Network Reduction for Coded Multiple-Hop Networks

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***Abstract*—Data transmission over multiple-hop networks is impaired by random deleterious events, and characterizing the probability of error for the end-to-end transmission is challenging as the size of networks grows. Adams et al. showed that, when re-encoding at intermediate nodes is enabled, coded transmission over tandem/parallel links can be reduced to a single equivalent link with a specified probability function. Although iterative application of the tandem/parallel reduction techniques in alternation can simplify the task, they are generally not sufficient to reduce an arbitrary network to a single link. In this paper, we propose upper- and lower- bounding processes to bound the end-to-end probability distribution of a network by combining the parallel/tandem link reduction with the structure of flows over the network. We evaluate the performance of the proposed bounding methods at the 99% success rate of end-to-end data transmission over randomly generated acyclic networks. The numerical results demonstrate that our bounding approaches enable us to characterize a network by a single probability function to a very good precision.**

***Index Terms*—network reduction, multi-hop networks, network coding, coded transmission**

I. Introduction

Data transmission over large networks is impaired by random deleterious events, such as packet losses caused by congestion and decoding errors owing to noise or collision. Reliable end-to-end transmission relies largely on various acknowledgement and retransmission schemes at the cost of transmission efficiency, since packet-wise (or block-wise) acknowledgement is needed on a end-to-end or even link-by-link basis. Numerous research efforts have been devoted to characterizing the fundamental limits of data transmission over lossy networks and to improving its efficiency from different aspects. We only list a very sparse sampling here. Dana *et al.* in [1] consider the use of linear network coding over packets and construct a network model based on correlated erasure links. Assuming the destination node has side information on each packet loss event and allowing packets broadcast from one node to its neighbors, the authors show that linear network codes achieve the capacity of such networks. Lun *et al.* in [2] propose a framework to translate a lossy unicast or multicast network into a lossless packet network by applying random linear network coding (RLNC) [3] and performing RLNC re-encoding at intermediate nodes. Assuming independent Poisson packet arrivals at each node and the number of packets is large, the probability of RLNC decoding error is characterized by the delay, rate, and the network capacity. Unlike [1], no side information is required to achieve capacity. While [2] does not consider the practical constraint of buffer size at intermediate nodes, Haeupler and Médard show in [4] that RLNC is asymptotically capacity-achieving even if intermediate nodes may only store one coded packet. Xiao *et al.* in [5] investigate the delay in packet erasure networks where RLNC is used in a rateless fashion, and its performance is upper bounded by a single packet erasure link generated based on all of the links in the minimum cut.

In general, if we choose a block of bits as the basic data unit for transmission, the behavior of each link in a network is simply characterized by a probability function that associates the probability of error with the number of data units transmitted within unit time. This data unit is most analogous to a packet, although the size of packets may vary depending on the link quality: a packet may carry several data units, and a data unit may be split into several packets. Hereafter we simply use packet when we actually refer to the data unit.

Considering the network abstraction where the maximum data rate and packet loss probability across each link are limited by local constraints, and assuming that packet losses are independent, it is interesting to ask whether the behavior of data transmission over an entire network or a collection of links can be fully described by a single probability function. In [6] Adams et al. have proven that links which are connected in tandem or in parallel can be reduced to a single equivalent link with a specified probability function. These two reduction operations serve to make networks simpler to model and study, but are not sufficient to reduce an arbitrary network. To tackle this difficulty, we propose upper- and lower- bounding methods by combining the parallel/tandem link reduction with the structure of data flows over the underlying networks. Numerical results over randomly generated networks demonstrate the effectiveness of the proposed methods.

The rest of this paper is organized as follows. In Sec. II we describe the system model and the parallel/tandem link reduction methods. We focus on flows over cuts in Sec. III to construct cut-based upper and lower bounds. Our node-reduction based upper bounds are presented in Sec. IV, and routing-based lower bounds are in Sec. V. We present the numerical results in Sec. VI and conclude in Sec. VII.

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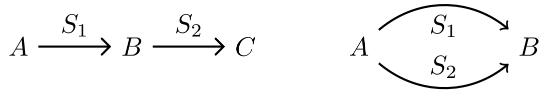

Fig. 1. Two links in tandem (left) and in parallel (right).

II. System Model

We represent a network by its underlying directed graph $\mathcal{N}\{\mathcal{V},\mathcal{E}\}$ where $\mathcal{V}$ is the set all vertices and $\mathcal{E}$ is the collection of edges. For some $u,v\in\mathcal{V}$, data transmission from u to v is possible if and only if $(u,v)\in\mathcal{E}$. If there are multiple edges among two nodes, we apply subscripts to distinguish them. For each $v\in\mathcal{V}$, we denote the **parents set** of v by

$$\mathcal{I}(v)\triangleq\{u\in\mathcal{V}|(u,v)\in\mathcal{E}\},$$

the **children set** of v by

$$\mathcal{O}(v)\triangleq\{u\in\mathcal{V}|(v,u)\in\mathcal{E}\},$$

and the set of edges incident to v by

$$\mathcal{E}(v)\triangleq\{e\in\mathcal{E}|e=(u,v),\text{ or }e=(v,u)\text{ for some }u\in\mathcal{V}\}.$$

In this paper we only focus on a single unicast transmission over acyclic networks. Extension to more general setups are left to future work. Under this scenario, the networks we are investigating in this paper have the following properties.

- For the **source** node v_s and the **destination** node v_d, we have $|\mathcal{I}(v_s)|=0, |\mathcal{O}(v_s)|>0$ and $|\mathcal{O}(v_d)|=0, |\mathcal{I}(v_d)|>0$.
- Every node is reachable from v_s and reverse-reachable from v_d (Otherwise the node and its connected edges can be removed from the graph).
- For link $e_i\in\mathcal{E}$, n_i denotes the maximum number of packets that can be transmitted over the link within the time constraint, and ξ_i is the probability that a packet is dropped independently at random.

Therefore data transmission over $e_i\in\mathcal{E}$ is characterized either by the Probability Mass Function (PMF) $\lambda_i\in[0,1]^{n_i+1}$ or by the Complementary Cumulative Distribution Function (CCDF) $\Lambda_i\in[0,1]^{n_i+1}$, where $\lambda_i(k)$ describes the probability that link e_i can successfully transmit exactly k packets per delay constraint, and $\Lambda_i(k)$ describes the probability that at least k packets are successfully transmitted. Given (n_i,ξ_i), the PMF λ_i is defined by a binomial distribution, i.e.,

$$\lambda_i(k)=\begin{cases}\binom{n_i}{k}\xi_i^{n_i-k}(1-\xi_i)^k, & \text{for } 0\le k\le n_i\\ 0, & \text{otherwise.}\end{cases} \tag{1}$$

We can easily convert PMFs to/from CCDFs via the following one-to-one mapping

$$\Lambda_i(k)=\sum_{j=k}^{n_i}\lambda_i(j),\quad \lambda_i(k)=\Lambda_i(k)-\Lambda_i(k+1), \tag{2}$$

where $\Lambda_i(k)=0,\forall k>n_i$ by default.

The tandem/parallel reduction techniques developed in [6] state that we can describe a two-link tandem/parallel network, as shown in Fig. 1, by a single PDF/CCDF. Define S, S_1, and S_2 the number of packets successfully transmitted across the network, and across the two individual links, respectively. The tandem network can be described by a single CCDF

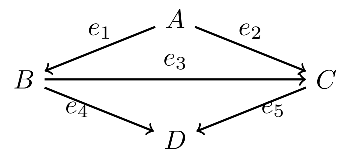

Fig. 2. The smallest network where the tandem/parallel link reduction fails.

$$\begin{aligned}\Lambda(k)&\triangleq P(S\ge k)=P(\min\{S_1,S_2\}\ge k)\\ &=P(S_1\ge k)P(S_2\ge k)=\Lambda_1(k)\Lambda_2(k),\end{aligned}$$

or equivalently, $\Lambda=\Lambda_1\odot\Lambda_2$ where $\odot$ denotes element-wise multiplication (Hadamard product) after zero-padding the shorter of the two CCDFs. Similarly, the parallel network with PMFs λ_1 and λ_2 can be described by a single PMF

$$\begin{aligned}\lambda(k)&\triangleq P(S=k)=P(S_1+S_2=k)\\ &=\sum_{j=0}^{k}P(S_1=j)P(S_2=k-j)=\sum_{j=0}^{k}\lambda_1(j)\lambda_2(k-j),\end{aligned}$$

or equivalently $\lambda=\lambda_1*\lambda_2$ where $*$ denotes convolution.

III. Flow-Cut Bounds

The two-link tandem/parallel reduction techniques can be straightforwardly extended to multiple links by induction. We can iteratively apply the tandem- and parallel-link reduction techniques in alternation to simplify the calculation of end-to-end PMFs/CCDFs. We should note, however, that the tandem/parallel reduction operations will not simplify arbitrary networks to a single distribution. Fig. 2 depicts a simple network that cannot be simplified in this way; no two links form a purely parallel or tandem structure, because of link e_3. Indeed, this network is the smallest network (in terms of the number of nodes/edges) that can't be fully reduced. As the number of nodes and edges grows, the possibility that the tandem/parallel reduction operations are sufficient for network reduction will decrease. We therefore need new approaches to tackle general network topologies that can't be fully reduced.

Definition 1 (Upper and Lower Bounds of a CCDF): Let S be the number of successfully delivered packets and Λ be its CCDF, if for all non-negative integers $k<E[S]$, we have

$$\Lambda_L(k)\le\Lambda(k)\le\Lambda_U(k), \tag{3}$$

then Λ_L is a lower bound and Λ_U is an upper bound.

This definition only focus on $k<E[S]$, motivated by the fact we are only interested in operation regimes where the rate of successful is large (say larger than 50%).

Definition 2 (Flow across a Cut): Let $\mathcal{C}$ be a cut of the directed acyclic graph and $\mathcal{E}(\mathcal{C})$ be the set of edges that cross $\mathcal{C}$ from the source side to the destination side. The flow across the cut $\mathcal{C}$, defined as the number of successfully delivered packets over the cut $\mathcal{C}$ within unit time, is therefore

$$S_{\mathcal{C}} = \sum_{i\in\mathcal{E}(\mathcal{C})} S_i.$$

Assuming $\mathcal{E}(\mathcal{C})=\{1,2,\ldots,m\}$, its associated PMF $\lambda_{\mathcal{C}}$ is

$$\lambda_{\mathcal{C}} = \lambda_1 * \lambda_2 * \cdots * \lambda_m, \tag{4}$$

where the equality is due to the parallel-link reduction by regarding links in $\mathcal{E}(\mathcal{C})$ as a parallel network.

Proposition 1: Let $\mathcal{C}_d$, $d = 1,2,\ldots,c$, be all the cuts separating the source node v_s and the destination node v_d, each associated with a flow $S_{\mathcal{C}_d}$ and a CCDF $\Lambda_{\mathcal{C}_d}$, and define

$$\Lambda_{all-cuts} \triangleq \Lambda_{\mathcal{C}_1} \odot \Lambda_{\mathcal{C}_2} \odot \cdots \odot \Lambda_{\mathcal{C}_c}.$$

$\Lambda_{v_s\to v_d}$, which describes the end-to-end data transmission, is lower bounded by $\Lambda_{all-cuts}$ and upper bounded by $\Lambda_{\mathcal{C}_d}, \forall d$.

Proof: Denoting D the number of successfully received data units at the destination node v_d, we have

$$D = \min\{S_{\mathcal{C}_1}, S_{\mathcal{C}_2}, \ldots, S_{\mathcal{C}_c}\},$$

where the equality comes from the fact that information passing through the network goes through every cut in order to to reach the destination. Therefore any cut $\mathcal{C}_d, \forall d$ provides a valid upper bound $\Lambda_{\mathcal{C}_d}$. To prove the lower bound, we need to show that for all non-negative integers $k < E[D]$ where $E[D]$ is the mean associated with Λ_D, we have

$$\Lambda_{all-cuts}(k) \le \Lambda_D(k). \tag{5}$$

Intuitively, $\Lambda_{all-cuts}$ neglects the dependence between all cuts and therefore represents a tandem network by connecting all cuts in serial. The formal proof of (5) is in Appendix A. ■

We refer to $\Lambda_{all-cuts}$ as the *All-Cuts* lower bound and $\Lambda_{\mathcal{C}_{min}}$ as the *Min-Cut* upper bound, where $\mathcal{C}_{min}$ is the cut whose average throughput is smallest among all cuts.

Remark 1: It is interesting to compare our *Min-Cut* upper bound to the one proposed in [5, Proposition 1], where all the erasure links (n_i, ξ_i) crossed by the minimum cut are modeled by a single erasure channel (n, ξ), where $n = \sum_i n_i$ and $\xi = \sum_i \frac{n_i}{n}\xi_i$. From Proposition 2 we can see that [5, Proposition 1] is accurate up to the first moment (the mean) $n(1-\xi)$ but provides a larger variance $n\xi(1-\xi)$.

Proposition 2: Given s independent binomial random variables $X_i \sim B(n_i, 1-\xi_i)$, $i = 1,\ldots,s$, and denoting

$$X = \sum_{i=1}^{s} X_i,\ n = \sum_{i=1}^{s} n_i,\ \xi = \sum_{i=1}^{s} \frac{n_i}{n}\xi_i,$$

we have

$$E(X) = n(1-\xi),\ \mathrm{Var}(X) \le n\xi(1-\xi), \tag{6}$$

where the equality holds if and only if $\xi_1 = \ldots = \xi_s$.

Proof: Since X_i are independent, we have

$$E(X) = \sum_{i=1}^{s} E(X_i) = \sum_{i=1}^{s} n_i(1-\xi_i) = n(1-\xi), \tag{7}$$

$$\mathrm{Var}(X) = \sum_{i=1}^{s} \mathrm{Var}(X_i) = \sum_{i=1}^{s} n_i\xi_i(1-\xi_i). \tag{8}$$

W.l.o.g., assuming $0 \le \xi_1 \le \cdots \le \xi_s \le 1$, we have $1 \ge 1-\xi_1 \ge \cdots \ge 1-\xi_s \ge 0$. By the *Chebyshev Sum Inequality*,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{s} n_i\xi_i(1-\xi_i) &\le \left(\frac{1}{n}\sum_{i=1}^{s} n_i\xi_i\right)\left(\frac{1}{n}\sum_{j=1}^{s} n_j(1-\xi_j)\right),\\ &= \xi\left(1 - \sum_{j=1}^{s}\frac{n_j}{n}\xi_j\right) = \xi(1-\xi),\end{aligned} \tag{9}$$

where the equality holds if and only if $\xi_1 = \cdots = \xi_s$. Substituting (8) into (9) and multiplying both sides by n, we get (6). ■

IV. Network Reduction Upper Bounds

We can also generate upper bounds by first altering the network structure while preserving its minimum cut, and then applying parallel/tandem link reduction. Firstly, we define the node reduction operations more precisely.

Definition 3: The **$1n$-node reduction function** $f(\mathcal{N}, v)$, where v has $\mathcal{I}(v) = \{v_{in}\}$ and $\mathcal{O}(v) = \{o_1, o_2, \cdots, o_n\}$, maps $\mathcal{N}\{\mathcal{V},\mathcal{E}\}$ to $\mathcal{N}'\{\mathcal{V}',\mathcal{E}'\}$, where

$$\mathcal{V}' = \mathcal{V}\setminus\{v\} \text{ and } \mathcal{E}' = (\mathcal{E}\setminus\mathcal{E}(v))\bigcup\Big(\bigcup_{i=1}^{n}(v_{in}, o_i)\Big).$$

We associate each new link (v_{in}, o_i) with a CCDF

$$\Lambda_{v_{in}\to o_i} = \Lambda_{v_{in}\to v} \odot \Lambda_{v\to v_i}.$$

Definition 4: The **$n1$-node reduction function** $g(\mathcal{N}, v)$, where v has $\mathcal{O}(v) = \{v_{out}\}$ and $\mathcal{I}(v) = \{i_1, i_2, \cdots, i_n\}$, maps $\mathcal{N}\{\mathcal{V},\mathcal{E}\}$ to $\mathcal{N}'\{\mathcal{V}',\mathcal{E}'\}$, where

$$\mathcal{V}' = \mathcal{V}\setminus\{v\} \text{ and } \mathcal{E}' = (\mathcal{E}\setminus\mathcal{E}(v))\bigcup\Big(\bigcup_{k=1}^{n}(i_k, v_{out})\Big).$$

We associate each new link (i_k, v_{out}) with a CCDF

$$\Lambda_{i_k\to v_{out}} = \Lambda_{i_k\to v} \odot \Lambda_{v\to v_{out}}.$$

A node v is said to be **$1n$-reducible** if it has $|\mathcal{I}(v)| = 1$, or **$n1$-reducible** if it has $|\mathcal{O}(v)| = 1$. The $1n$/$n1$-node reduction operations can be visualized as in Fig. 3, where a $1n$- ($n1$-)reducible node is firstly identified, then copied with its incoming (outgoing) edge, and finally removed by applying tandem-link reduction.

Proposition 3: For network $\mathcal{N}$, $f(\mathcal{N}, v)$ and $g(\mathcal{N}, v)$ provide upper bounds.

Proof: It suffices to notice that for any transmission task supported by $\mathcal{N}$, we can find a corresponding transmission protocol on the reduced network with the same or relaxed constraints, due to the underlying node/edge duplication operation by $f(\mathcal{N}, v)$ and $g(\mathcal{N}, v)$. ■

Proposition 4: Any unicast acyclic network $\mathcal{N}$ has at least one node v such that v is $1n$-reducible and at least one node u such that u is $n1$-reducible.

Proof: See the appendix B. ■

We can fully reduce a network to a single link by repeatedly applying $1n$/$n1$-node reduction and tandem/parallel link reductions, and hence provide upper bounds for the

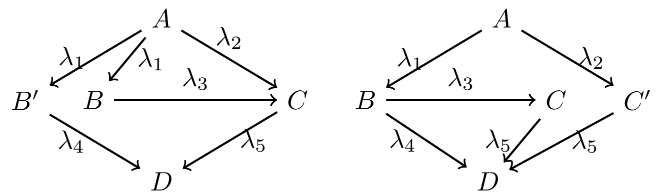

Fig. 3. The $1n$ node-reduction function $f(\mathcal{N}, B)$ (left) and the $n1$ node-reduction function $g(\mathcal{N}, C)$ (right) before removing the auxiliary nodes by tandem-link reduction.

original network. Disadvantages are the potentially relaxed transmission constraints at the reduced nodes and that it does not tell us in which order these $1n/n1$-reducible nodes should be reduced. We propose two strategies to select the node to be first reduced: the $1n/n1$-reducible node with the highest incoming/outgoing edge capacity (termed *NR-Abs* bound) or the one with highest ratio between incoming and outgoing link capacity (termed *NR-Ratio* bound).

Definition 5: The **capacity ratio** of a $1n/n1$-reducible node v is defined based on the average throughput across its incoming/outgoing edges, i.e.,

$$\overline{C_v} = \frac{C_{(v_{in},v)}}{\sum_{i=1}^{n} C_{(v,o_i)}} \text{ or } \overline{C_v} = \frac{\sum_{k=1}^{n} C_{(i_k,v)}}{C_{(v,v_{out})}}.$$

Proposition 5: If $\overline{C_v} \geq 1$ for some $1n/n1$-reducible node v, the minimum cut of the *NR-Ratio* reduced network is the same as that of the original network.
Proof omitted here since it is intuitive to see that the minimum cut is always preserved in *NR-Ratio* when $\overline{C_v} \geq 1$.

We can create a hybrid approach with some predetermined threshold t: applying *NR-Ratio* to reduce all nodes with $\overline{C_v} > t$ and then apply *NR-Abs* reduction for the remaining $1n/n1$-reducible nodes. Choosing $t = 0$ will be identical to *NR-Ratio* and choosing $t = \infty$ will be identical to *NR-Abs*.

V. Ford-Fulkerson Based Lower Bounds

The Ford-Fulkerson algorithm [7] computes the optimal routing paths in a flow network. To adapt the algorithm for our purposes, we use the average throughput of each link as the flow value, apply the Ford-Fulkerson algorithm to find all the feasible flows, and then split the network into disjoint paths using conservation of flow as shown in Fig. 4. Each path forms a tandem network, and all paths form a parallel network. Then we can use the basic reduction operations to reduce the network into a single link. Given a path with flow y_j in its tandem network, a link that originated from edge e_i inherits its erasure probability ξ_i and a share of its rate $n'_{i,j} = \left\lfloor \frac{y_j}{1-\xi_i} \right\rfloor$ to ensure integrality constraint. This approach is termed *FF-Flow*, as shown in Fig. 4 (middle). We can also split a link according to the shares of each flow that pass through it, i.e., $n'_{i,j} = \left\lfloor \frac{y_j}{\sum_k y_k} n_i \right\rfloor$ where $\sum_k y_k$ is the total flow that passes through the link e_i. This approach is named *FF-Split* and illustrated in Fig. 4 (right).

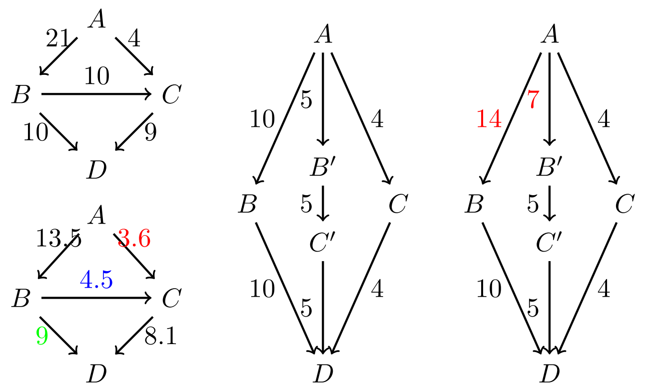

Fig. 4. An example network (upper left, n_i shown along the edges) with packet erasure probability $\xi = 0.1$ on all links, its flow graph generated by Ford-Fulkerson algorithm (lower left, with 3 disjoint flows $9, 4.5, 3.6$), and the reconstructed networks for the lower bounds *FF-Flow* (middle) and *FF-Split* (right, numbers in red indicate the difference).

Intuitively, the Ford-Fulkerson algorithm-based reduction yields lower bounds because the algorithm provides us with a network protocol on the original network that can produce the corresponding probability function.

VI. Numerical Illustration

To understand the general performance of all our proposed upper and lower bounds, we evaluate their end-to-end probability function over the 4-node irreducible network and over randomly generated networks. For a predefined network size (in nodes), the probability that there is a directed edge from one node to the other is set to $1/2$ and the rate of each link is uniformly chosen from $[100, 1000]$ with erasure probability randomly chosen from within a predefined range. Once the network has been populated, networks with cycles or isolated nodes will be discarded. Furthermore, we only focus on cases where tandem/parallel reduction are not sufficient as we would not need these bounding methods otherwise.

A. Error Probability over a Four-Node Test Network

We simulate the end-to-end error probability via 10^7 Monte-Carlo trials over the smallest irreducible network where all links have the same erasure probability ξ=0.1 but different rates, indicated by the numbers along the corresponding edges as in Fig. 5. In this test network, the *NR-Abs* bound and the *NR-Ratio* bound generated as in Fig. 3 are tighter than the *Min-Cut* upper bound. The *FF-Flow* lower bound is identical to *FF-Split* and they are better than the *All-Cuts* in some regions.

B. Random Networks: Gap from the Best Upper/Lower Bound

To evaluate the tightness of our bounds in general network settings, we compare their performance over randomly generated acyclic networks by the highest end-to-end data rate they can support with no less than 99% success probability. We call the corresponding rate K_{99}. In Fig. 6 we evaluate the K_{99} of all the lower bounds against the best upper bound (i.e., the smallest K_{99} produced by all the upper bounds) and

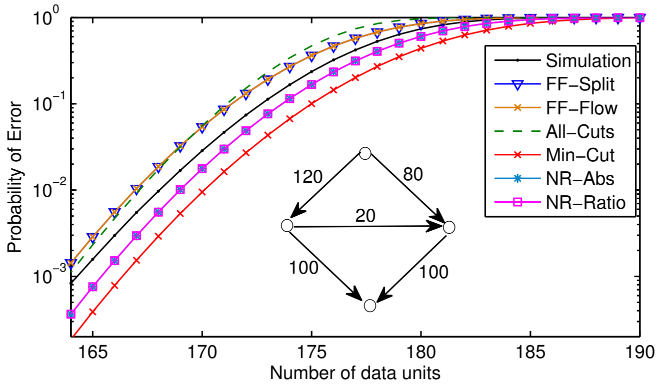

Fig. 5. Probability of end-to-end transmission error as a function of the number of data units over the test network shown, where all edges have the same erasure probability $\xi = 0.1$ but different rates (indicated by the number along the corresponding edge). The curve of simulation is generated by 10^7 Monte-Carlo simulations.

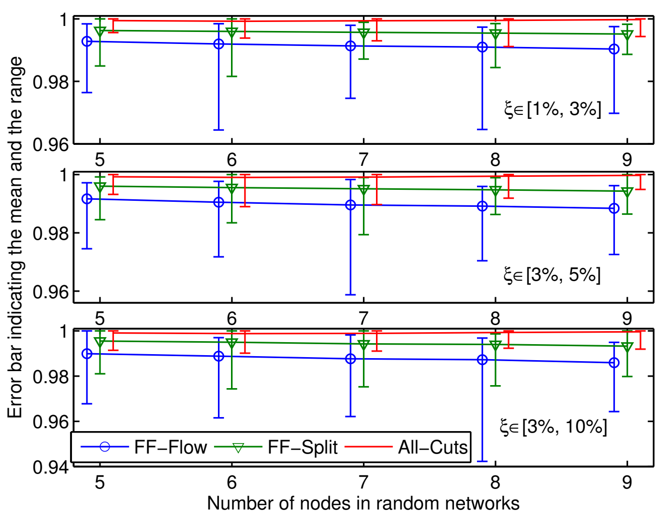

Fig. 6. Ratio between our three lower bounds and the best upper bound when evaluated at 99% success probability. Each error bar indicates the mean and the range of the corresponding ratio, which is based on 1000 trials over randomly generated acyclic networks. For each trial, the number of nodes is indicated on the abscissa and each directed edge e_i is generated at probability $1/2$ with randomly chosen rate $n_i \in [100, 1000]$ and erasure probability $\xi_i \in [1\%, 3\%]$ (upper), $\xi_i \in [3\%, 5\%]$ (middle), and $\xi_i \in [3\%, 10\%]$ (lower).

plot the mean value and the corresponding range based on 1000 trials for each network size and erasure probability range. The *All-Cuts* lower bound is always within 1% of the best upper bound over random networks with different sizes and erasure probability. The *FF-Split* lower bound improves *FF-Flow* uniformly. *FF-Split* (*FF-Flow*) provides a gap of less than 1% (2%) on average and less than 3% (6%) in the worst case[1], at least for the settings as we have demonstrated. Their performances degrade slightly with increasing network size and the erasure probability.

The *Min-Cut* upper bound provides a gap of less than one percent, which is much better than the *NR-Abs* and the *NR-Ratio* bounds over random networks as shown in Fig. 7, where

[1]The worse case is very rare since the variance is $O(10^{-6})$.

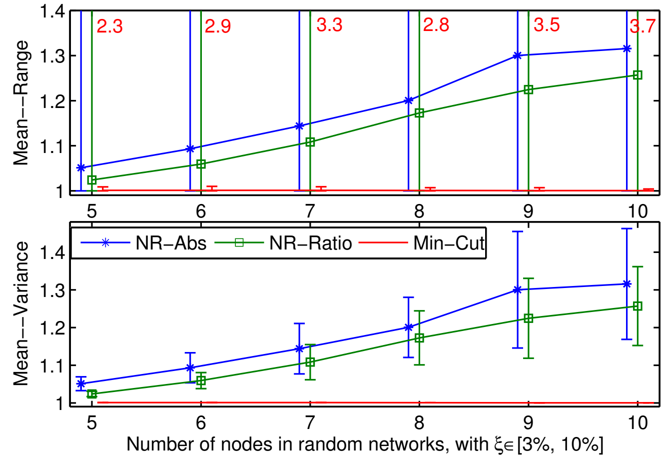

Fig. 7. Ratio between our three upper bounds and the best lower bound when evaluated at 99% success probability. Each error bar indicates the mean-range (upper) or the mean-variance (lower, $[\mu - \sigma^2, \mu + \sigma^2]$) of the corresponding ratio, which is based on 1000 trials over randomly generated acyclic networks with edge erasure probability $\xi_i \in [3\%, 10\%]$. Both the NR-Abs and NR-Ratio upper bounds may result in loose upper bounds, as indicated by their wide range (whose maximum value is shown in red on the top).

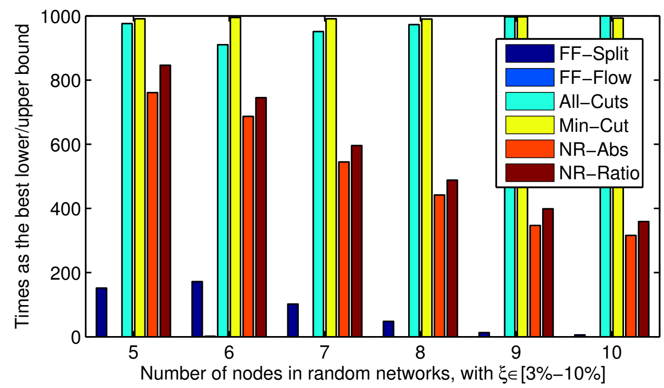

Fig. 8. Number of instances as the best upper/lower bound when evaluated at 99% success probability over 1000 randomly generated acyclic networks with edge erasure probability $\xi_i \in [3\%, 10\%]$. Multiple counts occur when several bounds are identical.

the erasure probability is chosen from the range $[3\%, 10\%]$. Although *NR-Abs* and *NR-Ratio* may produce better bounds than the *Min-Cut*, they may also relax the network constraints, since even a highly varied network often does not provide any nodes with a capacity ratio larger than 1. Therefore, they provide a loose bound, as indicated by the excessive range shown in Fig. 7 (above). Their mean and variance increase as the size of networks grows. Performance for other erasure probability ranges are similar and therefore omitted here.

C. Random Networks: Chances as the Best Bound

In Fig. 8 we count the instances in which each bounding method produces the best bound, as measured by K_{99}. With very high probility *All-Cuts* provides the best lower bound ($> 95\%$) and *Min-Cut* provides the best upper bound ($> 99\%$). The *FF-Split* produces the best lower bound with about 20% probability when network size is small. The *NR-Abs* and the *NR-Ratio* upper bounds are the best upper bound with high probability when the network constraints are preserved, although the probability to produce a loose bound is also large, as indicated by the large range in Fig. 7.

VII. Conclusions

In this work, we propose several lower and upper bounds to characterize the end-to-end transmission probability function. Our best lower bound yields a gap smaller than one percent in throughput from the best upper bound over randomly generated acyclic networks. This justifies our efforts by describing the end-to-end data transmission over lossy networks with a single probability function to high precision.

There are several ways to improve our proposed upper and lower bounds. For example, one can combine the *Min-Cut* and the *NR-Ratio* upper bounds to construct a new upper bound: we first reduce the network as in *NR-Ratio* until there is no node with input-output capacity ratio higher than 1, and then apply *Min-Cut* to the reduced network. We may also combine the $1n/n1$-node reduction and the *All-Cuts* lower bound to provide a good approximation that always falls between the *Min-Cut* upper bound and the *All-Cuts* lower bound. Other approximations can be found in [8].

Appendix A
Proof of the All-Cuts Lower Bound

If no edge appears in more than one cut, i.e., all $\mathcal{C}_d, \forall d$ are independent, all the cuts form a tandem network. By tandem-link reduction we have

$$\Lambda_D = \Lambda_{\mathcal{C}_1} \odot \cdots \odot \Lambda_{\mathcal{C}_c} = \Lambda_{all-cuts}.$$

Assuming $\mathcal{C}_1$ and $\mathcal{C}_2$ share some common links, we introduce three independent random variables Z_0, Z_1, Z_2 such that

$$S_{\mathcal{C}_1} = Z_0 + Z_1, \quad S_{\mathcal{C}_2} = Z_0 + Z_2,$$

where Z_0 represents the flow over the common links and Z_1 and Z_2 represent flows over the rest links in $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. We have

$$\begin{aligned}\Lambda_{\mathcal{C}_1}(k) &= P(Z_0+Z_1 \geq k) = \sum_i P(Z_1 \geq k-i | Z_0=i) P(Z_0=i) \\ &= \sum_i P(Z_1 \geq k-i) P(Z_0=i) = \sum_i P(Z_0=i) \Lambda_{Z_1}(k-i),\end{aligned}$$

where the second last equality comes from the fact that Z_0 and Z_1 are independent. Similarly we can show that

$$\Lambda_{\mathcal{C}_2}(k) = \sum_i P(Z_2 \geq k-i) P(Z_0=i) = \sum_i P(Z_0=i) \Lambda_{Z_2}(k-i).$$

On the other hand, denoting $S_m \triangleq \min\{S_{\mathcal{C}_1}, S_{\mathcal{C}_2}\}$, we have

$$\begin{aligned}\Lambda_{S_m}(k) &= P(S_m \geq k) = P(\min\{Z_1, Z_2\} + Z_0 \geq k) \\ &= \sum_i P(Z_1 \geq k-i, Z_2 \geq k-i | Z_0=i) P(Z_0=i) \\ &= \sum_i P(Z_0=i) \Lambda_{Z_1}(k-i) \Lambda_{Z_2}(k-i).\end{aligned}$$

Since $\Lambda_{Z_1}(j)$ and $\Lambda_{Z_2}(j)$ are monotonically decreasing, we can shown by following the *Chebyshev Sum Inequality* that for all feasible k (as long as $\Lambda(k) > 0$),

$$\Lambda_{\mathcal{C}_1}(k) \Lambda_{\mathcal{C}_2}(k) \leq \Lambda_{S_m}(k).$$

By grouping cuts that share common links and rearranging D in such a way that

$$D = \min\{\cdots \min\{\min\{S_{\mathcal{C}_1}, S_{\mathcal{C}_2}\}, S_{\mathcal{C}_3}\} \cdots S_{\mathcal{C}_c}\},$$

we can apply the above results iteratively and prove (5).

Appendix B
Proof of Proposition 4

Let us consider the nodes $v \in \mathcal{O}(v_s)$. If there is only one such node (call it v_0), it must be $1n$-reducible, since otherwise it would have an input which is not the source. Say it is connected to node u. If we trace the input of node u by traveling in reverse along the edges of the network, we must eventually reach the source, which means we must pass through v. Thus u, which has $v \in \mathcal{O}(u)$, is part of some path originating at v. This means we have found a cycle; a contradiction. Thus we need only consider the case in which there are multiple nodes $v \in \mathcal{O}(v_s)$. Using the same reasoning as above, for each such v we can trace a non-source input back to some other $v' \in \mathcal{O}(v_s)$. However, if we do the same for v', we find that any input not directly from the source must originate from some $v'' \in \mathcal{O}(v_s)$. If we continue this process, we must at some point reach a repeated node, since the network is finite. This means we have found a cycle, a contradiction. So there must be some $v \in \mathcal{O}(v_s)$ such that $\mathcal{I}(v) = \{v_s\}$. We can make the completely analogous "dual" argument using the sink to prove the existence of an $n1$-reducible node.

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