Exponential convergence of nonlinear Luenberger observers

Vincent Andrieu

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Abstract

In this paper, it is shown that under an extra observability assumption the nonlinear Luenberger observer as introduced recently in a previous publication may have an exponential convergence towards the state of the system. This version is the corrected version of the same paper in [1] in which there is mistake.

1 Introduction

State estimation is one of the main problem in engineering. In the deterministic framework, an algorithm which can solve this problem is called a *state observer*. This algorithm is based on the knowledge of a dynamical model with measured outputs representing in a good way the considered physical phenomena and the sensors available. Since 1964 and the seminal work of Luenberger in [10], designing an observer for detectable linear systems is now well known. The approach of Luenberger can be decomposed into two steps. In the first one, a linear dynamic extension which defines a contraction uniform in the measured output of the system is introduced. In the second step, based on some observability properties of the considered model, a linear map can be obtained such that when applied to the state of the dynamic extension a state observer is obtained.

For nonlinear models, the problem is much more complicated and many different routes have been followed in order to extend this strategy. Few years back, Shoshitaishvili in [17] and more recently Kazantzis and Kravaris in [7] (see also [9]) have introduced a nonlinear local extension of the linear Luenberger observer. With their approach, it was shown that the existence of an observer around an equilibrium was obtained assuming local observability.

Recently, the non-local version of this tool has been studied in [2]. The interest of this approach is that with a weak observability assumption (distinguishability of the state from the past output), a nonlinear Luenberger observer

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exists provided the trajectories of the system remain in a bounded forward invariant set.

However, although the observer of [2] ensures the asymptotic convergence of the estimate to the state of the system, no characterization of the convergence speed is given. In this paper, with an extra observability assumption it is shown that the convergence speed of a nonlinear Luenberger observer is exponential and that the argument of the exponential decay can be selected arbitrary large.

The paper is organized as follows. In Section 2, the nonlinear Luenberger observer as introduced in [17] is presented and one of the result obtained from [2] is given. Section 3 is devoted to the statement of the main result. The proof of this result is given in Section 4. Finally Section 5 gives the conclusion.

2 Existence of a Nonlinear Luenberger observer

Consider a nonlinear system described by the following equation[1]:

$$\dot{x} = f(x) \quad , \qquad y = h(x) \ . \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are two C^2 functions and where the initial value of the state is in a given compact set denoted $\mathcal{C}$. For all x in $\mathbb{R}^n$, the solution of System (1) initiated from x at time 0 is denoted $X(x,t)$.

For all x in a given open set $\mathcal{O}$ in $\mathbb{R}^n$, the maximal time interval of definition in $\mathcal{O}$ is denoted $(\sigma^-_{\mathcal{O}}(x), \sigma^+_{\mathcal{O}}(x))$. More precisely, for all x in $\mathcal{O}$, $X(x,t)$ is in $\mathcal{O}$ for all t in $(\sigma^-_{\mathcal{O}}(x), \sigma^+_{\mathcal{O}}(x))$. And if $X(x, \sigma^-_{\mathcal{O}}(x))$ (respectively $X(x, \sigma^+_{\mathcal{O}}(x))$) exists, then $X(x, \sigma^-_{\mathcal{O}}(x)) \notin \mathcal{O}$ (resp. $X(x, \sigma^+_{\mathcal{O}}(x)) \notin \mathcal{O}$).

The main structural assumption imposed on System (1) is the following:

Assumption 1 (Bounded forward invariant set) *There exists a forward invariant and compact set $\mathcal{I} \subset \mathbb{R}^n$ containing $\mathcal{C}$, the given set of initial value. In other words, $\mathcal{C} \subseteq \mathcal{I} \subset \mathbb{R}^n$ and for all x in $\mathcal{I}$ and all t in $\mathbb{R}_+$, $X(x,t)$ is in $\mathcal{I}$.*

Following [17, 7, 8, 2] a nonlinear Luenberger observer is a dynamical system of the form:

$$\dot{z} \ = \ A\, z \ + \ By \quad , \qquad \hat{x} \ = \ T^*(z) \ , \tag{2}$$

with state z (a complex vector) in $\mathbb{C}^{n+1}$, A is a diagonal Hurwitz matrix in $\mathbb{C}^{(n+1)\times(n+1)}$, B in $\mathbb{R}^{n+1}$ is defined as

$$B = (1, \ldots, 1)' \ , \tag{3}$$

and $T^* : \mathbb{C}^{n+1} \to \mathbb{R}^n$ is a continuous functions.

Note that in [2] the nonlinear Luenberger observer considered is slightly more general since the matrix B is a nonlinear function of the output. However due to the existence of a bounded invariant set (i.e. Assumption 1) no generality are lost by imposing this observer structure.

[1] In this paper, for the sake of clarity only time invariant systems are considered. However, following [13] it is possible to extend all these results to time varying systems provided all Assumptions imposed are uniform in the time.

The main interest of this approach is that with the only assumption that the past output path $t \mapsto h(X(x,t))$ restricted to the time in which the trajectory remains in a certain set is injective in x, it is sufficient to choose $n+1$ generic complex eigenvalues for A to get the existence of the function T^* making System (2) an observer which asymptotically estimates the state of System (1). The specific observability condition made is :

Assumption 2 (Backward distinguishability Property) *There exists two strictly positive real numbers $\delta_\Upsilon < \delta_d$ such that, for each pair of distinct points x_1 and x_2 in*[2] $\mathcal{I}+\delta_\Upsilon$*, there exists a negative time t in $\left(\max\left\{\sigma^-_{\mathcal{I}+\delta_d}(x_1), \sigma^-_{\mathcal{I}+\delta_d}(x_2)\right\}, 0\right]$ such that :*

$$h(X(x_1,t)) \neq h(X(x_2,t)) \ .$$

This distinguishability assumption says that the present state x can be distinguished from other states in an open set containing $\mathcal{I}$ by looking at the past output path restricted to the time in which the solution remains in $\mathcal{I}+\delta_d$.

With the existence of a forward invariant bounded set and the backward distinguishability property the following result can be obtained from[3] [2].

Theorem 1 ([2] Generic existence of Luenberger observer) *Assume System (1) satisfies Assumptions 1 and Assumption 2. Then there exists a negative real number ρ and zero Lebesgues measure subset $\mathcal{A}_d$ of*[4] $(\mathbb{C}_\rho)^{n+1}$ *such that for each $(\lambda_1,\ldots,\lambda_{n+1})$ in $(\mathbb{C}_\rho)^{n+1} \setminus \mathcal{A}_d$ there exists a function T^* such that for all x in $\mathcal{C}$ and all z in $\mathbb{C}^{n+1}$*

$$\lim_{t\to+\infty} \hat{X}(x,z,t) - X(x,t) = 0 \ ,$$

where,

$$\hat{X}(x,z,t) = T^*(Z(x,z,t)) \ ,$$

and where $(Z(x,z,t), X(x,t))$ is the solution of System (1) and (2) with $A = \mathtt{Diag}\{\lambda_1,\ldots,\lambda_{n+1}\}$.

In [2], this result was not stated in this way. However it is a direct consequence of the extra assumption made on the boundedness of the solution in positive time (i.e Assumption 1).

Consequently, with this result, as long as the trajectories of the system remain in a bounded set in forward time, there exists a nonlinear Luenberger

[2] Given a subset $S \subseteq \mathbb{R}^n$ and a strictly positive real real number δ, $\mathcal{S}+\delta$ is the open set defined as ,

$$\mathcal{S}+\delta = \{x \in \mathbb{R}^n, \exists\, x_\mathcal{S} \in \mathcal{S}, |x - x_\mathcal{S}| < \delta\} \ . \tag{4}$$

[3] Compare to the published version in [1] a mistake has been corrected by introducing a negative real number ρ.

[4] $\mathbb{C}_\mu$ is the open subset of $\mathbb{C}$ defined as

$$\mathbb{C}_\rho \ = \ \{\lambda \in \mathbb{C} : \mathtt{Re}(\lambda) \ < \ \rho\} \ , \tag{5}$$

where $\mathtt{Re}$ is the real part.

observer which provides an estimate converging asymptotically to the state. Note however that no characterization of the convergence speed is given. In the next Section a sufficient conditions is given under which exponential convergence of the estimation error towards the origin is obtained. In other words, the estimate satisfies an inequality like

$$|\hat{X}(x,z,t) - X(x,t)| \leq M(x,z)\exp(-ct) \ ,$$

where c is a positive real number.

3 Exponential convergence

3.1 Main result

In this section a sufficient condition guaranteeing exponential convergence of the observer (2) is given. This sufficient condition is an observability assumption which characterizes how a small change of the state modifies the backward output path restricted to the set $\mathcal{I} + \delta$. More precisely, in this Section the following observability assumption is imposed.

Assumption 3 (Locally linearly independent output) *There exists two strictly positive real numbers $\delta_\Upsilon < \delta_d$ such that, for all v in $\mathbb{R}^n/\{0\}$ and for all x in $\mathcal{I}$, there exists a negative time t in $\left(\sigma^-_{\mathcal{I}+\delta_d}(x), 0\right]$ such that*

$$\frac{\partial h(X(x,t))}{\partial x} v \ \neq \ 0 \ . \tag{6}$$

The main result of our paper can now be stated[5].

Theorem 2 (Exponential Luenberger observers) *Assume System (1) satisfies Assumptions 1, 2 and 3 (with the same δ_d and δ_Υ). Then there exist a negative real number ρ, a zero Lebesgues measure subset $\mathcal{A}_e$ of $(\mathbb{C}_\rho)^{n+1}$ such that for each $(\lambda_1, \ldots, \lambda_{n+1})$ in $(\mathbb{C}_\rho)^{n+1} \setminus \mathcal{A}_e$ there exists $T^* : \mathbb{C}^{n+1} \to \mathbb{R}^n$ and a function $M : \mathbb{R}^n \times \mathbb{C}^{n+1} \to \mathbb{R}_+$ such that for all (x,z) in $\mathcal{C} \times \mathbb{C}^{n+1}$*

$$|T^*(Z(x,z,t) - X(x,t)| \leq \qquad M(z,x)\exp\left(\max_i\{\mathtt{Re}(\lambda_i)\}t\right) \ , \tag{7}$$

and where $(Z(x,z,t), X(x,t))$ is the solution of System (1) and (2) with $A = \mathtt{Diag}\{\lambda_1, \ldots, \lambda_{n+1}\}$.

This result is proved in Section 4. The next Subsection contains some discussions about Assumptions 1, 2 and 3.

[5] Compare to the published version in [1] a mistake has been corrected by introducing a negative real number ρ.

3.2 Discussion on Assumptions

Note that requiring the existence of a bounded invariant set in positive time is the main restriction made on System (1). Note however that from a practical point of view, it is not surprising to require that the state solution is bounded in positive time.

Also, it is possible to modify the dynamics of the model (1) to fit in this context. For instance, assume we have an *a priori* knowledge of a compact set denoted $\mathcal{O} \subset \mathbb{R}^n$ which contains the state trajectory. In this case, one trick is to modify the dynamics of system (1) outside $\mathcal{O}$ to ensure the existence of a forward invariant compact set. More precisely, the following modified system is considered:

$$\dot{x} = \chi(x) f(x) \ , \tag{8}$$

where $\chi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that

$$\chi(x) = \begin{cases} 0 & x \notin \mathcal{O} + \delta_u \\ 1 & x \in \mathcal{O} \end{cases}$$

In this case, $\mathcal{I} := \mathcal{O} + \delta_u$ becomes invariant for trajectories of the modified system (8). Note however, that the validity of the observability assumptions, i.e. Assumptions 2 and 3 may be impacted by the use of this modification.

Assumptions 2 and 3 are observability assumptions. To describe these Assumptions with usual tools, assume that the output map h is sufficiently smooth so that the observability mapping of order p as defined in [5] by:

$$\mathcal{H}_p(x) = \left[h(x), L_f h(x), \dots, L_f^p h(x)\right]' \ ,$$

is properly defined. The following result can be obtained.

Proposition 1 ([5]) *If there exist a positive real number δ_d and an integer p such that $\mathcal{H}_p$ is injective in the set $\mathcal{I} + \delta_d$, then Assumption 2 is satisfied for System (1) for all $\delta_\Upsilon < \delta_d$.*

Proof : Assume Assumption 2 is not satisfied. Then for all δ_Υ such that $\delta_\Upsilon < \delta_d$ there exists x_1 and x_2 in $\mathcal{I} + \delta_\Upsilon$ such that:

$$h(X(x_1, t)) = h(X(x_2, t)) \ ,$$

for all t in $\left(\max\left\{\sigma^-_{\mathcal{I}+\delta_d}(x_1), \sigma^-_{\mathcal{I}+\delta_d}(x_2)\right\}, 0\right]$. This implies that the p first time derivatives of $h(X(x_1, t))$ and $h(X(x_2, t))$ are the same which implies that $\mathcal{H}_p$ is not injective. □

Note that a link between Assumption 3 and the observability mapping can be expressed as follows.

Proposition 2 *If there exist a positive real number δ_d and an integer p such that for all x in $\mathcal{I}+\delta_d$ and for all v in $\mathbb{R}^n \setminus \{0\}$,*

$$\frac{\partial \mathcal{H}_p}{\partial x}(x)v \neq 0$$

then Assumption 3 is satisfied for System (1).

Proof : Assume Assumption 3 is not satisfied. Then for all δ_Υ such that $\delta_\Upsilon < \delta_d$ there exists v in $\mathbb{R}^n/\{0\}$ and x in $\mathcal{I}+\delta_\Upsilon$ such that for all negative time t in $\left(\sigma^-_{\mathcal{I}+\delta_d}(x), 0\right]$

$$\frac{\partial h(X(x,t))}{\partial x} v \;=\; 0 \;. \tag{9}$$

This implies that,

$$\dot{\overbrace{\frac{\partial h(X(x,t))}{\partial x} v}} \;=\; \frac{\partial}{\partial x} L_f h(X(x,t))\, v \;=\; 0 \;. \tag{10}$$

By differentiating with time, it yields finally:

$$\frac{\partial \mathcal{H}(X(x,t))}{\partial x} v \;=\; 0 \;, \tag{11}$$

hence the result. □

In the context of Propositions 1 and 2, it is possible to apply the result presented in [15] to design a high-gain observer of dimension p employing embedding techniques which ensures exponential convergence of the estimate towards the state. Note moreover that it was shown in [4] that generically the context of Propositions 1 and 2 is satisfied by taking $p = 2n+1$.

4 Proof of Theorem 2

4.1 A constructive proposition

The proof of Theorem 2 is based on the following Proposition.

Proposition 3 *Assume that Assumption 1 is satisfied for system (1). If there exists a C^1 function $T : \mathbb{R}^n \to \mathbb{C}^{n+1}$ which satisfies the following three points:*

1. *T is solution of the partial differential equation*

$$\frac{\partial T}{\partial x}(x) f(x) = AT(x) + Bh(x) \;\forall\; x \in \mathcal{I} \;; \tag{12}$$

   where $A = \texttt{diag}\{\lambda_1, \ldots, \lambda_{n+1}\}$ and λ_i is in $\mathbb{C}_0$ (see the definition in (5)) and B is defined in (3).

2. *The function T is injective on $\mathcal{I}$;*

3. For all x in $\mathcal{I}$ the matrix $\overline{\frac{\partial T}{\partial x}(x)}' \frac{\partial T}{\partial x}(x)$ is positive definite;

then there exists $T^* : \mathbb{C}^{n+1} \to \mathbb{R}^n$ and $M : \mathbb{R}^n \times \mathbb{C}^{n+1} \to \mathbb{R}_+$ such that for all (x,z) in $\mathcal{I} \times \mathbb{C}^{n+1}$ equation (7) is satisfied.

Proof : Consider the function $\Delta : \mathcal{I} \times \mathcal{I} \to \mathbb{C}^{n+1}$ defined by,

$$\Delta(x_1, x_2) = T(x_1) - T(x_2) - \frac{\partial T}{\partial x}(x_2)(x_1 - x_2) \ .$$

The function T being C^1, this function is properly defined and moreover, for all x_2 in $\mathcal{I}$:

$$\lim_{x_1 \to x_2} \frac{\Delta(x_1, x_2)}{|x_1 - x_2|} = 0 \ . \tag{13}$$

Moreover, the function $\frac{\partial T}{\partial x}$ taking value in $\mathbb{C}^{(n+1)\times n}$ is continuous and by assumption full rank. Hence, the function R given by,

$$R(x) \ = \ \left(\overline{\frac{\partial T}{\partial x}(x)}' \frac{\partial T}{\partial x}(x)\right)^{-1} \left(\overline{\frac{\partial T}{\partial x}(x)}\right)'$$

is continuous and satisfies for all x in $\mathcal{I}$,

$$R(x) \neq 0 \ , R(x)\frac{\partial T}{\partial x}(x) = I_n \ ,$$

where I_n is the identity matrix in $\mathbb{R}^{n\times n}$. For all (x_1, x_2) in $\mathcal{I} \times \mathcal{I}$, it yields :

$$\begin{aligned} |x_1 - x_2| \ &\leq \ |R(x_2)| \left(|T(x_1) - T(x_2)| \ + \ |\Delta(x_1, x_2)|\right) \ , \\ &\leq \ R_{\max} \left(|T(x_1) - T(x_2)| \ + \ |\Delta(x_1, x_2)|\right) \ , \end{aligned} \tag{14}$$

where,

$$R_{\max} \ = \ \max_{x \in \mathcal{I}} R(x) \neq 0 \ , \tag{15}$$

It yields, for all (x_1, x_2) in $\mathcal{I} \times \mathcal{I}$

$$|x_1 - x_2| \left(1 - R_{\max} \frac{|\Delta(x_1, x_2)|}{|x_1 - x_2|}\right) \ \leq \ R_{\max} \ |T(x_1) - T(x_2)| \ ,$$

Moreover, with (13), for all a in $\mathcal{I}$, there exists $\delta(a) > 0$, such that, for all x_1 in[6] $\mathcal{B}_{\delta(a)}(a) \cap \mathcal{I}$, it gives :

$$|\Delta(x_1, a)| \ \leq \ \frac{1}{4R_{\max}} |x_1 - a| \ .$$

The function Δ being continuous in its second argument, for all a in $\mathcal{I}$, there exists a positive real number $\epsilon(a)$ such that, for all (x_1, x_2) in $\mathcal{B}_{\epsilon(a)}(a)^2 \cap \mathcal{I}^2$:

$$|\Delta(x_1, x_2)| \ \leq \ \frac{1}{2R_{\max}} |x_1 - x_2| \ .$$

[6] $\mathcal{B}_r(x_c)$ denotes the subset of $\mathbb{R}^n$: $\{x \in \mathbb{R}^n, |x - x_c| \leq r\}$

With (14) it yields that for all a in $\mathcal{I}$,

$$|x_1 - x_2| \;\leq\; 2\,R_{\max}\,|T(x_1) - T(x_2)| \;,\qquad \forall\,(x_1, x_2) \;\in\; \mathcal{B}_{\epsilon(a)}(a)^2 \cap \mathcal{I}^2 \;.$$

On another hand, $\{\mathcal{B}_{\frac{1}{2}\epsilon(a)}(a)\,,\;\; a \;\in\; \mathcal{I}\}$ is a covering by open subset of the compact subset $\mathcal{I}$. Hence, there exists $\{a_1, \ldots, a_N\}$ in $\mathcal{I}^N$ with N a positive integer, such that

$$\mathcal{I} \subseteq \cup_{i=1,\ldots,N} \mathcal{B}_{\frac{1}{2}\epsilon(a_i)}(a_i) \;.$$

Since the function T is injective on $\mathcal{I}$, it is possible to define the positive real number :

$$N_{\max} \;=\; \max_{(x_1,x_2)\,\in\,\Omega} \frac{|x_1 - x_2|}{T(x_1) - T(x_2)} \tag{16}$$

where Ω is the compact subset defined by,

$$\Omega \;=\; \{(x_1, x_2) \;\in\; \mathcal{I} \times \mathcal{I} \;:\; |x_1 - x_2| \;\geq\; \epsilon_{\min}\} \;, \tag{17}$$

where,

$$\epsilon_{\min} \;=\; \min_{i\,<\,N} \;\frac{1}{2}\epsilon(a_i) \;.$$

Consider now (x_1, x_2) in $\mathcal{I} \times \mathcal{I}$. Two cases can be distinguished:

1. $|x_1 - x_2| \;\leq\; \epsilon_{\min}$: since there exists $i < N$ such that $x_2 \;\in\; \mathcal{B}_{\frac{1}{2}\epsilon(a_i)}(a_i)$, it yields,

$$\begin{aligned} |x_1 - a_i| \;&\leq\; |x_1 - x_2| \;+\; |x_2 - a_i| \\ &\leq\; \epsilon_{\min} \;+\; \frac{1}{2}\epsilon(a_i) \\ &\leq\; \epsilon(a_i) \;. \end{aligned}$$

   Hence, $x_1 \;\in\; \mathcal{B}_{\epsilon(a_i)}(a_i)$, and consequently:

$$|x_1 - x_2| \;\leq\; 2\,R_{\max}\,|T(x_1) - T(x_2)| \;.$$

2. $|x_1 - x_2| \;\geq\; \epsilon_{\min}$: In this case (x_1, x_2) is in Ω and consequently :

$$|x_1 - x_2| \;\leq\; N_{\max}\,|T(x_1) - T(x_2)| \;. \tag{18}$$

Consequently, it yields that for all (x_1, x_2) in $\mathcal{I} \times \mathcal{I}$:

$$|x_1 - x_2| \;\leq\; K\,|T(x_1) - T(x_2)| \;,$$

with, $K \;=\; \max\{N_{\max}\,, 2\,R_{\max}\}$.

Hence, it is possible to define the function $T^{-1} : T(\mathcal{I}) \to \mathcal{I}$ and this one satisfies,

$$|T^{-1}(w_1) - T^{-1}(w_2)| \;\leq\; K\,|w_1 - w_2| \;,$$

for all (w_1, w_2) in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$. It yields that the function $T^{-1} : T(\mathcal{I}) \to \mathcal{I}$ is globally Lipschitz. Hence, the function $T^* \;:\; \mathbb{C}^{n+1} \;\to\; \mathcal{I}$ solution to our problem is a Lispchitz extension on the set $\mathbb{C}^{n+1}$ of this function. As exposed in [15] different solutions are possible. A constructive solution may be to use the Mc-Shane formula (see [12] and more recently [11]) and to introduce $T^* = (T_1^*, \ldots, T_n^*)$ as the function defined by:

$$T_i^*(w) = \inf_{z \in T(\mathcal{I})} \left\{ \left(T^{-1}(z)\right)_i + K|z - w| \right\} \ . \tag{19}$$

This function is such that,

$$T^*(T(x)) = x \ ,$$

and for all w in $\mathbb{C}^{n+1}$ it yields,

$$|T^*(w) - x| \ \leq \ nK \, |w - T(x)| \ .$$

This implies that the estimation error satisfies

$$|T^*(Z(x,z,t)) - X(x,t)| \leq nK \, |Z(x,z,t) - T(X(x,t))|$$

On another hand, the function T is solution of the partial differential equation (12), consequently, this implies that along the trajectories of system (1) and (2)

$$Z(x,z,t) - T(X(x,t)) = \exp(At)(z - T(x)) \ .$$

Note that since $A = \texttt{Diag}(\lambda_1, \ldots, \lambda_{n+1})$ with λ_i in $\mathbb{C}_0$, it yields that equation (7) holds with the function M defined as $M(x,z) = nK|z - T(x)|$ and concludes the proof of Proposition 3. $\square$

With Proposition 3 it can be checked that to prove Theorem 2, it is required to find an injective solution to the partial differential equation (12) for all x in $\mathcal{I}$ such that this one is injective in $\mathcal{I}$ and such that its gradient is full rank. In the rest of this Section, it is shown that this is indeed the case for *almost* all Hurwitz diagonal matrix A.

4.2 Solutions of the PDE given in (12)

As proposed in [2] (see also [8]), given $\delta_b > \delta_d$, a function $T : \mathbb{R}^n \to \mathbb{C}^{n+1}$ solution of the partial differential equation (12) can be simply expressed as,

$$T(x) \ = \ \int_{-\infty}^{0} \exp(-As) B h(\breve{X}(x,s)) ds \ , \tag{20}$$

where $\breve{X} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the solution of the modified system

$$\dot{x} = \chi(x) f(x) \ , \tag{21}$$

where $\chi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that

$$\chi(x) = \begin{cases} 0 & x \notin \mathcal{I} + \delta_b \\ 1 & x \in \mathcal{I} + \delta_d \end{cases}$$

In the case where the set $\mathcal{I}$ is not bounded, the existence of a solution to a partial differential equation similar to (12) can still be obtained provided linear vector B is replaced by a continuous function (see [2] for more details).

Moreover, when the set $\mathcal{I}$ is also backward invariant, it can be shown that the restriction of the solution of (12) to $\mathcal{I}$ is unique.

4.3 Generic properties of the solution of the PDE given in (12)

In the paper [2], it was shown that generically on the eigenvalues of the matrix A the function T defined in (20), solution of the PDE (12), is injective provided System (1) is backward distinguishable (i.e. Assumption 2 is satisfied). More precisely the result obtained in [2] is:

Theorem 3 (Generic Injectivity, [2]) *Assume that Assumption 2 is satisfied for System (1) for given positive real numbers δ_Υ and δ_d. Then there exist a negative real number ρ_d and a subset $\mathcal{A}_d \subset (\mathbb{C}_{\rho_d})^{n+1}$ of zero Lebesgue measure such that the function $T : \mathbb{R}^n \to \mathbb{C}^{n+1}$ defined by (20) (with same δ_d) is C^1 and injective on $\mathcal{I}$ provided A is a diagonal Hurwitz matrix with $n+1$ complex eigenvalues λ_i arbitrarily chosen in $(\mathbb{C}_{\rho_d})^{n+1} \setminus \mathcal{A}_d$.*

Consequently, to apply Proposition 3, it has to be shown that generically on A and under Assumption 3, the function T defined in (20) is such that for all x in $\mathcal{I}$ the matrix $\overline{\frac{\partial T}{\partial x}(x)}' \frac{\partial T}{\partial x}(x)$ is positive definite. This is proved by the following Theorem.

Theorem 4 (Generically a local embedding) *Assume that Assumption 3 is satisfied for system (1) for given positive real numbers δ_Υ and δ_d. Then there exist a negative real number ρ_{el} and a subset $\mathcal{A}_{le} \subset (\mathbb{C}_{\rho_{el}})^{n+1}$ of zero Lebesgue measure such that the function $T : \mathbb{R}^n \to \mathbb{C}^{(n+1)\times p}$ defined by (20) (with same δ_d) is C^2 and such that for all x in $\mathcal{I}$, $\frac{\partial T}{\partial x}(x)$ is full rank provided A is a diagonal matrix with $n+1$ complex eigenvalues λ_i arbitrarily chosen in $(\mathbb{C}_{\rho_{el}})^{n+1} \setminus \mathcal{A}_{le}$.*

Proof : The proof of this theorem follows the same line as the one of Theorem 3 (a proof of which is given in [2]) and is based on the use of Coron's Lemma:

Lemma 1 (Coron) *Let Γ and Υ be open subsets of $\mathbb{C}$ and $\mathbb{R}^{2n}$ respectively. Let $g : \Upsilon \times \Gamma \to \mathbb{C}^p$ be a function which is holomorphic in λ for each x in Υ and C^1 in x for each λ in Γ. If, for each pair (x, λ) in $\Upsilon \times \Gamma$ for which $g(x, \lambda)$ is zero it is possible to find, for at least one of the p components g_j of g, an integer k satisfying :*

$$\begin{aligned} \frac{\partial^i g_j}{\partial \lambda^i}(x,\lambda) &= 0 \ \forall i \in \{0,\ldots,k-1\} \ , \\ \frac{\partial^k g_j}{\partial \lambda^k}(x,\lambda) &\neq 0 \end{aligned} \tag{22}$$

then the following set has zero Lebesgue measure in $\mathbb{C}^{n+1}$:

$$\mathcal{A} = \bigcup_{x\in\Upsilon} \Big\{ (\lambda_1, \ldots, \lambda_{n+1}) \in \Gamma^{n+1} : \qquad g(x,\lambda_1) = \ldots = g(x,\lambda_{n+1}) = 0 \Big\} . \tag{23}$$

This result has been established by Coron in [3, Lemma 3.2] in a stronger form except for the very minor point that, here, g is not C^∞ in both x and λ. A proof of this specific result can be found in [2].

To show Theorem 4, the idea is to introduce an appropriate function g. Let Γ and Υ be open sets defined by:

$$\Gamma \;=\; \mathbb{C}_{\rho_{el}} \ , \tag{24}$$

where ρ_{el} is a negative real number defined later on and

$$\Upsilon \;=\; \{w \;=\; (x,v) \,\in\, \mathcal{I}+\delta_\Upsilon \times \mathbb{R}^n \;:\; v \,\neq\, 0\} \ . \tag{25}$$

With the fact that $\mathcal{I}+\delta_b$ is bounded and backward invariant for the modified system (21), it yields that for all (x,λ,t) in $\mathcal{I}+\delta_\Upsilon \times \Gamma \times (-\infty, 0]$,

$$|\exp(-\lambda t)h(\breve{X}(x,t))| \;\leq\; \exp([-\mathtt{Re}(\lambda)]t)c \ , \tag{26}$$

where c is a positive real number. By Lebesgue dominated convergence Theorem it yields that for all x in $\mathcal{I}+\delta_\Upsilon$, the function

$$T_\lambda(x) \;=\; \int_{-\infty}^{0} \exp(-\lambda s)h(\breve{X}(x,s))\,ds \ , \tag{27}$$

defines a continuous function $T_\lambda \,:\, \mathcal{I}+\delta_\Upsilon \to \mathbb{C}^{n+1}$.

Now following [14, Theorem 2.50], we show that by taking $\mathtt{Re}(\lambda)$ sufficiently negative, the function T_λ defined in (27) is C^2. First of all, for all x in $\mathcal{I}+\delta_\Upsilon$ and all s in $\mathbb{R}_-$ we have

$$\frac{\partial^2 \breve{X}}{\partial x \partial s}(x,s) = \frac{\partial \tilde{f}}{\partial x}(\breve{X}(x,s))\frac{\partial \breve{X}}{\partial x}(x,s)$$

where $\tilde{f}(x) = \chi(x)f(x)$. We can introduce the function U defined as

$$U(x,s) = \mathtt{trace}\left(\frac{\partial \breve{X}}{\partial x}(x,s)'\frac{\partial \breve{X}}{\partial x}(x,s)\right) \ .$$

Note that we have $U(x,0) = n$. Moreover for all x in $\mathcal{I}+\delta_\Upsilon$ and for all s in $\mathbb{R}_-$,

$$U(x,s) \geq \left|\frac{\partial \breve{X}}{\partial x}(x,s)\right|^2 \ .$$

Also, it satisfies for all s in $\mathbb{R}_-$

$$\frac{\partial U}{\partial s}(x,s) = \mathtt{trace}\left(\frac{\partial \breve{X}}{\partial x}(x,s)\left[\frac{\partial \tilde{f}}{\partial x}(\breve{X}(x,s)) + \frac{\partial \tilde{f}}{\partial x}(\breve{X}(x,s))\right]\frac{\partial \breve{X}}{\partial x}(x,s)\right) \ .$$

Hence, employing the fact that for all x in $\mathcal{I}+\delta_\Upsilon$ the trajectories $s \mapsto \breve{X}(x,s)$ are bounded it gives the existence of a negative real number ρ_1 such that for all x in $\mathcal{I}+\delta_\Upsilon$ and for all s in $\mathbb{R}_-$,

$$\frac{\partial U}{\partial s}(x,s) \leq -2\rho_1 U(x,s) \ .$$

Consequently, we obtain for all x in $\mathcal{I}+\delta_\Upsilon$ and for all s in $\mathbb{R}_-$,

$$\left|\frac{\partial \breve{X}}{\partial x}(x,s)\right| \leq \sqrt{n}\exp(\rho_1 s)\ .$$

Hence, employing the fact the trajectories $s\mapsto \breve{X}(x,s)$ is bounded in $\mathcal{I}+\delta_\Upsilon$ we can find a positive real number c such that for all x in $\mathcal{I}+\delta_\Upsilon$ and s in $\mathbb{R}_-$,

$$\left|\exp(-\lambda s)\frac{\partial h}{\partial x}(\breve{X}(x,s))\frac{\partial \breve{X}}{\partial x}(x,s)\right| \leq \exp([\rho_1 - \mathtt{Re}(\lambda)]s)c\ .$$

With Lebesgue dominate convergence Theorem, it can be established that the function

$$\frac{\partial T_\lambda}{\partial x}(x) \;=\; \int_{-\infty}^{0} \exp(-\lambda s)\frac{\partial h}{\partial x}(\breve{X}(x,s))\frac{\partial \breve{X}}{\partial x}(x,s)\,ds\ , \tag{28}$$

is continuous and properly defined provided $\mathtt{Re}(\lambda) < \rho_1$ and consequently the function T_λ is C^1. Similarly, it can be shown that this function is C^2 provided $\mathtt{Re}(\lambda) < \rho_{el}$ where ρ_{el} is a negative real number.

Now, consider the function $\mathcal{G}T\ :\Upsilon\times\Gamma\to\mathbb{C}^{n+1}$ defined by :

$$\mathcal{G}T(w,\lambda) \;=\; \frac{\partial T_\lambda}{\partial x}(x)\,v\ , \tag{29}$$

with $w=(x,v)$. This function is C^1 in w in Υ for all λ in Γ. Moreover, it can be shown in [16, chap 19, p. 367] that the Theorem of Morera and Fubini yields that this function is holomorphic in λ in Γ, for all w in Υ. Again, the set $\mathcal{I}+\delta_\Upsilon$ being bounded and backward invariant for System (21), it yields

$$\int_{-\infty}^{0} \exp(-2\mathtt{Re}(\lambda)s)\left|\frac{\partial h(\breve{X}(x,s))}{\partial x}\,v\right|^2\,ds \;<\; +\infty\ .$$

Consequently, Plancherel Theorem can be employed to get for all w in Υ,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}|\mathcal{G}T(w,\mathtt{Re}(\lambda)+is)|^2\,ds \;= \tag{30}$$

$$\int_{-\infty}^{0} \exp(-2\mathtt{Re}(\lambda)s)\left|\frac{\partial h(\breve{X}(x,s))}{\partial x}\,v\right|^2\,ds\ .$$

Now, for all w in Υ, exploiting Assumption 3 and the continuity with respect to the time, it yields the existence of an open interval (t_0,t_1) for which

$$\left|\frac{\partial h(X(x,s))}{\partial x}v\right| \;>\; 0 \qquad \forall s\in(t_0,t_1)\ , \tag{31}$$

with $\sigma^-_{\mathcal{I}+\delta_d}(x)\leq t_0 < t_1 \leq 0$. With the definition of the modified system (21), it yields

$$h(\breve{X}(x,s)) = h(X(x,s)) \qquad \forall s\in(t_0,t_1)\ .$$

Hence, with (30), the last equality and inequality (31) yield that:

$$\int_{-\infty}^{+\infty} |\mathcal{G}T(w,\mathtt{Re}(\lambda)+is)|^2\, ds \;>\; 0 \;.$$

This implies that for all w in Υ, the function $\lambda \mapsto \mathcal{G}T(w,\lambda)$ is not identically zero on Γ. Since this function is holomorphic, it yields that for all (w,λ) in $\Upsilon\times\Gamma$, there exists, for at least one of the $n+1$ components $\mathcal{G}T_j$ of $\mathcal{G}T$, an integer k which satisfies:

$$\left\{\begin{array}{l} \dfrac{\partial^i \mathcal{G}T_j}{\partial\lambda^i}(w,\lambda) \;=\; 0 \qquad \forall i\in\{0,\dots,k-1\}\;, \\[2ex] \dfrac{\partial^k \mathcal{G}T_j}{\partial\lambda^k}(w,\lambda) \;\neq\; 0\;. \end{array}\right.$$

Hence, employing Coron's Lemma with $\mathcal{G}$ as the g function, and by using (29), it allows to conclude that the set $\mathcal{A}_{le}$ defined by :

$$\mathcal{A}_{le} \;=\; \Bigg\{(\lambda_1,\dots,\lambda_{n+1})\in\Gamma^{n+1} \,:\, \exists\,(x,v)\in\Upsilon \,: \\ \frac{\partial T_{\lambda_i}}{\partial x}(x)v = 0 \;\forall i\in\{1,\dots,n+1\}\Bigg\}$$

has a zero Lebesgue measure in $\mathbb{C}^{n+1}$. □

4.4 Proof of Theorem 2

With Theorem 3 and 4 there exist a negative real number ρ, a subset $\mathcal{A}_e \subset \mathbb{C}^{n+1}$ of zero Lebesgue measure and defined as $\mathcal{A}_d \cup \mathcal{A}_{le}$ such that the function $T:\mathcal{I}\to\mathbb{C}^{(n+1)\times p}$ defined by (20) (with δ_d given in Assumption 2 and 3) is such that, provided A is a diagonal matrix with $n+1$ complex eigenvalues λ_i arbitrarily chosen in $(\mathbb{C}_\rho)^{n+1}\setminus\mathcal{A}_e$ the following holds.

1. For all x in $\mathcal{I}$, T is a C^2 solution of the PDE (12);
2. the function T it is injective in $\mathcal{I}$;
3. for all x in $\mathcal{I}$, $\frac{\partial T}{\partial x}(x)$ is full rank.

Consequently, given a matrix A with eigenvalues in $(\mathbb{C}_\rho)^{n+1}\setminus\mathcal{A}_e$ and with Proposition 3 the nonlinear Luenberger observer (2) estimates the state of System (1) and satisfies the exponential convergence property (7).

5 Conclusion

In this paper is presented a sufficient condition guaranteeing that a nonlinear Luenberger observer as introduced in [17], [7] and [2] converges exponentially towards the state of the model. This fact may be used to design some output feedback based on this observer. For instance some of these arguments have been used in output regulations in [6].

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