Hitting Times of Random Walks on Edge Corona Product Graphs

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Graph products have been extensively applied to model complex networks with striking properties observed in real-world complex systems. In this paper, we study the hitting times for random walks on a class of graphs generated iteratively by edge corona product. We first derive recursive solutions to the eigenvalues and eigenvectors of the normalized adjacency matrix associated with the graphs. Based on these results, we further obtain interesting quantities about hitting times of random walks, providing iterative formulas for two-node hitting time, as well as closed-form expressions for the Kemeny's constant defined as a weighted average of hitting times over all node pairs, as well as the arithmetic mean of hitting times of all pairs of nodes.

Keywords: Random walk, hitting time, normalized Laplacian spectrum, graph product

1. INTRODUCTION

Graph operations and products play an important role in network science, which have been used to model complex networks with the prominent scale-free [1] and small-world [2] properties as observed in various real-life networks [3]. Since diverse realistic large-scale networks consist of smaller pieces or patterns, such as communities [4], motifs [5], and cliques [6], graph operations and products are a natural way to generate a massive graph out of smaller ones. Furthermore, there are many advantages to using graph operations and products to create complex networks. For example, it allows analytical treatment for structural and dynamical aspects of the resulting networks. Thus far, a variety of graph operations and products have been introduced or proposed to construct models of complex networks, including triangulation [7, 8, 9, 10, 11], Kronecker product [12, 13, 14], hierarchical product [15, 16, 17, 18, 19], as well as corona product [20, 21, 22].

Recently, a class of iteratively growing network model was introduced, leveraging an edge operation on graphs [23]. This family of graphs exhibit the striking scale-free small-world properties as observed in diverse real systems. The degree distribution $P(d)$ of the graphs follows a power-law form $P(d) \sim d^{-\gamma_q}$ with the exponent γ_q lying in the interval $(2, 3)$. Their diameter scales logarithmically with the number of nodes. Moreover, their clustering coefficient is high. However, except some structural and combinatorial properties, the dynamical aspects on these networks are not well understood, for example, hitting times of random walks on this network family.

In this paper, we present an in-depth study on hitting time— a most relevant quantity about random walks on the iteratively growing networks [23]. We first give iterative formulas for eigenvalues and eigenvectors of normalized adjacency (or Laplacian) matrix for the networks, based on which we determine two-node hitting time and the Kemeny's constant for random walks. Also, we derive closed-form expressions for the sum of hitting times, additive-degree sum of hitting times, multiplicative-degree sum of hitting times over all pairs of nodes, as well as the arithmetic mean of hitting times of all node pairs.

2. PRELIMINARIES

In this section, we introduce some basic concepts for graphs and random walks on graphs.

2.1. Graph and Matrix Notation

Let $G(V, E)$ denote a simple connected graph with n nodes/vertices and m edges. Let $V(G) = \{1, 2, \ldots, n\}$ be the set of n nodes, and let $E(G) = \{e_1, e_2, \ldots, e_m\}$ be set of m edges.

Let A denote the adjacency matrix of G, the (i, j)th entry $A(i, j)$ of which is 1 (or 0) if nodes i and j are (not) adjacent in G. Let $\Psi(i)$ denote the set of neighbors for node i in graph G. Then the degree of node i is $d_i = \sum_{j\in\Phi(i)} A(i, j)$, which forms the ith diagonal entry of the diagonal degree matrix D for G. The incidence matrix B of G is an $n \times m$, where the (i, j)th entry $B(i, j) = 1$ (or 0) if node i is (not) incident with e_j.

LEMMA 2.1. [24] *Let G be a simple connected unbipartite graph with n nodes. Then the rank of its incidence matrix B is* $\mathrm{rank}(B) = n$.

LEMMA 2.2. [24] *Let G be a simple connected graph. Then its incidence matrix B, adjacency matrix A and diagonal degree matrix D satisfy*

$$BB^\top = A + D.$$

2.2. Random Walks on Graphs

For a graph G, one can define a discrete-time unbiased random walk running on it. At every time step, the walker jumps from its current location, node i, to an adjacent node j with probability $A(i, j)/d_i$. Such a random walk on G is a Markov chain [25] characterized by the transition probability matrix $T = D^{-1}A$, with its (i, j)th entry $T(i, j)$ being $A(i, j)/d_i$. For an unbiased random walk on unbipartite graph G with n nodes and m edges, its stationary distribution is an n-dimension vector $\pi = (\pi_1, \pi_2, \ldots, \pi_n)^\top = (d_1/2m, d_2/2m, \ldots, d_n/2m)^\top$.

In general, the transition probability matrix T of graph G is asymmetric. However, T is similar to a symmetric matrix P defined as

$$P = D^{-\frac{1}{2}}AD^{-\frac{1}{2}} = D^{\frac{1}{2}}TD^{-\frac{1}{2}},$$

which is often called the normalized adjacency matrix of G. By definition, the (i, j)th entry of P is $P(i, j) = \frac{A(i,j)}{\sqrt{d_i d_j}}$. Then, it is obvious that $P(i, j) = P(j, i)$. Let I be the identity matrix of approximate dimensions. Then, $I - P$ is the normalized Laplacian matrix [26] of graph G.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the n eigenvalues of matrix P. Then, these n eigenvalues can be listed in decreasing order as $1 = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n \geq -1$, with $\lambda_n = -1$ if and only if G is a bipartite graph. Let $v_1, v_2, \ldots, v_n$ be the orthonormal eigenvectors corresponding to the n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, where $v_i = (v_{i1}, v_{i2}, \ldots, v_{in})^\top$, $i = 1, 2, \ldots, n$. Then,

$$v_1 = \left(\sqrt{d_1/2m}, \sqrt{d_2/2m}, ..., \sqrt{d_n/2m}\right)^\top \tag{1}$$

and

$$\sum_{k=1}^{n} v_{ik}v_{jk} = \sum_{k=1}^{n} v_{ki}v_{kj} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

A key quantity associated with random walks is hitting time. The hitting time T_{ij} from one node i to another node j is defined as the expected number of jumps needed for a walker starting from node i to reach node j for the first time. The hitting time T_{ij} is encoded in the eigenvalues and eigenvectors of the normalized adjacency (or Laplacian) matrix P for graph G.

LEMMA 2.3. [27] *For random walks on a simple connected graph G with n nodes and m edges, the hitting time T_{ij} from one node i to another node j can be expressed in terms of the eigenvalues and their orthonormal eigenvectors for the normalized adjacency matrix P as*

$$T_{ij} = 2m \sum_{k=2}^{n} \frac{1}{1 - \lambda_k} \left(\frac{v_{kj}^2}{d_j} - \frac{v_{ki}v_{kj}}{\sqrt{d_i d_j}} \right).$$

The hitting time is relevant in various scenarios [28]. For example, it has been used to design clustering algorithm [29, 30], to measure the transmission costs in wireless networks [31, 32], as well as to evaluate the centrality of nodes in complex networks [33, 34].

For a graph G, T_{ij} is usually not equal to T_{ji}. However, the commute time between a pair of nodes can make up for this shortcoming. For two nodes i and j, their commute time C_{ij} is defined as the sum of T_{ij} and T_{ji}, namely, $C_{ij} = T_{ij} + T_{ji}$. Thus, the relation $C_{ij} = C_{ji}$ always holds for any pair of nodes nodes i and j.

LEMMA 2.4. [35] *Let G be a simple connected graph with n nodes and m edges. Then the sum of commute times C_{ij} between all the m pairs of adjacent nodes in G is equivalent to $2m(n - 1)$, i.e.*

$$\sum_{(i,j)\in E} C_{ij} = 2m(n - 1).$$

The symmetry of commute time makes it have many applications in different areas, such as link prediction [36] and graph embedding [37]. In addition to commute time, many other interesting quantities of graph G can be defined or derived from hitting times. For example, the mean hitting time $\bar{H}(G)$ of a graph G with n nodes is the average of hitting times over all $n(n - 1)$ node pairs:

$$\bar{H}(G) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij}. \tag{3}$$

The quantity of mean hitting time has been utilized as an indicator of mean cost of search in networks [38, 39] and global utility of social recommender networks [40].

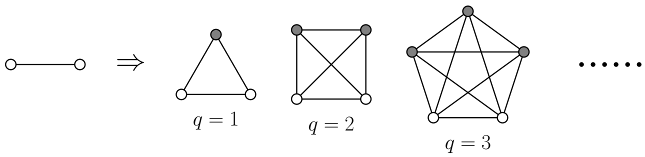

FIGURE 1. Network construction method. The next-iteration network is obtained from the current network by performing the operation on the right-hand side of the arrow for each existing edge.

Another quantity defined according to hitting times is the Kemeny's constant. For a graph G, its Kemeny's constant $K(G)$ is defined as the expected number of steps required for a walker starting from a node i to a destination node chosen randomly according to a stationary distribution of random walks on G [41], that is $K(G) = \sum_{j=1}^{n} \pi_j T_{ij}$. The Kemeny constant $K(G)$ is independent of the selection of starting node i [42], which means $\sum_{j=1}^{n} \pi_j T_{ij} = \sum_{j=1}^{n} \pi_j T_{kj}$ holds for an arbitrary pair of node i and k. Interesting, the Kemeny's constant of graph G is only dependent on the eigenvalues of matrix P.

Lemma 2.5. [43] *Let G be a simple connected graph with n nodes. Then, the Kemeny's constant $K(G)$ of G can be represented as*

$$K(G) = \sum_{j=1}^{n} \pi_j T_{ij} = \sum_{k=2}^{n} \frac{1}{1-\lambda_k}.. \tag{4}$$

The Kemeny constant has also found applications in diverse areas [41]. It has been widely used to characterize the criticality [44, 45] or connectivity [46] for a graph. Moreover, it can be applied to measure the efficiency of user navigation through the World Wide Web [42]. Finally, it was also exploited to quantify the performance of a class of noisy formation control protocols [47], and to gauge the efficiency of robotic surveillance in network environments [48]. Very recently, some properties and nearly linear time algorithms for computing the Kemeny's constant have been studied or developed [34, 49].

3. NETWORK CONSTRUCTION, PROPERTIES, AND IMPORTANT MATRICES

In this section, we introduce the construction and properties for the studied networks, and provide some relations among matrices related to the networks, which are very useful for deriving the properties of eigenvalues and eigenvectors of the normalized adjacency matrix, as well as the hitting times.

3.1. Network Construction and Properties

The network family studied here is proposed in [23] and constructed in an iterative way. It is controlled by two parameters q and g with $q \geq 1$ and $g \geq 0$. Let

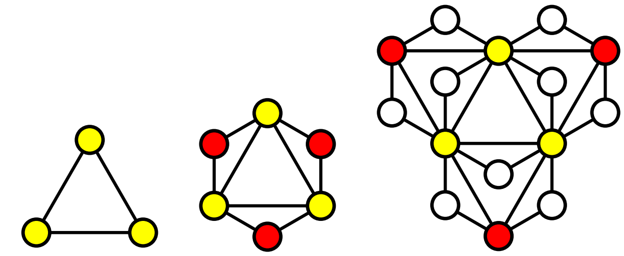

FIGURE 2. The networks of the first three iterations for $q = 1$.

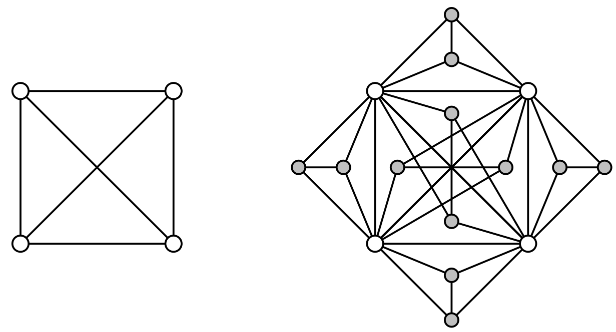

FIGURE 3. The networks of the first two iterations for $q = 2$.

$\mathcal{K}_q$ ($q \geq 1$) denote the complete graph with q nodes. For $q = 1$, suppose that $\mathcal{K}_1$ is a graph with an isolate node. Let $\mathcal{G}_q(g)$ be the network after g iterations. Then, $\mathcal{G}_q(g)$ is constructed as follows. For $g = 0$, $\mathcal{G}_q(0)$ is the complete graph $\mathcal{K}_{q+2}$. For $g > 0$, $\mathcal{G}_q(g+1)$ is obtained from $\mathcal{G}_q(g)$ by performing the operation shown in Fig. 1: for every existing edge of $\mathcal{G}_q(g)$, a complete graph $\mathcal{K}_q$ is introduced, every node of which is connected to both end nodes of the edge. Figures 2 and 3 illustrate the networks corresponding to two particular cases of $q = 1$ and $q = 2$.

For network $\mathcal{G}_q(g)$, let $\mathcal{V}_g$ and $\mathcal{E}_g$ denote its node set and edge set, respectively. And let $N_g = |\mathcal{V}_g|$ and $M_g = |\mathcal{E}_g|$ denote, respectively, the number of nodes and the number of edges in graph $\mathcal{G}_q(g)$. Then, for all $g \geqslant 0$,

$$M_g = \left(\frac{(q+1)(q+2)}{2}\right)^{g+1}, \tag{5}$$

$$N_g = \frac{2}{q+3}\left(\frac{(q+1)(q+2)}{2}\right)^{g+1} + \frac{2(q+2)}{q+3}. \tag{6}$$

The node set $\mathcal{V}_{g+1}$ of $\mathcal{G}_q(g+1)$ can be classified into two disjoint parts $\mathcal{V}_g$ and $\mathcal{W}_{g+1}$, where $\mathcal{V}_g$ is the set of old nodes belonging to $\mathcal{G}_q(g)$, while $\mathcal{W}_{g+1}$ is the set of new nodes generated in the process of performing aforementioned operation on $\mathcal{G}_q(g)$. Moreover, $\mathcal{W}_{g+1}$ can be further divided into q disjoint subsets $\mathcal{V}_{g+1}^{(1)}$, $\mathcal{V}_{g+1}^{(2)}$, …, $\mathcal{V}_{g+1}^{(i)}$ satisfying $\mathcal{W}_{g+1} = \mathcal{V}_{g+1}^{(1)} \cup \mathcal{V}_{g+1}^{(2)} \cup \cdots \cup \mathcal{V}_{g+1}^{(q)}$, with each $\mathcal{V}_{g+1}^{(i)}$ ($i = 1, 2, \ldots, q$) including M_g new nodes produced by M_g different edges in $\mathcal{G}_q(g)$. Hence, one has

$$\mathcal{V}_{g+1} = \mathcal{V}_g \cup \mathcal{V}_{g+1}^{(1)} \cup \mathcal{V}_{g+1}^{(2)} \cup ... \cup \mathcal{V}_{g+1}^{(q)}. \tag{7}$$

For any new node $x \in \mathcal{W}_{g+1}$, there are two old neighboring nodes in $\mathcal{V}_g$, the set of which is denoted by $\Gamma(x)$. By construction, for each old edge $uv \in \mathcal{E}_g$, there exists one and only one node x in each $\mathcal{V}_{g+1}^{(i)}$ ($i = 1, 2, \ldots, q$), satisfying $\Gamma(x) = \{u, v\}$. Therefore, for two different sets $\mathcal{V}_{g+1}^{(i)}$ and $\mathcal{V}_{g+1}^{(j)}$, their nodes have equivalent structural and dynamical properties.

Let $W_{g+1} = |\mathcal{W}_{g+1}|$ represent the number of those newly nodes generated at iteration $g+1$. Then,

$$W_{g+1} = q\left(\frac{(q+1)(q+2)}{2}\right)^{g+1}. \tag{8}$$

Let $d_v^{(g)}$ denote the degree of a node v in graph $\mathcal{G}_q(g)$, which was generated at iteration g_v. Then,

$$d_v^{(g)} = (q+1)^{g-g_v+1}. \tag{9}$$

In graph $\mathcal{G}_q(g)$, all simultaneously emerging nodes has the same degree. Thus, the number of nodes with degree $(q+1)^{g-g_v+1}$ is equal to $q+2$ and $q\left(\frac{(q+1)(q+2)}{2}\right)^{g_v}$ for $g_v = 0$ and $g_v > 0$, respectively.

The resulting family of networks is consist of cliques $\mathcal{K}_{q+2}$ or smaller cliques, and are thus called simplicial networks, characterized by a parameter q. These networks display some remarkable properties that are observed in most real networks [3]. They are scale-free, since their node degrees obey a power-law distribution $P(d) \sim d^{-\gamma_q}$ with $\gamma_q = 2 + \frac{\ln(q+2)}{\ln(q+1)} - \frac{\ln 2}{\ln(q+1)}$ [23]. They are small-world with their diameters increasing logarithmically with the number of nodes and their mean clustering coefficients converging to a large constant $\frac{q^2+3q+3}{q^2+3q+5}$ [23]. In addition, they have a finite spectral dimension $\frac{2(\ln(q^2+3q+3)-\ln 2)}{\ln(q+1)}$.

3.2. Relations among Various Matrices

Let A_g denote the adjacency matrix of graph $\mathcal{G}_q(g)$. The element $A_g(i,j)$ at row i and column j of A_g is defined as follows: $A_g(i,j) = 1$ if nodes i and j are directly connected by an edge in $\mathcal{G}_q(g)$, $A_g(i,j) = 0$ otherwise. Let B_g denote the incidence matrix of graph $\mathcal{G}_q(g)$. The element $B_g(i,j)$ at row i and column j of B_g is: $B_g(i,j) = 1$ if node i is incident with edge e_j in $\mathcal{G}_q(g)$, $B_g(i,j) = 0$ otherwise. Let D_g denote the diagonal degree matrix of matrix graph $\mathcal{G}_q(g)$, with the ith diagonal element being the degree $d_i^{(g)}$ of node i. And let $P_g = D_g^{-\frac{1}{2}} A_g D_g^{-\frac{1}{2}}$ denote the normalized adjacency matrix of graph $\mathcal{G}_q(g)$. Then for $\mathcal{G}_q(g+1)$, its adjacency matrix A_{g+1}, diagonal degree matrix D_{g+1} and normalized adjacency matrix P_{g+1}, can be expressed in terms of related matrices of $\mathcal{G}_q(g)$ as

$$A_{g+1} = \begin{pmatrix} A_g & B_g & B_g & \cdots & B_g \\ B_g^\top & O & I & \cdots & I \\ B_g^\top & I & O & \cdots & I \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_g^\top & I & I & \cdots & O \end{pmatrix},$$

$$D_{g+1} = \mathrm{diag}\{(q+1)D_g, \underbrace{(q+1)I, ..., (q+1)I}_{q}\},$$

and

$$P_{g+1} = D_{g+1}^{-\frac{1}{2}} A_{g+1} D_{g+1}^{-\frac{1}{2}} \tag{10}$$
$$= \frac{1}{q+1}\begin{pmatrix} P_g & D_g^{-\frac{1}{2}} B_g & D_g^{-\frac{1}{2}} B_g & \cdots & D_g^{-\frac{1}{2}} B_g \\ B_g^\top D_g^{-\frac{1}{2}} & O & I & \cdots & I \\ B_g^\top D_g^{-\frac{1}{2}} & I & O & \cdots & I \\ \vdots & \vdots & \vdots & \ddots & I \\ B_g^\top D_g^{-\frac{1}{2}} & I & I & \cdots & O \end{pmatrix}.$$

4. EIGENVALUES AND EIGENVECTORS OF NORMALIZED ADJACENCY MATRIX

In this section, we study the eigenvalues and eigenvectors of normalized adjacency matrix P_{g+1} for graph $\mathcal{G}_q(g+1)$, expressing both eigenvalues and eigenvectors for P_{g+1} in terms of those associated with graph $\mathcal{G}_q(g)$. Then we use these results to obtain the Kemeny's constant of graph $\mathcal{G}_q(g)$.

For the purpose of analyzing the eigenvalues and eigenvectors of matrix P_{g+1} precisely, we first study the orthonormal basis $\mathcal{Y}_g$ of the kernel space of the following matrix

$$C_g := \underbrace{\begin{pmatrix} B_g & B_g & \cdots & B_g \end{pmatrix}}_{q}.$$

Since $\mathcal{G}_q(g)$ is non-bipartite, by Lemma 2.1 one has $\mathrm{rank}(B_g) = N_g$ Thus, $\dim(\mathrm{Ker}(B_g)) = M_g - N_g$, $\mathrm{rank}(C_g) = N_g$, and $\dim(\mathrm{Ker}(C_g)) = qM_g - N_g$. Then, $\mathcal{Y}_g$ can be classified into two non-overlapping parts $\mathcal{Y}_g^{(1)}$ and $\mathcal{Y}_g^{(2)}$ obeying $\mathcal{Y}_g = \mathcal{Y}_g^{(1)} \cup \mathcal{Y}_g^{(2)}$, where $\mathcal{Y}_g^{(1)}$ has $M_g - N_g$ vectors, while $\mathcal{Y}_g^{(2)}$ has $(q-1)M_g$ vectors. Moreover, as will shown below, $\mathcal{Y}_g^{(1)}$ and $\mathcal{Y}_g^2$ can be constructed, respectively, by using the orthonormal basis vectors of the kernel space of matrix B_g and the column vectors of the $M_g \times M_g$ identity matrix I.

Let $\mathcal{X}_g = \{X_1(g), X_2(g), \ldots, X_{M_g-N_g}(g)\}$ denote the orthonormal basis of the kernel space of matrix B_g, and let $Z_i(g)$ denote the ith column vector of the $M_g \times M_g$ identity matrix I. Then the vectors in $\mathcal{Y}_g^{(1)}$ are

$$\frac{1}{\sqrt{q}}\begin{pmatrix} X_1^\top(g) \\ X_1^\top(g) \\ \vdots \\ X_1^\top(g) \end{pmatrix}, \frac{1}{\sqrt{q}}\begin{pmatrix} X_2^\top(g) \\ X_2^\top(g) \\ \vdots \\ X_2^\top(g) \end{pmatrix}, \cdots \frac{1}{\sqrt{q}}\begin{pmatrix} X_{M_g-N_g}^\top(g) \\ X_{M_g-N_g}^\top(g) \\ \vdots \\ X_{M_g-N_g}^\top(g) \end{pmatrix},$$

and the vectors in $\mathcal{Y}_g^{(2)}$ are

$$\begin{pmatrix} \frac{1}{\sqrt{2}}Z_i(g) \\ -\frac{1}{\sqrt{2}}Z_i(g) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}}Z_i(g) \\ \frac{1}{\sqrt{6}}Z_i(g) \\ -\frac{1}{\sqrt{3}}Z_i(g) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \frac{1}{\sqrt{q(q-1)}}Z_i(g) \\ \frac{1}{\sqrt{q(q-1)}}Z_i(g) \\ \frac{1}{\sqrt{q(q-1)}}Z_i(g) \\ \vdots \\ \frac{1}{\sqrt{q(q-1)}}Z_i(g) \\ -\sqrt{\frac{q-1}{q}}Z_i(g) \end{pmatrix},$$

where $i = 1, 2, \dots, M_g$.

Considering the process of network construction, we have the following lemmas.

LEMMA 4.1. *For any vector*

$$Y_i^{(2)}(g) = \begin{pmatrix} Y_{i1}^{(2)}(g) \\ Y_{i2}^{(2)}(g) \\ \vdots \\ Y_{iq}^{(2)}(g) \end{pmatrix}, i = 1, 2, \dots, (q-1)M_g,$$

in $\mathcal{Y}_g^{(2)}$, its components obey the following relation

$$Y_{i1}^{(2)}(g) + Y_{i2}^{(2)}(g) + \cdots + Y_{iq}^{(2)}(g) = 0. \tag{11}$$

LEMMA 4.2. *For any integer $j \in \{1, 2, \dots, qM_g\}$ and $\mathcal{Y}_g^{(2)} = \{Y_1^{(2)}(g), Y_2^{(2)}(g), \dots, Y_{(q-1)M_g}^{(2)}(g)\}$, we have*

$$\sum_{i=1}^{(q-1)M_g} \left(Y_{ij}^{(2)}(g)\right)^2 = 1 - \frac{1}{q}. \tag{12}$$

LEMMA 4.3. *Let $1 = \lambda_1(g) > \lambda_2(g) \geq \dots \geq \lambda_{N_g}(g) > -1$ be the eigenvalues of matrix P_g, and let $v_1(g), v_2(g), \dots, v_{N_g}(g)$ be their corresponding orthonormal eigenvectors. Then $\frac{\lambda_i(g)+q}{q+1}$, $i = 1$, 2,…, N_g, are eigenvalues of matrix P_{g+1}, and their corresponding orthonormal eigenvectors are*

$$\sqrt{\frac{\lambda_i(g)+1}{q+\lambda_i(g)+1}} \begin{pmatrix} v_i(g) \\ \frac{1}{\lambda_i(g)+1} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \\ \vdots \\ \frac{1}{\lambda_i(g)+1} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \end{pmatrix}; \tag{13}$$

$-\frac{1}{q+1}$'s are eigenvalues of matrix P_{g+1} with multiplicity $(q-1)M_g + N_g$, and the corresponding orthonormal eigenvectors are

$$\sqrt{\frac{q}{q+\lambda_i(g)+1}} \begin{pmatrix} v_i(g) \\ -\frac{1}{q} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \\ \vdots \\ -\frac{1}{q} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \end{pmatrix}, \tag{14}$$

$i = 1$, 2,…, N_g, and

$$\begin{pmatrix} 0 \\ Y_z^{(2)}(g) \end{pmatrix}, \ z = 1, \ 2, \dots, \ (q-1)M_g, \tag{15}$$

where $Y_z^{(2)}(g) \in \mathcal{Y}_g^{(2)}$; and $\frac{q-1}{q+1}$'s are eigenvalues of P_{g+1} having multiplicity $M_g - N_g$, with the corresponding orthonormal eigenvectors being

$$\begin{pmatrix} 0 \\ Y_z^{(1)}(g) \end{pmatrix}, \ z = 1, \ 2, \dots, \ M_g - N_g, \tag{16}$$

where $Y_z^{(1)}(g) \in \mathcal{Y}_g^{(1)}$.

Proof. For any eigenpair $\lambda_i(g)$ and $v_i(g)$, $P_g v_i(g) = \lambda_i(g) v_i(g)$ holds. Then, by Lemma 2.2 and Eq. (10), one has

$$P_{g+1} \begin{pmatrix} v_i(g) \\ \frac{1}{\lambda_i(g)+1} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \\ \vdots \\ \frac{1}{\lambda_i(g)+1} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \end{pmatrix} = \frac{1}{q+1} \begin{pmatrix} \lambda_i(g)v_i(g) + qv_i(g) \\ \frac{\lambda_i(g)+q}{\lambda_i(g)+1} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \\ \vdots \\ \frac{\lambda_i(g)+q}{\lambda_i(g)+1} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \end{pmatrix}$$

$$= \frac{\lambda_i(g)+q}{q+1} \begin{pmatrix} v_i(g) \\ \frac{1}{\lambda_i(g)+1} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \\ \vdots \\ \frac{1}{\lambda_i(g)+1} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \end{pmatrix}$$

and

$$P_{g+1} \begin{pmatrix} v_i(g) \\ -\frac{1}{q} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \\ \vdots \\ -\frac{1}{q} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \end{pmatrix} = \begin{pmatrix} \frac{\lambda_i(g)v_i(g) - (\lambda_i(g)+1)v_i(g)}{q+1} \\ \frac{1}{q(q+1)} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \\ \vdots \\ \frac{1}{q(q+1)} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \end{pmatrix}$$

$$= -\frac{1}{q+1} \begin{pmatrix} v_i(g) \\ -\frac{1}{q} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \\ \vdots \\ -\frac{1}{q} B_g^\top D_g^{-\frac{1}{2}} v_i(g) \end{pmatrix},$$

both of which lead to (13) and (14) through normalization.

In addition, according to Lemma 4.1, one has

$$P_{g+1} \begin{pmatrix} 0 \\ Y_z^{(2)}(g) \end{pmatrix} = \frac{1}{q+1} \begin{pmatrix} 0 \\ -Y_{i1}^{(2)}(g) + \sum_{k=1}^q Y_{ik}^{(2)}(g) \\ -Y_{i2}^{(2)}(g) + \sum_{k=1}^q Y_{ik}^{(2)}(g) \\ \vdots \\ -Y_{iq}^{(2)}(g) + \sum_{k=1}^q Y_{ik}^{(2)}(g) \end{pmatrix}$$

$$= \frac{1}{q+1} \begin{pmatrix} 0 \\ -Y_{i1}^{(2)}(g) \\ -Y_{i2}^{(2)}(g) \\ \vdots \\ -Y_{iq}^{(2)}(g) \end{pmatrix}$$

$$= -\frac{1}{q+1} \begin{pmatrix} 0 \\ Y_z^{(2)}(g) \end{pmatrix},$$

as claimed by (15).

Finally, for each $z = 1, \ 2, \dots, \ (q-1)M_g$,

$$P_{g+1} \begin{pmatrix} 0 \\ Y_z^{(1)}(g) \end{pmatrix} = P_{g+1} \begin{pmatrix} X_i(g) \\ X_i(g) \\ \vdots \\ X_i(g) \end{pmatrix}$$

$$= \frac{1}{q+1}\begin{pmatrix} 0 \\ (q-1)X_i(g) \\ (q-1)X_i(g) \\ \vdots \\ (q-1)X_i(g) \end{pmatrix}$$
$$= \frac{q-1}{q+1}\begin{pmatrix} 0 \\ Y_z^{(2)}(g) \end{pmatrix}.$$

Thus, we complete the proof. □

In fact, the orthonormal eigenvectors of $\mathcal{G}_q(g+1)$ can be expressed in more explicit forms. By Eq. (1) and Lemma 4.3, one can easily derive the following results.

Corollary 4.1. *Let $1 = \lambda_1(g) > \lambda_2(g) \geq ... \geq \lambda_{N_g}(g) > -1$ be the eigenvalues of matrix P_g, and let $v_1(g), v_2(g), ..., v_{N_g}(g)$ be their corresponding orthonormal eigenvectors. Then,*

1. *The eigenvectors corresponding to eigenvalues $\frac{\lambda_1(g)+q}{q+1} = 1$, the first $-\frac{1}{q+1}$ for matrix P_{g+1} are*

$$\left(\sqrt{\frac{d_1(g)}{M_g(q+2)}}, \cdots, \sqrt{\frac{d_{N_g}(g)}{M_g(q+2)}}, \frac{1}{\sqrt{M_g(q+2)}}, \cdots, \frac{1}{\sqrt{M_g(q+2)}}\right)^\top \tag{17}$$

and

$$\left(\sqrt{\frac{qd_1(g)}{2M_g(q+2)}}, \cdots, \sqrt{\frac{qd_{N_g}(g)}{2M_g(q+2)}}, -\sqrt{\frac{2}{qM_g(q+2)}}, \cdots, -\sqrt{\frac{2}{qM_g(q+2)}}\right)^\top, \tag{18}$$

respectively.

2. *The element of orthonormal eigenvectors for eigenvalues $\frac{\lambda_i(g)+q}{q+1}$, $i = 2, 3, \ldots, N_g$, corresponding to node j is*

$$\begin{cases} \sqrt{\frac{\lambda_i(g)+1}{\lambda_i(g)+q+1}}v_{ij}(g) & j \in \mathcal{V}_g \\ \sqrt{\frac{1}{(\lambda_i(g)+1)(\lambda_i(g)+q+1)}}\left(\frac{v_{is}(g)}{\sqrt{d_s(g)}} + \frac{v_{it}(g)}{\sqrt{d_t(g)}}\right) & j \in \mathcal{W}_{g+1}; \end{cases}$$

and the element of orthonormal eigenvectors for eigenvalues $-\frac{1}{q+1}$, $i = 2, 3, \ldots, N_g$, corresponding to node j is

$$\begin{cases} \sqrt{\frac{q}{\lambda_i(g)+q+1}}v_{ij}(g), & j \in \mathcal{V}_g, \\ \sqrt{\frac{1}{q(\lambda_i(g)+q+1)}}\left(\frac{v_{is}(g)}{\sqrt{d_s(g)}} + \frac{v_{it}(g)}{\sqrt{d_t(g)}}\right), & j \in \mathcal{W}_{g+1} \end{cases}$$

where $\Gamma(j) = \{s, t\}$;

3. *For orthonormal eigenvectors $\begin{pmatrix} 0 \\ Y_z(g) \end{pmatrix}$, $z = 1, 2, \ldots, qM_g - N_g$, of eigenvalues $0's$ of matrix P_{g+1}, we have*

$$\sum_{z=1}^{qM_g-N_g} Y_{zj}^2(g) = 1 - \frac{1}{qM_g} - \sum_{k=2}^{N_g} \frac{1}{(1+\lambda_k(g))q} \left(\frac{v_{ks}(g)}{\sqrt{d_s(g)}} + \frac{v_{kt}(g)}{\sqrt{d_t(g)}}\right)^2. \tag{19}$$

for each $j \in \mathcal{W}_{g+1}$ with $\Gamma(j) = \{s, t\}$.

5. TWO-NODE HITTING TIME AND KEMENY'S CONSTANT

Lemma 4.3 and Corollary 4.1 provide complete information about the eigenvalues and eigenvectors of matrix P_{g+1} in terms of those of matrix P_g of the previous iteration. In this section, we use this information to determine two-node hitting time and the Kemeny's constant for unbiased random walks on graph $\mathcal{G}_q(g+1)$.

5.1. Two-Node Hitting Time

We first present our results about hitting times for random walks on graph $\mathcal{G}_q(g)$. Let $T_{ij}(g)$ denote the hitting time from node i to node j in $\mathcal{G}_q(g)$.

Theorem 5.1. *For networks $\mathcal{G}_q(g)$ and $\mathcal{G}_q(g+1)$,*

1. *if $i, j \in \mathcal{V}_g$, then $T_{ij}(g+1) = (q+1)T_{ij}(g)$;*
2. *if $i \in \mathcal{W}_{g+1}$, $j \in \mathcal{V}_g$, $\Gamma(i) = \{s, t\}$, then*

$$\begin{aligned} T_{ij}(g+1) =& \frac{q+1}{2} + \frac{q+1}{2}\left(T_{sj}(g) + T_{tj}(g)\right), \\ T_{ji}(g+1) =& \frac{3(q+1)}{2}M_g - \frac{q+1}{2} + \frac{q+1}{4} \\ &\cdot \left(2\left(T_{js}(g) + T_{jt}(g)\right) - \left(T_{ts}(g) + T_{st}(g)\right)\right); \end{aligned}$$

3. *if $i, j \in \mathcal{W}_{g+1}$, (a) i is adjacent to j, then*

$$T_{ij}(g+1) = (q+1)M_g,$$

(b) else if i is not adjacent to j, $\Gamma(i) = \{s, t\}$, and $\Gamma(j) = \{u, v\}$, then

$$\begin{aligned} T_{ji}(g+1) =& \frac{3(q+1)}{2}M_g + \frac{q+1}{4}\left(T_{su}(g) + T_{tu}(g)\right. \\ &\left.+T_{sv}(g) + T_{tv}(g) - \left(T_{uv}(g) + T_{vu}(g)\right)\right). \end{aligned}$$

Proof. Note that $M_{g+1} = \frac{(q+1)(q+2)}{2}M_g$, $d_i(g+1) = (q+1)d_i(g)$ for $i \in \mathcal{V}_g$, and $d_i(g+1) = 2$ for $i \in \mathcal{W}_{g+1}$.

We first prove 1). By Lemmas 2.3 and 4.3, one has

$$
\begin{aligned}
&T_{ij}(g+1)\\
=&2M_q(g+1)\sum_{k=2}^{N_g}\left(\frac{1}{1-\frac{\lambda_k(g)+q}{q+1}}\frac{\lambda_k(g)+1}{\lambda_k(g)+q+1}\right.\\
&\left.+\frac{1}{1+\frac{1}{q+1}}\frac{q}{\lambda_k(g)+q+1}\right)\\
&\left(\frac{v_{kj}(g)^2}{(q+1)d_j(g)}-\frac{v_{ki}(g)v_{kj}(g)}{(q+1)\sqrt{d_i(g)d_j(g)}}\right)\\
=&2M_g\frac{(q+1)(q+2)}{2}\sum_{k=2}^{N_g}\frac{2q+2}{q+2}\frac{1}{1-\lambda_k(g)}\\
&\left(\frac{v_{kj}(g)^2}{(q+1)d_j(g)}-\frac{v_{kj}(g)v_{ki}(g)}{(q+1)\sqrt{d_i(g)d_j(g)}}\right)\\
=&(q+1)\cdot 2M_g\sum_{k=2}^{N_g}\frac{1}{1-\lambda_k(g)}\left(\frac{v_{kj}(g)^2}{d_j(g)}-\frac{v_{kj}(g)v_{ki}(g)}{\sqrt{d_i(g)d_j(g)}}\right)\\
=&(q+1)T_{ij}(g).
\end{aligned}
$$

Thus 1) is proved.

We continue to prove 2). Since $\Gamma(i)=\{s,t\}$,

$$
\begin{aligned}
T_{ij}(g+1)=&\frac{1}{q+1}\left(1+T_{sj}(g+1)+1+T_{tj}(g+1)\right.\\
&\left.+(q-1)(1+T_{ij}(g+1))\right)\\
=&\frac{q+1}{2}+\frac{1}{2}\left(T_{sj}(g+1)+T_{tj}(g+1)\right)\\
=&\frac{q+1}{2}+\frac{q+1}{2}\left(T_{sj}(g)+T_{tj}(g)\right).
\end{aligned}
$$

While for $T_{ji}(g+1)$, by Lemmas 2.3, 4.2, 4.3 and Corollary 4.1, one obtains

$$
\begin{aligned}
&T_{ji}(g+1)\\
=&2M_q(g+1)\left(\frac{1}{1+\frac{1}{q+1}}\frac{1}{q(q+1)M_g}+\sum_{k=2}^{N_g}\frac{1}{q+1}\right.\\
&\left(\frac{v_{ks}(g)}{\sqrt{d_s(g)}}+\frac{v_{kt}(g)}{\sqrt{d_t(g)}}\right)^2\left(\left(\frac{1}{1-\frac{\lambda_k(g)+q}{q+1}}\frac{1}{(\lambda_k(g)+1)(\lambda_k(g)+q+1)}\right)\right.\\
&\left.\left.+\left(\frac{1}{1-\frac{1}{q+1}}\frac{1}{q(\lambda_k(g)+q+1)}\right)\right)-\sum_{k=2}^{N_g}\frac{v_{kj}(g)}{\sqrt{(q+1)^2d_j(g)}}\right.\\
&\left(\frac{v_{ks}(g)}{\sqrt{d_s(g)}}+\frac{v_{kt}(g)}{\sqrt{d_t(g)}}\right)\left(\left(\frac{1}{1-\frac{\lambda_k(g)+q}{q+1}}\sqrt{\frac{1}{(\lambda_k(g)+1)(\lambda_k(g)+q+1)}}\right.\right.\\
&\left.\left.\sqrt{\frac{\lambda_k(g)+1}{\lambda_k(g)+q+1}}\right)-\left(\frac{1}{1+\frac{1}{q+1}}\sqrt{\frac{1}{q(\lambda_k(g)+q+1)}}\sqrt{\frac{q}{\lambda_k(g)+q+1}}\right)\right)\\
&\left.+\frac{3(q+1)}{2(q+2)}-\frac{1}{2qM_g}-\sum_{k=2}^{N_g}\frac{1}{2q(1+\lambda_k(g))}\left(\frac{v_{ks}(g)}{\sqrt{d_s(g)}}+\frac{v_{kt}(g)}{\sqrt{d_t(g)}}\right)^2\right)\\
=&\frac{3(q+1)}{2}M_g-\frac{q+1}{2}+(q+1)M_g\sum_{k=2}^{N_g}\frac{1}{1-\lambda_k(g)}\\
&\left(\left(\frac{v_{ks}(g)^2}{d_s(g)}-\frac{v_{ks}(g)v_{kj}(g)}{\sqrt{d_s(g)d_j(g)}}\right)+\left(\frac{v_{kt}(g)^2}{d_t(g)}-\frac{v_{kt}(g)v_{kj}(g)}{\sqrt{d_t(g)d_j(g)}}\right)\right.\\
&\left.-\frac{1}{2}\left(\frac{v_{ks}(g)}{\sqrt{d_s(g)}}-\frac{v_{kt}(g)}{\sqrt{d_t(g)}}\right)^2\right)\\
=&\frac{3(q+1)}{2}M_g-\frac{q+1}{2}+\frac{q+1}{4}\left(2\left(T_{js}(g)+T_{jt}(g)\right)\right.\\
&\left.-\left(T_{ts}(g)+T_{st}(g)\right)\right).
\end{aligned}
$$

We finally prove 3). (a) If i is adjacent to j, then $\Gamma(i)=\Gamma(j)=\{s,t\}$. In this case, we obtain

$$
\begin{aligned}
T_{ij}(g+1)=&\frac{1}{q+1}\left(1+T_{sj}(g+1)+1+T_{tj}(g+1)\right.\\
&\left.+q-1+(q-2)T_{ij}(g+1)\right)\\
=&\frac{q+1}{3}+\frac{1}{3}\left(T_{sj}(g+1)+T_{tj}(g+1)\right)\\
=&\frac{q+1}{6}\left(T_{ts}(g)+T_{st}(g)-(T_{st}(g)+T_{ts}(g))\right)\\
&+(q+1)M_g\\
=&(q+1)M_g.
\end{aligned}
$$

(b) If i is not adjacent to j, considering $\Gamma(i)=\{s,t\}$, $\Gamma(j)=\{u,v\}$, we obtain

$$
\begin{aligned}
T_{ij}(g+1)=&\frac{1}{q+1}\left(1+T_{sj}(g+1)+1+T_{tj}(g+1)\right.\\
&\left.+q-1+(q-1)T_{ij}(g+1)\right)\\
=&\frac{q+1}{2}+\frac{1}{2}\left(T_{sj}(g+1)+T_{tj}(g+1)\right)\\
=&\frac{q+1}{4}\left(T_{su}(g)+T_{tu}(g)+T_{sv}(g)+T_{tv}(g)\right.\\
&\left.-(T_{uv}(g)+T_{vu}(g))\right)+\frac{3(q+1)}{2}M_g.
\end{aligned}
$$

This completes the proof. $\square$

5.2. Kemeny's Constant

With Lemmas 2.5 and 4.3, the Kemeny's constant of $\mathcal{G}_q(g)$ can be determined explicitly.

THEOREM 5.2. *Let K_g be the Kemeny's constant for random walk in $\mathcal{G}_q(g)$. Then, for all $g \geq 0$,*

$$K_g = \left(\frac{(q+1)^2}{q+2} - \frac{3(q+1)}{2}\right)(q+1)^g + \frac{(q+1)(3q+7)}{2(q+3)}\left(\frac{(q+1)(q+2)}{2}\right)^g + \frac{q+1}{q+3}. \tag{20}$$

When $g \to \infty$,

$$\lim_{g\to\infty} K_g = \frac{3q+7}{2(q+2)} N_g. \tag{21}$$

Proof. Suppose that $1 = \lambda_1(g) > \lambda_2(g) \geq \dots \lambda_{N_g}(g) > -1$ are eigenvalues of the matrix P_g. By Lemmas 2.5 and 4.3, we obtain

$$\begin{aligned}
K_{g+1} &= \sum_{i=2}^{N_{g+1}} \frac{1}{1-\lambda_i(g+1)} \\
&= \sum_{i=2}^{N_g} \frac{1}{1-\frac{\lambda_i(g)+q}{q+1}} + \frac{(q-1)M_g + N_g}{1+\frac{1}{q+1}} + \frac{M_g - N_g}{1-\frac{q-1}{q+1}} \\
&= (q+1)\sum_{i=2}^{N_g} \frac{1}{1-\lambda_i(g)} + \frac{3q(q+1)}{2(q+2)}M_g - \frac{q(q+1)}{2(q+2)} \\
&= (q+1)K_g + \frac{3q(q+1)}{2(q+2)}M_g - \frac{q(q+1)}{2(q+2)}
\end{aligned} \tag{22}$$

With $M_g = \left(\frac{(q+1)(q+2)}{2}\right)^{g+1}$ and the initial condition $K_0 = \frac{(q+1)^2}{q+2}$, Eq. (22) is solved to obtain

$$K_g = \left(\frac{(q+1)^2}{q+2} - \frac{3(q+1)}{2}\right)(q+1)^g + \frac{(q+1)(3q+7)}{2(q+3)}\left(\frac{(q+1)(q+2)}{2}\right)^g + \frac{q+1}{q+3}, \tag{23}$$

which is exactly (20).

We continue to express the Kemeny's constant K_g in terms of the number of nodes N_g. From $N_g = \frac{2}{q+3}\left(\frac{(q+1)(q+2)}{2}\right)^{g+1} + \frac{2(q+2)}{q+3}$, we have $\left(\frac{(q+1)(q+2)}{2}\right)^g = \frac{q+3}{(q+1)(q+2)}N_g - \frac{2}{q+1}$ and $g = \ln\left(\frac{q+3}{(q+1)(q+2)}N_g - \frac{2}{q+1}\right) / \ln\left(\frac{(q+1)(q+2)}{2}\right)$. Inserting these two expressions into Eq. (23) results in

$$\begin{aligned}
K_g =& \frac{q+1}{q+3} + \left(\frac{(q+1)^2}{q+2} - \frac{3(q+1)}{2}\right) \\
& \left(\frac{q+3}{(q+1)(q+2)}N_g - \frac{2}{q+1}\right)^{\frac{\ln(q+1)}{\ln\left(\frac{(q+1)(q+2)}{2}\right)}} \\
& + \frac{(q+1)(3q+7)}{2(q+3)}\left(\frac{q+3}{(q+1)(q+2)}N_g - \frac{2}{q+1}\right).
\end{aligned}$$

Therefore, for $g \to \infty$,

$$\lim_{g\to\infty} K_g = \frac{3q+7}{2(q+2)} N_g.$$

This finishes the proof. □

Theorem 5.2 shows that for the whole family of networks $\mathcal{G}_q(g)$, the Kemeny's constant K_g grows as a linear function of N_g, the number of nodes, but the factor $(3q+7)/(2(q+2))$ is a decreasing function of q.

6. MEAN HITTING TIME

In this section, we study the mean hitting time for the studied networks with the remarkable scale-free small-world properties [23]. We will demonstrate that their mean hitting time also scales linearly with the number of nodes.

6.1. Some Definitions

Here we give definitions for some quantities related to network $\mathcal{G}_q(g)$.

DEFINITION 6.1. *For network $\mathcal{G}_q(g)$, the mean hitting time is*

$$\langle H_g \rangle = \frac{1}{N_g(N_g-1)} \sum_{i=1}^{N_g}\sum_{j=1}^{N_g} T_{ij}(g). \tag{24}$$

To obtain the explicit expression of the mean hitting time $\langle H_g \rangle$, we first determine three intermediary results for graph $\mathcal{G}_q(g)$, including the sum of hitting times, the additive-degree sum of hitting times, and the multiplicative-degree sum of hitting times.

For network $\mathcal{G}_q(g)$, the sum of hitting times is

$$H_g = \sum_{i=1}^{N_g}\sum_{j=1}^{N_g} T_{ij}(g); \tag{25}$$

the additive-degree sum of hitting times is

$$H_g^+ = \sum_{i=1}^{N_g}\sum_{j=1}^{N_g} (d_i(g) + d_j(g))\, T_{ij}(g); \tag{26}$$

and the multiplicative-degree sum of hitting times is

$$H_g^* = \sum_{i=1}^{N_g}\sum_{j=1}^{N_g} (d_i(g) \cdot d_j(g))\, T_{ij}(g). \tag{27}$$

LEMMA 6.1. *For any $g \geqslant 0$, the multiplicative-degree hitting time for graph $\mathcal{G}_q(g)$ is*

$$\begin{aligned}
H_g^* =& -\frac{(q+2)(q+4)(q+1)^3}{2}\left(\frac{(q+2)^2(q+1)^3}{4}\right)^g \\
& + \frac{(q+2)^2(q+1)^3}{q+3}\left(\frac{(q+1)(q+2)}{2}\right)^{2g} \\
& + \frac{(3q+7)(q+2)^2(q+1)^3}{2(q+3)}\left(\frac{(q+1)(q+2)}{2}\right)^{3g}.
\end{aligned}$$

Proof. By definition of the multiplicative-degree sum of hitting times, we have

$$H^*_{g+1} = \sum_{\{i,j\}\subseteq \mathcal{V}_g\cup\mathcal{W}_{g+1}} \left(d_i(g+1)d_j(g+1)\right) C_{ij}(g+1) =4M_g^2 K_g. \tag{28}$$

Using Theorem 5.2, the result is obtained. □

In what follows, we will determine the other two invariants H^+_g and H_g for network $\mathcal{G}_q(g)$.

6.2. Some Intermediary Results

Let $C_{ij}(g)$ be the commute time for any pair of nodes i and j in graph $\mathcal{G}_q(g)$. For any two subsets X and Y of set $\mathcal{V}_g$ of nodes in graph $\mathcal{G}_q(g)$, define

$$C_{X,Y}(g) = \sum_{i\in X, j\in Y} C_{ij}(g).$$

LEMMA 6.2. *For $g \geq 0$ and $Y \subseteq \mathcal{V}_g$,*

$$\sum_{i\in\mathcal{W}_{g+1}} C_{\Gamma(i),Y}(g+1) = \sum_{x\in\mathcal{V}_g} q d_x(g) C_{x,Y}(g+1). \tag{29}$$

Proof. For any node $x \in \mathcal{V}_g$, there are $d_x(g+1) - d_x(g) = qd_x(g)$ new nodes in $\mathcal{W}_{g+1}$ that are adjacent to i. So $C_{x,Y}(g+1)$ is summed $qd_x(g)$ times. □

LEMMA 6.3. *For any $g \geqslant 0$,*

$$\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g} C_{ij}(g+1) = \frac{q(q+1)}{4}H^+_q(g) + \frac{q(q+1)}{2}M_g \left(3M_gN_g - N_g^2 + N_g\right).$$

Proof. By Theorem 5.1, one obtains

$$\begin{aligned}
&\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g} C_{ij}(g+1)\\
=&\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g}\Bigg(\frac{3(q+1)}{2}M_g\\
&+\frac{q+1}{4}\left(2\left(C_{sj}(g)+C_{tj}g\right)-C_{st}(g)\right)\Bigg)\\
=&\frac{3q(q+1)}{2}M_g^2N_g + \frac{q+1}{2}\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g}\left(C_{sj}(g)+C_{tj}(g)\right)\\
&-\frac{q+1}{4}\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g} C_{st}(g).
\end{aligned} \tag{30}$$

For the second term on the right-hand side of the second equal sign in Eq. (30), we have

$$\begin{aligned}
&\frac{q+1}{2}\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g}\left(C_{sj}(g)+C_{tj}(g)\right)\\
=&\frac{q+1}{2}\sum_{i,j\in\mathcal{V}_g} d_i(g)C_{ij}(g)\\
=&\frac{q+1}{2}\frac{\sum_{i,j\in\mathcal{V}_g} d_i(g)C_{ij}(g)+\sum_{i,j\in\mathcal{V}_g} d_j(g)C_{ij}(g)}{2}\\
=&\frac{q+1}{4}\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g}(d_i(g)+d_j(g))C_{ij}(g)\\
=&\frac{q+1}{4}H^+_g.
\end{aligned} \tag{31}$$

With respect to the third term in Eq. (30), using Lemma 2.4, it can be rewritten as

$$\begin{aligned}
\frac{q+1}{4}\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g} C_{st}(g) =&\frac{q(q+1)}{4}N_g\sum_{st\in\mathcal{V}_g} C_{st}(g)\\
=&\frac{q(q+1)}{2}M_gN_g(N_g-1).
\end{aligned} \tag{32}$$

By plugging Eqs. (31) and (32) into Eq. (30), we obtain the desired result. □

LEMMA 6.4. *For any $g \geqslant 0$,*

$$\sum_{i,j\in\mathcal{W}_{g+1}} C_{ij}(g+1) = \frac{q^2(q+1)}{2}H^*_g + q(q+1)M_g^2 (3qM_g - qN_g - 2).$$

Proof. Suppose that $\Gamma(i) = \{s,t\}$ and $\Gamma(j) = \{u,v\}$. Note that for any two different nodes i and j in $\mathcal{W}_{g+1}$, if their old neighbors in $\mathcal{V}_g$ are the same, i.e., $\Gamma(i) = \Gamma(j) = \{s,t\}$, we use $i \sim j$ to denote this relation. Otherwise, if the sets of the old neighbors for i and j are different, we call $i \nsim j$. Then, by Theorem 5.1, we obtain

$$\begin{aligned}
&\sum_{i,j\in\mathcal{W}_{g+1}} C_{ij}(g+1)\\
=&\sum_{\substack{i,j\in\mathcal{W}_{g+1}\\ i\neq j, i\sim j}} C_{ij}(g+1) + \sum_{\substack{i,j\in\mathcal{W}_{g+1}\\ i\nsim j}} C_{ij}(g+1)\\
=&\sum_{\substack{i,j\in\mathcal{W}_{g+1}\\ i\nsim j}}\Bigg(3(q+1)M_g + \frac{q+1}{4}\Big(C_{su}(g)+C_{tu}(g)\\
&+C_{sv}(g)+C_{tv}(g)-\left(C_{uv}(g)+C_{st}(g)\right)\Big)\Bigg)\\
&+\sum_{\substack{i,j\in\mathcal{W}_{g+1}\\ i\neq j, i\sim j}} 2(q+1)M_g\\
=&3q^2(q+1)M_g^2(M_g-1) + 2q(q-1)(q+1)M_g^2\\
&+\frac{q+1}{4}\sum_{i,j\in\mathcal{W}_{g+1}}\left(C_{su}(g)+C_{tu}(g)+C_{sv}(g)+C_{tv}(g)\right)\\
&-\frac{q+1}{2}\sum_{\substack{i,j\in\mathcal{W}_{g+1}\\ i\sim j}} C_{st}(g)\\
&-\frac{q+1}{4}\sum_{\substack{i,j\in\mathcal{W}_{g+1}\\ i\nsim j}}\left(C_{st}(g)+C_{uv}(g)\right).
\end{aligned} \tag{33}$$

Below we evaluate the three sum terms on the right-hand side of the second equal sign in Eq. (33). By

Lemma 6.2 and Theorem 5.1, the first sum term can be computed as

$$\begin{aligned}
&\frac{q+1}{4}\sum_{i,j\in\mathcal{W}_{g+1}}\left(C_{su}(g)+C_{tu}(g)+C_{sv}(g)+C_{tv}(g)\right)\\
=&\frac{q+1}{4}\sum_{i,j\in\mathcal{W}_{g+1}}C_{\Gamma(i),\Gamma(j)}(g)\\
=&\frac{q+1}{4}\sum_{x,y\in\mathcal{V}_g}q^2d_x(g)d_y(g)C_{xy}(g)\\
=&\frac{q^2(q+1)}{2}H_g^*. \qquad (34)
\end{aligned}$$

We next compute the second sum term in Eq. (33). By Lemma 2.4, we have

$$\begin{aligned}
\frac{q+1}{2}\sum_{\substack{i,j\in\mathcal{W}_{g+1}\\ i\sim j}}C_{st}(g)=&\frac{q^2(q+1)}{2}\sum_{st\in\mathcal{E}_g}C_{st}(g)\\
=&q^2(q+1)M_g(N_g-1). \qquad (35)
\end{aligned}$$

We proceed to evaluate the third term in Eq. (33). According to Eq. (7), it follows that

$$\begin{aligned}
&\frac{q+1}{4}\sum_{\substack{i,j\in\mathcal{W}_{g+1}\\ i\nsim j}}\left(C_{st}(g)+C_{uv}(g)\right)\\
=&\frac{q+1}{4}\sum_{f=1}^{q}\sum_{i\in\mathcal{V}^{(f)}}\sum_{i\nsim j}\left(C_{st}(g)+C_{uv}(g)\right)\\
=&\frac{q+1}{4}q\sum_{st\in\mathcal{E}_g}q\sum_{\substack{uv\in\mathcal{E}_g\\ uv\neq st}}\left(C_{st}(g)+C_{uv}(g)\right)\\
=&\frac{q^2(q+1)}{4}\sum_{st\in\mathcal{E}_g}\sum_{\substack{uv\in\mathcal{E}_g\\ uv\neq st}}\left(C_{uv}(g)+C_{st}(g)\right). \qquad (36)
\end{aligned}$$

By Lemma 2.4, Eq. (36) can be recast as

$$\begin{aligned}
&\frac{q+1}{4}\sum_{i,j\in\mathcal{W}_{g+1}}\left(C_{st}(g)+C_{uv}(g)\right)\\
=&\frac{q^2(q+1)}{2}(M_g-1)\sum_{st\in\mathcal{E}_g}C_{st}(g)\\
=&q^2(q+1)M_g(M_g-1)(N_g-1). \qquad (37)
\end{aligned}$$

Plugging Eqs. (34), (35), and (37) into Eq. (33) gives the result. □

6.3. Addictive-Degree Sum of Hitting Times

We now determine the additive-degree hitting time for graph $\mathcal{G}_q(g)$.

Lemma 6.5. *For any $g \geqslant 0$, the additive-degree hitting time for graph $\mathcal{G}_q(g)$ is*

$$\begin{aligned}
H_g^+ =&\frac{2(q+2)^2(q+1)^3}{(q+3)^2}\left(\frac{(q+1)(q+2)}{2}\right)^{2g}\\
&+\frac{(q+2)(3q+7)(q+1)^3(q^3+8q^2+22q+20)}{(q+3)^2(q^2+5q+8)}\left(\frac{(q+1)(q+2)}{2}\right)^{3g}\\
&-\frac{(q+2)(q+4)(q+1)^3}{(q+3)}\left(\frac{(q+2)^2(q+1)^3}{4}\right)^{g}\\
&+\frac{(q+2)(q^2+9q+20)(q+1)^3}{(q+3)(q^2+5q+8)}\left(\frac{(q+2)(q+1)^2}{2}\right)^{g}\\
&+\frac{(q+2)(q+1)^3}{(q+3)^2}\left(\frac{(q+1)(q+2)}{2}\right)^{g}.
\end{aligned}$$

Proof. By definition of the additive-degree sum of hitting times, we have

$$\begin{aligned}
H_{g+1}^+ =&\sum_{i,j\in\mathcal{V}_g\cup\mathcal{W}_{g+1}}\left(d_i(g+1)+d_j(g+1)\right)C_{ij}(g+1)\\
=&\frac{1}{2}\sum_{i,j\in\mathcal{V}_g}\left(d_i(g+1)+d_j(g+1)\right)C_{ij}(g+1) \qquad (38)\\
&+\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g}\left(d_i(g+1)+d_j(g+1)\right)C_{ij}(g+1)\\
&+\frac{1}{2}\sum_{i,j\in\mathcal{W}_{g+1}}\left(d_i(g+1)+d_j(g+1)\right)C_{ij}(g+1).
\end{aligned}$$

We begin to compute the three sum terms for H_{g+1}^+ one by one.

By Theorem 5.1, the first sum term can be evaluated as

$$\begin{aligned}
&\frac{1}{2}\sum_{i,j\in\mathcal{V}_g}\left(d_i(g+1)+d_j(g+1)\right)C_{ij}(g+1)\\
=&\sum_{\{i,j\}\subseteq\mathcal{V}_g}(q+1)\left(d_i(g)+d_j(g)\right)(q+1)C_{ij}(g)\\
=&(q+1)^2H_g^+. \qquad (39)
\end{aligned}$$

For the second sum term, it can be computed as

$$\begin{aligned}
&\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g}\left(d_i(g+1)+d_j(g+1)\right)C_{ij}(g+1)\\
=&\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g}\left((q+1)+(q+1)d_j(g)\right)C_{ij}(g+1)\\
=&(q+1)\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g}C_{ij}(g+1) \qquad (40)\\
&+(q+1)\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g}d_j(g)C_{ij}(g+1),
\end{aligned}$$

where the two sum terms can be further computed as follows. First, by Lemma 6.3,

$$\begin{aligned}
&(q+1)\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g}C_{ij}(g+1)\\
=&(q+1)\Bigg(\frac{q(q+1)}{4}H_g^+ +\frac{q(q+1)}{2}M_g\\
&\left(3M_gN_g-N_g^2+N_g\right)\Bigg) \qquad (41)
\end{aligned}$$

$$=\frac{q(q+1)^2}{4}H_g^+ + \frac{q(q+1)^2}{2}M_g\left(3M_gN_g - N_g^2 + N_g\right).$$

On the other hand, by Lemma 2.4 and Theorem 5.1,

$$\begin{aligned}
&(q+1)\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g} d_j(g)C_{ij}(g+1)\\
=&(q+1)^2\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g} d_j(g)\Bigg(\frac{3}{2}M_g+\frac{1}{4}(2(C_{sj}(g)\\
&\qquad + C_{tj}(g)) - C_{st}(g))\Bigg)\\
=&(q+1)^2\cdot qM_g\cdot 2M_g\cdot\frac{3}{2}M_g\\
&+\frac{(q+1)^2}{2}\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g} d_j(g)(C_{js}(g)+C_{tj}(g))\\
&-\frac{(q+1)^2}{4}\sum_{j\in\mathcal{V}_g} d_j(g)C_{st}(g) \qquad (42)\\
=&3q(q+1)^2M_g^3\\
&+\frac{(q+1)^2}{2}\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g} d_j(g)(C_{js}(g)+C_{tj}(g))\\
&-\frac{(q+1)^2}{4}2M_g\cdot 2M_g(N_g-1)\\
=&3q(q+1)^2M_g^3\\
&+\frac{(q+1)^2}{2}\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g} d_j(g)(C_{js}(g)+C_{tj}(g))\\
&-q(q+1)^2M_g^2(N_g-1),
\end{aligned}$$

while the middle part can be computed to obtain

$$\begin{aligned}
&\frac{(q+1)^2}{2}\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g} d_j(g)\left(C_{js}(g)+C_{tj}(g)\right)\\
=&\frac{(q+1)^2}{2}q\sum_{i\in\mathcal{V}^{(1)}}\sum_{j\in\mathcal{V}_g} d_j(g)\left(C_{js}(g)+C_{tj}(g)\right)\\
=&\frac{q(q+1)^2}{2}\sum_{j\in\mathcal{V}_g}\sum_{i\in\mathcal{V}^{(1)}} d_j(g)\left(C_{js}(g)+C_{tj}(g)\right)\\
=&\frac{q(q+1)^2}{2}\sum_{j\in\mathcal{V}_g}\sum_{k\in\mathcal{V}_g} d_j(g)d_k(g)C_{kj}(g)\\
=&q(q+1)^2H_g^*. \qquad (43)
\end{aligned}$$

Combining Eqs. (40)-(43) yields

$$\begin{aligned}
&\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g}(d_i(g+1)+d_j(g+1))C_{ij}(g+1)\\
=&\frac{q(q+1)^2}{4}H_g^+ + q(q+1)^2H_g^* \qquad (44)\\
&+\frac{q(q+1)^2}{2}M_g\left(6M_g^2+M_gN_g-N_g^2+2M_g+N_g\right).
\end{aligned}$$

With regard to the third sum term in Eq. (38), by Lemma 6.4, we have

$$\begin{aligned}
&\sum_{i,j\in\mathcal{W}_{g+1}}(d_i(g+1)+d_j(g+1))C_{ij}(g+1)\\
=&2(q+1)\Bigg(\frac{q^2(q+1)}{2}H_g^* + q(q+1)M_g^2\\
&\qquad(3qM_g - qN_g - 2)\Bigg) \qquad (45)\\
=&q^2(q+1)^2H_g^* + 2q(q+1)^2M_g^2\left(3qM_g-qN_g-2\right).
\end{aligned}$$

Substituting Eqs. (39), (44) and (45) back into Eq. (38) gives

$$\begin{aligned}
H_{g+1}^+ =&\frac{(q+2)(q+1)^2}{2}H_g^+ + \frac{q(q+2)(q+1)^2}{2}H_g^*\\
&+\frac{1}{2}q(q+1)^2M_g(N_g+2M_g)\left(3M_g-N_g+1\right)\\
&+q(q+1)^2M_g^2\left(3qM_g-qN_g-2\right).
\end{aligned}$$

Considering the initial condition $H_0^+ = 2(q+2)(q+1)^3$, the above recursive relation is solved to yield the deriable result. □

6.4. Mean Hitting Time

We are now ready to present the result for mean hitting time of $\mathcal{G}_q(g)$, denoted by $\langle H_g\rangle$, and its dominant behavior.

THEOREM 6.1. *For any $g\geq 0$, the mean hitting time for graph $\mathcal{G}_q(g)$ is*

$$\begin{aligned}
&\langle H_g\rangle =\\
&\frac{(q+3)^2}{(q+1)^2(q+2)^2\left(\left(\frac{(q+1)(q+2)}{2}\right)^g+\frac{2}{q+1}\right)\left(\left(\frac{(q+1)(q+2)}{2}\right)^g+\frac{1}{q+2}\right)}\\
&\Bigg(\frac{(q+1)(q+2)^2(q^3+8q^2+15q+8)}{(q+3)^2(q^2+5q+8)}\left(\frac{(q+1)(q+2)}{2}\right)^{2g}\\
&+\frac{(q+4)(3q+7)(q+2)^2(q+1)^3}{2(q+3)^2(q^2+5q+8)}\left(\frac{(q+1)(q+2)}{2}\right)^{3g}\\
&-\frac{(q+2)(q+4)(q+1)^3}{2(q+3)^2}\left(\frac{(q+2)^2(q+1)^3}{4}\right)^g\\
&+\frac{(q+2)(q^2+9q+20)(q+1)^3}{(q+3)^2(q^2+5q+8)}\left(\frac{(q+2)(q+1)^2}{2}\right)^g\\
&+\frac{2(q+2)(q+1)^2(q+4)^2}{(q+3)^2(q^2+5q+8)}(q+1)^g\\
&-\frac{(q+2)(q+1)^2}{(q+3)^2}\left(\frac{(q+1)(q+2)}{2}\right)^g\Bigg). \qquad (46)
\end{aligned}$$

When $g\to\infty$,

$$\lim_{g\to\infty}\langle H_g\rangle = \frac{(q+3)(q+4)(3q+7)}{2(q+2)(q^2+5q+8)}N_g. \qquad (47)$$

Proof. Since $\langle H_g\rangle = H_g/(N_g(N_g-1))$, in order to determine $\langle H_g\rangle$, we first determine H_g. For network $\mathcal{G}_q(g+1)$, we have

$$\begin{aligned}
H_{g+1} =&\frac{1}{2}\sum_{i,j\in\mathcal{V}_g\cup\mathcal{W}_{g+1}} C_{ij}(g+1)\\
=&\frac{1}{2}\sum_{i,j\in\mathcal{V}_g} C_{ij}(g+1) + \sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g} C_{ij}(g+1)
\end{aligned}$$

$$+\frac{1}{2}\sum_{i,j\mathcal{W}_{g+1}} C_{ij}(g+1). \tag{48}$$

Below we will compute the three sum terms in Eq. (48). By Theorem 5.1, the first sum term can be evaluated as

$$\begin{aligned}\frac{1}{2}\sum_{i,j\in\mathcal{V}_g} C_{ij}(g+1) =& \sum_{\{i,j\}\subseteq\mathcal{V}_g}(q+1)C_{ij}(g)\\ =&(q+1)H_g. \end{aligned}\tag{49}$$

Using Lemma 6.3, the second sum term is determined as

$$\begin{aligned}\sum_{i\in\mathcal{W}_{g+1}}\sum_{j\in\mathcal{V}_g} C_{ij}(g+1) =&\frac{q(q+1)}{4}H_q^+(g) + \frac{q(q+1)}{2}M_g\\ &\left(3M_gN_g - N_g^2 + N_g\right). \end{aligned}\tag{50}$$

Finally, by Lemma 6.4, the third sum term is computed as

$$\begin{aligned}&\sum_{i,j\in\mathcal{W}_{g+1}} C_{ij}(g+1)\\ =&\frac{q^2(q+1)}{2}H_g^* + q(q+1)M_g^2(3qM_g - qN_g - 2). \end{aligned}\tag{51}$$

Plugging Eqs. (49)-(51) into Eq. (48) leads to

$$\begin{aligned}H_{g+1} =&(q+1)H_g + \frac{q(q+1)}{2}H_q^+(g) + \frac{q^2(q+1)}{4}H_q^*(g)\\ &+\frac{1}{2}q(q+1)M_gN_g(3M_g - N_g + 1)\\ &+\frac{1}{2}q(q+1)M_g^2(3qM_g - qN_g - 2).\end{aligned}$$

Considering the initial condition $H_0 = (q+2)(q+1)^2$, the recursive relation is solved to obtain

$$\begin{aligned}H_g =&\frac{(q+1)(q+2)^2(q^3+8q^2+15q+8)}{(q+3)^2(q^2+5q+8)}\left(\frac{(q+1)(q+2)}{2}\right)^{2g}\\ &+\frac{(q+4)(3q+7)(q+2)^2(q+1)^3}{2(q+3)^2(q^2+5q+8)}\left(\frac{(q+1)(q+2)}{2}\right)^{3g}\\ &-\frac{(q+2)(q+4)(q+1)^3}{2(q+3)^2}\left(\frac{(q+2)^2(q+1)^3}{4}\right)^{g}\\ &+\frac{(q+2)(q^2+9q+20)(q+1)^3}{(q+3)^2(q^2+5q+8)}\left(\frac{(q+2)(q+1)^2}{2}\right)^{g}\\ &+\frac{2(q+2)(q+1)^2(q+4)^2}{(q+3)^2(q^2+5q+8)}(q+1)^g\\ &-\frac{(q+2)(q+1)^2}{(q+3)^2}\left(\frac{(q+1)(q+2)}{2}\right)^{g}.\end{aligned}$$

Plugging this result to $\langle H_g\rangle = H_g/(N_g(N_g-1))$ gives (46).

In a similar way to that of Kemeny's constant K_g, we can represent mean hitting time $\langle H_g\rangle$ in terms of the number of nodes N_g, and obtain the leading term of $\langle H_g\rangle$ given by (47). □

Theorem 6.1 indicates that mean hitting time $\langle H_g\rangle$ of network $\mathcal{G}_q(g)$ scales linearly as N_g with the factor decreasing with q, which is similar to that for the Kemeny's constant K_g.

7. CONCLUSION

The edge corona product of a graph is a natural extension of traditional triangulation operation, which has been successfully applied to generate complex networks with prominent properties observed in various real-life systems. In this paper, we presented an extensive study of various properties for hitting times of random walks on a class of graphs, which are iteratively generated by edge corona product of complete graphs. We first deduced recursive formulas for the eigenvalues and eigenvectors of normalized adjacency matrix of the graphs under consideration. Using these results, we then determined a recursive expression for two-node hitting time from an arbitrary node to another. Also, we obtained exact solution to the Kemeny's constant, which is a weighted average of hitting times among all node pairs. Finally, we provided analytical formulas for the sum of hitting times, the sum of multiplicative-degree hitting times, and the sum of additive-degree hitting times.

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DATA AVAILABILITY STATEMENT

No new data were generated or analysed in support of this research.

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