Proceedings of the American Control Conference
Arlington, VA June 25-27, 2001

On the analysis of uncertain hybrid systems with estimated-state feedback

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Abstract

In this paper we present an analysis technique for hybrid linear systems using observers for state feedback control. We present a stability proof for such systems under sector bounded dynamic uncertainties, and discuss the implications of observer feedback for hybrid systems.

1 Introduction

Control techniques relying on logic-based switching have a long-standing history, from classical variable structure control [1, 2] to recent approaches to adaptive control [3,4]. Each of these techniques is limited in some way to a class of hybrid systems: it is well-known that there exists no universal analytic technique for control synthesis in arbitrary hybrid systems. In fact, even stability analysis is nontrivial for these systems [5], and can prove to be analytically intractable [6]. We will thus limit our domain of investigation to the field of linear hybrid systems. Such systems are made up of M subsystems Σ_i, given by:

$$\dot{x} = A_i x + B_i u_i + p,\ i \in \mathcal{I}_M \tag{1}$$

where the state is $x \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ is the control for the subsystem (which may be zero) and p is an uncertainty term including effects of parametric uncertainties and unmodeled dynamics. The current subsystem dynamics Σ_i determine the vector field for the state x at each time instant. We assume that, while switching is either controlled or implicit, it is state-based (as opposed to the time-based methods of some earlier work [7]). The switching arbiter can be modeled as a secondary discrete system with state κ, where the dynamics of the overall system are given by (A_i, B_i) when $\kappa = i$.

One common method for synthesizing controller switching laws in hybrid control literature depends on the use of Lyapunov theory. A Lyapunov function is generated for each subsystem of the hybrid system, from which a variety of switching methods can be developed [8, 9]. A common technique when using Lyapunov functions for hybrid systems is to rely on specialized classes of such functions that admit analytic techniques (see, e.g., [10, 11]). A promising approach relies on piecewise-quadratic Lyapunov functions [12], although existing results focus on stability of such systems under perfect state feedback, which is never the case in reality.

In a hybrid system whose switching surfaces are state-based, the lack of perfect state feedback (and other uncertainties) requires careful consideration, as the system dynamics and the estimates thereof may switch (according to the state-based switching surfaces) at different times. Under feedback control, errors in state estimates could therefore result in a controller designed for subsystem Σ_i begin implemented when the actual system dynamics are given by some other Σ_q, with potentially destabilizing effects. To address these concerns, we will focus our attention on stability analysis for linear hybrid systems using state feedback control and observers. We will utilize generalized Lyapunov functions as a basis for our initial investigations.

The paper is organized as follows. A complete problem formulation is offered in Section 2, followed by our main stability results in Section 3. A discussion of the implications of these results in hybrid control synthesis is given in Section 4.

2 Problem Statement

In this article, we address the stabilization of plants modeled as hybrid systems. Moreover, we consider two challenging issues associated with the above problem. First, we take into account the possibility that the switching hypersurfaces are uncertain by including a modeling error term, p, in the continuous state equations as follows:

$$\begin{aligned} \dot{x} &= A_i x + B_i u_i + p, \ i \in \mathcal{I}_M \\ y &= Cx \end{aligned} \tag{2}$$

where $p^T p \leq \mu^2 x^T x$. Second, when the state vector is unavailable for feedback, we employ the following hybrid system in the feedback path:

$$\begin{aligned} u_i &= K_i z \\ \dot{z} &= A_i z + B_i K_i z + L_i(Cz - y) \end{aligned} \tag{3}$$

whose discrete state i is synchronous with the one of the model (2).

3 Main Result

Let us consider the composite system

$$\begin{bmatrix} \dot{z} \\ \dot{z} - \dot{x} \end{bmatrix} = \begin{bmatrix} A_i + B_i K_i & L_i C \\ \mathbf{0}_{n\times n} & A_i + L_i C \end{bmatrix} \begin{bmatrix} z \\ z - x \end{bmatrix} + \begin{bmatrix} 0_n \\ -p \end{bmatrix}$$

which, for simplicity, we write as follows

$$\dot{\tilde{x}} = \tilde{A}_i \tilde{x} + \tilde{p} \tag{4}$$

where $\tilde{p}^T \tilde{p} \leq \tilde{\mu}^2 \tilde{x}^T \tilde{x}$, $\tilde{\mu} \geq \mu$. In the sequel, we analyze the stability of the composite hybrid system (4) using the following concept of the generalized Lyapunov function [13].

Definition 1 A continuous function V is a global generalized Lyapunov function, iff

(i) V is proper, i.e., the set $\{w \in \mathbb{R}^l | V(w) \leq a\}$ is compact for each $a > 0$

(ii) V is positive definite, i.e., $V(0) = 0$ and $V(w) > 0$ for each $w \in \mathbb{R}^l \backslash \{0\}$

(iii) For each $w \in \mathbb{R}^l \backslash \{0\}$, there exists some time $\sigma > 0$ such that, along the trajectory $\xi(t)$, where $\xi(0) = w$,

$$V(\xi(t)) < V(w), \ t \in (0, \sigma]$$

Let $\mathcal{I}_N$ be the index set $\{1, \ldots, N\}$ and consider a collection $\{\Omega_j\}_{j \in \mathcal{I}_N}$ of closed subsets of $\mathbb{R}^{2n}$ such that:

1. $0 \in \Omega_j$, for all $j \in \mathcal{I}_N$,
2. $\bigcup_{j=1}^{N} \Omega_j = \mathbb{R}^{2n}$,
3. Int $\Omega_j \bigcap$ Int $\Omega_k = \emptyset$, for $j, k \in \mathcal{I}_N$.

In the sequel, we refer to such collection as a partition of $\mathbb{R}^{2n}$. Let us define $\mathcal{S}_{jk} := \Omega_j \bigcap \Omega_k$ and note that $\mathcal{S}_{jk}$

$= \text{Bdy}\ \Omega_j \bigcap \text{Bdy}\ \Omega_k$; thus, $\mathcal{S}_{jk}$ is either a hypersurface in $\mathbb{R}^{2n}$ or the singleton $\{0\}$.

The following lemma applies to the composite hybrid system (4).

Lemma 1 Let $\{\Omega_j\}_{j \in \mathcal{I}_N}$ be a partition of $\mathbb{R}^{2n}$ and $V : \mathbb{R}^{2n} \to \mathbb{R}$ be a continuous function. V is a generalized Lyapunov function for the composite hybrid system (4), if it is continuously differentiable on $\bigcup_{j=1}^{N} \text{Int}\ \Omega_j$ and its restriction to Ω_j, $V_j : \Omega_j \to \mathbb{R}$, is proper, positive definite, and, for each $\tilde{x} \in \Omega_j$,

$$\nabla V_j(\tilde{x}) \cdot (\tilde{A}_i \tilde{x} + \tilde{p}) < 0, \ i \in \mathcal{I}_M \qquad (5)$$

Proof: Clearly, V is both proper and positive definite. Furthermore, for each $\tilde{x} \in \text{Int}\ \Omega_j$, $j \in \mathcal{I}_N$, we have $\nabla V(\tilde{x}) = \nabla V_j(\tilde{x})$. Therefore, from (5) it follows that property (iii), in Definition 1, holds for each initial state $\tilde{x} \in \bigcup_{j=1}^{N} \text{Int}\ \Omega_j$. The gradient, ∇V, however, is undefined on the hypersurfaces that form the Bdy Ω_j, i.e., $\mathcal{S}_{jk}$, $k \in \mathcal{I}_N$. Suppose that $\tilde{x} \in \mathcal{S}_{jk}$, then $\xi(t)$ evolves inside Ω_k, $k \in \mathcal{I}_N$, or on $\mathcal{S}_{jk}$ before it crosses Bdy Ω_k. In the former case, for small $\tau > 0$, $\xi(\tau) \in \text{Int}\ \Omega_k$. By continuity of ∇V_k on Int Ω_k, we are able to choose $\sigma > \tau$ small enough so that

$$\frac{dV(\xi(t))}{dt} = \nabla V_k(\xi(t)) \cdot (\tilde{A}_i \xi(\tau) + \tilde{p}) < 0, \ i \in \mathcal{I}_M$$

for all $t \in [\tau, \sigma]$. Therefore, $V(\xi(t)) < V(\xi(\tau))$ for all $t \in (\tau, \sigma]$. Moreover, as $\tau \to 0$, $\xi(\tau) \to \tilde{x}$ hence $V(\xi(\tau)) \to V(\tilde{x})$. In turn, $V(\xi(t)) < V(\tilde{x})$ for all $t \in (0, \sigma]$. A similar argument applies to the latter case, for $\xi(t)$ evolves on $\mathcal{S}_{jk}$ only if

$$\nabla V_j(\xi(t)) \cdot (\tilde{A}_i \xi(\tau) + \tilde{p}) = \nabla V_k(\xi(t)) \cdot (\tilde{A}_i \xi(\tau) + \tilde{p})$$

(the alternative implies that V is discontinuous). ■

Consider now a generalized Lyapunov function with quadratic restrictions on a certain partition $\{\Omega_j\}_{j \in \mathcal{I}_N}$.

Lemma 2 Let V be a continuous function and $V_j(\tilde{x}) := \tilde{x}^T \tilde{P}_j \tilde{x}$, $\tilde{P}_j \in \mathbb{R}^{2n \times 2n}$ and $\tilde{P}_j > 0$, its restriction to Ω_j. Suppose, for each $\tilde{x} \in \Omega_j$ and any $\tilde{p}$,

$$\begin{bmatrix} \tilde{x} \\ \tilde{p} \end{bmatrix}^T \begin{bmatrix} \tilde{A}_i^T \tilde{P}_j + \tilde{P}_j \tilde{A}_i + \tilde{\mu}^2 I_{2n} & \tilde{P}_j \\ \tilde{P}_j & -I_{2n} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{p} \end{bmatrix} < 0, \ i \in \mathcal{I}_M \qquad (6)$$

Then, the composite hybrid system (4) is globally asymptotically stable (g.a.s.)

Proof: Using the $\mathcal{S}$-procedure [14], the sufficient condition (5) in the hypothesis of Lemma 1 derives from (6). ■

Hereafter, the candidate generalized Lyapunov function, V, assumes the following form

$$V(\tilde{x}) = \max_{j \in \mathcal{I}_N} \{\tilde{x}^T \tilde{P}_j \tilde{x}\} \qquad (7)$$

By definition, the candidate function (7) has quadratic restrictions on the partition $\{\Omega_j\}_{j \in \mathcal{I}_N}$ where

$$\Omega_j = \{\tilde{x} \in \mathbb{R}^{2n} | \tilde{x}^T \tilde{P}_j \tilde{x} \geq \tilde{x}^T \tilde{P}_k \tilde{x}, \ k \in \mathcal{I}_N \backslash \{j\}\} \qquad (8)$$

Clearly, $\Omega_j \neq \{0\}$—otherwise the corresponding $\tilde{P}_j$ is superfluous in the RHS of (7). Moreover, for $\tilde{x} \in \mathcal{S}_{jk} \neq \{0\}$, it follows that $\tilde{x}^T \tilde{P}_j \tilde{x} = \tilde{x}^T \tilde{P}_k \tilde{x}$; thus, the candidate function (7) is continuous. It is worth noticing that, in general, $\tilde{P}_j \tilde{x} \neq \tilde{P}_k \tilde{x}$ on $\mathcal{S}_{jk}$; thus, the candidate function is nonsmooth.

Using the $\mathcal{S}$-procedure, the hypothesis of Lemma 2 holds if there exist $\tilde{P}_j$ and $\tau_{ijk} \geq 0$ such that

$$\begin{bmatrix} \tilde{A}_i^T \tilde{P}_j + \tilde{P}_j \tilde{A}_i + \tilde{\mu}^2 I_{2n} - \sum_{k=1}^{N} \tau_{ijk} (\tilde{P}_k - \tilde{P}_j) & \tilde{P}_j \\ \tilde{P}_j & -I_{2n} \end{bmatrix} < 0, \ i \in \mathcal{I}_M \qquad (9)$$

Let us consider positive definite matrices with the following structure:

$$\tilde{P}_j = \begin{bmatrix} Q_j^{-1} & \\ & P_j \end{bmatrix}, \ j \in \mathcal{I}_N$$

After manipulation, inequality (9) holds if there exist $P_j > 0, Q_j^{-1} > 0$, and $\tau_{ijk} > 0$ that satisfy the following simultaneous inequalities:

$$\begin{aligned} &R_{ij}(L_i, P_j) := \\ &(A_i + L_iC)^T P_j + P_j(A_i + L_iC) + \tilde{\mu}^2 I_n + P_j^2 \\ &-\sum_{k=1}^{N} \tau_{ijk}(P_k - P_j) < 0, \ i \in \mathcal{I}_M \end{aligned} \qquad (10)$$

$$\begin{aligned} &(A_i + B_iK_i)^T Q_j^{-1} + Q_j^{-1}(A_i + B_iK_i) + \tilde{\mu}^2 I_n + Q_j^{-2} \\ &-Q_j^{-1} L_i C R_{ij}^{-1}(L_i, P_j)(L_iC)^T Q_j^{-1} \\ &-\sum_{k=1}^{N} \tau_{ijk}(Q_k^{-1} - Q_j^{-1}) < 0, \ i \in \mathcal{I}_M \end{aligned} \qquad (11)$$

We formally state the above result as follows.

Theorem 1 Consider the hybrid system (2) and the observer-based state feedback (3) or, equivalently, the composite hybrid system (4). Suppose there exist $P_j > 0$, $Q_j > 0$, $j \in \mathcal{I}_N$, and $\tau_{ijk} \geq 0$, $i \in \mathcal{I}_M$, $k \in \mathcal{I}_N$, that satisfy the matrix inequalities (10) and (11). Then, the composite hybrid system (4) admits (7) as a global generalized Lyapunov function; thus, the closed loop (2)-(3) is g.a.s.

The above theorem casts the sufficient condition for the stability of the composite hybrid system (4) as an optimization problem, namely, an eigenvalue problem subject to the bilinear matrix inequalities (10) and (11) for the observer and controller design, respectively. Bilinear matrix inequalities have emerged in other areas of control theory and a wide range of approaches to the solution of the resulting optimization problems is available; see [15] and the references therein.

4 Conclusion

In summary, we consider hybrid systems with uncertainty in the switching of the discrete state variable. The control objective is to stabilize such systems when the continuous state is inaccessible. We propose to design the closed loop using an observer-based hybrid system in the feedback. In addition, we consider that the controller and observer gain matrices switch synchronously with the discrete state variable of the system in forward path. To assess the stability of such design, we follow the approach of piecewise quadratic Lyapunov functions, which leads to the formulation of the analysis problem as two coupled bilinear matrix inequalities. Characteristic of the proposed method is the independence of the Lyapunov function from the partition of the state space of the hybrid system in the forward path. As a result, one is able to formulate the design and analysis as synthesis step by synchronizing the hybrid system in the feedback (i.e., controller and observer gain matrices) with the partition that the piecewise quadratic Lyapunov function generates on the state space of the composite hybrid system. A detailed treatment of this idea, however, warrants further research.

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