RADON TRANSFORM INVERSION BASED ON HARMONIC ANALYSIS OF THE EUCLIDEAN MOTION GROUP

Can Evren Yarman, Birsen Yazıcı

Rensselaer Polytechnic Institute,
Electrical, Computer and System Engineering
Troy, NY

ABSTRACT

In this paper, we present a new derivation of the spherical harmonic decomposition of the projection slice theorem using harmonic analysis of the Euclidean motion group ($M(N)$). The Radon transform is formulated as a convolution integral over $M(N)$. Deconvolution using harmonic analysis of $M(N)$ leads to spherical harmonic decomposition of the projection slice theorem. The proposed method of decomposition leads to new algorithms for the inversion of the Radon transform.

1. INTRODUCTION

The Radon transform forms the backbone for majority of todays computerized tomography (CT) imaging systems. Reconstruction of images from the data collected by a CT imaging system, requires inversion of the Radon transform. Although an inversion method was first introduced by Radon [12], Cormack was the first to implement the inverse Radon transform with an independent inversion formula of the Radon transform based on the circular harmonic decomposition [2, 3]. The inversion algorithms based on the circular harmonic decompositions are comparable in efficiency with filtered back projection (FBP) and produce images superior than FBP [1, 10]. In this paper we consider the Radon transform of compactly supported functions on $\mathbb{R}^N$ and present an alternative method for the derivation of the spherical harmonic decomposition of the projection slice theorem using harmonic analysis over the Euclidean Euclidean motion group $M(N)$. This method of inversion leads to new algorithms for the inversion of the Radon transform. For the two dimensional case, proposed algorithms give inversion methods based on the circular harmonic decomposition. The reconstructed images for $N = 2$ are presented in Section 7 and compared with the standard filtered back projection method. The computational complexity of the proposed algorithm is $\mathcal{O}(S^2 logS)$, $\mathcal{O}(S)$ being the number of samples for each dimension of $M(2)$.

2. SPHERICAL HARMONIC DECOMPOSITION OF THE RADON TRANSFORM

The Radon transform of a function over $\mathbb{R}^N$ is defined as [4, 9]

$$\mathcal{R}f(\boldsymbol{\vartheta}, r) = \int_{\mathbb{R}^N} f(\boldsymbol{x})\delta(\boldsymbol{x}\cdot\boldsymbol{\vartheta} - r)d\boldsymbol{x}. \tag{1}$$

where $\boldsymbol{\vartheta} \in S^{N-1}$ and $t \in \mathbb{R}$. S^{N-1} is the unit sphere in $\mathbb{R}^N$. We assume that $f \in L^2(\mathbb{R}^N)$. Thus, $\mathcal{R}f \in L^2(S^{N-1} \times \mathbb{R})$.

Let

$$\widetilde{f}(\boldsymbol{\varepsilon}) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\boldsymbol{\varepsilon}\cdot\boldsymbol{x}}d\boldsymbol{x}, \quad \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \sigma) = \int_{\mathbb{R}} \mathcal{R}f(\boldsymbol{\vartheta}, r)\mathrm{e}^{-\mathrm{i}\sigma r}dr.$$

Since any function supported over the unit sphere can be expressed in terms of its spherical harmonic decomposition [1], by the projection slice theorem [2]

$$\widetilde{\mathcal{R}f}_n(\sigma) = \widetilde{f}_n(\sigma), \tag{2}$$

where

$$\widetilde{\mathcal{R}f}_n(\sigma) = \int_{S^{N-1}} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \sigma)S_m(\boldsymbol{\vartheta})d(\boldsymbol{\vartheta}) \tag{3}$$

and

$$\widetilde{f}_n(\sigma) = \int_{S^{N-1}} \widetilde{f}(\sigma\boldsymbol{\vartheta})S_m(\boldsymbol{\vartheta})d(\boldsymbol{\vartheta}). \tag{4}$$

Equation 2 will be referred as the spherical harmonic decomposition of the projection slice theorem. For $N = 2$, this decomposition was first acknowledged by Cormack [2, 3]. We will derive this relation from a convolution integral representation of the Radon transform over the Euclidean motion group by using the Euclidean motion group Fourier transform.

3. RADON TRANSFORM AS A CONVOLUTION OVER THE GROUP $M(N)$

The rigid motions of $\mathbb{R}^N$ form a group called the Euclidean motion group denoted by $M(N)$. The elements of the group are the $(N + 1) \times (N + 1)$ dimensional matrices of the form

$$(R_\theta, \boldsymbol{r}) = \begin{bmatrix} R_\theta & \boldsymbol{r} \\ \mathbf{0}^T & 1 \end{bmatrix}, \quad R_\theta \in SO(N),\ \boldsymbol{r} \in \mathbb{R}^N, \tag{5}$$

[3] parameterized by a rotation component θ and a translation component $\boldsymbol{r}$. For $SO(3)$, θ could be Euler angles, axis rotation parameters, or any other rotation parametrization. The group operation is the usual matrix multiplications and inverse of an element is obtained by matrix inversion as $(R_\theta, \boldsymbol{r})^{-1} = (R_\theta^{-1}, -R_\theta^{-1}\boldsymbol{r})$.

Let $g = (R_\theta, \boldsymbol{r})$ and $h = (R_\phi, \boldsymbol{x})$. Convolution over $M(N)$ is defined as

$$(f_1 *_{M(N)} f_2)(g) = \int_{M(N)} f_1(h)f_2(h^{-1}g)d(h) \tag{6}$$

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[1] $\varphi(\boldsymbol{\vartheta}) = \sum_m \left(\int_{S^{N-1}} \varphi(\boldsymbol{\omega})\overline{S_m(\boldsymbol{\omega})}d(\boldsymbol{\omega})\right) S_m(\boldsymbol{\vartheta})$, where $d(\boldsymbol{\omega})$ is the normalized measure on S^{N-1}.

[2] $\widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \sigma) = \widetilde{f}(\sigma\boldsymbol{\vartheta})$

[3] $SO(N)$, the special orthonormal group, is the group whose elements are $N \times N$ dimensional matrices with determinant equal to 1, with usual matrix multiplication. For $\int_{SO(N)} d(\phi)$ see Section VII.2 of [9].

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where $\int_{M(N)} d(h) = \int_{SO(N)} \int_{\mathbb{R}^N} d\boldsymbol{x} d(\phi)$. Then, the Radon transform of a real valued function f can be written as a convolution integral [4] over $M(N)$ as follows:

$$\begin{aligned}\mathcal{R}f(\boldsymbol{\vartheta}, r_1) &= (\Lambda *_{M(N)} f^*)(g) \\ &= \int_{M(N)} \Lambda(gh) f(h) d(h) \qquad (7)\end{aligned}$$

where $f^*(h) = \overline{f(h^{-1})}$, $\Lambda(h) = \delta(\boldsymbol{x} \cdot \boldsymbol{e_1})$, $f(h) = f(\boldsymbol{x})$, and $\boldsymbol{\vartheta} = -R_\theta^{-1}\boldsymbol{e_1}$, $r_1 = \boldsymbol{r} \cdot \boldsymbol{e_1}$.

We shall use the Fourier transform over the group $M(N)$ to express the Radon transform as a multiplication in the $M(N)$-Fourier domain.

4. $M(N)$-FOURIER TRANSFORM

The standard Fourier transform of a function, $f(x)$ over $\mathbb{R}$ is given by the projections of the function onto the irreducible unitary representations, $e^{i\lambda x}$, of the additive group $\mathbb{R}$, where λ is the frequency. Similarly, Fourier transform over Euclidean Euclidean motion group projects functions onto invariant subspaces of $M(N)$ in terms of a unitary representation of $M(N)$ parameterized by a frequency λ and an element g from $M(N)$. This decomposes a given function as a direct sum of its projections over the invariant subspaces of $M(N)$ [13]. This decomposition allows group convolution to be expressed as a multiplication in the Fourier domain.

Let $f \in L^2(M(N))$, then the $M(N)$-*Fourier transform* of f is defined as

$$\mathcal{F}(f)_{mn}(\lambda) = \widehat{f}_{mn}(\lambda) = \int_{M(N)} f(g) u_{mn}^{(\lambda)}(g^{-1}) d(g), \qquad (8)$$

for $\lambda > 0$, and the corresponding *inverse* $M(N)$-*Fourier transform* is given by

$$\mathcal{F}^{-1}(\widehat{f}_{mn})(g) = f(g) = \int_0^\infty \sum_{m,n} \widehat{f}_{mn}(\lambda) u_{nm}^{(\lambda)}(g) \lambda^{N-1} d\lambda, \quad (9)$$

where $u_{mn}^{(\lambda)}(g)$ are the matrix elements of the irreducible unitary representations of $M(N)$. They are given by [14]

$$u_{mn}^{(\lambda)}(g) = \int_{S^{N-1}} \overline{S_m(\boldsymbol{\omega})} e^{-i\lambda \boldsymbol{r}\cdot\boldsymbol{\omega}} S_n(R_\theta^{-1}\boldsymbol{\omega}) d(\boldsymbol{\omega}). \qquad (10)$$

Let $f,\ f_1,\ f_2 \in L^2(M(N))$. Then, $M(N)$-Fourier transform transforms satisfies the following properties:

1. Adjoint property:
$$\widehat{f^*}_{mn}(\lambda) = \overline{\widehat{f}_{nm}(\lambda)}, \qquad (11)$$
where $f^*(g) = \overline{f(g^{-1})}$.
2. Convolution property:
$$\mathcal{F}(f_1 * f_2)_{mn}(\lambda) = \sum_q \widehat{f}_{2mq}(\lambda)\ \widehat{f}_{1qn}(\lambda). \qquad (12)$$
3. If f is an $SO(N)$ invariant function over $M(N)$, i.e. $f(g) = f(\boldsymbol{r}) \in L^2(\mathbb{R}^N)$, then
$$\widehat{f}_{mn}(\lambda) = \delta_m \widetilde{f}_n(-\lambda) \qquad (13)$$
where δ_m is the Kronecker delta function.

[4] Pintsov was the first to recognize the Radon transform as a correlation integral over $M(N)$ [11].

5. SPHERICAL HARMONIC DECOMPOSITION OF THE PROJECTION SLICE THEOREM BY $M(N)$-FOURIER TRANSFORM

Using the convolution property of $M(N)$-Fourier transform given in Equation 12, we can express the convolution representation of the Radon transform given in Equation 7 in $M(N)$-Fourier domain as follows:

$$\widehat{\mathcal{R}f}_{mn}(\lambda) = \sum_q \widehat{f^*}_{mq}(\lambda)\widehat{\Lambda}_{qn}(\lambda). \qquad (14)$$

Note that, since $f \in L^2(\mathbb{R}^N)$, f can be treated as an $SO(N)$ invariant function over $M(N)$. Therefore, by the adjoint property (Equation 11) of $M(N)$-Fourier transform and Equation 13,

$$\widehat{f^*}_{mq} = \overline{\widehat{f}_{qm}(\lambda)} = \delta_q \overline{\widetilde{f}_m(-\lambda)}. \qquad (15)$$

As a result, Equation 14 becomes

$$\widehat{\mathcal{R}f}_{mn}(\lambda) = \overline{\widetilde{f}_m(-\lambda)}\widehat{\Lambda}_{0n}(\lambda). \qquad (16)$$

From Equation 16, as long as $\widehat{\Lambda}_{0n}(\lambda) \neq 0$, $\widetilde{f}_m(-\lambda)$ is given by

$$\widetilde{f}_m(-\lambda) = \overline{\left(\frac{\widehat{\mathcal{R}f}_{mn}(\lambda)}{\widehat{\Lambda}_{0n}(\lambda)}\right)}. \qquad (17)$$

Substituting the $M(N)$-Fourier transforms of $\mathcal{R}f$ and Λ in Equation 17,

$$\widehat{\mathcal{R}f}_{mn}(\lambda) = C_1 \frac{S_n(\boldsymbol{e_1}) + S_n(-\boldsymbol{e_1})}{\lambda^{N-1}} \int_{S^{N-1}} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \lambda) \overline{S_m(\boldsymbol{\vartheta})} d(\boldsymbol{\vartheta}),$$

$$\widehat{\Lambda}_{mn}(\lambda) = \delta_m C_1 \frac{S_n(\boldsymbol{e_1}) + S_n(-\boldsymbol{e_1})}{\lambda^{N-1}}, \qquad (18)$$

where $C_1 = (2\pi^{N-1})/\left|S^{N-1}\right|$, we obtain the spherical harmonic decomposition of projection slice theorem given by Equation 2.

This shows that we can compute the spherical harmonic decompositions of the Fourier transforms of the projections and hence the function f by Equation 17. Once the spherical harmonic decomposition of the Fourier transform of f is known, by Equation 13, we can reconstruct f by taking the inverse $M(N)$-Fourier transform.

In the following section we will present three new inversion algorithms based on the $M(N)$-Fourier transform.

6. INVERSION OF RADON TRANSFORM AND RECONSTRUCTION ALGORITHMS

Based on the results presented in Section 5, one can derive an inversion formula for the Radon transform as follows:

$$\begin{aligned} f(\boldsymbol{x}) &= f(h) = \mathcal{F}^{-1}\left(\widehat{f}_{km}(\lambda)\right) \\ &= \mathcal{F}^{-1}\left(\delta_k \widetilde{f}_m(-\lambda)\right) = \mathcal{F}^{-1}\left(\delta_k \overline{\left(\frac{\widehat{\mathcal{R}f}_{mn}(\lambda)}{\widehat{\Lambda}_{0n}(\lambda)}\right)}\right) \\ &= \int_0^\infty \sum_m \overline{\left(\frac{\widehat{\mathcal{R}f}_{mn}(\lambda)}{\widehat{\Lambda}_{0n}(\lambda)}\right)} u_{m0}^{(\lambda)}(h) \lambda^{N-1} d\lambda \qquad (19)\end{aligned}$$

Note that $f(h)$ is indeed independent of the rotation component of h since $u_{m0}^{(\lambda)}(h)$ is also independent. Back tracing Equation 19 leads to the following new reconstruction algorithms:

1. Compute $\widehat{\mathcal{R}f}_{mn}(\lambda)$, the $M(N)$-Fourier transform of $\mathcal{R}f$.
2. Compute $\widetilde{f}_m(-\lambda)$ by either of the following ways:
   (a) Choose an n_0 such that $\widehat{\Lambda}_{0n_0}(\lambda) \neq 0$ for each λ. Then, for each m compute $\widetilde{f}_m(-\lambda)$ by

$$\widetilde{f}_m(-\lambda) = \frac{\overline{\widehat{\mathcal{R}f}_{mn_0}(\lambda)}\widehat{\Lambda}_{0n_0}(\lambda)}{\overline{\widehat{\Lambda}_{0n_0}(\lambda)}\widehat{\Lambda}_{0n_0}(\lambda) + \sigma},$$

   where σ is a positive number close to zero.

   (b) For each λ, let $[\widehat{\Lambda}_0(\lambda)]$ and $[\widehat{\mathcal{R}f}_m(\lambda)]$ denote the row vectors with their elements given by $\widehat{\Lambda}_{0n}(\lambda)$ and $\widehat{\mathcal{R}f}_{mn}(\lambda)$, respectively. Then for each m compute $\widetilde{f}_m(-\lambda)$ by

$$\widetilde{f}_m(-\lambda) = \frac{[\overline{\widehat{\mathcal{R}f}_m(\lambda)}][\widehat{\Lambda}_0(\lambda)]^T}{[\overline{\widehat{\Lambda}_0(\lambda)}][\widehat{\Lambda}_0(\lambda)]^T + \sigma}$$

   where σ is a positive number close to zero.

   (c) For each λ, let $[\widehat{\Lambda}(\lambda)]$, $[\widehat{\mathcal{R}f}(\lambda)]$ and $[\widehat{f}(\lambda)]$ denote the matrices with their corresponding elements given by $\delta_m\widehat{\Lambda}_{0n}(\lambda)$, $\widehat{\mathcal{R}f}_{mn}(\lambda)$ and $\delta_m\widetilde{f}_n(-\lambda)$, where m and n denote the row and column number, respectively. Then,

$$[\widehat{f}(\lambda)] = [\widehat{\Lambda}(\lambda)]\left(\overline{[\widehat{\Lambda}(\lambda)]}^T[\widehat{\Lambda}(\lambda)] + \sigma I\right)^{-1}\overline{[\widehat{\mathcal{R}f}(\lambda)]}^T$$

   where σ is a positive number close to zero.
3. Using Equation 13, form $\widehat{f}_{mn}(\lambda)$ and take the inverse $M(N)$-Fourier transform to obtain f.

The first two inversion methods of the second step are based on Equation 16 whereas the third method is based on Equation 14. A variation of the third method was previously presented in our prior work [15].

The computational complexity of the algorithm is directly related with the complexity of the $M(N)$-Fourier transform. The $M(N)$-Fourier transform can be performed in four steps: *1.* Ordinary Fourier transform over $\mathbb{R}^N$; *2.* Interpolation from Cartesian to spherical coordinates; *3.* $SO(N)$-Fourier transform; *4.* Integration over the unit sphere, S^{N-1}. The inverse $M(N)$-Fourier transform is computed by reversing the order of steps in $M(N)$-Fourier transform.

For $N = 2$, if there are $\mathcal{O}(S)$ number of samples in each dimension of $M(2)$, i.e. $\mathbb{R}^2 \ltimes SO(2)$, and S^1, then the computational complexity of the $M(2)$-Fourier transform is $\mathcal{O}(S^3 \log S)$. Since the projections and the function do not depend on r_2 and $SO(2)$, respectively, complexity of computing the $M(2)$-Fourier coefficients of the projections and inverse $M(2)$-Fourier transform of $\widehat{f}_{mn}(\lambda)$ reduces to $\mathcal{O}(S^2 \log S)$. Therefore, computational complexity of methods *1.* and *2.* is $\mathcal{O}(S^2 \log S)$, whereas the computational complexity of method *3.* is $\mathcal{O}(S^3 \log S)$ due to matrix inversion and multiplication.

For the case of $N = 3$, a detailed discussion on the fast implementation of $M(3)$ and $SO(3)$-Fourier transforms can be found in [8, 6] and the references therein.

7. NUMERICAL RESULTS

The numerical implementations of the proposed algorithms were performed on a two dimensional modified Shepp-Logan phantom. A fast implementation of the $M(2)$-Fourier transform is implemented as described in [15]. All the numerical implementations were performed using MATLAB.

Figure 1 presents reconstructed images. In the first method (Fig. 1 (a)) $n = 0$. In the second and third methods (Fig. 1 (b), (c)) all the n's, for which the $M(2)$-Fourier coefficients are calculated, are used in reconstruction. In all three methods $\sigma = 10^{-5}$. Reconstruction using FBP is presented in (Fig. 1 (d)) The plots of the central row (below) and column (right hand side) for each of the reconstructed image (bold solid line) are presented overlaid on the corresponding phantom values. (thin solid line). Details of the reconstructed images are presented in Figure 2 for visual comparison. This example shows that the proposed reconstruction algorithms produce details better or comparable with that of FBP.

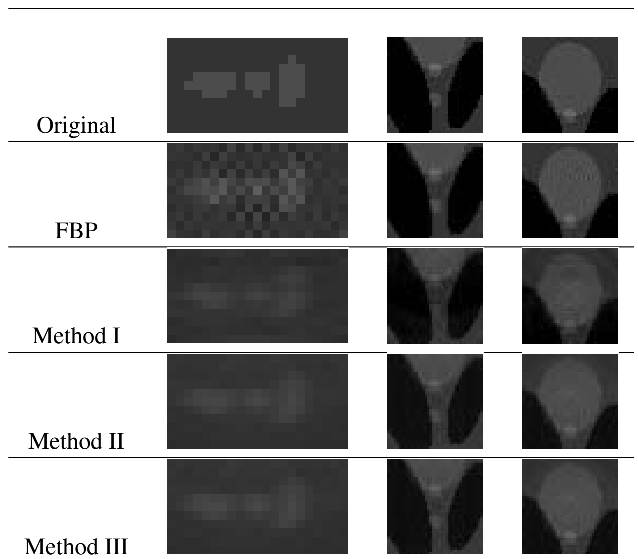

Fig. 2. Details from reconstructed images.

8. CONCLUSION

Our approach attempts to bridge group theoretical treatment of the Radon transform [7, 5] to the applied numerical treatment [9]. Although the relation between the Euclidean motion group symmetries and the Radon transform was studied before, the Radon transform was not treated as a convolution integral. We present a new derivation of the spherical harmonic decomposition of Radon transform using the convolution property of the $M(N)$-Fourier transform. This result can be viewed as a projection slice theorem over $M(N)$ which leads to new inversion algorithms.

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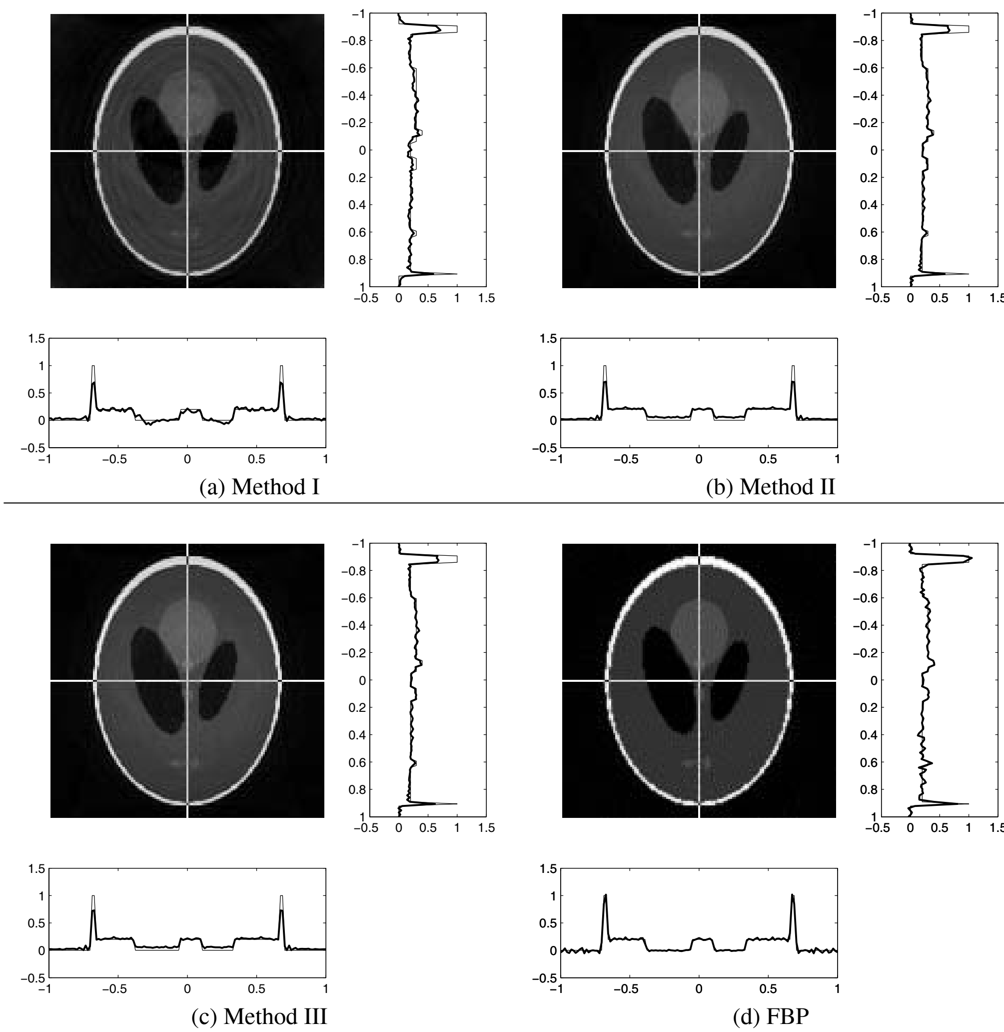

Fig. 1. Reconstruction of the modified Shepp-Logan phantom using **a.** first method, **b.** second method, **c.** third method, and **d.** FBP. For the first method $n = 0$. For second and third methods, all ns that the $M(2)$-Fourier coefficients are computed and $\sigma = 10^{-8}$ are used for reconstruction. Below and on the right are the plots of horizontal and vertical slices passing from the center of the original image (thin line) and the reconstructed image (bold line).

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