Sensitivity, Proximity and FPT Algorithms for Exact Matroid Problems

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Abstract

We consider the problem of finding a basis of a matroid with weight exactly equal to a given target. Here weights can be discrete values from $\{-\Delta, \ldots, \Delta\}$ or more generally m-dimensional vectors of such discrete values. We resolve the parameterized complexity completely, by presenting an FPT algorithm parameterized by Δ and m for arbitrary matroids. Prior to our work, no such algorithms were known even when weights are in $\{0, 1\}$, or arbitrary Δ and $m = 1$. Our main technical contributions are new proximity and sensitivity bounds for matroid problems, independent of the number of elements. These bounds imply FPT algorithms via matroid intersection.

1 Introduction

Matroids are one of the most fundamental abstractions of combinatorial structures and capture intricate set systems such as spanning trees and transversals while still offering tractability for many related problems[1]. The well known Greedy algorithm can find a minimum (or maximum) weight basis of a matroid. Inherently more difficult is the task of finding a basis B with weight $w(B) = \sum_{b \in B} w(b)$ exactly equal to some given target $\beta \in \mathbb{Z}$. A straightforward reduction from Subset Sum shows that the problem is weakly NP-hard even for the most trivial examples of matroids. Papadimitriou and Yannakakis [34] first mention this and observe that for $0, 1$ weights the problem can still be solved efficiently via matroid intersection. They also mention that this generalizes to a fixed number of equality constraints, that is, given m-dimensional integral weight vectors $W(e) \in \{-\Delta, \ldots, \Delta\}^m$ for each element e and a target $\boldsymbol{\beta} \in \mathbb{Z}^m$ the goal is to find a basis B with $W(B) = \sum_{b \in B} W(b) = \boldsymbol{\beta}$. Towards this, one can guess the correct number of elements for each distinct weight vector in time $O(n^{(2\Delta+1)^m})$ and then solve matroid intersection in polynomial time where an additional partition matroid dictates the correct number of elements of each weight vector. Papadimitriou and Yannakakis also asked whether a pseudopolynomial time algorithm exists for spanning trees. Via an algebraic algorithm that uses a variant of Kirchhoff's theorem this is indeed possible [4] and generalizes to all linear matroids [7]. This leads to an algorithm with running time $(n\Delta)^{O(m)}$ in the setting mentioned above. Algebraic methods to solve exact weight problems were also mentioned by Mulmuley, Vazirani, and Vazirani [32], who credit Lovász for

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[1]We refer the reader to Section 2 and to [36] for basic definitions and results regarding matroids.

describing an algorithm for the exact matching problem. In contrast, Doron-Arad, Kulik, and Shachnai [14] very recently showed that the problem cannot be solved in pseudopolynomial time for arbitrary matroids (in the standard independence oracle model).

Similar settings as the above have also been studied extensively in the field of approximation algorithms. Grandoni and Zenklusen [19] consider a range of multi-budgeted matroid problems, i.e., imposing linear inequalities with non-negative weights on different variants. As they observe, the problem of finding any basis subject to two such constraints is already weakly NP-hard. Thus, their focus is on finding any independent set and not only bases. For the problem of finding a maximum profit independent set subject to a fixed number of budget constraints Grandoni and Zenklusen derive a PTAS. EPTAS or FPTAS algorithms cannot exist due to a hardness result for 2-dimensional Knapsack [28]. Further, an EPTAS is known for a single budget constraint [11, 12]. We note that while in some problems, most famously Knapsack, approximation schemes can easily be derived from pseudopolynomial time algorithms, this is not true for multibudgeted independent set.

For the type of problems we mention above, the classical complexity and approximation algorithms are very well understood by now, but the status through the lens of parameterized algorithms[2] is still unsatisfying with answers being unknown even for the following basic questions:

> For $0, 1$ weights and graphic matroids, is there an FPT algorithm in parameter m?
>
> For a single equality constraint, is there an FPT algorithm in parameter Δ for arbitrary matroids?

We resolve the parameterized complexity completely by providing an FPT algorithm in parameters Δ and m for arbitrary matroids. This is an algorithm with a running time of the form $f(\Delta, m) \cdot \mathrm{poly}(n)$, where $f(\Delta, m)$ is a function depending on these parameters Δ and m only, see, e.g. [10]. In our case, $f(\Delta, m) = (m\Delta)^{O(\Delta)^m}$.

We remark the connection to binary integer linear programming of the form

$$\{x \in \{0,1\}^n : Ax = b\}, \tag{1}$$

where $A \in \{-\Delta, \dots, \Delta\}^{m\times n}$ and $b \in \mathbb{Z}^m$. This is a very simple example of a multidimensional exact matroid problem over the uniform matroid with $2n$ elements and rank n: elements $1, 2, \dots, n$ correspond to the variables $x_1, \dots, x_n$ with weight vector $W(i) = A_i$ and elements $n + 1, \dots, 2n$ that have a zero weight vector and are used to ensure a solution is a basis. Despite the simplicity of the matroid, it is already non-trivial to find FPT algorithms for integer program (1) in parameters Δ and m. Such results were obtained by Papadimitriou [33], using a slightly stronger parameterization, and by Eisenbrand and Weismantel [17] with only parameters Δ and m. The latter work is heavily based on proximity and sensitivity, which turn out to be the key elements also in our work. Throughout this document, we use the terms proximity and sensitivity to describe the distance between an optimal continuous solution and an optimal integer solution and between two integer solutions with a similar right-hand side. Bounds on these quantities are useful algorithmically, specifically to reduce the search space, but are also of independent interest, for example, to understand how severe the effects of uncertainty in data can be for decision making, see e.g. [22, 38]. The study of general proximity and sensitivity bounds in integer linear programming goes back to the seminal work by Cook, Gerards, Schrijver, and Tardos [9].

[2] We refer to [10] for background on parameterized (FPT) algorithms.

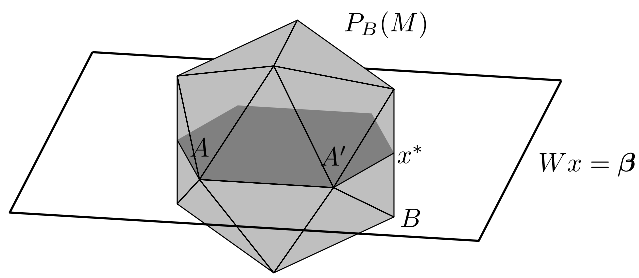

Figure 1: Schematic overview of proximity, sensitivity and their connection. The vertices of $P_B(M)$ (light gray) are the bases of M. The intersection (dark gray) of $P_B(M)$ with the affine subspace $\{x \in \mathbb{R}^E : Wx = \boldsymbol{\beta}\}$ may contain non-integral vertices. For such a vertex x^* by standard rounding there always exists a close by basis B with $W(B) \approx \boldsymbol{\beta}$. If there is a basis A with $W(A) = \boldsymbol{\beta}$ and the distance to B (equivalently, to x^*) is sufficiently large, then by sensitivity there is a closer basis A' also with $W(A') = \boldsymbol{\beta}$. This implies proximity.

1.1 Our contribution

Let $M = (E, \mathcal{I})$ be a matroid with (possibly multidimensional) weights $W(e) \in \{-\Delta, \dots, \Delta\}^m$. In the following, we denote by $A \oplus B = (A \setminus B) \cup (B \setminus A)$ the symmetric difference of sets A and B. We prove the following sensitivity result.

Theorem 1 (Sensitivity Theorem for Matroids)**.** *Let A, B be bases of M. Then there exists a basis A' with $W(A') = W(A)$ and*

$$|A' \oplus B| \leq (2m\Delta)^{12m} \cdot \|W(B) - W(A)\|_1.$$

Denote by $P_B(M) \subseteq [0,1]^E$ the matroid base polytope, that is, the convex hull of indicator vectors $\chi(B)$ of the bases B of M. In particular, there is a one-to-one correspondence between integral elements of $P_B(M)$ and bases of M. For convenience, we write $W \in \mathbb{Z}^{m \times n}$ as the matrix with columns $W(e)$ in the order the elements e appear as dimensions in $P_B(M)$. For $S \subseteq E$ we write $W(S) = \sum_{e \in S} W(e)$. We prove the following proximity result, see also Fig. 1.

Theorem 2 (Proximity Theorem for Matroids)**.** *Let A be a basis of M and let x^* be any vertex solution to the polytope*

$$\{x \in \mathbb{R}^n : x \in P_B(M),\, Wx = W(A)\}.$$

There exists a basis A' of M that satisfies $W(A') = W(A)$ and

$$\|x^* - \chi(A')\|_1 \leq (2m\Delta)^{13m}.$$

These sensitivity and proximity bounds are in the same order of magnitude as those known for (1), see e.g. [17]. From the proximity theorem we derive the following FPT algorithms.

Theorem 3. *For target $\boldsymbol{\beta} \in \mathbb{Z}^m$ there is an algorithm that finds a basis A of M with $W(A) = \boldsymbol{\beta}$, if one exists, in time*

$$\Delta^{O(\Delta)^m} \cdot n^{O(1)}\ .$$

Furthermore, if M is linear (with a given representation), it can be improved to

$$(m\Delta)^{O(m^2)} \cdot n^{O(1)}$$

randomized time.

Due to its generality, Theorem 3 can be used to obtain FPT algorithms for many concrete applications. To name a few, it improves an FPT algorithm for Feedback Edge Set with Budget Vectors due to Marx [31], generalizes a recent algorithm proposed by Liu and Xu [30] for Group-Constrained Matroid Base to arbitrary finite groups, and generalizes so-called combinatorial n-fold integer programs [27], which have a wide range of applications themselves. We refer to Section 6 for details.

There is a simple example based on long even cycles that shows that both sensitivity and proximity are unbounded for bipartite perfect matching, which is a special case of matroid intersection, even with a single $0, 1$ weight constraint. For details, see Section 7.

We remark that the proximity bounds in [9, 17] also hold in the optimization version, i.e., when we measure for some linear optimization direction the distance between optimal continuous solution and the closest optimal integer solution, and this optimization direction can have arbitrary real coefficients, i.e., they are not necessarily discrete as the equality constraints. This is interesting also from a polyhedral perspective. It remains open whether the optimization variant also admits the strong proximity in our setting. While this is not obviously connected to this question, there are also other disparities between feasibility and optimization in exact matroid problems. Namely, the algebraic techniques, which are the only known approach to solve exact matroid basis in pseudopolynomial time on linear matroids (and even graphical matroids), are also not capable of optimization.

1.2 Techniques

Our main technical contribution lies in the sensitivity and proximity bounds. We will briefly review previous techniques, specifically those by Cook et al. [9] and by Eisenbrand and Weismantel [17]. Suppose for some solution x to an integer linear program we want to prove a bound on the distance to the closest solution z, where the right-hand side is slightly perturbed. One can naturally decompose the change from z to x into atomic changes that involve the increase or decrease of a variable by 1. On a high level, Cook et al.'s approach can be summarized as defining small bundles of atomic changes that do not affect feasibility. In their case they use Hilbert basis elements of a carefully chosen cone and write $x - z$ as a (not necessarily integral) conic combination of these bundles. If the distance between z and x is sufficiently large, one of the bundles is taken at least once and then we can also apply only this bundle exactly once to z, proving that there is a closer solution. Crucial to this argument is that each bundle of atomic changes can be applied to z independently. In an inherently different approach, Eisenbrand and Weismantel arrange the atomic changes in a careful sequence given by the Steinitz Lemma and achieve that if the distance between x and z is sufficiently large, by pigeonhole principle there will be a "cycle" in the sequence, which is an interval of atomic changes that does not affect feasibility. This cycle is applied to z, which then also proves that there is a closer solution to x.

In the context of matroids, a natural candidate for an atomic change is the exchange of a pair of elements. Both of the approaches above are applicable to restrictive classes of matroids, specifically *strongly base-orderable* matroids, see [36, 42.6c]. Essentially, this limited class allows any subset of the atomic changes to be applied to z in isolation. Implementing this approach to general matroids seems to be elusive.

Our proof is based on a novel structural result on matroids. If two independent sets A and B are of the same cardinality and are roughly of the same weight, then there are large *unicolor* subsets of them of equal cardinality, that can be mutually exchanged (Lemma 8 and Corollary 9). A set is unicolor if all elements have the same weight. This is a key notion of this paper. Via identifying several such exchanges that nullify the total change of weight, and the theory of matroid intersection [16, 29], we can then derive the existence of another independent set of the matroid of equal cardinality that has the same weight as A. Specifically in higher dimension, we furthermore rely on polyhedral combinatorics like Carathéodory's theorem and bounds on the complexity of facets and vertices.

2 Preliminaries

A *matroid* $M = (E, \mathcal{I})$ is defined by a *ground set* $E = \{1, 2, \ldots, n\}$ and a collection $\mathcal{I} \subseteq 2^E$ of independent sets that satisfy the following properties:

(M1) $\emptyset \in \mathcal{I}$.

(M2) If $X \subseteq Y$ and $Y \in \mathcal{I}$, then $X \in \mathcal{I}$.

(M3) If $X, Y \in \mathcal{I}$ and $|X| < |Y|$, then there exists $e \in Y \setminus X$ such that $X \cup \{e\} \in \mathcal{I}$.

Condition (M3) is the *exchange property*. We often use the following elementary consequence.

Lemma 4 (Downsizing)**.** *Let $I \in \mathcal{I}$ be an independent set $A \subseteq I$ and $B \subseteq E \setminus I$ of equal cardinality $|B| = |A|$ and suppose that $(I \setminus A) \cup B \in \mathcal{I}$. Then, for each $A' \subseteq A$ there exists $B' \subseteq B$ with*

i) $|A'| = |B'|$ and

ii) $(I \setminus A') \cup B' \in \mathcal{I}$.

Similarly, for each $B' \subseteq B$ there exists $A' \subseteq A$ satisfying i) and ii)

Proof. The set $(I \setminus A')$ is an independent set. We can identify $|A'|$ elements of $((I \setminus A) \cup B) \setminus (I \setminus A') = B$ that can be added to this set. Similarly, $(I \setminus A) \cup B'$ is an independent set and can be extended to the cardinality of I. $\square$

A *basis* of M is an inclusionwise maximal set of $\mathcal{I}$. All the bases of a matroid have the same cardinality, denoted by $\operatorname{rank}(M)$, the *rank* of M. For a subset $S \subseteq E$, its *characteristic vector* $\chi(S) \in \{0, 1\}^n$ is defined as

$$\chi(S)_i = \begin{cases} 1, & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The *rank* of a subset $S \subseteq E$ is the rank of the matroid $(S, \mathcal{I}')$, where $\mathcal{I}' = \{I \cap S \colon I \in \mathcal{I}\}$ and we denote it by $\operatorname{rank}(S)$. The *matroid polytope* $P(M) \subseteq \mathbb{R}^n$ is the convex hull of the characteristic vectors $\chi(A) \in \{0, 1\}^n$ of independent sets $A \in \mathcal{I}$. Edmonds [15] has shown that $P(M)$ is described by the following set of inequalities

$$\begin{array}{ll} \sum_{e \in S} x_e \leq \operatorname{rank}(S), & S \subseteq E \\ x_e \geq 0, & e \in E. \end{array} \tag{2}$$

The convex hull of the characteristic vectors of bases of M is obtained from (2) by adding the equation $\sum_{e \in E} x_e = \operatorname{rank}(M)$ and we denote this *base polytope* by $P_B(M)$.

An *(m-dimensional) weight* of a matroid is given by a matrix $W \in \mathbb{Z}^{m \times n}$. The weight of a subset $S \subseteq E$ is defined as $W(S) = W \cdot \chi(S)$. A *target* is an integral vector $\boldsymbol{\beta} \in \mathbb{Z}^m$. A subset $S \subseteq E$ is *exact* for W and $\boldsymbol{\beta}$, if $W(S) = \boldsymbol{\beta}$. We also refer to the condition $W(S) = \boldsymbol{\beta}$ as imposing m *constraints*. The largest absolute value of an entry of the matrix $W \in \mathbb{Z}^{m \times n}$ is denoted by $\|W\|_\infty$.

3 FPT Algorithms

In this section we discuss how the proximity bound in Theorem 2 can be used to derive the FPT algorithms, thereby proving Theorem 3. Let $M = (E, \mathcal{I})$ be a matroid, $W \in \{-\Delta, \dots, \Delta\}^{m \times n}$ be a matrix and $\boldsymbol{\beta} \in \mathbb{Z}^m$ be a target vector. The goal is to find a basis $B \in \mathcal{I}$ of the matroid M with

$$W(B) = \boldsymbol{\beta},$$

or to assert that such a basis does not exist. Here $W(B)$ is the sum of all columns of the matrix W that correspond to elements of B. Thus $W(B) = W \cdot \chi(B)$, where $\chi(B) \in \{0,1\}^n$ denotes characteristic vector of the basis B.

We first assume that the matroid M is *implicitly given* by an *independence testing oracle* [36, Section 40.1]. This means that testing $S \in \mathcal{I}$ for a subset $S \subseteq E$ can be done in constant time and no other information on M can be queried. At the end of this section, we treat the second part of Theorem 3, where a linear matroid with explicit representation given and faster algorithms can be derived.

With the ellipsoid method, we compute a vertex x^* of the base polytope $P_B(M)$ intersected with the subspace $\{x \in \mathbb{R}^n \colon Wx = \boldsymbol{\beta}\}$ in time $\mathrm{poly}(n + m \log |\Delta|)$ using [26, 20]. Recall that the base polytope is the convex hull of incidence vectors of bases $P_B(M) = \mathrm{conv}\{\chi(B) \colon B \text{ basis of } M\}$. Theorem 2 implies that, if there exists a basis $A \in \mathcal{I}$ of the matroid M with weight $W(A) = \boldsymbol{\beta}$, then there exists a such basis with

$$\|\chi(A) - x^*\|_1 \le (m\Delta)^{O(m)}. \tag{3}$$

For $\boldsymbol{\alpha} \in \{-\Delta, \dots, \Delta\}^m$ and $S \subseteq E$, let $\ell_{\boldsymbol{\alpha}}(S) \in \mathbb{N}_0$ denote the number of elements of weight $\boldsymbol{\alpha}$ in S. In other words,

$$\ell_{\boldsymbol{\alpha}}(S) = |\{e \in E \colon W(e) = \boldsymbol{\alpha}\}|.$$

The proximity condition (3) implies

$$\begin{aligned} \sum_{e \in E \colon W(e) = \boldsymbol{\alpha}} x_e^* - (m\Delta)^{O(m)} &\le \ell_{\boldsymbol{\alpha}}(A) \\ &\le \sum_{e \in E \colon W(e) = \boldsymbol{\alpha}} x_e^* + (m\Delta)^{O(m)}. \end{aligned}$$

We can guess the correct value $\ell_{\boldsymbol{\alpha}}(A)$ out of a set of $(m\Delta)^{O(m)}$ candidates for each $\boldsymbol{\alpha}$. Since $\boldsymbol{\alpha} \in \{-\Delta, \dots, \Delta\}^m$ this leaves a total number of $(m\Delta)^{O(m) \cdot O(\Delta)^m} = \Delta^{O(\Delta)^m}$ vectors among one encodes the values $\ell_{\boldsymbol{\alpha}}(A)$ for each $\boldsymbol{\alpha}$. Assume that we have one such candidate. Observe that the condition

$$\sum_{\boldsymbol{\alpha}} \ell_{\boldsymbol{\alpha}}(A) = \mathrm{rank}(M)$$

must be satisfied. We next consider the partition matroid $M_p = (E, \mathcal{I}_p)$ with independent sets

$$\mathcal{I} = \{S \subseteq E : \ell_{\boldsymbol{\alpha}}(S) \le \ell_{\boldsymbol{\alpha}}, \boldsymbol{\alpha} \in \{-\Delta, \dots, \Delta\}^m\}.$$

We are looking for a basis A of M that is also contained in M_p. This is a *matroid intersection* problem for $M \cap M_p$ and can be solved in time polynomial in n [16, 29], see also [36]. All together, this shows that we can find a basis B of weight $W(B) = \boldsymbol{\beta}$ or assert that no such basis exists, in time $\Delta^{O(\Delta)^m} n^{O(1)}$. This proves the first part of Theorem 3.

3.1 Linear matroids

Linear matroids are matroids that can be defined by a matrix A over a field $\mathbb{F}$, such that the ground set E is the set of columns of A and X is an independent set if these columns are linearly independent. If a matroid can be defined by a matrix A over a field $\mathbb{F}$, then we say that the matroid is *representable* over $\mathbb{F}$. In this paper, when we consider linear matroids, we always assume that the matrix A that represents the matroid is also given on the input (along the independence testing oracle). We assume, that $\mathbb{F}$ is either a finite field or the rationals. For a detailed discussion regarding representation issues and its computational complexity we refer the reader to [31, Section 3].

Camerini et al. [7] presented a randomized, pseudopolynomial time algorithm to find a basis B of a linear matroid with weight $w(B)$ exactly β in $(\Delta n)^{O(1)}$ time. In our setting, this is the one-dimensional case, i.e., the case $m = 1$. This method does not immediately yield an FPT algorithm in m and Δ, but it can be combined with our proximity bound leading to one and improving on the running time of the previous method. This uses the pseudopolynomial time algorithm as a black box and for matroids where a deterministic variant is known, e.g. for graphic matroids [4], our algorithm is also deterministic. However, for general matroids given by an independence testing oracle, it is known that a pseudopolynomial time algorithm cannot exist [14].

We use a variant of a standard method of aggregating all m constraints into one single constraint $w(B) = \beta$, see, e.g. [25], which can be done more efficiently when the search space is bounded due to proximity. To this end, suppose there is an algorithm that, for a given matroid $M = (E, \mathcal{I})$ with weights $w : E \to \{-\Delta, \ldots, \Delta\}$ and a target $\alpha \in \mathbb{N}$ finds a basis B of M with $w(B) = \alpha$ in time $(n\Delta)^{O(1)}$. We will show that for the same matroid M, with a multidimensional weight $W : E \to \{-\Delta, \ldots, \Delta\}^m$ and a target $\boldsymbol{\beta} \in \mathbb{Z}^m$ one can find a basis B with $W(B) = \boldsymbol{\beta}$ in time $(m\Delta)^{O(m^2)} n^{O(1)}$. Let x^* be a vertex to the matroid base polytope $P_B(M)$ intersected with $\{x \in \mathbb{R}^E : Wx = \boldsymbol{\beta}\}$. Let A be the basis with $W(A) = \boldsymbol{\beta}$ that is close to x^* as guaranteed by Theorem 2. We write $\lfloor x^* \rceil$ for the vector derived from x^* by rounding each component to the closest integer and set

$$\Gamma := \|\chi(A) - \lfloor x^* \rceil\|_1 \le 2\|\chi(A) - x^*\|_1 \le (m\Delta)^{O(m)}.$$

The first inequality follows from the fact that when $|\chi(A)_i - x^*_i| < 1/2$ then rounding will decrease the distance in this dimension, and otherwise, it will increase it by at most 1. Since we do not know A, we also do not know the value of Γ, but we can obtain it through guessing.

Let B be any basis with $\|\chi(B) - \lfloor x^* \rceil\|_1 = \Gamma$ and define

$$\lambda := (1, (2\Gamma\Delta + 1), (2\Gamma\Delta + 1)^2, \ldots, (2\Gamma\Delta + 1)^{m-1}) \in \mathbb{Z}^m.$$

We argue that $W(B) = \boldsymbol{\beta}$ if and only if $\lambda^\mathsf{T} W(B) = \lambda^\mathsf{T} \boldsymbol{\beta}$. Since the other direction is trivial, assume that $\lambda^\mathsf{T} W(B) = \lambda^\mathsf{T} \boldsymbol{\beta}$. Inductively, one can conclude that also $W(B)_i = \boldsymbol{\beta}_i$ for $i = 1, 2, \ldots, m$: because all $j < i$ satisfy this equality by induction and all $j > i$ are multiplied by a higher power of $2\Gamma\Delta + 1$ than constraint i. Hence, one has

$$W(B)_i \equiv \boldsymbol{\beta}_i \mod 2\Gamma\Delta + 1.$$

Further, $W(B)_i$ and $\boldsymbol{\beta}_i = W(A)_i$ are both in $\{W\lfloor x^* \rceil - \Gamma\Delta, \ldots, W\lfloor x^* \rceil + \Gamma\Delta\}$ and therefore the modulo operator is a bijection and it follows that $W(B)_i = \boldsymbol{\beta}_i$.

It is not enough to run the pseudopolynomial time algorithm with $\lambda^\mathsf{T} W(B) = \lambda^\mathsf{T} \boldsymbol{\beta}$, since we also need to enforce that $\|\chi(B) - \lfloor x^* \rceil\|_1 = \Gamma$. Towards this, let $w(e) = w_1(e) + (2n + 1) w_2(e)$ be the one-dimensional weight of element e, where

$$w_1(e) = (1 - 2\lfloor x^*_e \rceil) \text{ and } w_2(e) = \lambda^\mathsf{T} W(e).$$

Further, let

$$\alpha := (\Gamma - \|\lfloor x^* \rceil\|_1) + (2n+1) \cdot \lambda^\mathsf{T} \boldsymbol{\beta}$$

be the target weight. Let B be a basis. We argue that $w(B) = \alpha$ if and only if $\|\chi(A) - \lfloor x^* \rceil\|_1 = \Gamma$ and $W(B) = \boldsymbol{\beta}$. One has the following connection between $w_1(B)$ and $\|\chi(B) - \lfloor x^* \rceil\|_1$.

$$\begin{aligned} w_1(B) + \|\lfloor x^* \rceil\|_1 &= \sum_{e \in E} \chi(B)_e \cdot (1 - 2\lfloor x_e^* \rceil) + \lfloor x_e^* \rceil \\ &= \sum_{e \in E} \chi(B)_e + \lfloor x_e^* \rceil - 2\chi(B)_e \lfloor x_e^* \rceil \\ &= \|\chi(B) - \lfloor x_e^* \rceil\|_1, \end{aligned}$$

where the last inequality follows because $a + b - 2ab = |a - b|$ for $a, b \in \{0, 1\}$. Thus, $\|\chi(A) - \lfloor x^* \rceil\|_1 = \Gamma$ and $W(B) = \boldsymbol{\beta}$ implies $w(B) = \alpha$. For the other direction assume that $w(B) = \alpha$. Then

$$w_1(B) \equiv w(B) \equiv \beta \equiv \Gamma - \|\lfloor x^* \rceil\|_1 \mod 2n+1.$$

Since both sides are in $\{-n, \ldots, n\}$, it follows that $w_1(B) = \Gamma - \|\lfloor x^* \rceil\|_1$ and further $\lambda^\mathsf{T} W(B) = w_2(B) = \lambda^\mathsf{T} \boldsymbol{\beta}$, which means that $W(B) = \boldsymbol{\beta}$. Thus, it suffices to run the pseudopolynomial time algorithm with w and α. The maximum weight given to the algorithm is $n \cdot O(\Delta\Gamma)^m = n \cdot (m\Delta)^{O(m^2)}$, which leads to the claimed running time.

4 From sensitivity to proximity

In this section, we show how to obtain the proximity bound in Theorem 2 from Theorem 1. We assume that $W \in \mathbb{Z}^{m \times n}$ is a matrix with $\|W\|_\infty \leq \Delta$. Let A be a basis of $M = (E, \mathcal{I})$ and $\boldsymbol{\beta} = W(A) \in \mathbb{Z}^m$. Furthermore, let $x^* \in \mathbb{R}^n$ be a vertex of the polytope

$$P_B(M) \cap \{Wx = \boldsymbol{\beta}\}. \tag{4}$$

The goal is to show that there exists a basis A' of M with weight $W(A') = \boldsymbol{\beta}$ that is close to x^*.

The next Lemma shows something weaker, namely that there exists a basis close to x^* whose weight might violate the target $\boldsymbol{\beta}$. But since it is close, it does not violate this target by much. The lemma essentially follows from the fact that the characteristic vectors of two bases $\chi(B_1)$ and $\chi(B_2)$ of M are neighboring vertices of $P_B(M)$ if and only if $|B_1 \cap B_2| = \text{rank}(M) - 1$, see, e.g. [36, Theorem 40.6].

Lemma 5. *Let $x^* \in P_B(M)$ and $F \subseteq P_B(M)$ be the unique face of $P_B(M)$ of minimal affine dimension containing x^*. Suppose $\dim(F) = d$. There exists a basis B of M with $\chi(B) \in F$ with*

$$\|\chi(B) - x^*\|_1 \leq d.$$

Proof. The proof is by induction on $\dim(F) = d$. If $d = 0$, then x is an integer vector and therefore the characteristic vector of a basis. Now, we suppose that $d \geq 1$. The face F is defined by setting all inequalities of the base polytope to equality that are satisfied by x^* with equality. The point x^* lies in the relative interior of F. This means that it satisfies all other inequalities of $P_B(M)$ strictly.

The face F contains two different adjacent vertices $u, v \in F \cap \mathbb{Z}^n$ of $P_B(M)$. Their difference $u - v \in \{0, \pm 1\}^n$ satisfies $\|u - v\|_1 = 2$, see, e.g. [36, Theorem 40.6]. We can assume that $\|u - x^*\|_1 \leq \|v - x^*\|_1$ holds, otherwise swap u and v. For

$\lambda > 0$ small enough, one has $x^* + \lambda(u-v) \in F$. Let $\lambda' > 0$ be maximal with $x^* + \lambda'(u-v) \in F$. Since also $x^* - \lambda'(u-v) \in F$ and $F \subseteq [0,1]^n$ one has $\lambda' \leq 1/2$. Furthermore, the point $y' = x^* + \lambda'(u-v)$ lies on a face $F' \subset F$ of $P_B(M)$ of dimension $\dim(F') \leq d-1$. By induction, there exists an integer point $z \in F' \cap \mathbb{Z}^n$ with $\|z - y'\|_1 \leq d-1$. The triangle inequality implies that the basis B with $\chi(B) = z$ satisfies the assertion. □

Proof of Theorem 2. Since x^* is a vertex solution, it must lie on a d dimensional face of $P_B(M)$ with $d \leq m$. By Lemma 5 there is an integer point $\chi(B) \in P_B(M) \cap \mathbb{Z}^n$ with $\|\chi(B) - x^*\|_1 \leq d \leq m$. Thus, by Theorem 1 there exists a basis A' with $W(A') = W(A)$ and $|A' \oplus B| \leq (2m\Delta)^{12m} \|W(B) - W(A)\|_1$ and by the triangle inequality

$$\begin{aligned} \|\chi(A') - x^*\|_1 &\leq \|\chi(A') - \chi(B)\|_1 + \|\chi(B) - x^*\|_1 \\ &\leq (2m\Delta)^{12m} \cdot (m\Delta) + m \\ &\leq (2m\Delta)^{13m} \end{aligned}$$ □

5 Sensitivity

We now show Theorem 1, the main result of this paper. We first show the 1-dimensional case, $m = 1$. The case $m \geq 2$ builds on the same important structural theorem on weighted matroids that is laid out in the first part of this section but also requires some further techniques from convex and polyhedral geometry.

5.1 One constraint

Recall that the elements of the ground set have integer weights $w\colon E \to \{-\Delta, \dots, \Delta\}$. We first explain how Theorem 1 follows from the following assertion on weighted matroids.

Theorem 6. *Let $A, B \in \mathcal{I}$ be disjoint with $|A| = |B|$, and let $w\colon E \to \{-\Delta, \dots, \Delta\}$ be integer weights with $|w(A) - w(B)| \leq \mu$. If $|A| = |B| \geq (2 \cdot \Delta + 1)^5 + \mu$ then there exists $A' \in \mathcal{I}$, $A' \neq A$ with*

$$w(A') = w(A) \quad \text{and} \quad |A'| = |A|.$$

Proof of Theorem 1 for $m = 1$. Given a bases A, B, let $\mu \in \mathbb{N}_0$ with $|w(A) - w(B)| \leq \mu$ and suppose that A has smallest symmetric difference with B among all bases of weight $w(A)$. Consider $A' = A \setminus (A \cap B)$ and $B' = B \setminus (A \cap B)$ and the minor $M' = (A' \cup B', \mathcal{I}')$, where

$$\mathcal{I}' = \{I \setminus (A \cap B) \colon I \in \mathcal{I},\, (A \cap B) \subseteq I \subseteq (A \cup B)\}.$$

The rank of M' is $|A \setminus B|$. If $|A \oplus B| \geq (2 \cdot \Delta + 1)^5 + \mu$, then Theorem 6 implies that there exists a basis C' of M' different from A' with weight $w(C') = w(A')$. This yields a basis

$$C' \cup (A \cap B) \quad \text{of weight} \quad w(C' \cup (A \cap B)) = w(A)$$

of the matroid M that has more elements in common with B than A. This is a contradiction to the minimality of the symmetric difference of A and B. □

The rest of this section is devoted to a proof of Theorem 6. We begin by showing a simple observation.

Proposition 7. *Let S be a finite set and $w\colon S \to \{-\Delta, \ldots, \Delta\}$ be integer weights with*

$$|w(S)| = \mu. \tag{5}$$

Denote the set of non-negative elements by $S^+ = \{s \in S\colon w(s) \geq 0\}$. Let $S_1, \ldots, S_\ell$ be a partitioning of S into subsets of S. There exists an index i such that

$$|S^+ \cap S_i| \geq \frac{|S| - \mu}{\ell \cdot (\Delta + 1)}.$$

Proof. Equation (5) implies

$$|S^+| \geq (|S| - \mu)/(\Delta + 1).$$

By an averaging argument, there exists a set S_i that contains at least $(|S| - \mu)/(\ell \cdot (\Delta + 1))$ many elements of S^+. ☐

Lemma 8. *Let $A, B \in \mathcal{I}$ be disjoint with $|A| = |B| = k$ and let $w\colon E \to \{-\Delta, \ldots, \Delta\}$ be integer weights with $|w(A) - w(B)| \leq \mu$. There exist subsets $A' \subseteq A$ and $B' \subseteq B$ of equal cardinality such that*

i) $A \setminus A' \cup B' \in \mathcal{I}$,

ii) $|A'| = |B'| \geq (k - \mu)/(2 \cdot \Delta + 1)^2$ *and*

iii) $w(a) \geq w(b)$ *for each* $a \in A'$ *and* $b \in B'$.

Proof. Let $a_1, \ldots, a_k$ be an ordering of A such that $w(a_i) \geq w(a_{i+1})$ for all i. Furthermore, let $A_\alpha = \{a \in A\colon w(a) = \alpha\}$ for $\alpha = -\Delta, \ldots, \Delta$. The A_α are a partitioning of $A = A_{-\Delta} \cup \cdots \cup A_\Delta$. From this we construct a partitioning $B_{-\Delta} \cup \cdots \cup B_\Delta$ of B such that, for each $j \in \{-\Delta, \ldots, \Delta\}$ one has

$$B_j \cup A_{j+1} \cup \cdots \cup A_\Delta \in \mathcal{I}.$$

The existence of such a partitioning $B_{-\Delta} \cup \cdots \cup B_\Delta$ of B is guaranteed by the exchange property of the matroid, specifically to the sets

$$A_{j+1} \cup \cdots \cup A_\Delta \in \mathcal{I} \text{ and } B \setminus (B_{-\Delta} \cup \cdots \cup B_{j-1}) \in \mathcal{I}. \tag{6}$$

If $B_{-\Delta}, \ldots, B_{j-1}$ have been constructed, then $B_j \subseteq B$ is a subset of cardinality $|A_j|$ of the right independent set in (6) that can be added to $A_{j+1} \cup \cdots \cup A_\Delta$.

Now let $b_1, \ldots, b_k$ be any ordering of B such that the elements of B_j come before the elements of B_{j+1} for each j. In the following, we refer to a tuple $a_i b_i$ as an *edge*. We apply Proposition 7 to the set of edges $S = \{a_i b_i\colon i = 1, \ldots, k\}$ and the weight function $w'\colon S \to \{-2 \cdot \Delta, \ldots, 2 \cdot \Delta\}$ defined by the difference of weights

$$w'(a_i b_i) = w(a_i) - w(b_i).$$

The partitioning of S is according to the value of the a_i. Formally,

$$S = S_{-\Delta} \cup \cdots \cup S_\Delta, \qquad \text{where} \qquad S_\alpha = \{a_i b_i\colon w(a_i) = \alpha\}.$$

Proposition 7 now shows that there exists an index $j \in \{-\Delta, \ldots, \Delta\}$ such that S_j contains at least

$$\frac{k - \mu}{(2\Delta + 1)^2}$$

non-negative edges $a_i b_i$. Let $B' \subseteq B_j$ be the corresponding end-nodes of these edges on the side of B. The following is an independent set

$$B' \cup A_{j+1} \cup \cdots \cup A_\Delta,$$

simply because $B_j \cup A_{j+1} \cup \cdots \cup A_\Delta \in \mathcal{I}$. Since A is independent, there exists a subset $\widetilde{A} \subseteq A_{-\Delta} \cup \cdots \cup A_j$ such that

$$\widetilde{A} \cup B' \cup A_{j+1} \cup \cdots \cup A_\Delta$$

is an independent set of cardinality $|A| = k$. Let

$$A' = A \setminus (\widetilde{A} \cup A_{j+1} \cup \cdots \cup A_\Delta).$$

Clearly, we have $|A'| = |B'| \geq (k - \mu)/(2\Delta + 1)^2$ and

$$(A \setminus A') \cup B' \in \mathcal{I}.$$

The crucial observation is that all elements in A' have weight at least j and all weights in B' have weights at most j. $\square$

A subset $S \subseteq E$ of the ground set is called *unicolor*, if $w(x) = w(y)$ for each $x, y \in S$. The following is a version of Lemma 8 guaranteeing an exchange with unicolor sets.

Corollary 9. *Let $A, B \in \mathcal{I}$ be disjoint with $|A| = |B| = k$ and let $w \colon E \to \{-\Delta, \ldots, \Delta\}$ be integer weights with $|w(A) - w(B)| \leq \mu$. There are unicolor subsets $A' \subseteq A$ and $B' \subseteq B$ of equal cardinality such that*

i) $A \setminus A' \cup B' \in \mathcal{I}$,

ii) $|A'| = |B'| \geq (k - \mu)/(2 \cdot \Delta + 1)^4$,

iii) $w(a) \geq w(b)$ *for each* $a \in A'$ *and* $b \in B'$.

Proof. Let $A^{(1)} \subseteq A$ and $B^{(1)} \subseteq B$ be the sets obtained from Lemma 8, which are not unicolor, but satisfy all other required properties. Further, $|A^{(1)}| = |B^{(1)}| \geq (k - \mu)/(2\Delta + 1)^2$. At least $|B^{(1)}|/(2 \cdot \Delta + 1)$ elements of $B^{(1)}$ have the same weight. By Lemma 4 we can thus restrict to this unicolor subset $B^{(2)} \subseteq B^{(1)}$ with a corresponding subset $A^{(2)} \subseteq A^{(1)}$ guaranteeing

$$|A^{(2)}| = |B^{(2)}| \geq (k - \mu)/(2\Delta + 1)^3.$$

Again, by restricting to the largest unicolor subset of $A^{(2)}$ we have $A^{(3)} \subseteq A^{(2)}$ and $B^{(3)} \subseteq B^{(2)}$ both unicolor and

$$|A^{(3)}| = |B^{(3)}| \geq (k - \mu)/(2\Delta + 1)^4. \qquad \square$$

Proof of Theorem 6. Corollary 9 implies that there exist two unicolor sets $A^+ \subseteq A$ and $B^+ \subseteq B$ of equal cardinality at least $2 \cdot \Delta$ such that

- $|A^+| = |B^+|$,
- $w(a) \geq w(b)$ for each $a \in A^+$ and $b \in B^+$, and
- $(A \setminus A^+) \cup B^+ \in \mathcal{I}$.

By symmetry, there also exist unicolor sets $A^- \subseteq A$ and $B^- \subseteq B$ of equal cardinality at least $2 \cdot \Delta$ such that

- $|A^-| = |B^-|$,
- $w(a) \leq w(b)$ for each $a \in A^+$ and $b \in B^+$, and
- $(A \setminus A^-) \cup B^- \in \mathcal{I}$.

Let $p = w(a^+) - w(b^+) \in \{0, \ldots, \Delta\}$ be the weight of the edges a^+b^+, $a^+ \in A^+$, $b^+ \in B^+$. Similarly, let $q \in \{0, \ldots, \Delta\}$ such that $-q$ is the common weight of the edges a^-b^-, $a^- \in A^-$, $b^- \in B^-$. If $p = 0$ then one has

$$w\left((A \setminus A^+) \cup B^+\right) = w(A).$$

This is an independent set of cardinality $|A|$ and weight $w(A)$, which is what we want. Similarly, when $q = 0$ then the assertion follows trivially. Hence, assume for the remainder of the proof that $p, q \geq 1$.

Since all four sets are of cardinality at least $2 \cdot \Delta$, we can assume, by downsizing (Lemma 4) if necessary, that

$$|A^+| = |B^+| = 2q \text{ and } |A^-| = |B^-| = 2p.$$

We next consider the fractional point y^*

$$\begin{aligned} y^* &= \frac{1}{2}\chi\left((A \setminus A^+) \cup B^+\right) + \frac{1}{2}\chi\left((A \setminus A^-) \cup B^-\right) \\ &= \chi(A) - \frac{1}{2}\left(\chi(A^+) - \chi(B^+) + \chi(A^-) - \chi(B^-)\right). \end{aligned}$$

Clearly, $y^* \in P_B(M)$ and the weight of y^* is the same as the weight of A, i.e.,

$$\begin{aligned} w^{\mathsf{T}} y^* &= w(A) - \frac{1}{2}(-p) \cdot 2 \cdot q - \frac{1}{2} q \cdot 2 \cdot p \\ &= w(A). \end{aligned}$$

Since $A^+, B^+, A^-, B^- \neq \emptyset$ one has $y^* \neq \chi(A)$. For $\alpha \in \{-\Delta, \ldots, \Delta\}$, let us denote the elements of weight α by $E_\alpha = \{e \in E \colon w(e) = \alpha\}$. We next argue that the sum of the components of y^* corresponding to E_α are integral for each $\alpha \in \{-\Delta, \ldots, \Delta\}$. This follows from the fact that the sets A^+, B^+, A^- and B^- are unicolor and of even cardinality, implying that

$$\chi(E_\alpha)^{\mathsf{T}}\left(\chi(A^+) - \chi(B^+) + \chi(A^-) - \chi(B^-)\right)$$

is an even integer. Consequently one has for each $\alpha \in \{-\Delta, \ldots, \Delta\}$

$$\chi(E_\alpha)^{\mathsf{T}} y^* \in \mathbb{N}_0.$$

We next consider the *partition matroid* $M_p = (E, \mathcal{I}_p)$ with

$$\mathcal{I}_p = \left\{I \subseteq E \colon |I \cap E_\alpha| \leq \chi(E_\alpha)^{\mathsf{T}} y^*,\ \alpha \in \{-\Delta, \ldots, \Delta\}\right\}.$$

The corresponding matroid polytope $P(M_p)$ is defined by the inequalities

$$\begin{array}{rcll} \sum_{e \in E_\alpha} x_e & \leq & \sum_{e \in E_\alpha} y_e^*, & \alpha \in \{-\Delta, \ldots, \Delta\} \\ x_e & \geq & 0, & e \in E. \end{array} \tag{7}$$

The point y^* satisfies all rank-constraints in (7) with equality. The crucial observation is now the following.

> Each $y \in P_B(M) \cap P_B(M_p)$ that satisfies the rank constraints in (7) with equality is of weight $w(y) = w(A)$.

The *matroid intersection polytope* $P_B(M) \cap P_B(M_p)$ is integral [16, 29], see also [36, Theorem 41.12]. The fractional point y^* is therefore in the convex hull of at least two integral vectors that are also tight at the rank inequalities in (7). One of them corresponds to the characteristic vector $\chi(A')$ of an independent set A' different from A. The cardinality of this independent set is equal to the one of A, since the sum of the right-hand-sides of the rank constraints in (7) is equal to $|A|$ and $\chi(A')$ satisfies all these constraints with equality. $\square$

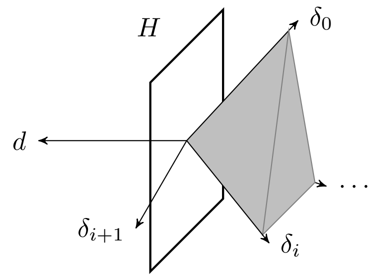

Figure 2: Visualization of the proof of Lemma 13

5.2 Several constraints

We start by stating two lemmas on cones generated by small discrete vectors. Recall that the convex cone generated by a set of vectors $X \in \mathbb{R}^m$ is defined as

$$\operatorname{cone}(X) = \left\{ \sum_{x \in X} \lambda_x x : \lambda_x \geq 0,\ x \in X \right\} \subseteq \mathbb{R}^m.$$

A cone is *pointed* if $\mathbf{0}$ is a vertex of the cone and *flat* otherwise.

Lemma 10. *Let $X \subseteq \{-\Delta, \dots, \Delta\}^m$ such that $C = \operatorname{cone}(X)$ is pointed. Then there is a halfspace $H = \{x \in \mathbb{R}^m : d^\intercal x \geq 0\}$ with $C \cap H = \{\mathbf{0}\}$ defined by some $d \in \mathbb{Z}^m$ with*

$$\|d\|_\infty \leq \Delta^m m^{m/2+1}.$$

Proof. One may assume without loss of generality that C has exactly $\dim(C) \leq m$ facets. Otherwise, the origin is a vertex of another cone containing C that is defined by $\dim(C)$ facets of C with linearly independent normals. The vector d can then be taken as the sum of the facet normals of C, each of which is up to scaling fully defined by being orthogonal to the $\dim(C) - 1$ vectors in X spanning the facet. By Cramer's rule and Hadamard's bound, for each face normal one can take an integer vector with entries bounded by $\Delta^m m^{m/2}$. □

Lemma 11. *Let $X \subseteq \{-\Delta, \dots, \Delta\}^m$ and $x \in X$ such that*

$$-x \in \operatorname{cone}(X \setminus \{x\}).$$

Then we can write $-\lambda_x x = \sum_{y \in X \setminus \{x\}} \lambda_y y$ for some $\lambda \in \mathbb{Z}^X_{\geq 0}$ with at most $m+1$ non-zero components and

$$\|\lambda\|_\infty \leq \Delta^m m^{m/2}.$$

Proof. By a variant of Carathéodory's Theorem, see [36, Theorem 5.2], we may assume without loss of generality that $X \setminus \{x\}$ are linearly independent, in particular, $|X| \leq m+1$. The assertion follows immediately from applying Cramer's rule and Hadamard's bound. □

In analogy to the one-dimensional case, we call a set $S \subseteq E$ *unicolor*, if $W(a) = W(b)$ for all $a, b \in S$. The following is a multidimensional analogue of Lemma 8.

Lemma 12. *Let $A, B \in \mathcal{I}$ be disjoint with $|A| = |B| = k$ and let $W : E \to \{-\Delta, \dots, \Delta\}^m$ be weight vectors with $\|W(A) - W(B)\|_1 \leq \mu$. Further, let $d \in \mathbb{Z}^m$. There exist unicolor sets $A' \subseteq A$, $B' \subseteq B$ of equal cardinality such that*

i) $A \setminus A' \cup B' \in \mathcal{I}$, *ii)* $|A'| = |B'| \geq \frac{k - \|d\|_1 \mu}{(2\|d\|_1\Delta+1)^2(2\Delta+1)^{2m}}$*, and*

iii) $d^{\mathsf{T}} W(a) \geq d^{\mathsf{T}} W(b)$ *for each* $a \in A'$ *and* $b \in B'$*.*

Proof. We apply Lemma 8 for the single-dimensional weight function

$$w(e) = d^{\mathsf{T}} W(e),\ e \in E.$$

Then $|w(A) - w(B)| \leq \|d\|_1 \cdot \mu$ and we obtain sets $A^{(1)} \subseteq A$ and $B^{(1)} \subseteq B$ of equal cardinality with

1) $A \setminus A^{(1)} \cup B^{(1)} \in \mathcal{I}$,

2) $|A^{(1)}| = |B^{(1)}| \geq (k - \|d\|_1\mu)/(2\|d\|_1\Delta + 1)^2$, and

3) $w(a) \geq w(b)$ for all $a \in A^{(1)}, b \in B^{(1)}$.

Note that there exists a $B^{(2)} \subseteq B^{(1)}$ with at least $|B^{(1)}|/(2\Delta + 1)^m$ elements e of the same weight $W(e)$. Via the downsizing (Lemma 4) we can restrict to $B^{(2)}$ and a corresponding subset of $A^{(2)} \subseteq A^{(1)}$ while still satisfying properties 1) and 3). Next, we observe that there exists $A^{(3)} \subseteq A^{(2)}$ with at least $|A^{(2)}|/(2\Delta + 1)^m$ elements e of the same weight $W(e)$. Downsizing again, we obtain two unicolor sets $A^{(2)}$, $B^{(2)}$ that satisfy properties 1) and 3) of cardinality

$$|A^{(2)}| = |B^{(2)}| \geq \frac{k - \|d\|_1\mu}{(2\|d\|_1\Delta + 1)^2 \cdot (2\Delta + 1)^{2m}}.$$

Thus, the sets $A^{(2)}, B^{(2)}$ satisfy all required properties. ☐

Lemma 13. *Let* $A, B \in \mathcal{I}$ *be disjoint with* $|A| = |B| = k$*, and* $\mu = \|W(A) - W(B)\|_1$*. Then there are sets*

$$A_0, A_1, \ldots, A_\ell \subseteq A \ \text{ and } B_0, B_1, \ldots, B_\ell \subseteq B, \tag{8}$$

all unicolor, such that

i) $A \setminus A_i \cup B_i \in \mathcal{I}$ *for all* i*,*

ii) $|A_i| = |B_i| \geq k/(2m\Delta)^{10m} - \mu$ *for all* i*,*

iii) $-\delta_0 \in \operatorname{cone}(\{\delta_1, \ldots, \delta_\ell\})$*, where* $\delta_i = W(a) - W(b)$ *for all* $a \in A_i, b \in B_i$*.*

Proof. We construct the sets iteratively. $A_0 \subseteq A$ and $B_0 \subseteq B$ can be created from Lemma 12 using $d = (0, 0, \ldots, 0)$.

Suppose we already have $A_0, \ldots, A_i$ and $B_0, \ldots, B_i$ and that these sets satisfy i) and ii). If $C = \operatorname{cone}(\{\delta_0, \ldots, \delta_i\})$ is flat then it contains some non-zero x, y with $x + y = 0$. In particular, there exists $\lambda \in \mathbb{R}^{i+1}_{\geq 0}$ with $\sum_{j=0}^{i} \lambda_j \delta_j = 0$ and $\lambda_k > 0$ for some $k \in \{0, 1, \ldots, i\}$. After swapping A_0, A_k and B_0, B_k in (8), the sequence of sets also satisfies iii).

Assume now that C is pointed. From Lemma 10 it follows that for some $d \in \mathbb{Z}^m$ with $\|d\|_\infty \leq (2\Delta)^m m^{m/2+1}$ the halfspace

$$H = \{x \in \mathbb{R} : d^{\mathsf{T}} x \geq 0\}$$

intersects C exactly in 0. By applying Lemma 12, we obtain unicolor sets A_{i+1}, B_{i+1} satisfying i) and

$$
\begin{aligned}
|A_{i+1}| = |B_{i+1}| &\geq \frac{k - \mu\|d\|_1}{(2\Delta\|d\|_1 + 1)^2(2\Delta + 1)^{2m}} \\
&\geq \frac{k}{(4\Delta\|d\|_1)^2(4\Delta)^{2m}} - \mu \\
&\geq \frac{k}{(4\Delta(2\Delta)^m m^{m/2+2})^2(4\Delta)^{2m}} - \mu \\
&\geq \frac{k}{(2\Delta m)^{10m}} - \mu
\end{aligned}
$$

thus also satisfying iii). Furthermore, if $\delta_{i+1} = 0$, then, after swapping A_0, A_{i+1} and B_0, B_{i+1} in (8), the sequence of sets also satisfies iii). Otherwise, $\delta_{i+1} \in H$ must be different from $\delta_1, \dots, \delta_i \notin H$. Since there are finitely many values for δ_i, the procedure will ultimately terminate and therefore eventually satisfy (iii). ☐

The proof of the multidimensional sensitivity theorem

As in the one-dimensional case, Theorem 1 reduces to the following statement by repeating the arguments from Section 5.1.

Lemma 14. *Let $A, B \in \mathcal{I}$ disjoint with $|A| = |B|$ and let $W : E \to \{-\Delta, \dots, \Delta\}^m$ be a multidimensional weight function with $\|W(A) - W(B)\|_1 \leq \mu$, where $\mu \in \mathbb{N}_+$. If*

$$|A| = |B| \geq (2m\Delta)^{12 \cdot m}\mu$$

then there exists $A' \in \mathcal{I}$, $A' \neq A$ with

$$W(A') = W(A) \quad \text{and} \quad |A'| = |A|.$$

Proof. From Lemma 13 we obtain unicolor sets A_i, B_i, $i = 0, 1, \dots, \ell$ with

1. $A \setminus A_i \cup B_i \in \mathcal{I}$,
2. $|A_i| = |B_i| \geq (2m\Delta)^{2m} - 1$,
3. $-\delta_0 \in \operatorname{cone}(\{\delta_1, \dots, \delta_\ell\})$, where $\delta_i = W(b) - W(a)$ for each $b \in B_i$ and $a \in A_i$.

Due to Lemma 11 we may assume that $\ell \leq m$ and there exists $\lambda \in \mathbb{Z}_{\geq 0}^{\ell+1} \setminus \{\mathbf{0}\}$ with $\sum_{i=0}^{\ell} \lambda_i \delta_i = 0$ and $\|\lambda\|_\infty \leq (2\Delta)^m m^{m/2}$. Apply downsizing (Lemma 4) to each A_i, B_i, to obtain an arbitrary $A'_i \subseteq A_i$, $|A'_i| = (\ell + 1)\lambda_i$ and a corresponding $B'_i \subseteq B_i$ with $|B'_i| = |A'_i|$ and $A \setminus A'_i \cup B_i \in \mathcal{I}$. We proceed as in the case with a single equality constraint and refer the reader to it for details. It holds that

$$y^* = \sum_{i=0}^{\ell} \frac{1}{\ell + 1} \chi(A \setminus A'_i \cup B'_i)$$

is in $P_B(M)$, satisfies $\sum_{e \in E} y^*_e = \operatorname{rank}(M)$, $Wy^* = W(A)$ and has an integral number of elements of each weight vector. Thus, y^* must be a convex combination of bases of M, all of which have weight $W(A)$ and since $y^* \neq \chi(A)$ not all of them can be equal to A. ☐

6 Applications

In this section we give specific examples of problems that can be cast as finding a basis of a matroid subject to m constraints, each of which have one of the following forms.

- Equality constraint: given $w : E \to \{-\Delta, \dots, \Delta\}$ and $\beta \in \mathbb{Z}$, require $w(B) = \beta$.
- Inequality constraint: given $w : E \to \{-\Delta, \dots, \Delta\}$ and $\beta \in \mathbb{Z}$, require $w(B) \leq \beta$, or alternatively $w(B) \geq \beta$.
- Congruence constraints: given $p \in \{1, 2, \dots, \Delta\}$, $w : E \to \{0, 1, \dots, p-1\}$, and $\beta \in \{0, 1, \dots, p-1\}$, require $w(B) \equiv \beta \mod p$.

While we proved our FPT algorithm only for the first type, it is easy to reduce the other two to it. This follows from standard constructions similar to slack variables. To this end, consider a matroid $M = (E, \mathcal{I})$. Suppose for a given weight function $w : E \to \{-\Delta, \dots, \Delta\}$ and $\beta \in \mathbb{Z}$, we are searching for a basis B with $w(B) \leq \beta$ and possibly other linear constraints (with one of the three types from above). We define M' as the direct sum of M and a rank n uniform matroid with $2n\Delta$ elements. The weight function w is extended by setting an arbitrary half of the elements in the uniform matroid to weight zero and the other half to weight 1. Any other linear constraint is extended with zero coefficients for the new elements, which means that they do not affect it. It follows easily that a basis B of the original matroid M can be extended to a basis B' of M' that satisfies $w(B') = \beta$ if and only if $w(B) \leq \beta$.

Now suppose we are given $p \in \{1, 2, \dots, \Delta\}$, $w : E \to \{0, 1, \dots, p-1\}$, and $\beta \in \{0, 1, \dots, p-1\}$ and are searching for a basis B with $w(B) \equiv \beta \mod p$ and possibly other constraints. Similar to before, we obtain M' as the direct sum of M with a uniform matroid of rank n over $2n$ elements. The weight function w is extended such that n many new elements have weight $-p$ and 0 each. Again, any other linear constraint is extended with zero coefficients for the new elements. Then a basis B of M is extendible to a basis B' of M' that satisfies $w(B') = \beta$ if and only if $w(B) \equiv \beta \mod p$.

Very recently, matroid problems with labels from an abelian group have gained some attention. Liu and Xu [30] study the following problem.

Group-Constrained Matroid Base **Input:** Matroid $M = (E, \mathcal{I})$, a labelling $\psi : E \to \Gamma$ for an abelian group $(\Gamma, \odot)$, and $g \in \Gamma$. **Task:** Find base B of M with $g = \psi(B) := \bigodot_{b \in B} \psi(b)$

Liu and Xu prove that if $\Gamma = \mathbb{Z}_m$ and m is either product of two primes or a prime power, then the problem can be solved in FPT time in m. Our result generalizes this to all finite abelian groups.

Corollary 15. *If $(\Gamma, \odot)$ is a finite abelian group then Group-Constrained Matroid Base can be solved in $f(m) \cdot n^{O(1)}$ time, for $m = |\Gamma|$.*

Proof. Every finite abelian group is isomorphic to the direct product of cyclic groups. Thus, we may assume without loss of generality that $\Gamma = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \dots \times \mathbb{Z}_{m_\ell}$, where $m = m_1 \cdot \ldots \cdot m_\ell$ for prime powers $m_1, m_2, \dots, m_\ell \leq m$. We can therefore model the problem using $\ell \leq \log_2 m$ congruency constraints with entries bounded by m. ☐

Similar to our results, Liu and Xu's techniques are based on proximity statements. However, their techniques rely on specific groups. More precisely, Liu and

Xu [30] prove that their techniques would work if a certain conjecture in additive combinatorics is true. The conjecture is proven when m is either product of two primes or a prime power, which allows them to obtain the result. In that matter, our techniques allow us to circumvent this issue.

Motivated by Liu and Xu's work, Hörsch et al. [21] considered the problem in the setting of non-finite groups. Among other problems, they show an FPT algorithm for the Group-Constrained Matroid Problem with $g = 0$ parameterized by $|\Gamma|$ in a special case when the matroid is GF(q)-representable for a prime power q. Similarly, to [30] their techniques also rely on an additive combinatorics result of Schrijver and Seymour [37], which prohibits their techniques from working in general finite abelian groups.

Budgeted matroid problems, in which one has to find an independent set subject to one or more budget constraints and possibly maximizing a profit function, have been studied with a great extend towards approximation schemes, see e.g. [19, 12, 13]. Finding bases is generally at least as hard as finding independent sets, since one can always fix the cardinality of the solution therefore reducing to bases. In the area of FPT algorithms, Marx [31] devised an algorithm for these type of problems, specifically motivated by the problem of finding a feedback edge set. Given a graph $G(V, E)$ a feedback edge set is a subset X of edges such that $G(V, E \setminus X)$ is acyclic.

Feedback Edge Set with Budget Vectors **Input:** A graph $G = (V, E)$, a vector $W(e) \in \mathbb{Z}^m_{\geq 0}$ for each $e \in E$, a budget $b \in \mathbb{Z}^m_{\geq 0}$. **Task:** Find a minimum cardinality feedback edge set X such that $W(X) \leq b$.

Note that for $m = 1$, this is a weighted variant of Feedback Edge Set which can be solved in polynomial time by a Greedy algorithm [31], however, for unbounded m and $\Delta = \|W\|_\infty$ the problem is NP-hard. Marx [31] presented a randomized FPT algorithm in the parameters m, Δ, and $|X|$. A direct application of our theorem is that a weaker parameterization that depends on just m and Δ suffices.

Corollary 16. *Feedback Edge Set with Budget Vectors can be solved in* $(m\Delta)^{O(m^2)} \cdot n^{O(1)}$ *randomized time.*

Proof. The problem is equivalent to finding a spanning forest F with $W(F) \geq W(E) - b$, which can be solved in the mentioned time using Theorem 3. ☐

Fairness considerations have inspired new variants of many problems, where elements belong to different groups and each group needs to be represented in the solution to some level [6, 8, 5, 3, 2, 23]. For example Abdulkadiroğlu and Sönmez [1] address the assignment of students to schools, where fairness constraints are selected to achieve racial, ethnic, and gender balance. For similar models see also [24, 35].

Matching with Group Fairness Constraints **Input:** A bipartite graph $G = (A \cup B, E)$, where each $b \in B$ belongs to a set of groups $G(b) \subseteq \{1, 2, \ldots, m\}$, and a quote $q_i \in \mathbb{Z}_{\geq 0}$ for each group $i = 1, 2, \ldots, m$. **Task:** Find a matching containing at least q_i many elements $b \in B$ with $i \in G(b)$ for $i = 1, 2, \ldots, m$.

To the best of our knowledge, FPT algorithms have not been considered for this problem before.

Corollary 17. *Matching with Group Fairness Constraints can be solved in* $(m\Delta)^{O(m^2)} \cdot n^{O(1)}$ *randomized time.*

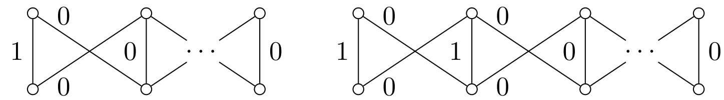

Figure 3: Example of high proximity and sensitivity in exact matroid intersection

Proof. Consider a transversal matroid defined on G with elements B. For each $b \in B$ let $W(b)$ be the indicator vector of $G(b)$ and $\boldsymbol{\beta} = q$. □

Partition matroids subject to equality constraints have been studied in the context of block structure integer linear programming, albeit without an explicit reference to matroids. Specifically, this variant was studied under the name *combinatorial n-fold* [27]. This has led to exponential improvements in the running time of FPT algorithms for problems in computational social choice, in string problems and more, see [27] for an overview. One concrete problem, that is captured by a partition matroid with equality constraints is the Closest String Problem. In this problem, one is given m strings, and the goal is to compute a string that minimizes the maximum Hamming distance to any of the input strings. Gramm et al. [18] design an FPT algorithm parameterized by m (see also [27]). Using our result, we can obtain an FPT algorithm for the following more general problem.

Closest Base
Input: A matroid $M = (E, I)$ with n elements and a subset of bases $S := \{B_1, \dots, B_m\}$.
Task: Find the basis B (not necessarily in S) that minimizes $\max_i |B \oplus B_i|$.

Corollary 18. *Closest Base on representable matroids can be solved in $m^{O(m^2)} n^{O(1)}$ randomized time.*

This running time matches the best known running time for the Closest String problem [27].

Proof. It is equivalent to minimize $H = \max_i |B \setminus B_i| = 1/2 \cdot \max_i |B \oplus B_i|$. We start by guessing H. Note that $H \leq |E|/2$, so only polynomially many guesses are required. The weight vector $W(e) \in \{0,1\}^m$ is defined as $W(e)_i = 0$ if $e \in B_i$ and 1 otherwise. Further, we define the target vector $\boldsymbol{\beta} = (H, H, \dots, H)$. We then use Theorem 3 to find a basis B with $W(B) \leq \boldsymbol{\beta}$. □

7 Lower bound for matroid intersection

In this section, we remark that low proximity and sensitivity do not generalize to matroid intersection, even for $m = 1$ and $\Delta = 1$. Our examples are based on matchings in a bipartite graph $G = (A \cup B, E)$. Although matching in a bipartite graph does not form an independent set in a single matroid, it can be represented as a matroid intersection of two partition matroids, where one partition matroid restricts the degree of each vertex in A to be at most one and the other does the same for B.

We start with the example of a high sensitivity for matroid intersection.

Theorem 19. *For infinitely many $n \in \mathbb{N}$ there exist matroids $M = (I, E), M' = (I', E)$ over the same set of elements and a weight function $w : E \to \{0, 1\}$, such that there exist exactly two two common bases B and B' of both matroids that satisfy:*

1. $w(B) = 0$ *and* $w(B') = 1$,

2. $B \cap B' = \emptyset$ *and* $|B| = |B'| = n/2$*.*

Proof. For n even, we create an instance of bipartite matching consisting of a cycle of length n that has a single edge of weight 1 and all others of weight 0, see also left cycle in Figure 3. There are exactly two perfect matchings, which correspond to the common bases of the two matroids and trivially satisfy the properties stated in the theorem. □

Next, we show an example of a high proximity for matroid intersection.

Theorem 20. *For infinitely many* $n \in \mathbb{N}$ *there exist matroids* $M = (I, E), M' = (I', E)$ *and a weight function* $w : E \to \{0, 1\}$ *with a vertex solution* x^* *to the continuous relaxation* $x \in P_B(M) \cap P_B(M'), w^\intercal x = 1$ *such that there is a unique common basis* B *of both* M *and* M' *with* $w(B) = 1$ *and* B *satisfies* $\|x^* - \chi(B)\|_1 = 3/4 \cdot n$*.*

Proof. For n a multiple of 4, consider an instance of bipartite matching similar to the proof of Theorem 19, but now on a graph with two disjoint cycles of length $n/2$ each, each of which we can think of as the union of two perfect matchings. The first cycle has a single edge of weight 1 and the second cycle has two edges of weight 1, which appear in the same perfect matching. All other edges have weight zero, see also Figure 3. Now suppose we want to find a perfect matching with total weight equal to 1. The only such perfect matching, i.e., the only common basis of both matroids, takes the perfect matching with the weight 1 edge in the first cycle and the all-zero perfect matching in the second cycle. This is also a vertex solution for the continuous relaxation, but there is a second vertex solution that takes each edge of the second cycle 1/2 times and the all-zero perfect matching of the first cycle. The distance between the two solutions is $3/4 \cdot n$. □

References

[1] Atila Abdulkadiroğlu and Tayfun Sönmez. School Choice: A Mechanism Design Approach. *American economic review*, 93(3):729–747, 2003.

[2] Georg Anegg, Haris Angelidakis, Adam Kurpisz, and Rico Zenklusen. A Technique for Obtaining True Approximations for k-Center with Covering Constraints. *Mathematical Programming*, pages 1–25, 2022.

[3] Sayan Bandyapadhyay, Tanmay Inamdar, Shreyas Pai, and Kasturi Varadarajan. A Constant Approximation for Colorful k-Center. In *Proceedings of ESA*, pages 12:1–12:14, 2019.

[4] Francisco Barahona and William R. Pulleyblank. Exact arborescences, matchings and cycles. *Discrete Applied Mathematics*, 16(2):91–99, 1987.

[5] Suman K. Bera, Shalmoli Gupta, Amit Kumar, and Sambuddha Roy. Approximation Algorithms for the Partition Vertex Cover Problem. *Theoretical Computer Science*, 555:2–8, 2014.

[6] Tolga Bolukbasi, Kai-Wei Chang, James Y. Zou, Venkatesh Saligrama, and Adam T. Kalai. Man is to Computer Programmer as Woman is to Homemaker? Debiasing Word Embeddings. *Proceedings of NIPS*, 29:4349–4357, 2016.

[7] Paolo M. Camerini, Giulia Galbiati, and Francesco Maffioli. Random Pseudo-Polynomial Algorithms for Exact Matroid Problems. *Journal of Algorithms*, 13(2):258–273, 1992.

[8] L. Elisa Celis, Damian Straszak, and Nisheeth K. Vishnoi. Ranking with Fairness Constraints. In *Proceedings of ICALP*, pages 28:1–28:15, 2018.

[9] W. Cook, A.M.H. Gerards, A. Schrijver, and É. Tardos. Sensitivity Theorems in Integer Linear Programming. *Mathematical Programming*, 34:251–264, 1986.

[10] Marek Cygan, Fedor V Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized algorithms*, volume 5. Springer, 2015.

[11] Ilan Doron-Arad, Ariel Kulik, and Hadas Shachnai. An EPTAS for Budgeted Matching and Budgeted Matroid Intersection via Representative Sets. In *Proceedings of ICALP*, pages 49:1–49:16, 2023.

[12] Ilan Doron-Arad, Ariel Kulik, and Hadas Shachnai. An EPTAS for Budgeted Matroid Independent Set. In *Proceedings of SOSA*, pages 69–83. SIAM, 2023.

[13] Ilan Doron-Arad, Ariel Kulik, and Hadas Shachnai. Budgeted Matroid Maximization: a Parameterized Viewpoint. In *Proceedings of IPEC*, pages 13:1–13:17, 2023.

[14] Ilan Doron-Arad, Ariel Kulik, and Hadas Shachnai. Lower Bounds for Matroid Optimization Problems with a Linear Constraint. In *Proceedings of ICALP*, pages 56:1–56:20, 2024.

[15] Jack Edmonds. Submodular Functions, Matroids, and Certain Polyhedra. In *Combinatorial structures and their applications*. Gordon and Breach, 1970.

[16] Jack Edmonds. Matroid intersection. In *Annals of discrete Mathematics*, volume 4, pages 39–49. Elsevier, 1979.

[17] Friedrich Eisenbrand and Robert Weismantel. Proximity Results and Faster Algorithms for Integer Programming Using the Steinitz Lemma. *ACM Transactions on Algorithms (TALG)*, 16(1):1–14, 2019.

[18] Jens Gramm, Rolf Niedermeier, and Peter Rossmanith. Fixed-Parameter Algorithms for Closest String and Related Problems. *Algorithmica*, 37:25–42, 2003.

[19] Fabrizio Grandoni and Rico Zenklusen. Approximation Schemes for Multi-Budgeted Independence Systems. In *Proceedings of ESA*, pages 536–548, 2010.

[20] Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2. Springer Science & Business Media, 2012.

[21] Florian Hörsch, András Imolay, Ryuhei Mizutani, Taihei Oki, and Tamás Schwarcz. Problems on group-labeled matroid bases. In *Proceedings of ICALP*, pages 86:1–86:20, 2024.

[22] David Rios Insua and Simon French. A Framework for Sensitivity Analysis in Discrete Multi-Objective Decision-Making. *European journal of operational research*, 54(2):176–190, 1991.

[23] Xinrui Jia, Kshiteej Sheth, and Ola Svensson. Fair Colorful k-Center Clustering. *Mathematical Programming*, 192(1):339–360, 2022.

[24] Yuichiro Kamada and Fuhito Kojima. Stability and Strategy-Proofness for Matching with Constraints: A Problem in the Japanese Medical Match and Its Solution. *American Economic Review*, 102(3):366–370, 2012.

[25] Ravindran Kannan. Polynomial-Time Aggregation of Integer Programming Problems. *Journal of the ACM (JACM)*, 30(1):133–145, 1983.

[26] Leonid G. Khachiyan. Polynomial Algorithms in Linear Programming. *USSR Computational Mathematics and Mathematical Physics*, 20(1):53–72, 1980.

[27] Dušan Knop, Martin Koutecký, and Matthias Mnich. Combinatorial n-fold integer programming and applications. *Mathematical Programming*, 184(1):1–34, 2020.

[28] Ariel Kulik and Hadas Shachnai. There is No EPTAS for Two-dimensional Knapsack. *Information Processing Letters*, 110(16):707–710, 2010.

[29] Eugene L. Lawler. Matroid intersection algorithms. *Mathematical programming*, 9(1):31–56, 1975.

[30] Siyue Liu and Chao Xu. On the congruency-constrained matroid base. In *Proceedings of IPCO*, pages 280–293, 2024.

[31] Dániel Marx. A parameterized view on matroid optimization problems. *Theoretical Computer Science*, 410(44):4471–4479, 2009.

[32] Ketan Mulmuley, Umesh V. Vazirani, and Vijay V. Vazirani. Matching is as easy as matrix inversion. *Combinatorica*, 7(1):105–113, 1987.

[33] Christos H. Papadimitriou. On the Complexity of Integer Programming. *Journal of the ACM (JACM)*, 28(4):765–768, 1981.

[34] Christos H. Papadimitriou and Mihalis Yannakakis. The Complexity of Restricted Spanning Tree Problems. *Journal of the ACM (JACM)*, 29(2):285–309, 1982.

[35] G.S. Sankar, A. Louis, M. Nasre, and P. Nimbhorkar. Matchings with Group Fairness Constraints: Online and Offline Algorithms. In *Proceedings of IJCAI*, pages 377–383, 2021.

[36] Alexander Schrijver. *Combinatorial Optimization: Polyhedra and Efficiency*, volume 24. Springer, 2003.

[37] Alexander Schrijver and Paul D. Seymour. Spanning Trees of Different Weights. *Polyhedral combinatorics*, 1:281–288, 1990.

[38] Evangelos Triantaphyllou and Alfonso Sánchez. A Sensitivity Analysis Approach for some Deterministic Multi-Criteria Decision-Making Methods. *Decision sciences*, 28(1):151–194, 1997.