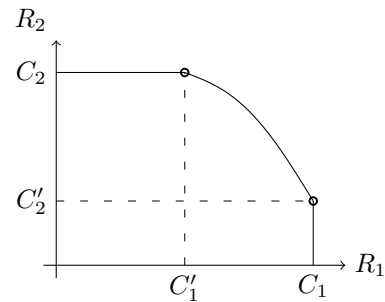


Fig. 2: Corner points of the capacity region.

Because  $n$  is taken arbitrarily large, it is convenient to use the following notation.

**Definition 1** (Asymptotic Almost Inequalities). Let  $\epsilon(n)$  denote any positive function of  $n$  which tends to  $0^+$  as  $n \rightarrow +\infty$  (thus we can write, for example,  $\epsilon(n) + \epsilon(n) = \epsilon(n)$ ). Given real number sequences  $a_n, b_n$ , we write  $a_n \lesssim b_n$  ( $a_n$  is *almost less than*  $b_n$ ) if  $a_n \leq b_n + n\epsilon(n) \iff b_n \geq a_n - n\epsilon(n)$ . We also write  $b_n \gtrsim a_n$  ( $b_n$  is *almost greater than*  $a_n$ ).

That  $(C_1, C'_2)$  is a corner point is established by showing that it is almost achievable and that for any  $(R_1, R_2)$  for which the associated probability of error tends to 0 as  $n \rightarrow +\infty$ ,

$$nR_1 \gtrsim nC_1 \implies nR_2 \lesssim nC'_2. \quad (1a)$$

That  $(C'_1, C_2)$  is a corner point is similarly characterized by:

$$nR_2 \gtrsim nC_2 \implies nR_1 \lesssim nC'_1. \quad (1b)$$

Achievability is generally not a problem and is done using classical ingredients such as random coding, onion peeling and rate splitting. Therefore, in this paper, we focus exclusively on the derivation of the converse, that is, of (1). We provide exact proofs for corner points in all situations except for the case of the “missing corner point” for which our proof is almost complete. All these proofs rely on the following notions.

### II. ALMOST GAUSSIANNES AND LOSSLESSNESS

Throughout the paper we assume that the average powers of the considered random vectors are bounded by constants (independent of  $n$ ). Thus, to any random vector  $X \in \mathbb{R}^n$  we assume that  $P = \sup_n \frac{1}{n} \mathbb{E}\{\|X\|^2\}$  is finite. The maximum entropy (MaxEnt) property states that  $h(X) \leq \frac{n}{2} \log(2\pi eP)$  with equality iff  $X$  is white Gaussian with power  $P$ .

**Definition 2** (AG).  $X$  is *almost (white) Gaussian* (AG) if

$$h(X) \gtrsim \frac{n}{2} \log(2\pi eP). \quad (2)$$

Let  $Z, Z', Z'', \dots$  be mutually independent (not necessarily Gaussian) vectors, independent of  $X$  and consider the Markov chain of cumulative additions:

$$X \rightarrow (X + Z) \rightarrow (X + Z + Z') \rightarrow (X + Z + Z' + Z'') \rightarrow \dots$$

By the data processing inequality (DPI), each addition decreases mutual information, e.g.,  $I(X; X + Z + Z') \leq I(X; X + Z)$ .

**Definition 3 (AL).** The addition of  $Z'$  in  $X + Z + Z'$  is *almost lossless* (AL) with respect to  $X$  if mutual information is almost nondecreasing:

$$I(X; X + Z + Z') \gtrsim I(X; X + Z). \quad (3)$$

We say that  $X + Z + Z'$  is *almost lossless* compared to  $X + Z$  with respect to  $X$ , or more briefly that  $(X + Z) + Z'$  is AL (w.r.t.  $X$ ).

**Lemma 1 (Fork Lemma).** *Let  $X_1, X_2$  and  $Z$  be independent. If  $X_1 + X_2 + Z$  is AL compared to  $X_1 + Z$  w.r.t.  $X_1$ , then it is also AL compared to  $X_2 + Z$  w.r.t.  $X_2$ .*

*Proof of Lemma 1:*  $I(X_2; X_1 + X_2 + Z) - I(X_2; X_2 + Z) = h(X_1 + X_2 + Z) - h(X_1 + Z) - h(X_2 + Z) + h(Z) = I(X_1; X_1 + X_2 + Z) - I(X_1; X_1 + Z)$ . ■

Letting  $Y = X_1 + X_2 + Z$ , this means that if adding  $X_1$  is almost lossless with respect to the transmission from  $X_2$  to  $Y$ , then adding  $X_2$  is also almost lossless with respect to the transmission from  $X_1$  to  $Y$ .

The usefulness of the AG and AL properties for determining corner points is given by the following proposition.

**Proposition 1.** *The condition  $nR_1 \gtrsim nC_1$  in (1a) implies<sup>3</sup>*

- (a)  $X_1 + Z$  is AG;
- (b) adding interference  $bX_2$  in  $Y_1 = X_1 + bX_2 + Z$  is AL compared to  $X_1 + Z$  w.r.t.  $X_1$ ;

The symmetrical proposition holds for transmission 2.

*Proof of Prop. 1:* By the classical derivation of the converse theorem:

$$nR_1 = H(W_1) \lesssim I(W_1; Y_1) \quad (\text{Fano}) \quad (4a)$$

$$\leq I(X_1; Y_1) \quad (\text{DPI}) \quad (4b)$$

$$\leq I(X_1; X_1 + Z) \quad (\text{DPI again}) \quad (4c)$$

$$= h(X_1 + Z) - h(Z) \quad (4d)$$

$$\leq nC_1 \quad (\text{MaxEnt}) \quad (4e)$$

Thus  $nR_1 \gtrsim nC_1$  amounts to saying that all quantities in (4) are at distance  $\leq n\epsilon(n)$ . This implies, in particular, (a) from (4e) and (b) from (4c). The only remaining condition is  $I(W_1; Y_1) \gtrsim I(X_1; Y_1)$  which holds (with equality) if the encoder mapping is invertible. ■

Notice that (b) becomes vacuous in the case of no interference ( $b = 0$ ). If  $b \neq 0$ , by the fork lemma (Lemma 1), condition (b) is equivalent to

- (b') adding  $X_1$  in  $Y_1 = X_1 + bX_2 + Z$  is AL compared to  $bX_2 + Z$  w.r.t.  $X_2$ ;

In the following, we examine the case of a Gaussian  $Z$ -interference channel with one of the interference parameters (e.g.,  $b$ ) equal to zero, illustrated in Fig. 3. The general determination of corner points will follow in the general case of two-sided interference by noting that removing an interference link can only enlarge the capacity region, as explained in [2, Table I].

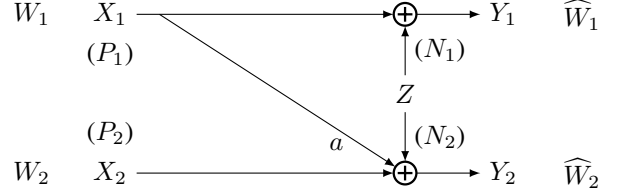


Fig. 3: Gaussian  $Z$ -interference channel.

The *very strong interference* case ( $a^2 \geq 1 + P_2/N$ ) is well-known [3]. One has ( $C'_1 = C_1, C'_2 = C_2$ ) and in this case there is no need to prove (1).

### III. THE STRONG INTERFERENCE CASE

For strong interference ( $1 \leq a^2 \leq 1 + P_2/N$ ) the corner points are known and given by (5) below. The usual derivation follows from that of the capacity region of the multiple access channel and from the result of Han and Kobayashi [4] and Sato [5], who showed that both receivers should be able to decode both messages  $W_1$  and  $W_2$ . In comparison, the following (converse) proof is almost trivial.

**Proposition 2.** *For the strong  $Z$ -interference Gaussian channel,*

$$\begin{aligned} C'_1 &= \frac{1}{2} \log_2 \left( 1 + \frac{a^2 P_1 + P_2}{N} \right) - C_2 \\ &= \frac{1}{2} \log_2 \left( 1 + \frac{a^2 P_1}{P_2 + N} \right) \end{aligned} \quad (5a)$$

$$\begin{aligned} C'_2 &= \frac{1}{2} \log_2 \left( 1 + \frac{a^2 P_1 + P_2}{N} \right) - C_1 \\ &= \frac{1}{2} \log_2 \left( 1 + \frac{(a^2 - 1)P_1 + P_2}{P_1 + N} \right). \end{aligned} \quad (5b)$$

**Lemma 2.** *Let  $X \perp\!\!\!\perp Z$  where  $Z$  is Gaussian. For any  $a^2 \geq 1$ ,*

$$h(aX + Z) \geq h(X + Z), \quad (6a)$$

that is,

$$I(X; aX + Z) \geq I(X; X + Z). \quad (6b)$$

*Proof:* Let  $Z'$  be an independent copy of  $Z$  and set  $a'$  such that  $1/a^2 + 1/a'^2 = 1$ . By the DPI,  $I(X; aX + Z) = I(X; X + Z/a) \geq I(X; X + Z/a + Z'/a') = I(X; X + Z)$  since  $Z/a + Z'/a'$  is identically distributed as  $Z$ . ■

*Proof of Proposition 2:* First suppose that  $nR_1 \gtrsim nC_1$ . From Proposition 1,  $X_1 + Z$  is AG. Therefore, from (4a)–(4b) where index 1 is replaced by 2,

$$nR_2 \lesssim I(X_2; Y_2) \quad (7a)$$

$$= h(X_2 + aX_1 + Z) - h(aX_1 + Z) \quad (7b)$$

$$\leq h(X_2 + aX_1 + Z) - h(X_1 + Z) \quad (\text{Lemma 2}) \quad (7c)$$

$$\lesssim h(X_2 + aX_1 + Z) - h(Z) - nC_1 \quad (\text{AG}) \quad (7d)$$

$$\leq nC'_2 \quad (\text{MaxEnt}) \quad (7e)$$

<sup>3</sup>Equivalence holds if encoder 1 is invertible.

which proves that  $nR_2 \lesssim nC'_2$  (cf. (1a)).

Next suppose that  $nR_2 \gtrsim nC_2$ . From Proposition 1 written for transmission 2,  $X_2 + Z$  is AG and adding interference  $aX_1$  in  $Y_2 = aX_1 + X_2 + Z$  is AL w.r.t.  $X_2$ . Since  $a \neq 0$ , by Lemma 1, this implies that adding  $X_2$  in  $Y_2 = aX_1 + X_2 + Z$  is AL w.r.t.  $X_1$ . Therefore, from (4a)–(4b),

$$nR_1 \lesssim I(X_1; Y_1) = I(X_1; X_1 + Z) \quad (8a)$$

$$\leq I(X_1; aX_1 + Z) \quad (\text{Lemma 2}) \quad (8b)$$

$$\lesssim I(X_1; aX_1 + X_2 + Z) \quad (\text{AL}) \quad (8c)$$

$$= h(aX_1 + X_2 + Z) - h(X_2 + Z) \quad (8d)$$

$$\lesssim h(aX_1 + X_2 + Z) - h(Z) - nC_2 \quad (\text{AG}) \quad (8e)$$

$$\leq nC'_1 \quad (\text{MaxEnt}) \quad (8f)$$

which proves that  $nR_1 \leq nC'_1$  (cf. (1b)).  $\blacksquare$

#### IV. A CONSEQUENCE OF THE EPI ON AGNESS

In terms of its entropy power, the AG condition for  $X$  is easily shown to be equivalent to

$$\frac{1}{2\pi e} e^{\frac{2}{n}h(X)} \geq P - \epsilon(n). \quad (9)$$

**Proposition 3.** *If  $X, Y$  are independent and AG, then  $X + Y$  is also AG.*

*Proof:* Assume (9) and the corresponding condition for  $Y$  with average power  $Q$ . Since the average power constraint associated to  $X + Y$  is  $P + Q$ , the assertion follows from the EPI:

$$\frac{1}{2\pi e} e^{\frac{2}{n}h(X+Y)} \geq \frac{1}{2\pi e} e^{\frac{2}{n}h(X)} + \frac{1}{2\pi e} e^{\frac{2}{n}h(Y)} \quad (10)$$

$$\geq P + Q - \epsilon(n) \quad (11)$$

where we have used that  $\epsilon(n) + \epsilon(n) = \epsilon(n)$ .  $\blacksquare$

Prop. 3 will be essentially useful in the following particular case, which is essentially the almost obvious fact that adding white Gaussian noise preserves the AG property.

**Corollary 1.** *Let  $X \perp\!\!\!\perp Z$  where  $Z$  is Gaussian. For any  $a^2 \leq 1$ , if  $X + Z$  is AG then  $X + Z/a$  is also AG.*

*Proof:* The AG property depends on the random vector only through its distribution. Let  $Z'$  be an independent copy of  $Z$  and set  $a'$  such that  $1/a^2 = 1 + 1/a'^2$ . Then  $X + Z/a$  is identically distributed as  $X + Z + Z'/a'$  and the assertion follows from Proposition 3.  $\blacksquare$

#### V. SATO'S CORNER POINT

For weak interference  $a^2 \leq 1$ , Sato [6] has found that the first corner point is given by (12) below. The usual derivation follows from the equivalence between Gaussian Z-interference channel and a “fully” degraded version proved in [2], the fact that it can be considered as a broadcast channel with input power given by  $P_1 + P_2$  [6], and Bergmans' derivation of the capacity region of the Gaussian (degraded) broadcast channel [7]. In comparison the following proof is almost trivial.

**Proposition 4.** *For the weak Z-interference Gaussian channel,*

$$C'_2 = \frac{1}{2} \log \left( 1 + \frac{P_2}{a^2 P_1 + N} \right). \quad (12)$$

*Proof:* Suppose that  $nR_1 \gtrsim nC_1$ . From Proposition 1,  $X_1 + Z$  is AG. Multiplying by  $a$ ,  $aX_1 + aZ$  is AG. By Corollary 1,  $aX_1 + Z$  is also AG. Therefore, from (4a)–(4b) written for  $i = 2$ ,

$$nR_2 \lesssim I(X_2; Y_2) = h(Y_2) - h(aX_1 + Z) \quad (13a)$$

$$\lesssim h(Y_2) - \frac{n}{2} \log(2\pi e(a^2 P_1 + N)) \quad (\text{AG}) \quad (13b)$$

$$\leq nC'_2 \quad (\text{MaxEnt}) \quad (13c)$$

which proves that  $nR_2 \lesssim nC'_2$  (cf. (1a)).  $\blacksquare$

#### VI. A CONSEQUENCE OF THE EPIC ON ALNESS

Let  $Z_t = \sqrt{t}Z$ . The entropy power inequality of concavity (EPIC) [8] states that if  $t' = \lambda t + (1 - \lambda)t''$ , then

$$e^{\frac{2}{n}h(X+Z_{t'})} \geq \lambda e^{\frac{2}{n}h(X+Z_t)} + (1 - \lambda) e^{\frac{2}{n}h(X+Z_{t''})}. \quad (14)$$

We first rewrite this in terms of mutual informations.

**Lemma 3.** *For any fixed  $t \leq t' \leq t''$ , there exists  $\mu \in [0, 1]$  independent of  $n$  such that*

$$e^{\frac{2}{n}I(X; X+Z_{t'})} \geq \mu e^{\frac{2}{n}I(X; X+Z_t)} + (1 - \mu) e^{\frac{2}{n}I(X; X+Z_{t''})} \quad (15)$$

*Proof:* Writing  $h(X + Z) = I(X; X + Z) + h(Z)$  in (14) gives  $t' e^{\frac{2}{n}I(X; X+Z_{t'})} \geq \lambda t e^{\frac{2}{n}I(X; X+Z_t)} + (1 - \lambda)t'' e^{\frac{2}{n}I(X; X+Z_{t''})}$ . The result follows by letting  $\mu = \lambda t/t' \in [0, 1]$ .  $\blacksquare$

Despite appearances, this lemma does *not* imply that  $\exp\left(\frac{2}{n}I(X; X + Z_t)\right)$  is concave in  $t$  since in general  $\mu t + (1 - \mu)t''$  will not be equal to  $t'$ .

Again let  $Z, Z', Z''$  be independent Gaussian, independent of  $X$ .

**Proposition 5.** *If  $X + Z + Z' + Z''$  is AL compared to  $X + Z + Z'$ , then it is also AL compared to  $X + Z$ .*

In other words, the AL property is preserved if we “remove” Gaussian noise of finite power.

*Proof:* Let  $t \leq t' \leq t''$ . Since the mutual information  $I(X; X + Z)$  depends on  $Z$  only through its distribution, it is easily seen that it is equivalent to prove that  $I(X; X + Z_{t''}) \gtrsim I(X; X + Z_{t'})$  implies  $I(X; X + Z_{t'}) \gtrsim I(X; X + Z_t)$ . Taking exponentials it suffices to prove that  $e^{\frac{2}{n}I(X; X+Z_{t''})} \geq e^{\frac{2}{n}I(X; X+Z_{t'})} - \epsilon(n)$  implies  $e^{\frac{2}{n}I(X; X+Z_{t'})} \geq e^{\frac{2}{n}I(X; X+Z_t)} - \epsilon(n)$ . But this is a trivial consequence of Lemma 3.  $\blacksquare$

The following particular case is related to our problem.

**Corollary 2.** *Let  $a^2 \leq 1$ . If  $aX_1 + X_2 + Z$  is almost lossless compared to  $aX_1 + Z$ , and if  $X_2 + Z$  is white Gaussian, then  $aX_1 + X_2 + Z$  is almost lossless compared to  $X_1 + Z$  w.r.t.  $X_1$ .*

*Proof:* Since  $X_2 + Z$  is white Gaussian and  $a^2 \leq 1$ ,  $aX_1 + X_2 + Z$  is AL compared to  $aX_1 + aZ$ , that is, AL compared to  $X_1 + Z$ .  $\blacksquare$

We shall use the following almost identical version of Corollary 2, which differs from it only by the addition of the word “almost”:

**Conjecture 1 (Almost-Conjecture).** *Let  $a^2 \leq 1$ . If  $aX_1 + X_2 + Z$  is almost lossless compared to  $aX_1 + Z$ , and if  $X_2 + Z$  is almost white Gaussian, then  $aX_1 + X_2 + Z$  is almost lossless compared to  $X_1 + Z$  w.r.t.  $X_1$ .*

## VII. THE ALMOST MISSING CORNER POINT

For weak interference  $a^2 \leq 1$ , Costa [2] has stated that the second corner point is given by (16) below. A problematic issue in the proof was detected by Sason [9] and the corner point has since been dubbed “missing” [10]. We provide an almost trivial proof, but based on the Almost-Conjecture.

**Proposition 6** (conjectural). *For the weak Z-interference Gaussian channel,*

$$C'_1 = \frac{1}{2} \log \left( 1 + \frac{a^2 P_1}{P_2 + N} \right). \quad (16)$$

*Proof Using the Almost-Conjecture:* Suppose that  $nR_2 \gtrsim nC_2$ . From Proposition 1 written for transmission 2,  $X_2 + Z$  is AG and adding interference  $aX_1$  in  $Y_2 = aX_1 + X_2 + Z$  is AL w.r.t.  $X_2$ . Since  $a \neq 0$ , by the Fork Lemma (Lemma 1), this implies that adding  $X_2$  in  $Y_2 = aX_1 + X_2 + Z$  is AL compared to  $aX_1 + Z$  w.r.t.  $X_1$ . Using the Almost Conjecture, this would imply that  $aX_1 + X_2 + Z$  is AL compared to  $X_1 + Z$  w.r.t.  $X_1$ . Therefore, from (4a)–(4b),

$$nR_1 \lesssim I(X_1; Y_1) = I(X_1; X_1 + Z) \quad (17a)$$

$$\lesssim I(X_1; aX_1 + X_2 + Z) \quad (\text{AL}) \quad (17b)$$

$$= h(aX_1 + X_2 + Z) - h(X_2 + Z) \quad (17c)$$

$$\lesssim h(aX_1 + X_2 + Z) - h(Z) - nC_2 \quad (\text{AG}) \quad (17d)$$

$$\leq nC'_1 \quad (\text{MaxEnt}) \quad (17e)$$

which proves that  $nR_1 \lesssim nC'_1$  (cf. (1b)). ■

Notice that this proof is almost identical to the one given above in the strong interference case. The only difference is the problematic step that  $I(X_1; Y_1) = I(X_1; X_1 + Z) \lesssim I(X_1; aX_1 + X_2 + Z) = I(X_1; Y_2)$ .

## VIII. TOWARDS A PROOF OF THE ALMOST-CONJECTURE

Let  $a^2 \leq 1$  and assume, as above, that  $X_2 + Z$  is AG and  $aX_1 + X_2 + Z$  is AL compared to  $aX_1 + Z$ , that is,  $I(X_1; aX_1 + X_2 + Z) \gtrsim I(X_1; aX_1 + Z)$ . Also let  $X_2^G$  be white Gaussian of variance  $P_2$ , independent of the other random vectors. The fact that  $X_2 + Z$  is AG amounts to saying that

$$h(X_2^G + Z) \lesssim h(X_2 + Z) \leq h(X_2^G + Z). \quad (18)$$

On the other hand, we have the following, which is essentially the content of [11, § 3].

**Lemma 4.** *If  $aX_1 + X_2 + Z$  is AL compared to  $aX_1 + Z$  w.r.t.  $X_1$ , then  $I(X_1; aX_1 + X_2 + Z) \gtrsim I(X_1; aX_1 + X_2^G + Z)$ .*

*Proof:*  $I(X_1; aX_1 + X_2 + Z) \gtrsim I(X_1; aX_1 + Z) \geq I(X_1; aX_1 + X_2^G + Z)$  where the latter inequality is a DPI. ■

What we need to prove the Almost-Conjecture is, in fact, the opposite almost inequality, which is essentially the problematic Lemma 1 in [2]:

**Conjecture 2** (Entropy-Conjecture). *If  $X_2 + Z$  is AG, then*

$$h(aX_1 + X_2 + Z) \lesssim h(aX_1 + X_2^G + Z), \quad (19)$$

*that is,  $I(X_1; aX_1 + X_2 + Z) \lesssim I(X_1; aX_1 + X_2^G + Z)$ .*

This conjecture essentially states that adding  $aX_1$  almost does not increase the difference of the differential entropies between  $X_2 + Z$  and  $X_2^G + Z$ .

**Proposition 7.** *The Entropy-Conjecture implies the Almost-Conjecture.*

*Proof:* With the above assumptions,  $I(X_1; aX_1 + X_2^G + Z) \gtrsim I(X_1; aX_1 + X_2 + Z) \gtrsim I(X_1; aX_1 + Z)$ . Since  $X_2^G$  is white Gaussian, it follows from Corollary 2 that  $I(X_1; aX_1 + X_2^G + Z) \gtrsim I(X_1; aX_1 + aZ) = I(X_1; X_1 + Z)$ . But from Lemma 4,  $I(X_1; aX_1 + X_2 + Z) \gtrsim I(X_1; aX_1 + X_2^G + Z) \gtrsim I(X_1; X_1 + Z)$ . Therefore,  $aX_1 + X_2 + Z$  is almost lossless compared to  $X_1 + Z$  w.r.t.  $X_1$ , which proves the assertion of the Almost-Conjecture. ■

Some partial results towards proving the Entropy-Conjecture are given in [12]. On the other hand, a direct proof of the Almost Conjecture would perhaps require an extension of the EPIC in the form of Lemma 3 for almost Gaussian  $Z$ .

## ACKNOWLEDGMENTS

We are grateful to Michèle Wigger for her valuable suggestions. We also thank Chandra Nair, Igal Sason, Gerhard Kramer, Lihong Zheng and Shlomo Shamai for inspiring discussions. Additionally, we would like to cite two recent contributions to this long standing problem, the papers by Sason [13] and by Bustin et al. [14].

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