The Potential Effectiveness of the Detection of Pulsed Signals in the Non-Uniform Sampling

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Abstract— **We consider a non-uniform time quantization of the optimal form of the signal by shifting one of the samples in the neighborhood of the point at which the minimum eigenvalue of the covariance matrix of the noise is equal to zero. Shown, that in testing simple hypotheses, arbitrarily large value of the signal-to-noise- ratio is achieved by this shift on the output of the discrete matched filter at finite energy of the signal and noise power. We discuss some aspects of the ill conditioning of the problem and the a priori of uncertainty.**

Keywords: *Detection, eigenvalue, SNR, signal detection*

I. INTRODUCTION

The task of improving the efficiency of detection of pulsed signals is relevant to many applications of statistical radio engineering. Detection of a deterministic signal **S** with known arrival time in the additive stationary Gaussian noise **X** with correlation matrix **B** is described by a discrete matched filter [1] **BG = S**. Statistic (**Z = X + AS** - signal on the input with a hypothesis $\mathbf{H_1}$, **A** – amplitude) $\boldsymbol{\alpha} = \mathbf{G^T Z}$ provides the signal-to-noise ratio of the discrete matched filter: $\mathbf{d^2 = G^T B G = S^T B^{-1} S}$. Its dependence on the shape of the signal allows optimizing the efficiency of detection of the appointment of the optimal signal: $\mathbf{S_{opt} = AU_{min}}$, $\mathbf{U_{min}}$ - Eigenvector of the matrix corresponding to the minimum eigenvalue $\boldsymbol{\alpha_{min} > 0}$. This delivers the signal-to-noise ratio: $\mathbf{d^2 = A^2 U^T_{min} B^{-1} U_{min} = A^2 U^T_{min} U \Lambda^{-1} U^T U_{min} = E / \lambda_{min}}$, **U, Λ** - matrices of eigenvectors and eigenvalues, $\mathbf{E = A^2}$ - the signal energy [1,2]. Inverse proportionality of the signal-to-noise ratio to the minimum eigenvalue means that when $\boldsymbol{\lambda_{min} \to 0}$ the probability of detection is very high. Obviously, this condition is satisfied when there are two samples with interval **Δ:** eigenvectors and eigenvalues of the correlation matrix of the standard noise: $B = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$ is equal to the following expression:

$$U = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \Lambda = \sigma^2 \begin{bmatrix} 1-\rho & 0 \\ 0 & 1+\rho \end{bmatrix};$$

increasing the value of $\boldsymbol{\Delta \to 0}$, implies $\boldsymbol{\rho \to}$ 1and $\boldsymbol{\lambda_{min} \to 0}$.

The need to expand bandwidth $\mathbf{\Delta F} \to \infty$ which is followed from $\boldsymbol{\Delta \to 0}$ makes this example not very interesting. But there is another case to be considered when there are many samplings available. It can be shown that if one sample is shifted, which implements a non-uniform signal sampling in time, then in this case the value of $\boldsymbol{\alpha_{min}}$ will be much less when compared to uniform sampling.

II. SIMPLE HYPOTHESIS

Let's assume that vector $\mathbf{X^T = [x_1,\ldots,x_n]}$ consists of sampling of stationary Gaussian process **x(t)** .Singular value decomposition [3] of the correlation matrix B_x will be as following: $\mathbf{B_x = U \Delta U^T}$. Eigenvectors **U** of this matrix are the directions of the ellipsoid axes of dispersion and eigenvalues Δ are lengths of its axes. But at the same time eigenvalues are the roots $\lambda_i > 0$ of the characteristic equation: det $(B_x-\alpha I) = \lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} - \ldots + (-1)^n \lambda_n = 0$,
in which free term is described as following:

$$a_n = \det B_x = \prod_{i=1}^{n} \alpha_i .$$

Let's now assume that vector X can be modified in some way that one of the eigenvalues $\lambda_k \to 0$. Then $a_n \to 0$, ellipsoid degenerates dispersion by bringing one of the diameters to zero. At the same time having a condition that $\lambda_k \to 0$ when the signal is defined as $\mathbf{S_{opt} = U_k}$ which according to [2] is equivalent to $d^2 = 1/\lambda_k \to \infty$. A non-uniform sampling can implement such a transformation. For example, let's take six samples $\mathbf{X^T = [x_0,\ldots,x_5]}$with a correlation function:

$R(\tau) = \exp(-\lambda|\tau|)(\cos \beta\tau + \lambda\beta^{-1}\sin\beta|\tau|)$, $\lambda = 0.5$, $\beta = \pi$, sampling interval Δ= 1, then the characteristic equation will be as following: $\lambda^6 - 6\lambda^5 + 12.4266\, \lambda^4 - 11.6602\lambda^3 + 5.3878\lambda^2 -$

-1.1930λ + 0.1009 = 0 has a minimum solution $\lambda_1 = 0.26$ with corresponding signal: $\mathbf{S}^{\mathbf{T}}_{\mathbf{opt}} = \mathbf{U_1}^{\mathbf{T}}$ = [0.1747; 0.4120; 0.5475; 0.5475; 0.4120; 0.1747] which carries out a signal-to-noise ratio $d^2 = 3.85$. This value determines a potential effectiveness of detection [4] with uniform sampling. But if $t_3 = 3$ will be changed to $t_3 = 3.75$, then the characteristic equation, by having $\lambda_1 = 0.0013$ will be as following:

$\lambda^6 \ -6\lambda^5 \ +12.5520\lambda^4 \ -11.0818\lambda^3 + \ 3.8685\lambda^2 - \ 0.3407\lambda +$ $+4.3634=0$ which carries out a signal-to-noise ratio for the following signal:

S_{opt} = [0.3132; 0.5646; 0.2824; -0.2824; 0.5646; -0.3132]

Then a matrix of eigenvalues for a non-uniform sampling will be as following:

$$U=\begin{bmatrix} 0.3132 & 0.3670 & -0.3969 & -0.5617 & 0.4558 & -0.2939 \\ 0.5646 & 0.0300 & -0.5206 & 0.0980 & -0.4775 & 0.4143 \\ 0.2884 & -0.6036 & -0.2672 & 0.4182 & 0.2534 & -0.4919 \\ -0.2884 & -0.6036 & -0.2672 & -0.4182 & 0.2534 & 0.4919 \\ -0.5646 & 0.0300 & -0.5206 & 0.0980 & -0.4772 & -0.4143 \\ -0.3132 & 0.3670 & -0.3969 & 0.5617 & 0.4558 & 0.2939 \end{bmatrix}$$

Figure 1 shows a dependency $\lambda_1 = \varphi(t_1)$, which was obtained on the interval $0<t_1<2$ for the correlation function:

$R(\tau) = \exp(-\lambda|\tau|)(\cos \beta\tau + \lambda\beta^{-1}\sin\beta|\tau|)$, $\lambda = 0.5$, $\beta = \pi$, eigenvalue at point t_0 changes a sign, then $\lambda_1(t_0)=0$, and therefore $d^2 = 1/\lambda_k$ at point t_0 undergoes a discontinuity of the second kind.

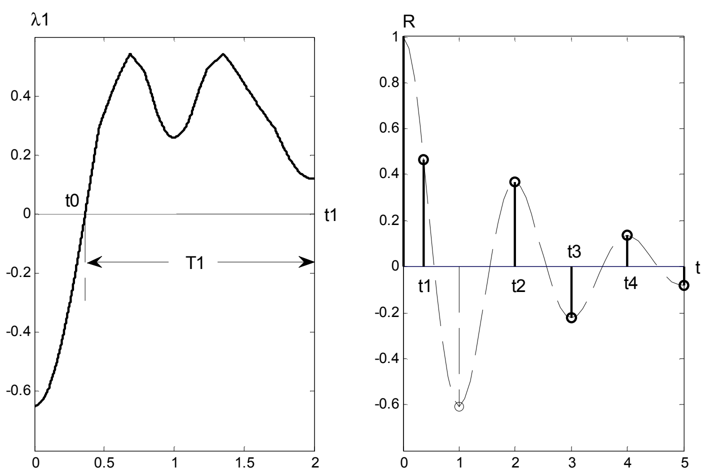

Figure 1. Minimal Eigen value, sampling nodes

The signal-to-noise ratio d^2 around point t_0 changes very steeply. But in the Figure 2 (d^2 = D) the value of the function $d^2(t_0)$ is not defined. A steep growth of the signal-to-noise ratio when $t_0 \leftarrow t_1$ defines a potential effectiveness of the detection of the signal of optimal form when having a non-uniform sampling. Potential property $d^2 \rightarrow \infty$ with finite energy of the signal can be called the non-uniform sampling method "super detection".

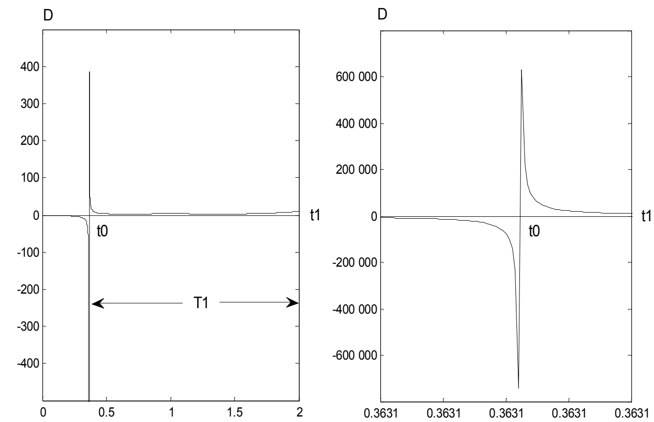

Figure 2. Signal-to-noise ratio

By modifying the vectors of the correlation matrix $\mathbf{B_x}$ with operator $\mathbf{A} = \mathbf{B}^{1/2}\mathbf{B_x}^{-1/2}$[2], we find out that due to the structure of the matrix **B** there is a sequence of complex numbers generated. When you generate the "semi-infinite" sequence with a given correlation vector painting of a discrete white noise linear system with the weight vector, which is the discrete solution of the integral equation [1,5] below:

$$\int_0^{\infty} h(t)h(t+\tau)dt = R(\tau)$$

Figure 3 shows the samplings of positive definite function and its reproduction with solving $\hat{\mathbf{h}}(\mathbf{t})$ with uniform sampling with the interval Δ =1.

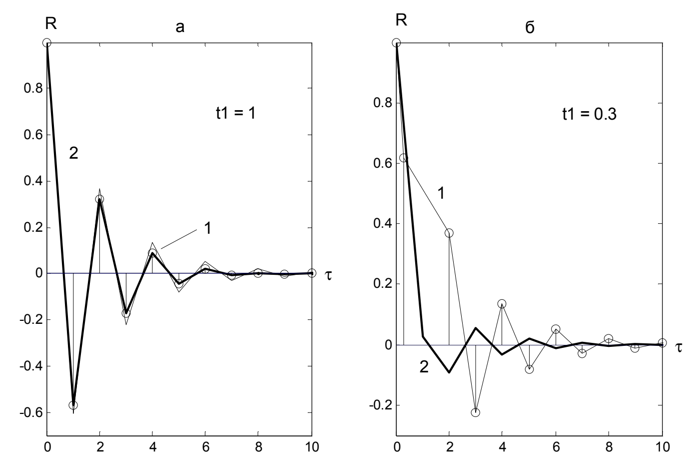

Figure 3. Correlation function

The gap of the function $\mathbf{d^2(t)}$ causes poor conditionality of method of non-uniform sampling. For example, signals:

$\mathbf{S_1^T}$=[0.3234; -0.5596; 0.2687; 0.2867; -0.5596; 0.3234],
$\mathbf{S_2^T}$=[0.3234; -0.5600; 0.2873; 0.2873; -0.5600; 0.3234]
correspond to the signal-to-noise ratio $\mathbf{d_1^2 = -265.4}$ and $\mathbf{d_2^2 = 265.4}$ In this case, super detection can be obtained near points $\mathbf{t_{21}, t_{22}; t_{31}, t_{32};}$ t_{41} when and others. Figure 4 shows the minimal eigenvalues and examples of the optimal non-uniform sampling with the following samplings received with sampling interval $\mathbf{\Delta t = 0.001}$. Sampling nodes $t_2 \in T_2$, $t_3 \in T_3$, $t_4 \in T_4$.

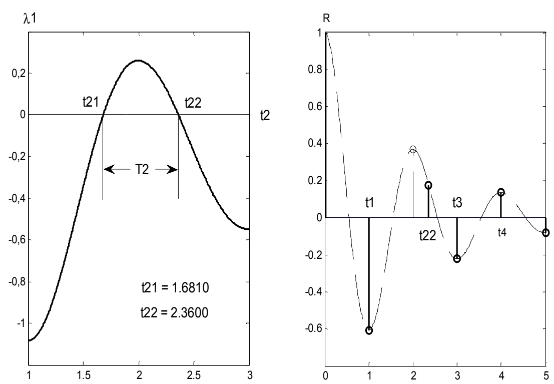

Figure 4. Minimal Eigenvalues sampling nodes

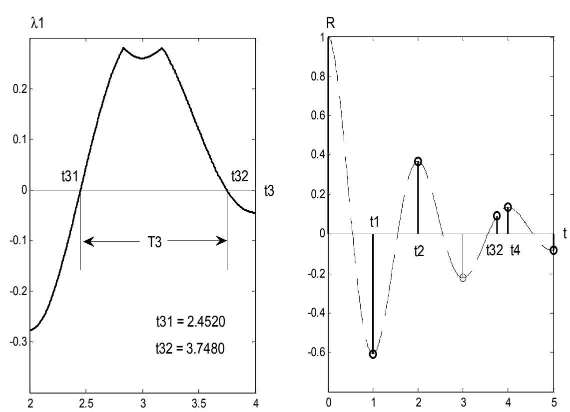

Figure 5. Minimal Eigenvalues, sampling nodes

The same type of modeling ($\Delta t = 0.001$, $n = 6$) was done for the noise with the following corresponding correlation functions:

$R(\tau) = \exp(-\lambda|\tau|)(\cos \beta\tau + \lambda\beta^{-1}\sin\beta|\tau|)$, $R(\tau) = \exp(-\lambda|\tau|)+$
$+\cos \beta\tau$
Table 1 corresponds to the results of modeling.

Table 1. Modeling function using different parameters

# function	t_{11}	t_{21}	t_{22}	t_{31}	t_{32}	t_{41}
(3)	0.364	1.681	2.360	2.452	3.748	3.049
(6)	0.272	1.574	2.269	2.356	3.635	3.011
(7)	0.316	1.629	2.313	2.403	3.695	3.020

The minimal eigenvalue changes its sign with the first sampling node is shifted near the point $t_{11} = 0.5720$, which is shown in Figure 6.

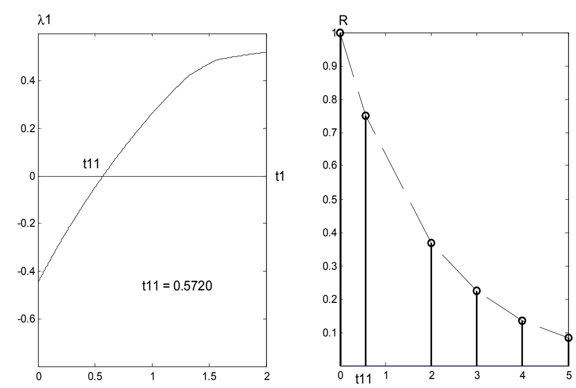

Figure 6. Minimal Eigenvalues, sampling nodes

The shifting of other sampling nodes leaves $\lambda > 0$ and at the same time $d^2 \leq 21$. If the coefficient $\boldsymbol{\alpha}$ changes the value of $\mathbf{t_{11}}$ to zero, then the signal-to-noise ratio $d^2 = 1$.

It is very important to note that the method of non-uniform sampling requires an expended, but finite frequency bandwidth of the detector.

III. Complex Hypothesis

Practical implementation of the "super detection" requires a special study related to the specific problems. In radio engineering, as a rule, the noise is stationary at limited intervals of time; this is why adaptive systems with channel measuring noise characteristics are used. Error of measurement of the correlation function or the correlation matrix will lead to errors in setting the waveform (eigenvector of the correlation matrix). You need to research the effectiveness of the detection signal with distorted shapes to define the requirements for the measuring channel. The reported results of the statistical analysis of the eigenvectors and eigenvalues associated with the method of principal component analysis [5], which allocates the maximum value. Statistical studies of the minimal eigenvalue and the corresponding eigenvector in the problem of detection of independent interest. The following example describes this kind of issues that are in this kind of research should be allowed.

Let us now assume that the correlation function has the following form and the following coefficients are defined as following:

$R(\tau) = \exp(-\alpha|\tau|)(\cos \beta\tau + \alpha\beta^{-1}\sin\beta|\tau|)$, $\alpha = 0.5$, $\beta = \pi$,

$\alpha^* \in N(0.5,\ 0.01)$, $\beta^* \in N(\pi,\ 0.06)$, the arrival time and sampling nodes are available. Let's define a sampling interval $\Delta = 1$, the second sample will be moved by point t_{21} (shown in Fig. 4) and is equal to $t_2 = 1.71$. If the coefficients were known, then the signal-to-noise ratio would be equal to $\mathbf{d^2 = 22.3}$. The results of modeling such signals **S** with random parameters chosen above are shown in the Figure 7.

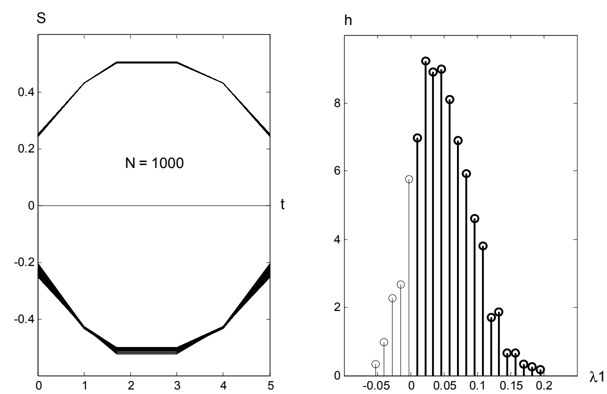

Figure 7. Signals, histogram of Eigenvalue

The histogram shows that with the random parameters chosen above; detecting at least 14% of signal with $\alpha_1 < 0$ is impossible. The average value of positive values of signal-to-noise ratio is equal to $\mathbf{d^2 \approx 50}$ is not a sign that the detection is more efficient because among all the values that were obtained when $d^2 > 4.5$, some values are in the range $d^2 > 5000$.

This example clearly shows that an attempt to get close to a super detection, for example, in synchronous system can be accompanied by a set of theoretical and engineering tasks in the wide range including but not limited to the following: correlation properties, organizing system etc.

There could be a different limitation in the asynchronous systems which is defined by discontinuity of a function $d^2 = 1/\lambda_1(t)$ – need of synchronizing of the detector for maximizing the signal-to-noise ratio by approaching a sample node to the point of discontinuity.

If we assume that the third sample node of the correlation function is optimized by moving around point $t_0 \approx t_{32} = 3.748$ (Figure 5). Let us now assign $t_3 = 3.7$, then the corresponding signal will be as following:

S^T_{opt} = A [0.3137; 0,5645; 0.2880; -0.2880; -0.5645;

-0.3137], which carries out the signal-to-noise ratio $d^2 = 54.04$. If the synchronizer has an error $\delta \in N(0,\sigma)$, $\sigma = 0.012$ (normal density is multiplied by 50 in Figure 8), the average signal-to-noise ratio will be $d^2 = 58.8$. Whine uniform discretization is applied, then $d_r^2 = 3.85$, then non-uniform discretization is 11.8 db more efficient. In this case modeling a detector is provided for NN = 3000 for random values of t_3. The value $d^2 = 6$ states for the signal amplitude A = 0.3333. Figure 9 shows the characteristics obtained for signal with $t_3 = 3.7$: instability of the signal generator does not affect the signal efficiency detection. Thus, the a priori uncertainty of the effectiveness of the method of non-uniform discretization can be significantly reduced when compared with the potentially attainable efficiency, however, it is expected that it will overrun the method of uniform discretization.

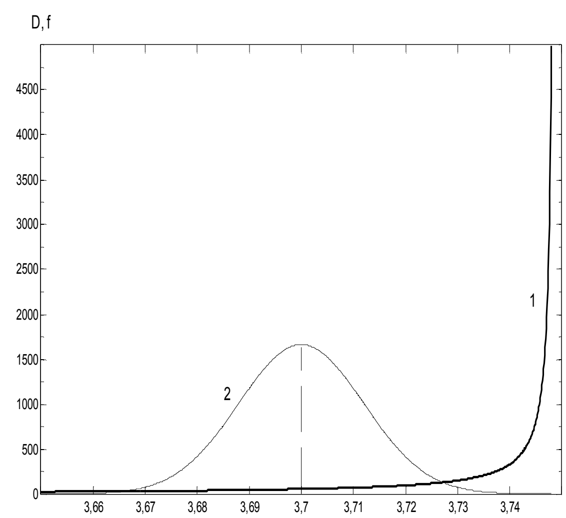

Figure 8. Averaging signal-to-noise ratio

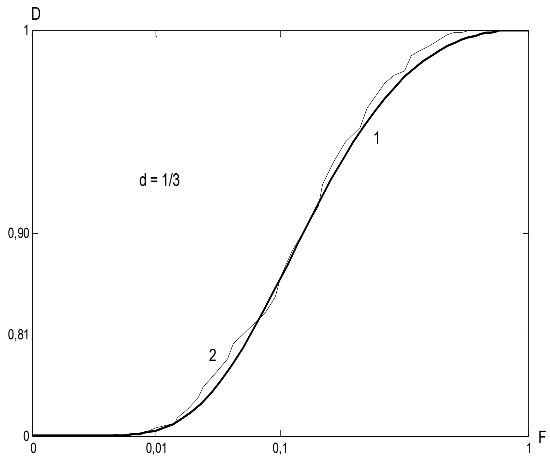

Figure 9. Signal characteristics

IV. Conclusions

In terms of additive stationary Gaussian noise, optimum signal is in the form of the eigenvector corresponding to its minimum eigenvalue. When testing simple hypotheses and initial uniform sampling signal, one of the samples can be shifted to the point where the minimum eigenvalue of the correlation matrix of the noise is zero. This achieves the potential effectiveness of detection - the output of a discrete matched filter can be obtained by arbitrarily high signal-to - noise ratio of the signal with unit energy and the noise from the unit variance.

At point t_0, the signal-to-noise ratio undergoes a discontinuity of the second kind which defines a bad conditionality of detection with non-uniform discretization. This conditionality complicates testing the hypothesis.

Acknowledgments

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