# Gradient-like observer design on the Special Euclidean group SE(3) with system outputs on the real projective space

Minh-Duc Hua, Tarek Hamel, Robert Mahony, Jochen Trumpf

Abstract—A nonlinear observer on the Special Euclidean group SE(3) for full pose estimation, that takes the system outputs on the real projective space directly as inputs, is proposed. The observer derivation is based on a recent advanced theory on nonlinear observer design. A key advantage with respect to existing pose observers on SE(3) is that we can now incorporate in a unique observer different types of measurements such as vectorial measurements of known inertial vectors and position measurements of known feature points. The proposed observer is extended allowing for the compensation of unknown constant bias present in the velocity measurements. Rigorous stability analyses are equally provided. Excellent performance of the proposed observers are shown by means of simulations.

#### I. INTRODUCTION

The development of a robust and reliable estimator of the pose (i.e. position and attitude) of a rigid body is a key requirement for robust and high performance control of robotic vehicles. Pose estimation is a highly nonlinear problem in which the sensors normally utilized are prone to non-Gaussian noise [2]. Classical approaches for state estimation are based on nonlinear filtering techniques such as extended Kalman filters, unscented Kalman filters or particle filters. However, nonlinear observers have become an alternative to these classical techniques, starting with the work of Salcudean [23] for attitude estimation and subsequent contributions over the last two decades [3], [7]–[12], [17], [19]–[22], [24]–[28]. Early nonlinear attitude observers have been developed on the basis of Lyapunov analysis. Recently, the attitude estimation problem has motivated the development of theories on invariant observers for systems endowed with symmetry properties [1], [5], [6], [15]–[18], [26]. For instance, complementary nonlinear attitude observers exploiting the underlying Lie group structure of the Special Orthogonal group SO(3) are derived in [17] with proofs of almost global stability of the error system. A symmetry-preserving nonlinear observer design based on the Cartan moving-frame method is proposed in [5], [6], which is locally valid for arbitrary Lie groups. A gradient-like observer design technique for invariant systems on Lie groups is proposed in [16], leading to almost global convergence provided that a non-degenerate Morse-Bott cost function is used. More recently, an observer design method directly on

the homogeneous output space for the kinematics of mechanical systems is proposed in [18], leading to autonomous error evolution and strong convergence properties. Finally, [15] extends the observer design methodology proposed in [18] in order to deal with the case where the measurement of system input is corrupted by an unknown constant bias.

Full pose observer design, although less studied than attitude observer design, has recently attracted more attention [2]–[4], [14], [21], [27], [28]. For instance, observers designed directly on SE(3) have been proposed using both full state feedback [4] or bearing measurements of known landmarks [3]. An observer on  $SO(3) \times \mathbb{R}^3$  is proposed in [27], using full range and bearing measurements of known landmarks and achieving almost global asymptotic stability. In a prior work by the authors [14], a nonlinear observer on SE(3) is proposed using directly position measurements in the body-fixed frame of known inertial feature points or landmarks, with motivation strongly related to robotic vision applications using either stereo camera or Kinect sensor. The observer derivation is based on the gradient-like observer design technique proposed in [16], and the almost global asymptotic stability of the error system is proved by means of Lyapunov analysis.

In this paper, we consider the question of deriving a nonlinear observer on SE(3) for full pose estimation that takes the system outputs on the real projective space  $\mathbb{RP}^3$ directly as inputs. A key advance on our prior work [14] is the possibility of incorporating "naturally" in a sole observer both vectorial measurements (provided e.g. by magnetometers or inclinometers) and position measurements of known inertial feature points (provided e.g. by stereo camera). In addition, sharing the same robustness property with the observer proposed in [14], the algorithm here proposed is also well-posed even when there is insufficient data for full pose reconstruction using algebraic techniques. In such situations, the proposed observer continues to operate, incorporating what information is available and relying on propagation of prior estimates where necessary. Finally, as a complementary contribution, a modified version of the basic observer is proposed so as to deal with the case where bias is present in the velocity measurements.

The remainder of this paper is organised as follows. Section II formally introduces the problem of pose estimation on SE(3) along with the notation used. In Section III, based on a recent advanced theory for nonlinear observer design directly on the output space [18], a nonlinear observer on SE(3) is proposed using direct body-fixed measurements of known inertial elements of the real projective space  $\mathbb{RP}^3$  and

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the knowledge of the group velocity. Stability analysis is also provided in this section. Then, in Section IV the proposed basic observer is extended using Lyapunov theory in order to cope with the case where the measurement data of the group velocity are corrupted by an unknown constant bias. In Section V, the performance of the proposed observers are validated by means of simulation. Finally, concluding remarks are given in Section VI.

#### II. PRELIMINARY MATERIAL

# A. Notation

Let  $\{\mathcal{A}\}\$  and  $\{\mathcal{B}\}\$  denote an inertial frame and a bodyfixed frame attached to a vehicle moving in 3D-space, respectively. The vehicle's position, expressed in the frame  $\{\mathcal{A}\}\$ , is denoted as  $p \in \mathbb{R}^3$ . The attitude of the vehicle is represented by a rotation matrix  $R \in SO(3)$  of the frame  $\{\mathcal{B}\}\$  relative to the frame  $\{\mathcal{A}\}\$ . Let  $V \in \mathbb{R}^3$  and  $\Omega \in \mathbb{R}^3$ denote the vehicle's translational and angular velocities, both expressed in  $\{\mathcal{B}\}\$ .

In this paper, we consider the problem of estimating the vehicle's pose, which can be represented by an element of the Special Euclidean group SE(3) given by the matrix

$$X := \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \operatorname{SE}(3) \subset \mathbb{R}^{4 \times 4}.$$
 (1)

This representation, known as homogeneous coordinates, preserves the group structure of SE(3) with the GL(4) operation of matrix multiplication, i.e.  $X_1X_2 \in SE(3)$ ,  $\forall X_1, X_2 \in SE(3)$ . Now let us recall some common definitions and notation.

• The Lie-algebra  $\mathfrak{se}(3)$  of the group SE(3) is defined as

$$\mathfrak{se}(3) := \left\{ A \in \mathbb{R}^{4 \times 4} \mid \exists \, \Omega, V \in \mathbb{R}^3 \, : \, A = \begin{bmatrix} \Omega_{\times} & V \\ 0 & 0 \end{bmatrix} \right\},\$$

with  $\Omega_{\times}$  denoting the skew-symmetric matrix associated with the cross product by  $\Omega$ , i.e.  $\Omega_{\times}v = \Omega \times v$ ,  $\forall v \in \mathbb{R}^3$ . The adjoint operator is a mapping  $Ad : SE(3) \times \mathfrak{se}(3) \to \mathfrak{se}(3)$ defined as  $Ad_XA := XAX^{-1}$ , with  $X \in SE(3), A \in \mathfrak{se}(3)$ . • For any two matrices  $M_1, M_2 \in \mathbb{R}^{n \times n}$ , the Euclidean matrix inner product and Frobenius norm are defined as

$$\langle M_1, M_2 \rangle := \operatorname{tr}(M_1^{\top} M_2), \ \|M_1\| := \sqrt{\langle M_1, M_1 \rangle}$$

Let  $\mathbf{P}_a(M)$ ,  $\forall M \in \mathbb{R}^{n \times n}$ , denote the anti-symmetric part of M, i.e.  $\mathbf{P}_a(M) := (M - M^{\top})/2$ . Let  $\mathbf{P} : \mathbb{R}^{4 \times 4} \to \mathfrak{se}(3)$  denote the unique orthogonal projection of  $\mathbb{R}^{4 \times 4}$  onto  $\mathfrak{se}(3)$  with respect to the inner product  $\langle \cdot, \cdot \rangle$ , i.e.  $\forall A \in \mathfrak{se}(3)$ ,  $M \in \mathbb{R}^{4 \times 4}$ , one has

$$\langle A, M \rangle = \langle A, \mathbf{P}(M) \rangle = \langle \mathbf{P}(M), A \rangle.$$

It is verified that for all  $M_1 \in \mathbb{R}^{3 \times 3}, m_{2,3} \in \mathbb{R}^3, m_4 \in \mathbb{R}$ ,

$$\mathbf{P}\left(\begin{bmatrix}M_1 & m_2\\m_3^\top & m_4\end{bmatrix}\right) := \begin{bmatrix}\mathbf{P}_a(M_1) & m_2\\0 & 0\end{bmatrix}.$$
 (2)

• For all  $X \in SE(3)$ ,  $A_1, A_2 \in \mathfrak{se}(3)$ , the following equation defines a *right-invariant Riemannian metric*  $\langle \cdot, \cdot \rangle_X$ :

$$\langle A_1 X, A_2 X \rangle_X := \langle A_1, A_2 \rangle$$

• For any  $x \in \mathbb{R}^4$  (or  $\in \mathbb{RP}^3$ ), the notation  $\underline{x} \in \mathbb{R}^3$  denotes the vector of first three components of x and the notation  $x_i$ stands for the i-th component of x. Thus, it can be written as  $x = [\underline{x} \quad x_4]^{\mathsf{T}}$ .

# B. System equations and measurements

The vehicle's pose  $X \in SE(3)$ , defined by (1), satisfies the kinematic equation

$$X = F(X, A) := XA,$$
(3)

with group velocity  $A \in \mathfrak{se}(3)$ . System (3) is *left invariant* in the sense that it preserves the (Lie group) invariance properties with respect to constant translation and constant rotation of the body-fixed frame  $\{\mathcal{B}\} X \mapsto X_0 X$ .

Assume that the group velocity A (i.e.  $\Omega$  and V) is bounded, continuous, and available to measurement. Moreover,  $N \in \mathbb{N}^+$  constant elements of the real projective space  $\hat{y}_i \in \mathbb{RP}^3$   $(i = 1, \dots, N)$ , known in the inertial frame  $\{A\}$ , are assumed to be measured in the body-fixed frame  $\{B\}$  as

$$y_i = h(X, \mathring{y}_i) := \frac{X^{-1} \mathring{y}_i}{|X^{-1} \mathring{y}_i|} \in \mathbb{RP}^3, \quad i = 1, \cdots, N.$$
 (4)

Note that the Lie group action  $h : SE(3) \times \mathbb{RP}^3 \to \mathbb{RP}^3$  is a *right group action* in the sense that for all  $X_1, X_2 \in SE(3)$  and  $y \in \mathbb{RP}^3$ , one has  $h(X_2, h(X_1, y)) = h(X_1X_2, y)$ . For later use, define

$$Y := (y_1, \cdots, y_N), \quad \dot{Y} := (\dot{y}_1, \cdots, \dot{y}_N).$$
 (5)

**Remark 1** Interestingly, by considering the measurement data in the real projective space  $\mathbb{RP}^3$ , we are able to combine in a sole pose observer various types of measurements coming from sensors of different nature. For instance, from a stereo camera or a Kinect sensor we can obtain a matching of  $N_1 \in \mathbb{N}^+$  feature points whose position coordinates are known in both the inertial reference frame  $\{\mathcal{A}\}$  and the current body-fixed frame  $\{\mathcal{B}\}$ , i.e. one has

$$p_i = R^{+}(\mathring{p}_i - p), \quad i = 1, \cdots, N_1,$$

with  $\mathring{p}_i, p_i \in \mathbb{R}^3$  the position coordinates of the feature points expressed in the frames  $\{A\}$  and  $\{B\}$ , respectively. Then, the following simple transformations:

$$\begin{split} \underline{\mathring{y}}_i &:= \frac{\mathring{p}_i}{\sqrt{|\mathring{p}_i|^2 + 1}}, \quad \mathring{y}_{i,4} &:= \frac{1}{\sqrt{|\mathring{p}_i|^2 + 1}}, \\ \underline{y}_i &:= \frac{p_i}{\sqrt{|p_i|^2 + 1}}, \quad y_{i,4} &:= \frac{1}{\sqrt{|p_i|^2 + 1}}, \end{split}$$

yield the following relations in the form (4):

$$y_i = \frac{X^{-1} \mathring{y}_i}{|X^{-1} \mathring{y}_i|} = h(X, \mathring{y}_i), \quad i = 1, \cdots, N_1,$$

with  $\mathring{y}_i = [\mathring{y}_i \quad \mathring{y}_{i,4}]^\top \in \mathbb{RP}^3$  and  $y_i = [\underbrace{y}_i \quad y_{i,4}]^\top \in \mathbb{RP}^3$ . Such a transformation provides a "natural" scaling of the position measurements of known inertial feature points so that the measurement of a very far feature point will act closely to a vectorial measurement. On the other hand, assume also that the vehicle is equipped with  $N_2 \in \mathbb{N}^+$  vectorial sensors (e.g. magnetometer or inclinometer) so as to provide the measurements  $v_j \in \mathbb{R}^3$  in the body-fixed frame  $\{\mathcal{B}\}$  of  $N_2$  Euclidean vectors (given for example by the geomagnetic field or the gravity field) whose coordinates  $\mathring{v}_j \in \mathbb{R}^3$  in the inertial frame  $\{\mathcal{A}\}$  are known. Then, one verifies that  $v_j = R^{\top} \mathring{v}_j$  and deduces the following relations in the form (4):

$$y_j = \frac{X^{-1} \mathring{y}_j}{|X^{-1} \mathring{y}_j|} = h(X, \mathring{y}_j), \quad j = N_1 + 1, \cdots, N_1 + N_2,$$
  
with  $\mathring{y}_j = [\mathring{y}_j \quad 0]^\top \in \mathbb{RP}^3, \quad y_j = [\underbrace{y}_j \quad 0]^\top \in \mathbb{RP}^3, \quad \mathring{y}_j := \frac{\mathring{y}_j}{|\mathring{v}_j|} \text{ and } \underbrace{y}_j := \frac{v_j}{|v_j|}.$ 

We verify that SE(3) is a symmetry group with group actions  $\phi : SE(3) \times SE(3) \longrightarrow SE(3), \psi : SE(3) \times \mathfrak{se}(3) \longrightarrow$  $\mathfrak{se}(3)$  and  $\rho : SE(3) \times \mathbb{RP}^3 \longrightarrow \mathbb{RP}^3$  defined by

$$\begin{array}{llll} \phi(Q,X) &:= & XQ, \\ \psi(Q,A) &:= & Ad_{Q^{-1}}A = Q^{-1}AQ, \\ \rho(Q,y) &:= & \frac{Q^{-1}y}{|Q^{-1}y|}. \end{array}$$

Indeed, it is straightforward to verify that  $\phi$ ,  $\psi$ , and  $\rho$  are right group actions in the sense that  $\phi(Q_2, \phi(Q_1, X)) = \phi(Q_1Q_2, X), \quad \psi(Q_2, \psi(Q_1, A)) = \psi(Q_1Q_2, A), \text{ and } \rho(Q_2, \rho(Q_1, y)) = \rho(Q_1Q_2, y), \text{ for all } Q_1, Q_2, X \in \text{SE}(3), A \in \mathfrak{se}(3), \text{ and } y \in \mathbb{RP}^3.$  Clearly, one has

$$\rho(Q, h(X, \mathring{y}_i)) = \frac{Q^{-1} \frac{X^{-1} \mathring{y}_i}{|X^{-1} \mathring{y}_i|}}{|Q^{-1} \frac{X^{-1} y_i}{|X^{-1} \mathring{y}_i|}|} = h(\phi(Q, X), \mathring{y}_i),$$
  
$$d\phi_Q(X)[F(X, A)] = XAQ = (XQ)(Q^{-1}AQ)$$
  
$$= F(\phi(Q, X), \psi(Q, A)).$$

Thus, the kinematics (3) are *right equivariant* in the sense of [18, Def. 2]. This is a condition allowing us to apply the theory proposed in [18] for nonlinear observer design directly on the output space. Note also that the system under consideration belongs to type I systems (see [18]) where both the velocity sensors and the state sensors are attached to the body-fixed frame.

# III. GRADIENT-LIKE OBSERVER DESIGN

Denote by  $\hat{X}(t) \in SE(3)$  the estimate of the pose X(t)and denote by  $\hat{R}$  and  $\hat{p}$  the estimates of R and p, respectively. One has  $\hat{X} = \begin{bmatrix} \hat{R} & \hat{p} \\ 0 & 1 \end{bmatrix}$ . Define the group error

$$E(\hat{X}, X) := \hat{X} X^{-1} \in SE(3),$$
 (6)

which is *right invariant* in the sense that for all  $X, X, Q \in$  SE(3), one has  $E(\hat{X}Q, XQ) = E(\hat{X}, X)$ . From now on, without confusion the shortened notation E is used for  $E(\hat{X}, X)$ . The group error E converges to the identity element  $I_4 \in$  SE(3) iif  $\hat{X}$  converges to X. For later use, define also the output errors  $e_i \in \mathbb{RP}^3$ , with  $i = 1, \dots, N$ , as

$$e_i := h(\hat{X}^{-1}, y_i) = \frac{Xy_i}{|\hat{X}y_i|} = \frac{E\dot{y}_i}{|E\dot{y}_i|}.$$
 (7)

Note that  $e_i$   $(i = 1, \dots, N)$  can be viewed as the estimates of  $\mathring{y}_i$ , since they converge to  $\mathring{y}_i$  when E converges to  $I_4$ . Note also that  $e_i$  are computable by the observer.

We now proceed the observer design. As proposed by [18], the observer takes the form

$$\hat{X} = \hat{X}A - \Delta(\hat{X}, Y)\hat{X}, \quad \hat{X}(0) \in \text{SE}(3), \tag{8}$$

where  $\Delta(\hat{X}, Y) \in \mathfrak{se}(3)$ , which is a matrix-valued function of  $\hat{X}$  and Y with Y defined by (5), is the innovation term to be designed hereafter and must be *right equivariant* in the sense that  $\forall Q \in SE(3)$ :

$$\Delta(\phi(Q, \hat{X}), \rho(Q, Y)) = \Delta(\hat{X}, Y),$$

with  $\rho(Q, Y) := (\rho(Q, y_1), \dots, \rho(Q, y_N))$ . Interestingly, if the innovation term  $\Delta(\hat{X}, Y)$  is right equivariant, the dynamics of the group error E are autonomous [18, Th. 1]:

$$\dot{E} = -\Delta(E, \mathring{Y})E.$$
(9)

In order to determine the innovation term  $\Delta(\hat{X}, Y)$ , the following cost function is considered:

$$\mathcal{C} \colon \operatorname{SE}(3) \times (\mathbb{RP}^3 \times \cdots \times \mathbb{RP}^3) \longrightarrow \mathbb{R}^+, \\ (\hat{X}, Y) \quad \mapsto \mathcal{C}(\hat{X}, Y) := \sum_{i=1}^N \frac{k_i}{2} \left| \frac{\hat{X}y_i}{|\hat{X}y_i|} - \mathring{y}_i \right|^2$$
(10)

with positive constant parameters  $k_i$ . It is easily verified that the cost function  $C(\hat{X}, Y)$  is *right invariant* in the sense that  $C(\phi(Q, \hat{X}), \rho(Q, Y)) = C(\hat{X}, Y)$  for all  $Q \in SE(3)$ . From here, the innovation term  $\Delta(\hat{X}, Y)$  is computed as [18, Eq. (40)]:

$$\Delta(\hat{X}, Y) := (\operatorname{grad}_1 \mathcal{C}(\hat{X}, Y)) \hat{X}^{-1}, \tag{11}$$

where  $\text{grad}_1$  is the *gradient* in the first variable, using a right-invariant Riemannian metric on SE(3).

**Lemma 1** The innovation term  $\Delta(\hat{X}, Y)$  defined by (11) is right equivariant and explicitly given by

$$\Delta(\hat{X}, Y) = -\mathbf{P}\left(\sum_{i=1}^{N} k_i \left(I_4 - e_i e_i^{\top}\right) \mathring{y}_i e_i^{\top}\right), \qquad (12)$$

with  $e_i$  considered as functions of  $\hat{X}$  and  $y_i$ , i.e.  $e_i = \frac{\hat{X}y_i}{|\hat{X}y_i|}$ .

**Proof:** The proof for  $\Delta(\hat{X}, Y)$  given by (12) to be right equivariant is straightforward. Now, using standard rules for transformations of Riemannian gradients and the fact that the Riemannian metric is right invariant, one obtains

$$\mathcal{D}_{1}\mathcal{C}(\hat{X},Y)[U\hat{X}] = \langle \operatorname{grad}_{1}\mathcal{C}(\hat{X},Y), U\hat{X} \rangle_{X}$$
  
=  $\langle \operatorname{grad}_{1}\mathcal{C}(\hat{X},Y)\hat{X}^{-1}\hat{X}, U\hat{X} \rangle_{X}$   
=  $\langle \operatorname{grad}_{1}\mathcal{C}(\hat{X},Y)\hat{X}^{-1}, U \rangle$   
=  $\langle \Delta(\hat{X},Y), U \rangle,$  (13)

with some  $U \in \mathfrak{se}(3)$ . On the other hand, using (10) one deduces

$$\mathcal{D}_{1}\mathcal{C}(X,Y)[UX] = d_{1}\mathcal{C}(X,Y)[UX]$$

$$= \sum_{i=1}^{N} k_{i} \left(\frac{\hat{X}y_{i}}{|\hat{X}y_{i}|} - \mathring{y}_{i}\right)^{\mathsf{T}} \left(I_{4} - \frac{(\hat{X}y_{i})(\hat{X}y_{i})^{\mathsf{T}}}{|\hat{X}y_{i}|^{2}}\right) \frac{(U\hat{X})y_{i}}{|\hat{X}y_{i}|}$$

$$= \sum_{i=1}^{N} k_{i}(e_{i} - \mathring{y}_{i})^{\mathsf{T}} (I_{4} - e_{i}e_{i}^{\mathsf{T}})Ue_{i}$$

$$= \operatorname{tr} \left(\sum_{i=1}^{N} k_{i}(I_{4} - e_{i}e_{i}^{\mathsf{T}})(e_{i} - \mathring{y}_{i})e_{i}^{\mathsf{T}}U^{\mathsf{T}}\right)$$

$$= \left\langle -\sum_{i=1}^{N} k_{i}(I_{4} - e_{i}e_{i}^{\mathsf{T}})\mathring{y}_{i}e_{i}^{\mathsf{T}}, U \right\rangle$$

$$= \left\langle -\mathbf{P} \left(\sum_{i=1}^{N} k_{i}(I_{4} - e_{i}e_{i}^{\mathsf{T}})\mathring{y}_{i}e_{i}^{\mathsf{T}}\right), U \right\rangle.$$
(14)

Finally, the expression of  $\Delta(\hat{X}, Y)$  given by (12) is directly obtained from (13) and (14).

Using the definition (2) of the projection  $\mathbf{P}(\cdot)$ , the innovation term  $\Delta(\hat{X}, Y)$  given by (12) can be rewritten in matrix form as follows:

$$\Delta(\hat{X}, Y) = \begin{bmatrix} -\frac{1}{2} \sum_{i=1}^{N} k_i (\underline{e}_i \times \underline{\mathring{y}}_i)_{\times} & \sum_{i=1}^{N} k_i e_{i,4} ((e_i^{\top} \underline{\mathring{y}}_i) \underline{e}_i - \underline{\mathring{y}}_i) \\ 0 & 0 \end{bmatrix}$$
(15)

Using (9), (11) and (12), one deduces the error system

$$\dot{E} = -\operatorname{grad}_{1}\mathcal{C}(E, \mathring{Y})$$
$$= \mathbf{P}\left(\sum_{i=1}^{N} k_{i} \left(I_{4} - e_{i}e_{i}^{\top}\right) \mathring{y}_{i}e_{i}^{\top}\right) E$$
(16)

with  $e_i$  considered as functions of E and  $\mathring{y}_i$ , i.e.  $e_i = \frac{E\mathring{y}_i}{|E\mathring{y}_i|}$ 

For the sake of analysis purposes, the following assumption is introduced.

**Assumption 1** (Observability) The set  $\{ \mathring{y}_i \in \mathbb{RP}^3, i = 1, \cdots, N \}$  satisfies one of the three following cases:

- Case 1 (at least 2 vectorial and 1 position measurements): There exist two different points ŷ<sub>i1</sub> and ŷ<sub>i2</sub> with ŷ<sub>i1,4</sub> = ŷ<sub>i2,4</sub> = 0 and one point ŷ<sub>j1</sub> such that ŷ<sub>j1,4</sub> ≠ 0.
- Case 2 (at least 1 vectorial and 2 position measurements): There exist one point  $\mathring{y}_{i_1}$  with  $\mathring{y}_{i_1,4} = 0$  and two different points  $\mathring{y}_{j_1}$  and  $\mathring{y}_{j_2}$  (i.e.,  $\mathring{y}_{j_1} \neq \mathring{y}_{j_2}$ ) with  $\mathring{y}_{j_1,4} \neq 0$  and  $\mathring{y}_{j_2,4} \neq 0$ . Furthermore, the vector  $\mathring{y}_{i_1}$ and the resultant vector  $v_{j_{12}} := \mathring{y}_{j_2,4} \mathring{y}_{j_1} - \mathring{y}_{j_1,4} \mathring{y}_{j_2}$  are non-collinear.
- Case 3 (at least 3 position measurements): There exist three different points  $\mathring{y}_{j_1}$ ,  $\mathring{y}_{j_2}$  and  $\mathring{y}_{j_3}$  such that  $\mathring{y}_{j_1,4} \neq 0$ ,  $\mathring{y}_{j_2,4} \neq 0$  and  $\mathring{y}_{j_3,4} \neq 0$ . Furthermore, the resultant vectors  $v_{j_{12}} := \mathring{y}_{j_2,4} \mathring{y}_{j_1} - \mathring{y}_{j_1,4} \mathring{y}_{j_2}$ ,  $v_{j_{23}} := \mathring{y}_{j_3,4} \mathring{y}_{j_2} - \mathring{y}_{j_2,4} \mathring{y}_{j_3}$  and  $v_{j_{31}} := \mathring{y}_{j_1,4} \mathring{y}_{j_3} - \mathring{y}_{j_3,4} \mathring{y}_{j_1}$  are not all collinear.

From here, the first result of this paper is stated.

**Theorem 1** Consider the kinematics (3). Consider the observer (8) with the innovation term  $\Delta(\hat{X}, Y)$  given by (12).

Assume that Assumption 1 is satisfied. Then, the equilibrium  $E = I_4$  of the error system (16) is locally asymptotically stable.

**Proof:** Since the right-hand side of (16) is a gradient flow of C, in order to prove the local asymptotic stability of  $E = I_4$ , it suffices to prove that C(E, Y) is minimal when  $E = I_4$ . Note that

$$\mathcal{C}(E, \mathring{Y}) = \mathcal{V}(E) := \frac{1}{2} \sum_{i=1}^{N} k_i \left| \frac{E \mathring{y}_i}{|E \mathring{y}_i|} - \mathring{y}_i \right|^2.$$
(17)

Let us prove that the function  $\mathcal{V}(E)$  has a unique global minimum at  $E = I_4$ , i.e.  $\mathcal{V}(E) = 0 \Leftrightarrow E = I_4$ .

First, it is straightforward to verify that  $\mathcal{V}(I_4) = 0$ . Denote  $E = \begin{bmatrix} R_e & p_e \\ 0 & 1 \end{bmatrix}$ , with  $R_e \in \mathrm{SO}(3), p_e \in \mathbb{R}^3$ . Now assuming that  $\mathcal{V}(E) = 0$ , we only have to prove that  $E = I_4$  or, equivalently,  $R_e = I_3$  and  $p_e = 0$ . In view of (17) and  $\mathcal{V}(E) = 0$ , one deduces that  $E\mathring{y}_i = |E\mathring{y}_i|\mathring{y}_i, \forall i$ , i.e.

$$\begin{cases} R_e \underline{\mathring{y}}_i + p_e \mathring{y}_{i,4} = \sqrt{\mathring{y}_{i,4}^2 + |R_e \underline{\mathring{y}}_i + p_e \mathring{y}_{i,4}|^2} \ \underline{\mathring{y}}_i & (18a) \\ \\ \mathring{y}_{i,4} = \sqrt{\mathring{y}_{i,4}^2 + |R_e \underline{\mathring{y}}_i + p_e \mathring{y}_{i,4}|^2} \ \mathring{y}_{i,4} & (18b) \end{cases}$$

Let us consider all the three cases of Assumption 1.

• Case 1 of Assumption 1: Since  $\mathring{y}_{i_1,4} = \mathring{y}_{i_2,4} = 0$ , one has  $|\mathring{\underline{y}}_{i_1}| = |\mathring{\underline{y}}_{i_2}| = 1$ . Then, one deduces from (18a) that  $R_e \mathring{\underline{y}}_{i_1} = \mathring{\underline{y}}_{i_1}$  and  $R_e \mathring{\underline{y}}_{i_2} = \mathring{\underline{y}}_{i_2}$ . These equalities and the non-collinearity of  $\mathring{\underline{y}}_{i_1}$  and  $\mathring{\underline{y}}_{i_2}$  allows one to deduce that  $R_e = I_3$ . Since  $\mathring{y}_{j_1,4} \neq 0$ , (18b) implies that  $|E\mathring{y}_{j_1}| = 1$ . As a consequence, one deduces from (18a) that  $p_e = 0$ .

• Case 2 of Assumption 1: Analogously to case 1, one deduces that  $R_e \underline{\mathring{y}}_{i_1} = \underline{\mathring{y}}_{i_1}$ . Now, since  $\mathring{y}_{j_1,4} \neq 0$  and  $\mathring{y}_{j_2,4} \neq 0$ , (18b) implies that  $|E\mathring{y}_{j_1}| = |E\mathring{y}_{j_2}| = 1$ . Then, from (18a) one obtains

$$\begin{cases} (R_e - I_3)\underline{\mathring{y}}_{j_1} + p_e \mathring{y}_{j_1,4} = 0\\ (R_e - I_3)\underline{\mathring{y}}_{j_2} + p_e \mathring{y}_{j_2,4} = 0 \end{cases}$$

From here, simple combination yields  $R_e v_{j_{12}} = v_{j_{12}}$ , with  $v_{j_{12}}$  defined in Assumption 1. It is easily verified that  $v_{j_{12}} \neq 0$  using the fact that  $\mathring{y}_{j_1}$  and  $\mathring{y}_{j_2}$  are non-collinear by assumption. Furthermore, since  $\mathring{\underline{y}}_{i_1}$  and  $v_{j_{12}}$  are non-collinear by assumption, relations  $R_e \mathring{\underline{y}}_{i_1} = \mathring{\underline{y}}_{i_1}$  and  $R_e v_{j_{12}} = v_{j_{12}}$  obtained previously imply that  $R_e = I_3$ . From here, it is straightforward to deduce that  $p_e = 0$ .

• Case 3 of Assumption 1: Analogously to case 2, one deduces from (18) that  $|E\mathring{y}_{j_1}| = |E\mathring{y}_{j_2}| = |E\mathring{y}_{j_3}| = 1$  and

$$\begin{cases} (R_e - I_3)\underline{\mathring{y}}_{j_1} + p_e \mathring{y}_{j_1,4} = 0\\ (R_e - I_3)\underline{\mathring{y}}_{j_2} + p_e \mathring{y}_{j_2,4} = 0\\ (R_e - I_3)\underline{\mathring{y}}_{j_3} + p_e \mathring{y}_{j_3,4} = 0 \end{cases}$$

From here, analogously to case 2 one deduces that  $R_e v_{j_{12}} = v_{j_{12}}$ ,  $R_e v_{j_{23}} = v_{j_{23}}$ ,  $R_e v_{j_{31}} = v_{j_{31}}$ , and that  $v_{j_{12}}$ ,  $v_{j_{23}}$  and  $v_{j_{31}}$  are not null. Then, using the non-collinearity assumption of the vectors  $v_{j_{12}}$ ,  $v_{j_{23}}$  and  $v_{j_{31}}$ , it is easily deduced that  $R_e = I_3$  and, consequently, that  $p_e = 0$ .

## IV. OBSERVER DESIGN WITH VELOCITY BIAS COMPENSATION

The observer developed in the previous section will be extended in order to cope with the case where the measurement  $A_y \in \mathfrak{se}(3)$  of the group velocity  $A \in \mathfrak{se}(3)$  is corrupted by an unknown constant bias  $b_A \in \mathfrak{se}(3)$ , i.e.  $A_y = A + b_A$ .

**Assumption 2** Assume that the following matrices  $\hat{G} \in \mathbb{R}^{3\times 3}$  and  $\hat{H} \in \mathbb{R}^{3\times 3}$  are full rank:

$$\begin{split} \mathring{G} &:= \sum_{i=1}^{N} k_i (\underline{\mathring{y}}_{i\times})^2 \\ \mathring{H} &:= \left( \sum_{i=1}^{N} k_i \underline{\mathring{y}}_{i,4} \underline{\mathring{y}}_{i\times} \right) \mathring{G}^{-1} \left( \sum_{i=1}^{N} k_i \underline{\mathring{y}}_{i,4} \underline{\mathring{y}}_{i\times} \right) \\ &- \sum_{i=1}^{N} k_i \underline{\mathring{y}}_{i,4}^2 (I_3 - \underline{\mathring{y}}_i \underline{\mathring{y}}_i^\top) \end{split}$$

The condition on the set  $\{ \hat{y}_i \in \mathbb{RP}^3, i = 1, \dots, N \}$ evoked in Assumption 1 ensures that it is always possible to choose a set of parameters  $\{k_i, i = 1, \dots, N\}$  such that  $\mathring{G}$  and  $\mathring{H}$  are full rank (i.e. invertible). Now, the second result of this paper is stated.

**Proposition 1** Consider the observer system

$$\hat{X} = \hat{X}(A_y - \hat{b}_A) - \Delta(\hat{X}, Y)\hat{X}$$
(21a)

$$\begin{cases} \dot{\hat{b}}_A = -k_b \mathbf{P} \left( \hat{X}^\top \sum_{i=1}^N k_i \left( I_4 - e_i e_i^\top \right) \mathring{y}_i e_i^\top \hat{X}^{-\top} \right) & (21b) \\ \hat{X}(0) \in \mathrm{SE}(3), \quad \hat{b}_A(0) \in \mathfrak{se}(3) \end{cases}$$

with  $\Delta(\hat{X}, Y)$  given by (12). Assume that Assumptions 1 and 2 are satisfied. Assume also that A and X are bounded for all time. Then, the equilibrium  $(E, \tilde{b}_A) = (I_4, 0)$  of the dynamics of  $(E, \tilde{b}_A)$ , with  $\tilde{b}_A := b_A - \hat{b}_A$ , is locally asymptotically stable.

*Proof:* It is easily verified that  $\tilde{b}_A = -\hat{b}_A$  and  $\hat{X} = \hat{X}(A + \tilde{b}_A) - \Delta(\hat{X}, Y)\hat{X}$ . Then, one deduces

$$\dot{E} = \left(Ad_{\hat{X}}\tilde{b}_A - \Delta(E, \mathring{Y})\right)E.$$
(22)

Consider the candidate Lyapunov function

$$\mathcal{V}_b(E, \tilde{b}_A) := \frac{1}{2} \sum_{i=1}^N k_i \left| \frac{E \dot{y}_i}{|E \dot{y}_i|} - \dot{y}_i \right|^2 + \frac{1}{2k_b} \|\tilde{b}_A\|^2.$$
(23)

Analogously to the proof of Theorem 1, it can be verified that  $\mathcal{V}_b(E, \tilde{b}_A)$  is locally positive-definite and has a unique global minimum at  $(E, \tilde{b}_A) = (I_4, 0)$ , i.e.  $\mathcal{V}_b(E, \tilde{b}_A) = 0 \Leftrightarrow$  $(E, \tilde{b}_A) = (I_4, 0)$ .

The time-derivative of  $\mathcal{V}_b$  satisfies

$$\begin{aligned} \dot{\mathcal{V}}_{b} &= \left\langle -\sum_{i=1}^{N} k_{i} \left( I_{4} - e_{i} e_{i}^{\top} \right) \mathring{y}_{i} e_{i}^{\top}, A d_{\hat{X}} \widetilde{b}_{A} - \Delta(E, \mathring{Y}) \right\rangle \\ &- \frac{1}{k_{b}} \left\langle \dot{\tilde{b}}_{A}, \widetilde{b}_{A} \right\rangle \\ &= - \left\| \mathbf{P} \left( \sum_{i=1}^{N} k_{i} \left( I_{4} - e_{i} e_{i}^{\top} \right) \mathring{y}_{i} e_{i}^{\top} \right) \right\|^{2} \\ &= - \| \Delta(E, \mathring{Y}) \|^{2}. \end{aligned}$$

$$(24)$$

Since the dynamics of  $(E, \tilde{b}_A)$  are not autonomous, LaSalle's theorem does not apply to deduce the convergence of  $\dot{\mathcal{V}}_b$  to zero. Thus, the next step of the proof consists in proving that  $\dot{\mathcal{V}}_b$  is (locally) uniformly continuous in order to deduce, by application of Barbalat's lemma, the convergence of  $\dot{\mathcal{V}}_b$  to zero. To this purpose it suffices to prove that  $\ddot{\mathcal{V}}_b$  is bounded. In view of (24),  $\ddot{\mathcal{V}}_b$  is bounded if  $\dot{e}_i$   $(i = 1, \dots, N)$  are bounded, where (using (22) and the relation  $e_i = \frac{E\hat{y}_i}{|E\hat{y}_i|}$ )

$$\dot{e}_i = (I_4 - e_i e_i^{\top}) (Ad_{\hat{X}} \tilde{b}_A - \Delta(E, \mathring{Y})) e_i.$$

According to Assumption 2, there exists at least one point  $\dot{y}_i$ such that its fourth component  $\mathring{y}_{i,4}$  is not null. This indicates that for a given small number  $\varepsilon > 0$  there exists  $\delta_{\varepsilon} > 0$  such that if  $|p_e| > \delta_{\varepsilon}$  or  $|b_A| > \delta_{\varepsilon}$  then  $\mathcal{V}_b(E, b_A) > \varepsilon$ . Therefore, there exists a small enough neighborhood  $\mathfrak{B}_{\varepsilon} \in SE(3) \times \mathbb{R}^3$ of the point  $(I_4, 0)$  such that if  $(E(0), \tilde{b}_A(0)) \in \mathfrak{B}_{\varepsilon}$  then  $\mathcal{V}_b(E(0), \hat{b}_A(0)) < \varepsilon$ . Since  $\mathcal{V}_b(E, \hat{b}_A)$  is non-increasing, one has  $\mathcal{V}_b(E(t), b_A(t)) < \varepsilon, \forall t \leq 0$ . This implies that E and  $\tilde{b}_A$  remain bounded. Since X is bounded by assumption, one deduces from the boundedness of E that  $\hat{X}$  is also bounded, which in turn implies the boundedness of  $\dot{E}$  and  $\dot{e}_i$ . This concludes the proof of (local) uniform continuity of  $\dot{\mathcal{V}}_b$  and the convergence of  $\dot{\mathcal{V}}_b$  to zero. One easily verifies that  $(E, b_A) = (I_4, 0)$  is an equilibrium of the error system. Let us prove the local stability of this equilibrium. To this purpose let us first prove that  $\forall (E, \tilde{b}_A) \in \mathfrak{B}_{\varepsilon}$ :

$$\begin{cases} \dot{\mathcal{V}}_b(E, \tilde{b}_A) = 0 & \text{if} \quad E = I_4 \\ \dot{\mathcal{V}}_b(E, \tilde{b}_A) < 0 & \text{if} \quad E \neq I_4 \end{cases}$$

Consider a first order approximation of  $E = \begin{bmatrix} R_e & p_e \\ 0 & 1 \end{bmatrix}$  around  $I_4$  as

$$\begin{cases} p_e &= \varepsilon_p \\ R_e &= I_3 + \varepsilon_{r\times} \end{cases}$$

with  $\varepsilon_p, \varepsilon_r \in \mathbb{R}^3$ . We only need to prove that

$$\dot{\mathcal{V}}_b(E, b_A) = 0 \iff \varepsilon_p = \varepsilon_r = 0.$$

Note that (24) and (15) indicate that the relation  $V_b = 0$  is equivalent to

$$\begin{cases} \sum_{i=1}^{N} k_i \underline{e}_i \times \underline{\mathring{y}}_i &= 0\\ \sum_{i=1}^{N} k_i ((e_i^\top \mathring{y}_i) \underline{e}_i - \underline{\mathring{y}}_i) e_{i,4} &= 0 \end{cases}$$
(25)

In first order approximations, one verifies that

$$E\mathring{y}_{i} = \begin{bmatrix} \underline{\mathring{y}}_{i} + \varepsilon_{r \times} \underline{\mathring{y}}_{i} + \mathring{y}_{i,4} \varepsilon_{p} \\ \underline{\mathring{y}}_{i,4} \end{bmatrix}, \quad |E\mathring{y}_{i}| = 1 + \mathring{y}_{i,4} \varepsilon_{p}^{\top} \underline{\mathring{y}}_{i},$$

and, thus,

$$e_i = \frac{E\mathring{y}_i}{|E\mathring{y}_i|} = \begin{bmatrix} \underline{\mathring{y}}_i + \varepsilon_{r\times}\underline{\mathring{y}}_i + \mathring{y}_{i,4}(I_3 - \underline{\mathring{y}}_i\underline{\mathring{y}}_i^\top)\varepsilon_p \\ \underline{\mathring{y}}_{i,4} - \underline{\mathring{y}}_{i,4}^2\varepsilon_p^\top\underline{\mathring{y}}_i \end{bmatrix}.$$

Therefore, in first order approximations the equalities in (25) can be rewritten as

$$\left(\sum_{i=1}^{N} k_i \mathring{y}_{i,4} \mathring{\underline{y}}_{i,\times}\right) \varepsilon_p = \left(\sum_{i=1}^{N} k_i (\mathring{\underline{y}}_{i\times})^2\right) \varepsilon_r \tag{26a}$$

$$\left( \left( \sum_{i=1}^{N} k_i \mathring{y}_{i,4}^2 (I_3 - \underline{\mathring{y}}_i \underline{\mathring{y}}_i^\top) \right) \varepsilon_p = \left( \sum_{i=1}^{N} k_i \mathring{y}_{i,4} \underline{\mathring{y}}_{i\times} \right) \varepsilon_r \quad (26b)$$

Since  $\mathring{G}$  is full rank according to Assumption 2, it is deduced from (26a) that  $\varepsilon_r = \mathring{G}^{-1} \left( \sum_i k_i \mathring{y}_{i,4} \mathring{\underline{y}}_{i,\times} \right) \varepsilon_p$ . This relation along with (26b) yields  $\mathring{H}\varepsilon_p = 0$ . Since  $\mathring{H}$  is also full rank by Assumption 2, it is deduced that  $\varepsilon_p = 0$  and, consequently,  $\varepsilon_r = 0$ .

It remains to prove to convergence of  $\tilde{b}_A$  to zero. From the convergence of E to  $I_4$  (proven previously) and (22), the application of Barbalat's lemma yields the convergence of  $\dot{E}$ to zero. Finally, Eq. (22) and the convergence of  $\dot{E}$  and of  $\Delta(E, \mathring{Y})$  to zero imply the convergence of  $\tilde{b}_A$  to zero.

The estimate  $\hat{b}_A$  plays the role of integral correction for the error dynamics (22), allowing for the compensation of the unknown constant bias  $b_A$ . It may, however, grow arbitrarily large, resulting in slow convergence and sluggish dynamics of the error evolution. This leads us to replace hereafter the integral term  $\hat{b}_A$ , with dynamics given by (21b), by an "antiwindup" integrator similar to the one proposed in [11], [13]. More precisely, by decomposing  $\hat{b}_A$  as  $\hat{b}_A = \begin{bmatrix} (\hat{b}_{\Omega})_{\times} & \hat{b}_V \\ 0 & 0 \end{bmatrix}$ with  $\hat{b}_{\Omega}, \hat{b}_V \in \mathbb{R}^3$ , one rewrites the dynamics (21b) of the estimated bias  $\hat{b}_A$  as

$$\begin{cases} \dot{\hat{b}}_{\Omega} = k_b \hat{R}^{\top} \left( \Omega_{\Delta} + \frac{1}{2} V_{\Delta} \times \hat{p} \right) \\ \dot{\hat{b}}_V = k_b \hat{R}^{\top} V_{\Delta} \end{cases}$$

with  $V_{\Delta} := \sum_{i=1}^{N} k_i e_{i,4}((e_i^{\top} \mathring{y}_i) \underline{e}_i - \underline{\mathring{y}}_i)$  and  $\Omega_{\Delta} := -\frac{1}{2} \sum_{i=1}^{N} k_i \underline{e}_i \times \underline{\mathring{y}}_i$ . From here, the following modified dynamics of  $\hat{b}_A$  (i.e. of  $\hat{b}_{\Omega}$  and  $\hat{b}_V$ ) are proposed:

$$\begin{cases} \dot{\hat{b}}_{\Omega} = k_b \hat{R}^{\top} (\Omega_{\Delta} + \frac{1}{2} V_{\Delta} \times \hat{p}) - \kappa_{\Omega} (\hat{b}_{\Omega} - \operatorname{sat}_{\delta_{\Omega}} (\hat{b}_{\Omega})) \quad (28a) \\ \dot{\hat{b}}_{M} = k_b \hat{R}^{\top} V_{\Omega} = \kappa_{M} (\hat{b}_{M} - \operatorname{sat}_{\delta_{\Omega}} (\hat{b}_{M})) \quad (28b) \end{cases}$$

$$\left(b_V = k_b R^\top V_\Delta - \kappa_V (b_V - \operatorname{sat}_{\delta_V}(b_V))\right) \tag{28b}$$

with initial conditions  $|b_{\Omega}(0)| \leq \delta_{\Omega}$  and  $|b_{V}(0)| \leq \delta_{V}$ ;  $\kappa_{\Omega}$  and  $\kappa_{V}$  two positive numbers;  $\delta_{\Omega}$  and  $\delta_{V}$  two positive parameters associated with the classical functions  $\operatorname{sat}_{\delta_{\Omega}}(\cdot)$ and  $\operatorname{sat}_{\delta_{\Omega}}(\cdot)$  defined by  $\operatorname{sat}_{\delta}(x) = x \min(1, \delta/|x|), \forall x \in \mathbb{R}^{3}$ .

**Corollary 1** Consider the observer (21a)+(28). Assume that  $\delta_{\Omega}$  and  $\delta_{V}$  are chosen such that  $|b_{\Omega}| \leq \delta_{\Omega}$  and  $|b_{V}| \leq \delta_{V}$ . Then, provided that all the assumptions of Proposition 1 are satisfied, the local asymptotic stability property of Proposition 1 holds.

**Proof:** Based on the inequality  $|b-\operatorname{sat}_{\delta}(b-\tilde{b})| \leq |\tilde{b}|$  for all  $\tilde{b} \in \mathbb{R}^3$  and provided that  $\delta \geq |b|$  (see e.g. [11]), it can be easily proved that the time-derivative of  $\mathcal{V}_b$  defined by (23) satisfies  $\dot{\mathcal{V}}_b \leq -||\Delta(E, \mathring{Y})||^2$ . Therefore, the local asymptotic stability property given in Proposition 1 still holds when the dynamics of  $\hat{b}_A$  given by (21b) is replaced by (28).

## V. SIMULATION RESULTS

In this section, the performance of observer (21a)+(28) is illustrated by simulations. The angular and translational velocity measurements are corrupted by the following constant biases:

$$b_{\Omega} = \begin{bmatrix} -0.02 & 0.02 & 0.01 \end{bmatrix}^{\top} (rad/s), b_{V} = \begin{bmatrix} 0.2 & -0.1 & 0.1 \end{bmatrix}^{\top} (m/s).$$

We consider the three following cases where only three system outputs  $y_i \in \mathbb{RP}^3$  of known inertial elements  $\mathring{y}_i \in \mathbb{RP}^3$  (i = 1, 2, 3) are available to measurement:

- Case 1: corresponds to Case 1 of Assumption 1, in which two vectorial measurements  $v_1, v_2 \in \mathbb{R}^3$  and the position measurement  $p_1 \in \mathbb{R}^3$  of one feature point are available, where  $v_1 = R^{\top} \dot{v}_1$ ,  $v_2 = R^{\top} \dot{v}_2$ ,  $p_1 = R^{\top} (\dot{p}_1 p)$ , with  $\dot{v}_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\top}$ ,  $\dot{v}_2 = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \end{bmatrix}^{\top}$  and  $\dot{p}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\top}$ .
- Case 2: corresponds to Case 2 of Assumption 1, in which one vectorial measurement  $v_1 \in \mathbb{R}^3$  and the position measurements  $p_1, p_2 \in \mathbb{R}^3$  of two feature points are available, where  $v_1 = R^{\top} \dot{v}_1, p_1 = R^{\top} (\dot{p}_1 p), p_2 = R^{\top} (\dot{p}_2 p)$ , with  $\dot{v}_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\top}$ ,  $\dot{p}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\top}$  and  $\dot{p}_2 = \begin{bmatrix} -1/2 & \sqrt{3}/2 & 0 \end{bmatrix}^{\top}$ .
- Case 3: corresponds to Case 3 of Assumption 1, in which the position measurements  $p_1, p_2, p_3 \in \mathbb{R}^3$  of three feature points are available, where  $p_1 = R^{\top}(\mathring{p}_1 p)$ ,  $p_2 = R^{\top}(\mathring{p}_2 p)$ ,  $p_3 = R^{\top}(\mathring{p}_3 p)$ , with  $\mathring{p}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\top}$ ,  $\mathring{p}_2 = \begin{bmatrix} -1/2 & \sqrt{3}/2 & 0 \end{bmatrix}^{\top}$  and  $\mathring{p}_3 = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \end{bmatrix}^{\top}$ .

Recall that Remark 1 explains how to transform a vector or a position of a feature point into a corresponding element of  $\mathbb{RP}^3$ .

The gains and parameters involved in the proposed observer are chosen as follows:

$$k_1 = k_2 = k_3 = 2, \ k_b = 1,$$
  
 $\kappa_{\Omega} = \kappa_V = 10, \ \delta_{\Omega} = 0.052, \ \delta_V = 0.346.$ 

For each simulation run, the proposed filter is initialized at the origin (i.e.  $\hat{R} = I_3$ ,  $\hat{p} = 0$ ,  $\hat{b}_{\Omega} = 0$ ,  $\hat{b}_V = 0$ ) while the true trajectories are initialized differently. Combined sinusoidal inputs are considered for both the angular and translational velocity inputs of the system kinematics. The rotation angle associated with the axis-angle representation is used to represent the attitude trajectory. One can observe from Figure 1 that the observer trajectories converge to the true trajectories after a short transition period for all the three considered cases. Figure 2 shows that the norms of the estimated velocity bias errors  $|\tilde{b}_{\Omega}|$  and  $|\tilde{b}_V|$  converge to zero, which means that the group velocity bias  $b_A$  is also correctly estimated.

#### VI. CONCLUSIONS

In this paper, we propose a nonlinear observer on SE(3)for full pose estimation that takes the system outputs on the real projective space  $\mathbb{RP}^3$  directly as inputs. The observer derivation is based on a recent observer design technique directly on the output space, proposed in [18]. An advantage with respect to our prior work [14] is that we can now incorporate in a unique observer different types of measurements such as vectorial measurements of known inertial vectors and position measurements of known feature points. The proposed observer is also extended on  $SE(3) \times \mathfrak{se}(3)$ so as to compensate for unknown additive constant bias in the velocity measurements. Rigorous stability analyses are



Fig. 1. The rotation angle and the position tracking performance of the proposed observer. Note that the dashed lines are the estimated trajectories (for Cases 1 (green), Case 2 (blue), Case 3 (red)) while the solid line (black) represents the true trajectory.



Fig. 2. The norms of the estimated velocity bias errors  $|\tilde{b}_{\Omega}|$  and  $|\tilde{b}_V|$  vs. time.

equally provided. Excellent performance of the proposed observers are justified through simulations.

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