THE UNIVALENCE AXIOM IN POSETAL MODEL CATEGORIES

MISHA GAVRILOVICH, ASSAF HASSON*, AND ITAY KAPLAN

ABSTRACT. In this note we interpret Voevodsky's Univalence Axiom in the language of (abstract) model categories. We then show that any posetal locally Cartesian closed model category $\mathcal{Q}t$ in which the mapping $Hom^{(w)}(Z \times B, C) : \mathcal{Q}t \longrightarrow Sets$ is functorial in Z and represented in $\mathcal{Q}t$ satisfies our homotopy version of the Univalence Axiom, albeit in a rather trivial way. This work was motivated by a question reported in [Gar11], asking for a model of the Univalence Axiom not equivalent to the standard one.

1. INTRODUCTION

Though the notion of a categorical model of dependent type theory was known for quite some time now, it is only in recent years that it was realized that the extra categorical structure required to model the structure of equality in dependent type theory corresponds to the structure of weak factorization equivalence, occurring in Quillen's model categories ([Gar11, p.2]). This connection is the basis for V. Voevodsky project known as *univalent foundations* whose main objective is to give a foundation of mathematics based on dependent type theory, which is intrinsically homotopical, in which types are interpreted not as sets, but rather as homotopy types (cf.). A central ideal in Voevodsky's univalent foundations is the extension of Martin-Löf's dependent type theory by a "homotopy theory reflection principle", known as the *Univalence Axiom*. Roughly speaking, the Univalence Axiom is the condition that the identity type between two types is naturally weakly equivalent to the type of weak equivalences between these types (V. Voevodsky, talk at UPENN, May 2011).

Within the category (or, rather, the model category) of simplicial sets $sSets$, Voevodsky constructs a model of Martin-Löf dependent type theory, satisfying also the Univalence Axiom. The models constructed in this way are called the standard univalent models (cf. Definition 3.2). During a mini-workshop around these developments held in Oberwolfach in 2010 the following question was raised: "Does UA have models in other categories (e.g., 1-topoi) not equivalent to the standard one?", [Gar11, p.27]. Though this question is probably referring to a univalent universe (for type theory), it seems to be meaningful also if taken literally. It turns out that the Univalence Axiom can be given a precise meaning in the framework of Quillen's model categories (provided they are locally Cartesian closed). It is then meaningful to ask whether such a model category satisfies the Univalence Axiom.

There are two main parts to this note. In the first of these parts (Section 3) we give an interpretation of the notion of a univalent fibration in a purely category theoretic language. To formulate this notion we introduce, for a model category $\mathfrak{C}$, a correspondence $Hom^{(w)}(Z \times B, C) : \mathfrak{C} \longrightarrow Sets$, intended to capture the class of weak equivalences between given fibrant objects $B, C \in \mathcal{O}b\mathfrak{C}$. We then show that, given a fibration $p : C \xrightarrow{(f)} B$, if $Hom^{(w)}_{B \times B}(- \times B \times C, C \times B)$ is a representable functor (in the slice category $\mathfrak{C}/B \times B$), the "obvious" morphism (in $\mathfrak{C}/B \times B$) from the diagonal B_δ to $Hom_{B \times B}(B \times C, C \times B)$ factors "naturally" (and uniquely in that sense) through the object representing this functor, $((C \times B)^{B \times C})_w$. We can then define the fibration p to be *univalent* if the morphism $m : B_\delta \longrightarrow ((C \times B)^{B \times C})_w$ is a weak equivalence. This construction (or a closely related one) is probably known to experts in the field, but since we could not find any reference suitable for our purposes we give it in textbook detail.

We then introduce the notion of a locally (w/f)-Caretsian closed model category, which is a locally Cartesian closed model category with the additional property that $Hom^{(w)}_{B \times B}(- \times B \times C, C \times B)$ is a representable functor for any fibrant objects B, C and fibration $p : C \longrightarrow B$. We observe that in a posetal (w/f)-Cartesian closed model category all fibrations are univalent in the above sense. This is, of

---

course, to be expected in view of Voevodsky's informal description of a univalent fibration as "...one of which every other fibration is a pullback in at most one way (up to homotopy)". Apparently, this should suffice to assure that any posetal (w/f)-Cartesian closed model category satisfies the Univalence Axiom. But this does not provide, to our taste, a satisfying analogy with Voevodsky's construction. Such an analogy should have a natural interpretation of all the key features in Voevodsky's construction. To our understanding one such feature of univalent models is that they come equipped with a universal (univalent) fibration, of which all "small" fibrations are a pullback (in a unique way), [Voe10, Theorem 3.5]. So our aim is to show, in addition, that such a universal fibration exists in our model category (with respect to an appropriate notion of smallness).

The second of the main parts of the paper (Section 4) is dedicated to a self-contained construction of a posetal locally (w/f)-Cartesian closed model category, QtNaamen_c. This construction is a special case of a more general construction introduced in [GH10]. In [Gar11, p.8] Voevodsky writes: "Now for any $A, B : U$, it is possible to construct a term $\theta : pathsU(A, B) \longrightarrow weq(A, B)$... The Univalence Axiom states that the map θ should itself be a weak equivalence for every $A, B : U$". The map θ in Voevodsky's quote corresponds (to the best of our understanding) to the morphism m appearing in the factorisation of the "obvious morphism" mentioned above. It is now obvious that this formulation of the Univalence Axiom is satisfied (in a rather trivial sense) in QtNaamen_c. To fulfill our goals, it remains to construct an analogue in QtNaamen_c of Voevodsky's universe of "small" fibration (those fibrations all of whose fibers are of cardinality smaller than α for some cardinal α). To that end we suggest a (possibly over-simplified) notion of smallness for fibrations in a posetal model category, and show that with this definition QtNaamen_c admits a universe of small fibration (which is automatically univalent).

Admittedly, the model category QtNaamen_c may be too simple an object to be of real interest. In [GH10] we suggest a construction of a (c)-(f)-(w)-labelled category analogous to that of QtNaamen_c resulting in a non-posetal category whose slices are equivalent to those of QtNaamen_c. This category satisfies axioms (M1)-(M5) of Quillen's model categories, but does not have products (and co-products). We ask whether this richer category can be embedded in a model category, and whether such a model category would satisfy the Univalence Axiom, as formulated in this note.

It should be made clear that none of the authors of this note is familiar with type theory and its categorical models. When we realized, moreover, that formally accurate literature on Voevodsky's univalent foundations exists only in the form of Coq code, we decided to base our homotopy theoretic interpretation of the Univalence Axiom on the somewhat less formal presentation appearing, e.g., in [Voe10], [Gar11] and similar sources whose language is closer to the categorical language for which we were aiming. To compensate for the lack of precise references, we have taken some pains to give a detailed formal account of our interpretation of those sources. M. Warren's comments and clarifications, [War], were of great help to us, but all mistakes, are - of course - ours.

A couple of words concerning terminology and notation are in place. In this text we refer to Quillen's axiomatization of model categories, as it appears in [Qui67]. Our usage of "Axiom (M0)...(M5)" refers to Quillen's enumeration of his axioms in that book. Our commutative diagram notation is pretty standard, and is explained in detail in [GH10]. The labeling of arrows, (c) for co-fibrations, (f) for fibrations and (w) for weak equivalences, is borrowed from N. Durov.

2. CARTESIAN CLOSED POSETAL CATEGORIES

Given a category $\mathfrak{C}$ and $B, C \in \mathcal{O}b\mathfrak{C}$, it is often desirable to treat $Hom(B, C)$ as an object of the category: this is a natural requirement, as it is inconvenient, while working in $\mathfrak{C}$, to be constantly required to work with elements external to $\mathfrak{C}$, namely, working with Hom-sets merely as sets. A category $\mathfrak{C}$ is Cartesian closed if it is closed under "exponentiation", namely, that given $B, C \in \mathcal{O}b\mathfrak{C}$, an object C^B (satisfying certain category theoretic properties to be explained shortly) exists. As notation suggests, the object C^B is supposed to represent the set $Hom(B, C)$. We will now explain this in more detail:

A category $\mathfrak{C}$ (with finite limits, or at least binary products) is called *Cartesian closed* if for every $B, C \in \mathcal{O}b\mathfrak{C}$ there exist an object, denoted C^B, and an arrow $\epsilon : C^B \times B \longrightarrow C$ such that for every object

D and arrow $D \times B \xrightarrow{g} C$, there is a unique arrow $f : D \longrightarrow C^B$ such that $D \times B \xrightarrow{f \times \mathrm{id}_B} C^B \times B \xrightarrow{\epsilon} C$ and $D \times B \xrightarrow{g} C$ coincide [Awo10, 6.2,p.108].
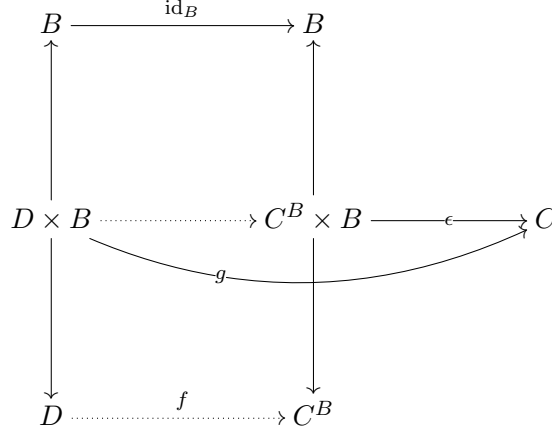

FIGURE 1. The existence of the arrow $D \xrightarrow{f} C^B$ assures the existence of the arrow $D \times B \xrightarrow{f \times \mathrm{id}_B} C^B \times B$ by the universal property of $C^B \times B$.

Given $B, C \in \mathfrak{C}$, Figure 1 implies the existence of a bijective correspondence, functorial in D, between $Hom(D \times B, C)$ and $Hom(D, C^B)$, which we can write as:

$$(*) \qquad Hom(D \times B, C) \equiv Hom(D, C^B)$$

In the above equation we say that C^B is an *object representing* the functor $Hom(- \times B, C) : \mathfrak{C} \longrightarrow Sets$, i.e. that this functor is naturally equivalent to the functor $Hom(-, C^B)$. This equation is of importance not only in understanding the ideology behind the definition of a Cartesian closed category, but will also play an important role in our interpretation of the Univalence Axiom in a model category.

To see why C^B can be, in many cases, identified with the set $Hom(B, C)$ consider, in the equation $(*)$ the terminal object, $\top$ (for the variable D). We get an equivalence of categories:

$$Hom(B, C) \equiv Hom(\top \times B, C) \equiv Hom(\top, C^B)$$

which, in many cases (e.g., in the category $Sets$, or in the category Top where $\top$ is a point) gives:

$$Hom(B, C) = Hom(\top, C^B) = C^B$$

This explains the general category theoretic convention of identifying exponents with Hom-sets.

In this note we will be interested, mainly, in posetal model categories. We conclude this section with a discussion of a posetal category being Cartesian closed. Recall that a category $\mathfrak{C}$ is *posetal* if arrows are unique whenever they exist. Namely, given $B, C \in \mathcal{O}b\mathfrak{C}$ there exists at most one $f \in \mathcal{M}or\mathfrak{C}$ such that $B \xrightarrow{f} C$. Thus, in a posetal category all diagrams are commutative, and therefore, as can be seen in Figure 1, if $\mathfrak{C}$ is posetal, to verify that $\mathfrak{C}$ is Cartesian closed it is enough to verify that for any $B, C \in \mathcal{O}b\mathfrak{C}$ there exists an object C^B such that $C^B \times B \longrightarrow C$ and such that for every object D, $D \times B \longrightarrow C$ implies $D \longrightarrow C^B$.

Given a category $\mathfrak{C}$ and $A \in \mathcal{O}b\mathfrak{C}$, the *slice of $\mathfrak{C}$ over A*, denoted $\mathfrak{C}/A$ is the category of arrows $B \longrightarrow A$: its objects are arrows $B \longrightarrow A$ in $\mathfrak{C}$ and an arrow from $B \longrightarrow A$ to $C \longrightarrow A$ is an arrow in $\mathfrak{C}$ making the triangular diagram commute. For a posetal category the slice $\mathfrak{C}/A$ can be identified with the full sub-category whose objects are all $B \in \mathcal{O}b\mathfrak{C}$ such that $B \longrightarrow A$.

A category is *locally Cartesian closed* if $\mathfrak{C}/A$ is Cartesian closed for all $A \in \mathcal{O}b\mathfrak{C}$, [Awo10, Prop.9.20,p.206]. Observe that a posetal category with a terminal object is Cartesian closed if and only if it is locally Cartesian closed. Indeed, $\mathfrak{C}$ has a terminal object $\top$ and $\mathfrak{C}/\top$ — which is, by assumption, Cartesian closed - is merely $\mathfrak{C}$, so locally Cartesian closed implies Cartesian closed. In the other direction, if $\mathfrak{C}$ is

Cartesian closed and $A \in \mathcal{Ob}\mathfrak{C}$ is any object, $B, C \in \mathcal{Ob}\mathfrak{C}/A$ then $C^B \times A \longrightarrow A$. Thus, $C^B \times A \in \mathcal{Ob}\mathfrak{C}/A$. So it remains to verify that for any $D \in \mathcal{Ob}\mathfrak{C}/A$, if there exist an arrow $D \times B \longrightarrow C$ then there exist an arrow $D \longrightarrow C^B \times A$. By definition, there is an arrow $D \longrightarrow C^B$, and since $D \in \mathcal{Ob}\mathfrak{C}/A$ there is an arrow $D \longrightarrow A$. By the universal property of $C^B \times A$ this means that there is an arrow $D \longrightarrow C^B \times A$, as required.

3. The Univalence Axiom

As explained in the introduction, the original formulation of the Univalence Axiom is given in the language of type theory (and, apparently, its precise formulation exists only in Coq code). The axiom asserts that, given a universe of type theory, the homotopy theory of the types in this universe should be fully and faithfully reflected by the equality on the universe. To prove that the universes of type theory he constructs in the category $sSets$ are univalent, Voevodsky proves, [Voe10, Theorem 3.5], that there is a fibration universal for the class of *small* fibrations, and that this fibration is *univalent*. Apparently, this statement is the right reformulation of the Univalence Axiom in the context of the model category of simplicial sets.

In order to generalize the Univalence Axiom to arbitrary (locally Cartesian closed) model categories, one has to explain what it means for a fibration to be univalent, and to define a suitable notion of smallness. Our first step is to define (and explain) what is a univalent fibration in an arbitrary locally Cartesian closed model category. We then show using this definition, that if our locally Cartesian closed model category, $\mathfrak{C}$, is posetal, then every fibration is univalent. Thus, to show that such a model category $\mathfrak{C}$ meets the Univalence Axiom (for a suitable notion of smallness) it remains to show that a universal fibration for all small fibrations exists. This section is concluded with the observation that this is indeed the case for a natural (though somewhat trivial) notion of smallness, provided $\mathfrak{C}$ is posetal.

3.1. **A model category object for weak equivalences.** Recall that Voevodsky's formulation of the Univalence Axiom takes place in the category of simplicial sets. In order to reformulate this axiom in the more general setting of model categories we have to set up a dictionary between Voevodsky's terminology and the common terminology of model categories. Apparently, such a translation is folklore to the experts, but since we were unable to find a precise formulation meeting the level of generality need for this note, we give the details. The main difficulty in this translation is the definition of a univalent fibration. Since there is no literature on the subject, our translation of this notion relies almost entirely on Voevodsky's notes, [Voe10], and some clarifications corresponded to us by Warren, [War].

Let us recall Voevodsky's definition of a univalent fibration in the category $sSets$ of simplicial sets, [Voe10, p.7]:

> For any morphism $q : E \to B$ consider the simplicial set $\underline{Hom}_{B \times B}(E \times B, B \times E)$. If q is a fibration then it contains, as a union of connected components, a simplicial subset $weq(E \times B, B \times E)$ which corresponds to morphisms which are weak equivalences. The obvious morphism from the diagonal $\delta : B \to B \times B$ to $\underline{Hom}_{B \times B}(E \times B, B \times E)$ over $B \times B$ factors uniquely through a morphism $m_q : B \to weq(E \times B, B \times E)$.

In this terminology the fibration $q : E \longrightarrow B$ is *univalent* if the morphism $m_q : B \longrightarrow weq(E \times B, B \times E)$ is a weak equivalence (cf. Definition 3.4 [ibid.])

Voevodsky's text translates readily into the language of Cartesian closed model categories, with the possible exception of the definition of the object $weq(E \times B, B \times E)$. In this subsection we perform this translation, focusing on the model categorical definition of $weq(E \times B, B \times E)$. As we will see, the object $weq(C, B)$ has much in common with the exponential C^B, it is therefore convenient to introduce:

Notation 1. *Given a model category $\mathfrak{C}$ and $B, C \in \mathcal{Ob}\mathfrak{C}$, the object $weq(C, B)$ will be denoted C_w^B.*

For the sake of clarity, we explain the above text word for word. So let $\mathfrak{C}$ be a locally Cartesian closed model category, $E, B \in \mathcal{Ob}\mathfrak{C}$ and $q : E \longrightarrow B$ a fibration. Let $E \times B$ be the product of E and B in $\mathfrak{C}$. This objects comes with two morphisms: $E \times B \xrightarrow{\mathrm{pr}_E^{E \times B}} E$ and $E \times B \xrightarrow{\mathrm{pr}_B^{E \times B}} B$. Since a morphism into a product is uniquely determined by a pair of morphisms into its components the following defines

a unique morphism: $(q, id) : E \times B \xrightarrow{q \circ \mathrm{pr}_E^{E \times B} \times \mathrm{pr}_B^{E \times B}} B \times B$. In set-theoretic notation, the morphism $q \circ \mathrm{pr}_E^{E \times B} \times \mathrm{pr}_B^{E \times B}$ defined above is given by the mapping $(e, b) \mapsto (q(e), b)$.

As an object of $\mathfrak{C}/B \times B$ this morphism is denoted by Voevodsky $E \times B$. In order to define the object $B \times E \in \mathcal{Ob}\mathfrak{C}/B \times B$ observe that (in $\mathfrak{C}$) the object $B \times B$ comes equipped with two morphisms $\mathrm{pr}_1, \mathrm{pr}_2$ into each of its components. Thus, there is a morphism (in $\mathfrak{C}$) $\tau_{B \times B} : B \times B \xrightarrow{\mathrm{pr}_2 \times \mathrm{pr}_1} B \times B$ (which can be thought of us the morphism permuting the factors of the product). In set-theoretic notation $\tau_{B \times B}(b_1, b_2) = (b_2, b_1)$. The morphism $\tau \circ (q, id)$ as an object of $\mathfrak{C}/B \times B$ is denoted by Voevodsky $B \times E$.

Thus, we have interpreted $B \times E$ and $E \times B$ as objects in the slice category $\mathfrak{C}/B \times B$. In view of our discussion of exponentials in Section 2, this allows us to identify $\underline{Hom}_{B \times B}(E \times B, B \times E)$ with the object $((B \times E)^{E \times B})_{B \times B}$ (recall that our assumption that $\mathfrak{C}$ is locally Cartesian closed assures that such an object exists).

Recall that, in Voevodsky words, "[the object $weq(E \times B, B \times E)$] corresponds to morphisms [i.e., elements of $Hom_{B \times B}(E \times B, B \times E)$] which are weak equivalences [in the slice category $\mathcal{Q}t/B \times B$]". Let us now try to understand, in more generality, given a Cartesian closed model category $\mathfrak{C}$ and objects $B, C \in \mathcal{Ob}\mathfrak{C}$ what should be the object C_w^B. In the terminology used in Section 2 Voevodsky's text should mean that the object C_w^B *represents* the set of morphisms from B to C satisfying the additional requirement that these morphisms are weak equivalences. In a Cartesian closed (model) category we identified $Hom(B, C)$ with the functor $Hom(- \times B, C)$. Since, in Voevodsky's text C_w^B is a sub-object of the exponential C^B, it is natural to try and identify C_w^B with a sub-functor, let us denote it $Hom^{(w)}(- \times B, C)$, of $Hom(- \times B, C)$. Moreover, any $Z \in \mathcal{Ob}\mathfrak{C}$ and morphism $f : B \longrightarrow C$ induces a morphism $Z \times B \longrightarrow C$ given by $f \circ \mathrm{pr}_B^{Z \times B}$. It is, therefore, reasonable to require that the same be true for the sub-functor $Hom^{(w)}(- \times B, C)$. The "obvious" choice of letting $Hom^{(w)}(Z \times B, C)$ be the set of all morphisms $h : Z \times B \xrightarrow{(w)} C$ does not have this property. So the next best choice seems to be:

Notation 2. *Given $Z, B, C \in \mathcal{Ob}\mathfrak{C}$, let*

$$Hom^{(w)}(Z \times B, C) := \{h : Z \times B \longrightarrow C \mid (\mathrm{pr}_Z^{Z \times B} \times h) : Z \times B \xrightarrow{(w)} Z \times C\}.$$

Observe that $Hom^{(w)}(Z \times B, C) \subseteq Hom(Z \times B, C)$ for all Z. We do not know, however, whether — in general — it is functorial in Z. We leave it as an exercise to the reader to show that if $\mathfrak{C}$ is right proper (i.e., if the base change of a weak equivalence along a fibration is again a weak equivalence), then $Hom(- \times B, C)$ is indeed functorial provided that $B \xrightarrow{(f)} \top$ and $C \xrightarrow{(f)} \top$. We remind that $sSets$ is right proper, and so is Top — and, more generally, any model category all of whose objects are fibrant (see below) is right proper.

At all events, if $Hom^{(w)}(- \times B, C)$ is a functor, and as such it is represented in $\mathfrak{C}$ we let C_w^B denote the representing object. In particular we obtain:

$(\ast\ast)$ $\qquad\qquad\qquad\qquad Hom^{(w)}(Z \times B, C) \equiv Hom(Z, C_w^B)$

Definition 3. *Let $\mathfrak{C}$ a model category. Say that $\mathfrak{C}$ is (w/f)-Cartesian closed, if it is Cartesian closed and, in addition, $Hom^{(w)}(- \times B, C) : \mathfrak{C} \longrightarrow Sets$ is represented (in the sense of $(\ast\ast)$ above) for all fibrant $B, C \in \mathcal{Ob}\mathfrak{C}$ (i.e., the morphisms $B \xrightarrow{(f)} \top$ and $C \xrightarrow{(f)} \top$ into the terminal object, $\top$ are fibrations). Say that $\mathfrak{C}$ is locally (w/f)-Cartesian closed, if for any $X \in \mathcal{Ob}\mathfrak{C}$ the slice category $\mathfrak{C}/X$ is (w/f)-Cartesian closed.*

Remark 4. *The above definition is our straightforward interpretation of Voevosky's words in the langauges of Cartesian closed model categories. This definition is sufficeint for our purposes, as it is met by the model category $QtNaamen_c$, constructed in the last section of this note. It is conceivable that, in the general setting, a more accurate definition will be required.*

We will now show that if $\mathfrak{C}$ is a (locally) Cartesian closed model category such that $Hom_{B \times B}^{(w)}(- \times (E \times B), B \times E)$ is represented in $\mathfrak{C}/B \times B$ then the object representing this functor satisfies the requirement

in Voevodsky's text, namely, the "obvious" morphism from the diagonal to $\underline{Hom}_{B\times B}(E\times B, B\times E)$ factors uniquely through this representing object.

Before we proceed, some explanations are needed. Recall that we are working in the slice category $\mathfrak{C}/B\times B$. Thus, the diagonal $\delta : B \to B\times B$ is an object of $\mathfrak{C}/B\times B$, which we denote B_δ. To avoid confusion, we denote $E\underline{\times}B := E\times B$ and $B\underline{\times}E := B\times E$ (viewed as objects of $\mathfrak{C}/B\times B$, as explained above). We shall also let, given an object $X \in \mathcal{Ob}\mathfrak{C}/B\times B$, $X_s \in \mathcal{Ob}\mathfrak{C}$ denote the source object of the morphism (in $\mathfrak{C}$) corresponding to X. We have already explained in Section 2 in what sense $\underline{Hom}_{B\times B}(E\underline{\times}B, B\underline{\times}E)$ can be viewed as an object of $\mathfrak{C}/B\times B$. So in order to make Voevodsky's statement clear we only have to explain what is the "obvious morphism" from the diagonal to $\underline{Hom}_{B\times B}(E\underline{\times}B, B\underline{\times}E)$.

Consider the product $B_\delta \times_{B\times B} E\underline{\times}B$, by definition it is the pullback of the morphisms $B \xrightarrow{\delta} B\times B$ and $E\times B \xrightarrow{(q,id)} B\times B$.

$$
\begin{array}{ccc}
(B_\delta \times_{B\times B} E\underline{\times}B)_s & \xrightarrow{\ \text{pr}_2\ } & E\times B \\
\downarrow{\scriptstyle \text{pr}_1} & & \downarrow{\scriptstyle (q,\mathrm{id}_B)} \\
B & \xrightarrow{\ \delta\ } B\times B \xrightarrow{\ \tau\ } & B\times B
\end{array}
$$

$$B \xrightarrow{\qquad \delta \qquad} B\times B$$

FIGURE 2

It follows immediately from the fact that δ is the diagonal morphism and from the definition of τ that $\tau\circ\delta\circ\text{pr}_1 = \delta\circ\text{pr}_1$. The right hand side morphism in the above equality corresponds in $\mathfrak{C}/B\times B$ to the object $B_\delta \times (E\underline{\times}B)$, while the composition $\tau\circ(q,\mathrm{id}_B)$ corresponds, by definition, to the object $B\underline{\times}E$. The commutativity of the diagram of Figure 2 implied by the above equality means, by definition of $\mathfrak{C}/B\times B$ that the morphism pr_2 corresponds in $\mathfrak{C}/B\times B$ to a morphism $h : B_\delta \times (E\underline{\times}B) \longrightarrow B\underline{\times}E$ in $\mathfrak{C}/B\times B$. By $(*)$ the morphism h corresponds to a morphism $\bar{m}_q : B_\delta \longrightarrow \underline{Hom}_{B\times B}(E\underline{\times}B, B\underline{\times}E)$. The morphism $\bar{m}_q$ is *the obvious morphism from the diagonal $\delta : B \to B\times B$ to $\underline{Hom}_{B\times B}(E\underline{\times}B, B\underline{\times}E)$ over $B\times B$.*

Let us denote π_1 and π_2 the morphisms in $\mathfrak{C}/B\times B$ corresponding to the morphisms pr_1 and pr_2 respectively (see Figure 2). Note that these morphisms can be identified with the two canonical morphisms associated to $B_\delta \times (E\underline{\times}B)$ as the pullback of B_δ and $E\underline{\times}B$. Thus, we have a morphism (in $\mathfrak{C}/B\times B$) $\pi_1 \times (\tau\circ\pi_2) : B_\delta \times (E\underline{\times}B) \longrightarrow B_\delta \times (B\underline{\times}E)$. This morphism, by definition of $\mathfrak{C}/B\times B$, arises from a morphism $\sigma : ((B_\delta \times (E\underline{\times}B))_s \longrightarrow ((B_\delta \times (B\underline{\times}E))_s$ in $\mathfrak{C}$. This morphism in $\mathfrak{C}$ is readily seen to be an isomorphism, and therefore a weak equivalence. By the definition of the model structure on $\mathfrak{C}/B\times B$, a morphism $X \longrightarrow Y$ in $\mathfrak{C}/B\times B$ is labelled (w), (f) or (c) if and only if the corresponding morphism $X_s \longrightarrow Y_s$ in $\mathfrak{C}$ is labelled (w), (f) or (c) respectively. Thus σ (or, rather, $(\pi_1, \tau\circ\pi_2)$) is a weak equivalence also in $\mathfrak{C}/B\times B$. By definition this means precisely that $h \in Hom^{(w)}(B_\delta \times (E\underline{\times}B), B\underline{\times}E)$. Therefore, $(**)$ implies that h corresponds to a unique morphism $m_q : B_\delta \longrightarrow ((E\underline{\times}B)^{(B\underline{\times}E)})_w$.

Finally, observe that in any (w/f)-Cartesian closed model category, $\mathfrak{C}$, and for all fibrant $B, C \in \mathcal{Ob}\mathfrak{C}$ setting $D = C_w^B$ in $(**)$, we get that the identity $\mathrm{id} : C_w^B \longrightarrow C_w^B$ is an element of $Hom(C_w^B, C_w^B)$ and therefore also of $Hom^{(w)}(C_w^B \times B, C)$. Because $Hom^{(w)}(D\times B, C) \subseteq Hom(D\times B, C)$ for all D, we get that $\mathrm{id} : C_w^B \longrightarrow C_w^B$ is an element of $Hom(C_w^B \times B, C) \equiv Hom(C_w^B, C^B)$. Thus, $\mathrm{id} : C_w^B \longrightarrow C_w^B$ induces, through this last equivalence of functors, a morphism $\mathrm{id}_*^{(B,C)} : C_w^B \longrightarrow C^B$ and a natural transformation $\mathrm{id}_*^{(B,C)} : Hom^{(w)}(-\times B, C) \equiv Hom(-, C_w^B)$ coinciding with the natural transformation provided by the inclusion $Hom^{(w)}(-\times B, C) \subseteq Hom(-\times B, C)$. Combining this with the conclusion of the previous paragraph, we get that $\bar{m}_q = \mathrm{id}_*^{(B,C)} \circ m_q$. The identification of $\mathrm{id} : C_w^B \longrightarrow C_w^B$ as an element of $Hom(-\times B, C)$ is natural in that sense. Requiring that $\bar{m}_q$ factors *naturally* through C_w^B

amounts, therefore, to the requirement that this factorization is obtained via $\mathrm{id}_*^{(B,C)}$. We observe that with this additional requirement this factorization is unique.

3.2. The Univalence Axiom in posetal model categories.
Having defined the object C_w^B for a locally (w/f)-Cartesian closed model category $\mathfrak{C}$, we can define a fibration $p : E \longrightarrow B$ to be *univalent* if the morphism $\bar{m}_q : B_\delta \longrightarrow ((E\underline{\times}B)^{(B\underline{\times}E)})_w$ is a weak equivalence. In this subsection we prove:

Lemma 5. *Let $\mathfrak{C}$ be a locally (w/f)-Cartesian closed posetal model category. Then every fibration is univalent.*

Proof. First, observe that since $\mathfrak{C}$ is posetal for any object $B \in \mathcal{O}b\mathfrak{C}$ the product $B \times B$ is isomorphic to B. Indeed, by the universal property of $B \times B$ there is a morphism $B \longrightarrow B \times B$, and since $B \times B \longrightarrow B$ we get that $B \cong B \times B$ (because $\mathfrak{C}$ is posetal).

Recall that since $\mathfrak{C}$ is posetal, for any object B the slice category $\mathfrak{C}/B$ can be identified with the full subcategory whose objects are $\{A \in \mathcal{O}b\mathfrak{C} : A \longrightarrow B\}$. Namely, the morphism $A \longrightarrow B$, as an object in $\mathfrak{C}/B$ can be identified with the object A in $\mathfrak{C}$. In particular B_δ can be identified with the object B, and given a fibration $E \xrightarrow{(f)} B$ the objects $E\underline{\times}B$ and $B\underline{\times}E$ in $\mathfrak{C}/B \times B \equiv \mathfrak{C}/B$ are isomorphic and both can be identified with the object $E \times B$ of $\mathfrak{C}$ (indeed, in a posetal model category $E\underline{\times}B$ and $B\underline{\times}E$ are the same object since $(E\underline{\times}B)_s = (B\underline{\times}E)_s$). So $\underline{Hom}_{B\times B}(E\underline{\times}B, B\underline{\times}E)$ is isomorphic to the object $(E \times B)^{E\times B}$.

It will suffice to show that for any object $C \in \mathcal{O}b\mathfrak{C}$ the exponent C^C is isomorphic to $\top$, the terminal object of $\mathfrak{C}$. Indeed, then $\underline{Hom}_{B\times B}(B\underline{\times}E, E\underline{\times}B) = ((E \times B)_{B\times B}^{(E\times B))}) = \top_{B\times B}$. But the terminal object of $\mathfrak{C}/B \times B$ is $B \times B = B$. We get that the "obvious morphism" $h : B_\delta \longrightarrow \underline{Hom}_{B\times B}(E\underline{\times}B, B\underline{\times}E)$ defined in the previous subsection corresponds to the arrow $B \longrightarrow B$, so it is an isomorphism, and therefore a weak equivalence. But in a posetal model category, if $X \longrightarrow Y$ is an isomorphism then for all Z, if $X \longrightarrow Z \longrightarrow Y$ then $X \longrightarrow Z$ and $Z \longrightarrow Y$ are both isomorphisms. In particular, the morphism $\bar{m}_q : B_\delta \longrightarrow ((E\underline{\times}B)^{(B\underline{\times}E)})_w$ is an isomorphism, and therefore a weak fibration.

It remains, therefore, to show that in a posetal Cartesian closed model category $C^C \cong \top$ for all $C \in \mathfrak{C}$. Indeed, $\top \times C \cong C$, implying $\top \times C \longrightarrow C$. So by Figure 2 (with $D = \top$ and $B = C$) we get an arrow $\top \longrightarrow C$, and $\mathfrak{C}$ being posetal we get $C^C \cong \top$. $\qquad\square$

Having seen that in posetal locally Cartesian closed model categories the notion of univalent fibrations degenerates, it remains to show that there exists a fibration p universal for the class of *small* fibrations. Of course, the notion of smallness in this context should be defined as well.

Definition 6. *Let $\mathfrak{C}$ be a model category, Fix a morphism $\tilde{U} \xrightarrow{p} U$. A morphism $Y \xrightarrow{f} X$ is p-small if $Y \xrightarrow{f} X$ fits in a pull-back square:*

$$
\begin{array}{ccc}
Y & \dashrightarrow & \tilde{U} \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p} \\
X & \xrightarrow{f_p} & U
\end{array}
$$

FIGURE 3. This is a pullback square, if for any morphisms $Z \longrightarrow X$ and $Z \longrightarrow \tilde{U}$ making the diagram commute there is an arrow $Z \longrightarrow Y$ making the diagram commute.

Say that p is universal *(with respect to a pre-defined class of* small *fibrations) if the class of p-small fibrations contains all small fibrations.*

Observe that in a posetal category, given morphisms p and f as in the above definition, the morphism $X \xrightarrow{f_p} U$ is unique if it exists. Therefore, $Y \xrightarrow{f} X$ is p-small if and only if $X \longrightarrow U$ and $Y = \tilde{U} \times X$.

Lemma 7. *Let $\mathcal{Q}t$ be a posetal model category. Consider the unique morphism $\varnothing \longrightarrow \top$ and let $\tilde{U}$ be the unique object such that $\varnothing \xrightarrow{(wc)} \tilde{U} \xrightarrow{(f)} \top$. Let p denote the fibration $\tilde{U} \xrightarrow{(f)} \top$. Assume, in addition, that all morphisms in $\mathcal{Q}t$ are co-fibrations. Then a fibration $f : Y \longrightarrow X$ is p-small iff $\emptyset \xrightarrow{(wc)} Y$.*

Proof. The key to the proof is the following observation:

Claim If $Z \longrightarrow \tilde{U}$ then $Z \xrightarrow{(wc)} \tilde{U}$.

Proof. Let $Z \xrightarrow{(wc)} Z_{wc} \xrightarrow{(f)} \tilde{U}$ It will suffice to prove that $\tilde{U} \longrightarrow Z$, since then $Z_{(wc)}$ is isomorphic to $\tilde{U}$ (member that $\mathcal{Q}t$ is posetal). Indeed, consider the following diagram:

$$
\begin{array}{ccc}
\bot & \longrightarrow & Z_{wc} \\
\downarrow {\scriptstyle(wc)} & \nearrow & \downarrow {\scriptstyle(f)} \\
\tilde{U} & \longrightarrow & \tilde{U}
\end{array}
$$

Finishing the proof of the claim $\qquad\qquad \square_{\text{Claim}}$

Now, if $\bot \xrightarrow{(wc)} Y \xrightarrow{(f)} X$ (where $\bot$ is the initial object), then $\bot \longrightarrow Y \curlywedge \tilde{U} \longrightarrow \top$, giving $Y \longrightarrow \tilde{U}$. So it suffices to show that $Y = \tilde{U} \times X$. Because $\bot \xrightarrow{(wc)} Y \curlywedge \tilde{U} \xrightarrow{(f)} \top$ we know that $Y \longrightarrow \tilde{U}$. So $Y \longrightarrow X \times \tilde{U}$. Let $Y \xrightarrow{(wc)} Y_{wc} \xrightarrow{(f)} X \times \tilde{U}$. By the above claim $X \times \tilde{U} \xrightarrow{(wc)} \tilde{U}$ and $Y_{wc} \xrightarrow{(wc)} \tilde{U}$. So by (M5):

$$
\begin{array}{ccc}
 & Y_{wc} & \\
{\scriptstyle \therefore (w)} \swarrow & & \searrow {\scriptstyle (wc)} \\
X \times \tilde{U} & \xrightarrow{\quad (wc) \quad} & \tilde{U}
\end{array}
$$

FIGURE 4. By (M5) the arrow $Y_{wc} \longrightarrow X \times \tilde{U}$ is a weak equivalence.

But, by assumption all arrows in $\mathcal{Q}t$ are co-fibrations, and we chose Y_{wc} so that $Y_{wc} \xrightarrow{(f)} X \times \tilde{U}$. so $Y_{wc} \xrightarrow{(wcf)} X \times \tilde{U}$, and since $\mathcal{Q}t$ is posetal, this implies that Y_{wc} is isomorphic to $X \times \tilde{U}$. We conclude that $Y \xrightarrow{(wc)} X \times \tilde{U}$. Therefore $Y \longrightarrow X \times \tilde{U} \curlywedge Y \longrightarrow X$, giving an arrow $X \times \tilde{U} \longrightarrow Y$, with the conclusion that Y is isomorphic to the product, as required.

In the other direction. If $Y \xrightarrow{(f)} X$ is p-small then $Y \longrightarrow \tilde{U}$, and by the claim $Y \xrightarrow{(wc)} \tilde{U}$. Similarly, if $\bot \xrightarrow{(wc)} Y_{wc} \xrightarrow{(f)} Y$ then $Y_{wc} \xrightarrow{(wc)} \tilde{U}$. So (M5), applied to the triangle $\tilde{U} \longleftarrow Y_{wc} \longrightarrow Y \longrightarrow \tilde{U}$, assures that $Y_{wc} \xrightarrow{(w)} Y$. Since, by assumption, all arrows are co-fibrations, we get $Y_{wc} \xrightarrow{(wcf)} Y$, with the conclusion that $\bot \xrightarrow{(wc)} Y$, as required. $\qquad \square$

In order to conclude we have to give a reasonable notion of smallness — namely, to define when is a fibration $f : X \to Y$ in an arbitrary model category *small*. For reasons to be explained below we do not attempt to give a definition of a small fibration in that generality. Rather, our goal is find some (minimal) necessary conditions that such a class of fibrations should satisfy. Since we are trying to interpret the Univalence Axiom, as it is discussed in [Voe10], it is natural that our analysis of the notion of smallness be based on the definition of a universal fibration introduced there ([Voe10], p.6). In the category of simplicial sets a fibration $f : X \longrightarrow Y$ is *small* (for some fixed cardinality α) if all its fibers are of cardinality smaller than α.

Observe that (for most cardinalities) Voevodsky's definition of small fibrations depends, e.g., on the choice of model of ZFC. This suggests that there is no natural category theoretic counterpart exactly capturing this definition. So, let us consider some obvious properties of Voevodsky's definition:

(1) The class of small fibrations is closed under finite products and co-products.
(2) Since co-fibrations are injective, if $f : X \longrightarrow Y$ is a small fibration, and $g : X' \longrightarrow X$ is a fibration and a co-fibration then also $f \circ g : X' \longrightarrow Y$ is small.

Thus, by the second point above, if $\mathcal{Q}t$ is a posetal model category all of whose morphisms are co-fibrations, then for any small fibration $X \xrightarrow{(f)} Y$ if $\bot \xrightarrow{(wc)} X_{wc} \xrightarrow{(f)} X$, then $X_{wc} \xrightarrow{(f)} Y$ should also be a small fibration. Therefore, in any such model category, under any non-trivial definition of small fibrations, some small fibrations will be of the form $X_{wc} \xrightarrow{(f)} Y$ where $\bot \xrightarrow{(wc)} X_{wc}$. Moreover, in order to satisfy the first of the above points we have to close the collection of trivial co-fibrant objects, X_{wc}, such that there exists some small fibration $X_{wc} \xrightarrow{(f)} Y$ under finite limits and co-limits. The properties of $\mathcal{Q}t$ assure that this is still a collection of trivial co-fibrant objects (that the co-base change of a weak co-fibration is a weak equivalence - and therefore in $\mathcal{Q}t$ a weak co-fibration - follows from Axiom (M4) of model categories; that the product of two trivial co-fibrant objects is a trivial co-fibrant objects is proved precisely as in the claim of Lemma 7). Let $\mathcal{S}$ denote the collection of trivial co-fibrant objects thus obtained. Consider $\mathcal{Q}t^{\mathcal{S}}$, the "co-slice category over $\mathcal{S}$", i.e., $\mathcal{Q}t^{\mathcal{S}}$ is the full sub-category whose objects are all those $X \in \mathcal{O}b\mathcal{Q}t$ such that $S \xrightarrow{(f)} X$ for some $S \in \mathcal{S}$. Then $\mathcal{Q}t^{\mathcal{S}}$ is still a model category (one only needs to check that $\mathcal{Q}t^{\mathcal{S}}$ is closed under finite limits and co-limits, which is obvious). Moreover, as can be readily checked in Figure 1, since $\mathcal{Q}t$ is Cartesian closed, so is $\mathcal{Q}t^{\mathcal{S}}$. Being posetal, $\mathcal{Q}t^{\mathcal{S}}$ is also locally Caretsian closed. In addition, if $\mathcal{Q}t$ is (locally) (w/f)-Cartesian closed then, by putting $Z = C$ in ($**$) we see that $\mathcal{Q}t^{\mathcal{S}}$ is also (locally) Cartesian closed.

It follows that, replacing $\mathcal{Q}t$ with $\mathcal{Q}t^{\mathcal{S}}$ we obtain a locally Cartesian closed posetal model category all of whose trivial co-fibrant objects are small, in the sense that whenever $\bot \xrightarrow{(wc)} X \xrightarrow{(f)} Y$ the fibration $X \xrightarrow{(f)} Y$ is small. Of course, the model category $\mathcal{Q}t^{\mathcal{S}}$ may be less interesting than the original category $\mathcal{Q}t$. But the above argument shows that - at least for posetal model categories all of whose morphisms are co-fibrations - it is possible to have a notion of smallness which corresponds exactly to a fibration $X \xrightarrow{(f)} Y$ being small when $\bot \xrightarrow{(wc)} X$. Since, as explained above, there cannot be a natural category theoretic definition of smallness capturing precisely Voevodsky's notion of small fibration, we believe that, given the level of generality we are working in, the above is as good an approximation of this notion as could be expected.

We conclude that:

Proposition 8. *Let $\mathcal{Q}t$ be a posetal model category all of whose morphisms are co-fibrations. Let $\bot \xrightarrow{(wc)} \tilde{U} \xrightarrow{(f)} \top$, and define a fibration $Y \xrightarrow{(f)} X$ to be small if $\bot \xrightarrow{(wc)} Y$. Then the fibration $p : \tilde{U} \longrightarrow \top$ is universal. If, in addition, $\mathcal{Q}t$ is locally (w/f)-Cartesian closed then $\mathcal{Q}t$ meets the Univalence Axiom, with respect to the above notion of small fibrations.*

In the next section we give an example of a non-trivial model category satisfying all the assumptions of Proposition 8.

4. The model category $\mathrm{QtNaamen}_c$

The main result of [GH10] is the construction of a (non-trivial) posetal model category of classes of sets, QtNaamen. In this section we show that the full sub-category, $\mathrm{QtNaamen}_c$, of all co-fibrant objects meets all the assumptions of Proposition 8. Indeed, this follows almost immediately from the results of [GH10], but for the sake of completeness, we give a self-contained proof.

To simplify the exposition, and in order to avoid irrelevant foundational issues, we give a slightly simplified version of the model category $\mathrm{QtNaamen}_c$. Let $\mathrm{QtNaamen}_c$ be the category whose objects are the members of $\mathbb{P}(\mathbb{P}(\mathbb{N})) := \{X \subseteq \{M \subseteq \mathbb{N}\}\}$ and for $X, Y \in \mathcal{O}b\mathrm{QtNaamen}_c$ let $X \longrightarrow Y$ precisely

when for every $x \in X$ there exists $y \in Y$ such that $x \subseteq y$. We leave it as an easy exercise for the reader to verify that this is indeed a (posetal) category.

Claim 9. *The category QtNaamen$_c$ has limits. Direct limits are given by unions $X \vee Y = X \cup Y$, and inverse limits are given by pointwise intersection, namely $X \times Y = \{x \cap y : x \in X, y \in Y\}$. The same formulas hold for infinite limits.*

Proof. This is straightforward. Assume, e.g. that we are given X, Y and $Z \longrightarrow X$, $Z \longrightarrow Y$. By definition, this means that for all $z \in Z$ there are $x \in X$, $y \in Y$ such that $z \subseteq x$ and $z \subseteq y$. This means that for all $z \in Z$ there are $x \in X$ and $y \in Y$ such that $z \subseteq x \cap y$. This proves that $X \times Y$ as defined above is the inverse limit of X and Y. The proof for direct limits is similar. $\square$

Now we endow QtNaamen$_c$ with a model structure. We do not attempt to justify the intuition behind these definitions - this is done in some detail in [GH10]. In order to meet the assumptions of Proposition 8, we must require that all morphisms are labelled (c). So we now proceed to the (w) and (f) labels.

For the definition of weak equivalences it is convenient to denote for $X, Y \in \mathcal{O}b\text{QtNaamen}_c$, $X \longrightarrow^* Y$ if for all $x \in X$ there exists $y \in Y$ such that $|y \setminus x| < \aleph_0$. We now set $X \xrightarrow{(w)} Y$ if $X \longrightarrow Y$ and $Y \longrightarrow^* X$. This definition obviously satisfies Axiom (M5) (2 out of 3). Also, if $Z \xleftarrow{(wc)} X \longrightarrow Y$ then $Y \xrightarrow{(wc)} Z \vee Y$. Indeed, if $r \in Z \vee Y$ then either $r \in Y$ in which case $r \setminus r = \varnothing$ or $r \in Z$, in which case there is $x \in X$ such that $|r \setminus x| < \aleph_0$ but $X \longrightarrow Y$, so there is $y \in Y$ such that $x \subseteq y$ and $|r \setminus y| \leq |r \setminus x| < \aleph_0$. This shows that the (wc)-part of Axiom (M4) is met by this notation.

It remains to define the (f)-labelling: an arrow $X \longrightarrow Y$ is labelled (f) if and only if for every $x \in X \cup \{\emptyset\}$, $y \in Y$ and a finite subset $\{b_1, \ldots, b_n\} \subseteq y$ there exists $x' \in X$ such that $(x \cap y) \cup \{b_1, ..., b_n\} \subseteq x'$.

First, we observe:

Claim 10. *If $X \xrightarrow{(wcf)} Y$ then $Y \longrightarrow X$.*

Proof. Let $y \in Y$. We have to show that there exists $x \in X$ such that $y \subseteq x$. Let $x_0 \in X$ be such that $z := y \setminus x$ is finite, as provided by the (w)-label. So the (f)-label, applied for x_0, y and $z \subseteq y$ assures the existence of x with the desired property. $\square$

This claim gives us, automatically, one part of (M1) - any arrow right-lifts with respect to an isomorphism - one part of (M2) - any arrow $X \xrightarrow{(c)} Y$ decomposes as $X \xrightarrow{(c)} Y \xrightarrow{(wf)} Y$ and (M3) (it remains only to verify that fibrations are stable under base-change). Axiom (M4) is also automatic. So we are left with the $(wc) \curlywedge (f)$ part of (M1), the (wc)-(f) decomposition of (M2) and the stability of fibrations under base change. All computations are trivial, so we will be brief.

Let $X \xrightarrow{(wc)} Y$ and $W \xrightarrow{(f)} Z$ be such that $X \longrightarrow W$ and $Y \longrightarrow Z$. We have to show that $Y \longrightarrow W$. So let $y \in Y$. Let $x \in X$ be such that $b := y \setminus x$ is finite. Let $w \in W$ be such that $x \subseteq w$. Let $z \in Z$ be such that $y \subseteq z$. Apply the definition of (f)-arrows with respect to w, z and b. Then there exists $w' \in W$ such that $(w \cap z) \cup b \subseteq w'$. So $y \subseteq w'$, as required. An essentially similar argument shows that fibrations are stable under base-change.

To prove (M2), let $X \longrightarrow Y$ be any arrow. Let

$$X_{wc} := \{x \cup y_0 : x \in X, (\exists y \in Y)(y_0 \subseteq y), y_0 \text{ finite}\}.$$

Then $X \xrightarrow{(wc)} X_{wc} \xrightarrow{(f)} Y$, as can be readily checked.

We conclude that QtNaamen$_c$ is a posetal model category all of whose arrows are co-fibrations. It is not trivial (in the sense that not all arrows are fibrations) because $\varnothing \xrightarrow{(wc)} X$ is not a fibration unless $X = \varnothing$. Since it is posetal, to show that it is locally Cartesian closed, it suffices to show that it is Cartesian closed.

Define, for $C, B \in \mathcal{O}b\text{QtNaamen}_c$:

$$C^B := \bigcup \{Z : Z \times B \longrightarrow C, A \longrightarrow Z\}.$$

This is, obviously, an object in QtNaamen$_c$, so we need only check that $C^B \times B \longrightarrow C$ and that for every object Z, $Z \longrightarrow A$ and $Z \times B \longrightarrow C$ implies $Z \longrightarrow C^B$. The latter is immediate by definition of C^B. The former requires a little argument. We need to check that for every $d \in C^B$ and $b \in B$ there is a morphism $\{d \cap b\} \longrightarrow C$. By definition of C^B, there exists Z such that $Z \times B \longrightarrow C$ and $d \in Z$, i.e. $\{d\} \longrightarrow Z$. By definition of the product $Z \times B$, this implies $\{d \cap b\} \longrightarrow C$, as required.

Remark 11. *Note that the above shows that QtNaamen$_c$ is, in particular, a logical model category in the sense of* [GK95, Definition 23]. *Consequently (cf. Theorem 26) QtNaamen$_c$ admits a sound interpretation of the syntax of type theory (though the lack of non-trivial sections probably makes this interpretation trivial).*

All of the above shows that QtNaamen$_c$ is a posetal locally Cartesian closed model category which is non-trivial (in the sense that not all morphisms are labelled (fc)). So in order to apply Proposition 8 it remains to show that it is locally (w/f)-Cartesian closed. We prove:

Claim 12. $Z \times B \xrightarrow{(wc)} Z \times C$ *iff for all* $\{z\} \longrightarrow Z$, $\{z\} \times B \xrightarrow{(wc)} \{z\} \times C$

Proof. The right to left direction is immediate from the definition of (wc)-arrows, so we prove the other direction. The arrow $Z \times B \xrightarrow{(wc)} Z \times C$ means that:

 → for any $z \in Z$, $b \in B$ exists $z' \in Z$ and $c' \in C$ such that $\{z \cap b\} \longrightarrow \{z' \cap c'\}$; and
 → for any $z'' \in Z$, $c'' \in C$ exists $z \in Z$, $b \in B$ such that $\{z'' \cap c''\} \longrightarrow^* \{z \cap b\}$.

Observe that the first bullet (for fixed $z \in Z, b \in B$) gives $z \cap b \subseteq z' \cap c'$, implying that $z \cap b \subseteq z \cap z' \cap c' \subseteq z \cap c'$, therefore $\{z\} \times B \longrightarrow \{z\} \times C$.

Analogously, for fixed $z'' \in Z, c'' \in C$ the assumption $\{z'' \cap c''\} \longrightarrow^* \{z \cap b\}$ implies $\{z'' \cap c''\} \longrightarrow^*$ $\{z'' \cap z \cap b\} \longrightarrow \{z'' \cap b\}$. Combining these two observations we get $\{z\} \times B \xrightarrow{(wc)} \{z\} \times C$. $\qquad\square$

Now, given $A \in \mathcal{O}b\mathrm{QtNaamen}_c$ and $B \longrightarrow A$, $C \longrightarrow A$, we define

$$(C_w^B)/A = \bigcup \{Z : Z \times B \xrightarrow{(w)} Z \times C\} \times A$$

and show that this is an object representing $Hom_A^{(w)}(- \times B, C)$ (QtNaamen$_c$ is trivially right proper, so this is indeed a functor). More precisely:

Claim 13. *For all* $Z \longrightarrow A$, *we have* $Z \longrightarrow (C_w^B)/A$ *if and only if* $Z \times B \xrightarrow{(w)} Z \times C$.

Proof. The Right to left direction is immediate from the definition. So suppose $Z \longrightarrow (C_w^B)/A$. We need to show that $Z \times B \xrightarrow{(w)} Z \times C$. By Claim 12, this happens if for all $\{z\} \longrightarrow Z$, $\{z\} \times B \xrightarrow{(w)} \{z\} \times C$. But our assumption that $Z \longrightarrow (C_w^B)/A$ implies that $Z \longrightarrow Z'$ for some Z' such that $Z' \times B \xrightarrow{(w)} Z' \times C$. So $\{z\} \longrightarrow Z'$, by Claim 12 we are done. $\qquad\square$

Combining everything together we get:

Theorem 14. *There exists a non-trivial posetal model category satisfying the Univalence Axiom.*

Before continuing we remark that QtNaamen$_c$, as presented in this note is a full sub-category of the category of co-fibrant objects in the model category QtNaamen defined in [GH10]. The exact same proof of the above theorem would work for the full category of co-fibrant objects in QtNaamen. Of course this category captures the full homotopy structure of QtNaamen, and may - therefore - be a more interesting example. We remark also that there does not seem to be anything spcial about $\mathbb{N}$ or about $\aleph_0$ in the above construction (or in the more genreal construction of QtNaamen). Apparently, the exact same construction could be achieved for any regular cardinal λ (in place of $\aleph_0$) replacing, throughout "finite" by "less than λ". This gives — in addition to the formal discussion of smallness in the previous sub-section — an analogy with Voevodsky's notion of small fibrations: it is not unreasonable (see the following paragraph) to think of the morphisms in the resulting model category as a class of injections

(satisfying certain compatibility conditions), our definition of smallness implies that a fibration is small precisely when every memebr of the class of these injections has a domain smaller than λ.

To conclude, let us consider the category $\mathfrak{C}$, whose objects are $\mathcal{O}b\text{QtNaamen}_c$ and such that $\mathcal{M}or(X, Y)$ consists of the arrows $X \xrightarrow{\sigma} Y$ for $X, Y \in \mathcal{O}b\mathfrak{C}$ such that $\sigma : \bigcup X \longrightarrow \bigcup Y$ and $\sigma(X) \longrightarrow Y$ is an arrow in $\mathcal{M}or\text{QtNaamen}_c$ (where $\sigma(X) := \{\{\sigma(a) : a \in x\} : x \in X\}$). The category $\mathfrak{C}$ is, on the one hand, obvioulsy richer than QtNaamen_c (it is not posetal). But, on the other hand, it is readily seen that any slice of $\mathfrak{C}$ is (naturally) equivalent to the corresponding slice of QtNaamen_c. This local model structure induces naturally a (c)-(f)-(w) labeling on $\mathcal{M}or(\mathfrak{C})$ (see [GH10] for the details) satsifying Quillen's axioms (M1)-(M5). But the category $\mathfrak{C}$ does not have products and co-products. So we ask:

Question: Is there a model category $\mathfrak{C}'$ such that the labeled category $\mathfrak{C}$ described above embeds in $\mathfrak{C}'$? Does $\mathfrak{C}'$ satisfy the Univalence Axiom?

References

[Awo10] Steve Awodey. *Category theory*, volume 52 of *Oxford Logic Guides*. Oxford University Press, Oxford, second edition, 2010.

[Gar11] Richard Garner, editor. *Mini workshop: The homotopy interpretation of constructive type theory*, Oberwolfach meeting Report No. 11/2011, 2011.

[GH10] M. Gavrilovich and Assaf Hasson. Exercises de style: a homotopy theory for set theory I. Available on arXiv, 2010.

[GK95] T. E. Gallivan and Arie Kapulkin. Failure of universality in noncompact lattice field theories. *J. Math. Phys.*, 36(5):2341–2353, 1995.

[Qui67] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.

[Voe10] Vladimir Voevodsky. Univalent foundations project (a modified version of an nsf grant application). Available at http://www.math.ias.edu/~vladimir/Site3/Univalent_Foundations.html, 2010.

[War] Michael Warren. Univalen axiom as a property of a model category? http://mathoverflow.net/questions/73086/univalent-axiom-as-a-property-of-a-model-category.

Department of mathematics, Ben Gurion University of the Negev, Be'er Sheva,, Israel
E-mail address: hassonas@math.bgu.ac.il

WWU M unster, Mathematisches Institut, Einsteinstrae 62, 48149 Mnster, Germany
E-mail address: itay.kaplan@uni-muenster.de