

The Riemann hypothesis and holomorphic index in complex dynamics

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Abstract

We give an interpretation of the Riemann hypothesis in terms of complex and topological dynamics. For example, the Riemann hypothesis is affirmative and all zeros of the Riemann zeta function are simple if and only if a certain meromorphic function has no attracting fixed point. To obtain this, we use holomorphic index (residue fixed point index), which characterizes local properties of fixed points in complex dynamics.

1 The Riemann zeta function

For $s \in \mathbb{C}$, the series

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

converges if $\operatorname{Re} s > 1$. Indeed, $\zeta(s)$ is analytic on the half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > 1\}$ and continued analytically to a meromorphic function (a holomorphic map) $\zeta : \mathbb{C} \rightarrow \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with only one pole at $s = 1$, which is simple. This is *the Riemann zeta function*.

It is known that $\zeta(s) = 0$ when $s = -2, -4, -6, \dots$. These zeros are called *trivial zeros* of the Riemann zeta function. We say the other zeros are *non-trivial*.

The *Riemann hypothesis*, which is the most important conjecture on the Riemann zeta function, concerns the alignment of the non-trivial zeros of ζ :

The Riemann hypothesis. *All non-trivial zeros lie on the vertical line $\{s \in \mathbb{C} : \operatorname{Re} s = 1/2\}$.*

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The line $\{s \in \mathbb{C} : \operatorname{Re} s = 1/2\}$ is called *the critical line*. It is numerically verified that the first 10^{13} zeros (from below) lie on the critical line [G].

It is also conjectured that every zero of ζ is simple. We refer to this conjecture as *the simplicity hypothesis* after some literature. (See [RS] or [Mu] for example. One may find some related results and observations in [T, §10.29, §14.34, §14.36].)

The aim of this note is to translate the Riemann hypothesis in terms of complex and topological dynamical systems. More precisely, we translate the locations of the non-trivial zeros into some dynamical properties of the fixed points of a meromorphic function of the form

$$\nu_g(z) = z - \frac{g(z)}{z g'(z)},$$

where g is a meromorphic function on \mathbb{C} that shares (non-trivial) zeros with ζ . For example, we will set $g = \zeta$ or the *Riemann xi function*

$$\xi(z) := \frac{1}{2} z(1-z) \pi^{-z/2} \Gamma\left(\frac{z}{2}\right) \zeta(z),$$

etc. The function $\nu_g(z)$ is carefully chosen so that

- If $g(\alpha) = 0$ then $\nu_g(\alpha) = \alpha$; and
- The *holomorphic index* (or the *residue fixed point index*, see §2) of ν_g at α is α itself when α is a simple zero of $g(z)$.

See §3 for more details.

For a given meromorphic function $g : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$, we say a fixed point α of g is *attracting* if $|g'(\alpha)| < 1$, *indifferent* if $|g'(\alpha)| = 1$, and *repelling* if $|g'(\alpha)| > 1$. Here are some possible translations of the Riemann hypothesis (plus the simplicity hypothesis) by ν_ζ :

Theorem 1 *The following conditions are equivalent:*

- (a) *The Riemann hypothesis is affirmative and every non-trivial zero of ζ is simple.*
- (b) *Every non-trivial zero of ζ is an indifferent fixed point of the meromorphic function*

$$\nu_\zeta(z) := z - \frac{\zeta(z)}{z \zeta'(z)} .$$
- (c) *The meromorphic function ν_ζ above has no attracting fixed point.*
- (d) *There is no topological disk D with $\overline{\nu_\zeta(D)} \subset D$.*

Note that (d) is a topological property of the map $\nu_\zeta : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$, in contrast to the analytic (or geometric) nature of (a). We will present some variants of this theorem in §4.

§5 is devoted for some numerical observations and questions on linearization problem. §6 is an appendix: we apply Newton’s method to the Riemann zeta function. We will also give a “semi-topological” criteria for the Riemann hypothesis in terms of the Newton map (Theorem 14).

Remarks.

- We can apply the method of this note to the L -functions without extra effort.
- The following well-known facts are implicitly used in this paper:
 - Every non-trivial zero is located in the *critical stripe*

$$\mathcal{S} := \{s \in \mathbb{C} : 0 < \operatorname{Re} s < 1\}.$$

- The functional equation $\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s)$ implies that if α is a non-trivial zero of $\zeta(s)$, then so is $1 - \alpha$ and they have the same order.

Hence the non-trivial zeros are symmetrically arrayed with respect to $s = 1/2$. By these properties, we will mainly consider the zeros which lie on the upper half of the critical stripe \mathcal{S} . See [E, T] for more details.

- We used *Mathematica* 10.0 for all the numerical calculation.

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2 Fixed points and holomorphic indices

Multiplier. Let g be a holomorphic function on a domain $\Omega \subset \mathbb{C}$. We say $\alpha \in \Omega$ is a *fixed point* of g with *multiplier* $\lambda \in \mathbb{C}$ if $g(\alpha) = \alpha$ and $g'(\alpha) = \lambda$. The multiplier λ is the primary factor that determines the local dynamics near α . In fact, the Taylor expansion about α gives a representation of the local action of g :

$$g(z) - \alpha = \lambda(z - \alpha) + O(|z - \alpha|^2). \tag{1}$$

Hence the action of g near α is locally approximated by $w \mapsto \lambda w$ in the coordinate $w = z - \alpha$. We say the fixed point α is *attracting*, *indifferent*, or *repelling* according to $|\lambda| < 1$, $|\lambda| = 1$, or $|\lambda| > 1$.

Topological characterization. Attracting and repelling fixed points of holomorphic mappings have purely topological characterizations (cf. Milnor [Mi, §8]):

Proposition 2 (Topological characterization of fixed points) *Let g be a holomorphic function on a domain $\Omega \subset \mathbb{C}$. The function g has an attracting (resp. repelling) fixed point if and only if there exists a topological disk $D \subset \Omega$ such that $\overline{g(D)} \subset D$ (resp. $g|_D$ is injective and $\overline{D} \subset g(D) \subset \Omega$).*

The condition that g is holomorphic is essential. For example, the proposition is false if we only assume that g is C^∞ .

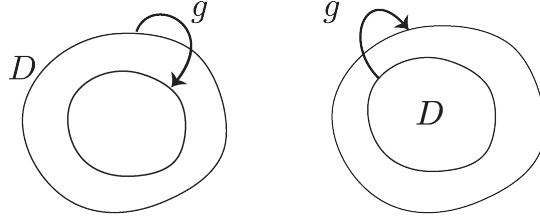


Figure 1: Topological disks which contain an attracting or a repelling fixed point.

Proof. Suppose that g has an attracting fixed point $\alpha \in \Omega$. By the Taylor expansion (1) as above, we have $g(z) - \alpha = (\lambda + o(1))(z - \alpha)$ near α and thus a small circle around α is mapped strictly inside the circle. By the maximum principle, this circle bounds a round disk D such that $\overline{g(D)} \subset D$.

Conversely, if a topological disk D in Ω satisfies $\overline{g(D)} \subset D$, we may observe the map $g : D \rightarrow g(D) \subset D$ via the Riemann map and it is enough to consider the case of $D = \mathbb{D}$, the unit disk. By the Schwarz-Pick theorem (see [Ah2, §1]) the map is strictly contracting with respect to the distance $d(z, w) := |z - w|/|1 - \bar{z}w|$ on \mathbb{D} . Hence there exists a fixed point α by the fixed point theorem. It must be an attracting fixed point by the Schwarz lemma.

The repelling case is analogous. ■

Holomorphic index. Let α be a fixed point of a holomorphic function $g : \Omega \rightarrow \mathbb{C}$. We define the *holomorphic index* (or *residue fixed point index*) of α by

$$\iota(g, \alpha) := \frac{1}{2\pi i} \int_C \frac{1}{z - g(z)} dz,$$

where C is a small circle around α with counterclockwise direction. The holomorphic index is mostly determined by the multiplier:

Proposition 3 *If the multiplier $\lambda := g'(\alpha)$ is not 1, then we have $\iota(g, \alpha) = \frac{1}{1 - \lambda}$.*

See [Mi, Lem.12.2] for the proof.

Remark. Any complex number K may be the holomorphic index of a fixed point of multiplier 1. For example, the polynomial $g(z) = z - z^2 + Kz^3$ has a fixed point at zero with $g'(0) = 1$ and $\iota(g, 0) = K$.

Since the Möbius transformation $\lambda \mapsto \frac{1}{1-\lambda} = \iota$ sends the unit disk to the half-plane $\{\iota \in \mathbb{C} : \operatorname{Re} \iota > 1/2\}$, fixed points are classified as follows:

Proposition 4 (Classification by index) *Suppose that the multiplier $\lambda = g'(\alpha)$ is not 1. Then α is attracting, repelling, or indifferent if and only if the holomorphic index $\iota = \iota(g, \alpha)$ satisfies $\operatorname{Re} \iota > 1/2$, $< 1/2$, or $= 1/2$ respectively.*

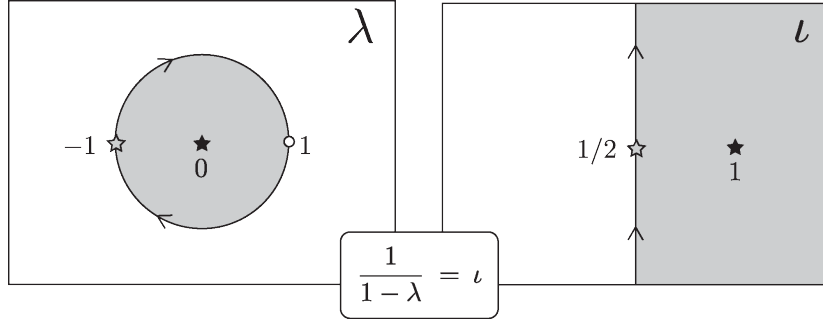


Figure 2: Multipliers and holomorphic indices

Note that the indifferent fixed points whose multiplier is not 1 correspond to the “critical line” in the ι -plane.

3 The nu function

Let $g : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ be a non-constant meromorphic function. (We regard such a meromorphic function as a holomorphic map onto the Riemann sphere.) We define the *nu function* $\nu_g : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ of g by

$$\nu_g(z) := z - \frac{g(z)}{zg'(z)}.$$

This is also a non-constant meromorphic function on \mathbb{C} . We first check:

Proposition 5 (Fixed points of ν_g) *Suppose that $\alpha \neq 0$. Then α is a fixed point of ν_g if and only if α is a zero or a pole of g . Moreover,*

- if α is a zero of g of order $m \geq 1$, then $\nu_g'(\alpha) = 1 - \frac{1}{m\alpha}$ and $\iota(\nu_g, \alpha) = m\alpha$; and
- if α is a pole of g of order $m \geq 1$, then $\nu_g'(\alpha) = 1 + \frac{1}{m\alpha}$ and $\iota(\nu_g, \alpha) = -m\alpha$.

If 0 is a zero or a pole of g of order of $m \geq 1$, then $\nu_g(0) = 1/m$ or $-1/m$ respectively. In particular, 0 is not a fixed point of ν_g .

An immediate corollary is:

Corollary 6 *The function ν_g has no fixed point of multiplier 1.*

Hence we can always apply Proposition 3 to the fixed points of ν_g .

Proof of Proposition 5. The point $\alpha \in \mathbb{C}$ is a fixed point of $\nu_g(z) = z - g(z)/(zg'(z))$ if and only if $g(\alpha)/(\alpha g'(\alpha)) = 0$. This implies that α must be a zero or a pole of g .

Conversely, suppose that

$$g(z) = C(z - \alpha)^M(1 + O(z - \alpha))$$

for some $M \in \mathbb{Z} - \{0\}$ and $C \in \mathbb{C} - \{0\}$ near α .

Suppose in addition that $\alpha \neq 0$. Then the expansion of ν_g about α is

$$\nu_g(z) = \alpha + \left(1 - \frac{1}{M\alpha}\right)(z - \alpha) + O((z - \alpha)^2)$$

and thus α is a fixed point with multiplier $\nu'_g(\alpha) = 1 - 1/(M\alpha) \neq 1$. By Proposition 3, its holomorphic index is $1/(1 - \nu'_g(\alpha)) = M\alpha$.

If $\alpha = 0$, the expansion of ν_g about $\alpha = 0$ is

$$\nu_g(z) = \frac{1}{M} + O(z).$$

This implies that $\nu_g(0) = 1/M \neq 0$. ■

Remark. There is another way to calculate the holomorphic index. (This is actually how the author found the function ν_g .) Suppose that α is a zero of g of order m . A variant of the argument principle ([Ah1, p.153]) yields that for any holomorphic function ϕ defined near α , we have

$$\frac{1}{2\pi i} \int_C \phi(z) \frac{g'(z)}{g(z)} dz = m\phi(\alpha)$$

where C is a small circle around α . Set $\phi(z) := z$. Then the equality above is equivalent to

$$\frac{1}{2\pi i} \int_C \frac{1}{z - \nu_g(z)} dz = m\alpha.$$

The same argument is also valid when α is a pole.

Example. Consider a rational function $g(z) = \frac{(z + 1)(z - 1/2)^2}{z^3(z - 1)}$. By Proposition 4 and Proposition 5, the zeros -1 and $1/2$ of g are repelling and attracting fixed points of ν_g respectively. The pole 1 is a repelling fixed point, though 0 is not a fixed point of ν_g .

4 The Riemann hypothesis

Let us consider the Riemann zeta function and prove Theorem 1.

The Riemann zeta. Set $g = \zeta$, the Riemann zeta function. It is known that the trivial zeros $\alpha = -2, -4, \dots$ of ζ are all simple. By Proposition 5, they are fixed points of ν_ζ of multiplier $1 - 1/\alpha \neq 1$ and their holomorphic indices are α itself. Hence by Proposition 4, they are all repelling fixed points of ν_ζ .

Similarly the unique pole $z = 1$ of ζ is simple and it is a repelling fixed point of ν_ζ with multiplier $1 + 1/1 = 2$. Hence we have:

Proposition 7 *Every fixed point α of ν_ζ off the critical stripe \mathcal{S} is repelling.*

Let α be a non-trivial zero of order $m \geq 1$ in the critical strip \mathcal{S} . (Under the simplicity hypothesis m is always 1.) By Proposition 5, α is a fixed point of ν_ζ with multiplier $\lambda := 1 - 1/(m\alpha) \neq 1$, and its holomorphic index is $\iota := m\alpha$.

If the Riemann hypothesis holds, $\text{Re } \iota = \text{Re } m\alpha = m/2$. Thus $\text{Re } \iota = 1/2$ if $m = 1$ and $\text{Re } \iota \geq 1$ if $m \geq 2$. Since “ $\text{Re } \iota \geq 1$ in the ι -plane” is equivalent to “ $|\lambda - 1/2| \leq 1/2$ (and $\lambda \neq 1$) in the λ -plane”, we have:

Proposition 8 *Under the Riemann hypothesis, any fixed point α of ν_ζ in \mathcal{S} is a zero of ζ of some order $m \geq 1$ that lies on the critical line. Moreover,*

- *when $m = 1$, α is indifferent with multiplier $\lambda \neq 1$; and*
- *when $m \geq 2$, α is attracting with multiplier λ satisfying $|\lambda - 1/2| \leq 1/2$.*

In particular, if the simplicity hypothesis also holds, all non-trivial zeros of ζ are indifferent fixed point of ν_ζ .

Hence (a) implies (b) in Theorem 1. Now we show the converse.

Proposition 9 *If the fixed points of ν_ζ in the critical stripe \mathcal{S} are all indifferent, then both the Riemann hypothesis and the simplicity hypothesis are affirmative.*

Proof. Let α be an indifferent fixed point of ν_ζ in the critical stripe \mathcal{S} . Since ζ has no pole in \mathcal{S} , α is a zero of some order $m \geq 1$ of ζ and the holomorphic index is $\iota(\nu_\zeta, \alpha) = m\alpha$ by Proposition 5. The point $1 - \alpha$ is also a zero of ζ of order m contained in \mathcal{S} , and the holomorphic index is $\iota(\nu_\zeta, 1 - \alpha) = m(1 - \alpha)$.

By assumption, both α and $1 - \alpha$ are indifferent fixed points of ν_ζ . Hence by Proposition 4, the real parts of $\iota(\nu_\zeta, \alpha) = m\alpha$ and $\iota(\nu_\zeta, 1 - \alpha) = m(1 - \alpha)$ are both $1/2$. This happens only if $m = 1$ and $\text{Re } \alpha = 1/2$. ■

Let us finish the proof of Theorem 1:

Proof of Theorem 1. The equivalence of (a) and (b) is shown by Proposition 8 and Proposition 9 above. The condition (b) implies (c) since the fixed points of ν_ζ off the critical stripe are all repelling by Proposition 7.

Suppose that (c) holds. Then any fixed point α of ν_ζ in the critical stripe is repelling or indifferent, and it is also a zero of ζ of some order m by Proposition 5. Hence the holomorphic index $\iota(\nu_\zeta, \alpha) = m\alpha$ satisfies $\operatorname{Re} m\alpha \in (0, 1/2]$. Moreover, $1 - \alpha$ is also a zero of ζ with the same order m and $\iota(\nu_\zeta, 1 - \alpha) = m(1 - \alpha)$ satisfies $\operatorname{Re} m(1 - \alpha) \in [m - 1/2, m)$.

If $m \geq 2$, then $\operatorname{Re} m(1 - \alpha) > 1/2$ and thus $1 - \alpha$ is attracting. This is a contradiction. If $m = 1$ and α is repelling, then $1 - \alpha$ is an attracting fixed point. This is also a contradiction. Hence $m = 1$ and α is indifferent. This implies $\operatorname{Re} \alpha = 1/2$ and we conclude that (c) implies (a).

The equivalence of (c) and (d) comes from Proposition 2. ■

Remark. We used the functional equation to show the equivalence of (a), (b), and (c).

A more topological version. Note again that (d) of Theorem 1 is a purely topological condition for the function ν_ζ . Even if one observe the dynamics (the action) of $\nu_\zeta : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ through *any* homeomorphism on the sphere, this condition will be preserved (Figure 3). More precisely, we have:

Theorem 10 *The conditions (a) - (d) of Theorem 1 are equivalent to:*

- (e) *For any homeomorphism $h : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with $h(\infty) = \infty$, the continuous function $\nu_{\zeta, h} := h \circ \nu_\zeta \circ h^{-1}$ has no topological disk D with $\overline{\nu_{\zeta, h}(D)} \subset D$.*

The proof is a routine. We may regard the dynamics of the map $\nu_{\zeta, h}$ as a topological deformation of the original dynamics of ν_ζ . Remark that the critical line may not be a “line” any more when it is mapped by a homeomorphism. (That may even have a positive area!)

Variants. Next we consider ν_g for $g = \xi$, the *Riemann xi function* given in the first section. It is known that $\xi : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function whose zeros are exactly the non-trivial zeros of the Riemann zeta. Moreover, we have a functional equation $\xi(z) = \xi(1 - z)$.

Now we have a variant of Theorem 1:

Theorem 11 (Interpretation by xi) *The following conditions are equivalent:*

- (a') *The Riemann hypothesis is affirmative and all zeros of ζ are simple.*
 (b') *All the fixed points of the meromorphic function $\nu_\xi(z) := z - \frac{\xi(z)}{z\xi'(z)}$ are indifferent.*

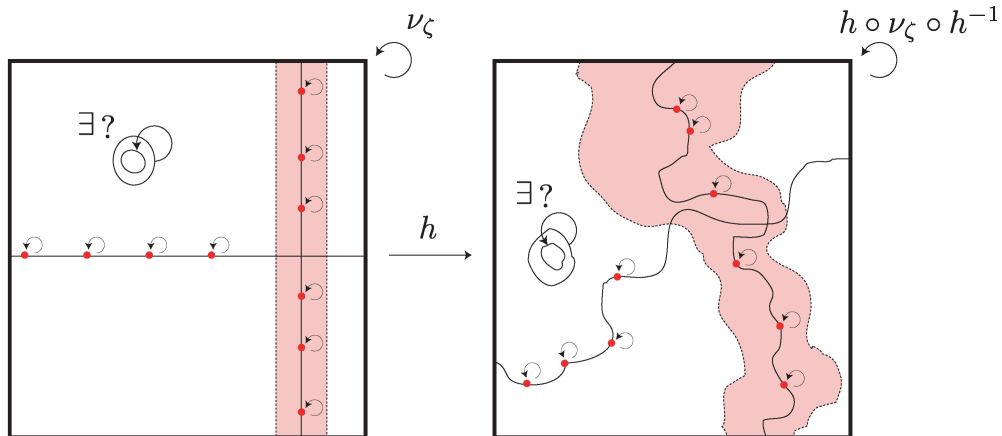


Figure 3: Topological deformation of the dynamics of ν_ζ . By Theorem 10, we can still state the Riemann hypothesis in the deformed dynamics.

(c') *The meromorphic function ν_ξ above has no attracting fixed point.*

(d') *There is no topological disk D with $\overline{\nu_\xi(D)} \subset D$.*

The proof is just analogous to the case of ν_ζ . (It is simpler because ξ has neither trivial zeros nor poles.)

Here is another example. To have an entire function it is enough to consider the function

$$\eta(z) := (z - 1)\zeta(z),$$

which seems simpler than $\xi(z)$. In this case $\eta(z)$ and $\zeta(z)$ share all the zeros and we will have a similar theorem to Theorem 1.

Yet another interesting entire function is

$$\chi(z) := (z - a)^m \eta(z) = (z - a)^m (z - 1)\zeta(z)$$

where $m \in \mathbb{N}$ and $a \in \mathbb{C}$ satisfy $\operatorname{Re} ma > 1/2$. It has an extra zero at $z = a$ of order m which is an attracting fixed point of ν_χ by Proposition 5. The virtue of this family with parameters m and a is that we can characterize the chaotic locus (the *Julia set* defined in the next section) of the dynamics of ν_χ as the boundary of the *attracting basin* of a (i.e., the set of points z whose orbit $z, \nu_\chi(z), \nu_\chi(\nu_\chi(z)), \dots$ converges to the attracting fixed point a .) In general, this property helps us to draw the pictures of the chaotic locus.

5 Global dynamics and linearization problem

If any of conditions in Theorem 1 is true, then all the fixed points of ν_ζ (and ν_ξ) are indifferent. In this section we give some brief observations on these fixed points.

Global dynamics of ν_g . Let $g : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ be a non-constant meromorphic function and consider the dynamics given by iteration of g :

$$\mathbb{C} \xrightarrow{g} \widehat{\mathbb{C}} \xrightarrow{g} \widehat{\mathbb{C}} \xrightarrow{g} \widehat{\mathbb{C}} \xrightarrow{g} \dots$$

(See [Mi] or [B] for basic results in complex dynamics.) We use a conventional notation g^n ($n \geq 0$) to denote n times iteration of g . More precisely, we set $g^0 := \text{id}$ and $g^n := g \circ g^{n-1}$ for $n \in \mathbb{N}$. The *Fatou set* $F(g)$ of g is defined by the set of $z \in \mathbb{C}$ which has a neighborhood U where g^n is defined for all $n \in \mathbb{N}$ and the family $\{z \mapsto g^n(z)\}_{n \in \mathbb{N}}$ is normal (or equivalently, equicontinuous with respect to the spherical metric of $\widehat{\mathbb{C}}$). The complement $\widehat{\mathbb{C}} - F(g)$ is called the *Julia set* of g and denoted by $J(g)$. We regard the Fatou set and the Julia set as the stable and chaotic parts of the dynamics. For example, the attracting (resp. repelling) fixed points (periodic points, more generally) belong to the Fatou (resp. Julia) set.

Indifferent fixed points may be in the Fatou set or the Julia set, and this is a source of the *linearization problem*.

Classification of indifferent fixed point. (See [Mi, §10, §11] for more details.)

Let $g : \Omega \rightarrow \mathbb{C}$ be a holomorphic function on a domain Ω and α an indifferent fixed point of multiplier $g'(\alpha) = e^{2\pi i\theta}$ ($\theta \in \mathbb{R}$). In the theory of complex dynamics, we say α is *parabolic* or *rationally indifferent* if $\theta \in \mathbb{Q}$, and *irrationally indifferent* otherwise.

Let α be an irrationally indifferent fixed point of g with $g'(\alpha) = e^{2\pi i\theta}$ ($\theta \in \mathbb{R}$). It is called *linearizable* if there exists a holomorphic homeomorphism ϕ defined on a neighborhood U of α such that $\phi(\alpha) = 0$; $\phi(U)$ is a round disk centered at 0; and $\phi \circ g \circ \phi^{-1}(w) = e^{2\pi i\theta}w$. In other words, the local action near a is conjugate to a rotation of angle $2\pi\theta$. If this is the case, the largest possible neighborhood U with these properties is called a *Siegel disk*. The dynamics of g on the Siegel disk is equicontinuous and thus U is contained in the Fatou set. In fact, it is known that an indifferent fixed point belongs to the Fatou set if and only if it is linearizable.

It is also known that for Lebesgue almost every $\theta \in [0, 1) - \mathbb{Q}$, the irrationally indifferent fixed point of multiplier $e^{2\pi i\theta}$ is linearizable. For example, suppose that $\theta \in [0, 1) - \mathbb{Q}$ has the continued fraction expansion

$$\theta = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}} =: [a_1, a_2, a_3, \dots],$$

where the coefficients $a_i \in \mathbb{N}$ are uniformly bounded. Such a θ is called of *bounded type*. Then the irrationally indifferent fixed points of multiplier $e^{2\pi i\theta}$ with bounded type θ is always linearizable. However, there is also a dense subset of $\theta \in [0, 1) - \mathbb{Q}$ that gives non-linearizable fixed points. (Unfortunately, the multiplier $e^{2\pi i\theta}$ is not the only factor for the precise condition of linearizability.)

Does ν_ζ has Siegel disks? Under the Riemann hypothesis and the simplicity hypothesis, each non-trivial zero of ζ is of the form $\alpha = 1/2 + \gamma i$ ($\gamma \in \mathbb{R}$) and α is an indifferent fixed point of ν_ζ (or ν_ξ , etc.).

Now it is easy to check:

Proposition 12 *Under the Riemann hypothesis and the simplicity hypothesis, non-trivial zero $\alpha = 1/2 + \gamma i$ ($\gamma \in \mathbb{R}$) is an indifferent fixed point of ν_ζ with multiplier $e^{2\pi i \theta}$ where the values γ and θ are related by*

$$\gamma = \frac{1}{2 \tan \pi \theta} \iff \theta = \frac{1}{\pi} \arctan \frac{1}{2\gamma}.$$

Note that $\theta \rightarrow 0$ as $\gamma \rightarrow \pm\infty$.

Proof. By Proposition 5, we have $\nu'_\zeta(\alpha) = 1 - 1/\alpha$ and thus $e^{2\pi i \theta} = 1 - 1/(1/2 + \gamma i)$. This is equivalent to the formulae above. ■

Now it is natural to ask:

Linearization problem. *Can θ be a rational number? Is α linearizable? That is, can ν_ζ has a Siegel disk?*

Numerical observation. As we have mentioned, randomly chosen θ gives a Siegel disk with probability one. Table 1 gives the continued fractions up to the 50th term for some zeros. (Here $1/2 + \gamma_n i$ ($\gamma_n > 0$) is the n th non-trivial zero of ζ from below and $\gamma_n = 1/(2 \tan \pi \theta_n)$.) It seems very unlikely that these zeros have rational θ .

Figure 4 shows some recurrent orbits near the first 4 zeros of ζ in the dynamics of ν_ζ and ν_ξ .

6 Appendix: Newton's method

There are many root finding algorithms, but the most famous one would be *Newton's method*. Let us apply it to the Riemann zeta and its variants. The aim of this Appendix is to describe it in terms of holomorphic index.

Relaxed Newton maps and fixed points. Let $g : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ be a non-constant meromorphic function and κ a complex constant with $|\kappa - 1| < 1$. We define its *relaxed Newton's map* $N_g = N_{g,\kappa} : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ by

$$N_g(z) = z - \kappa \frac{g(z)}{g'(z)},$$

which is again meromorphic. (See [B].) When $\kappa = 1$ the map $N_g = N_{g,1}$ is traditional *Newton's map* for Newton's method. Here is a version of Proposition 5 (and Corollary 6) for N_g :

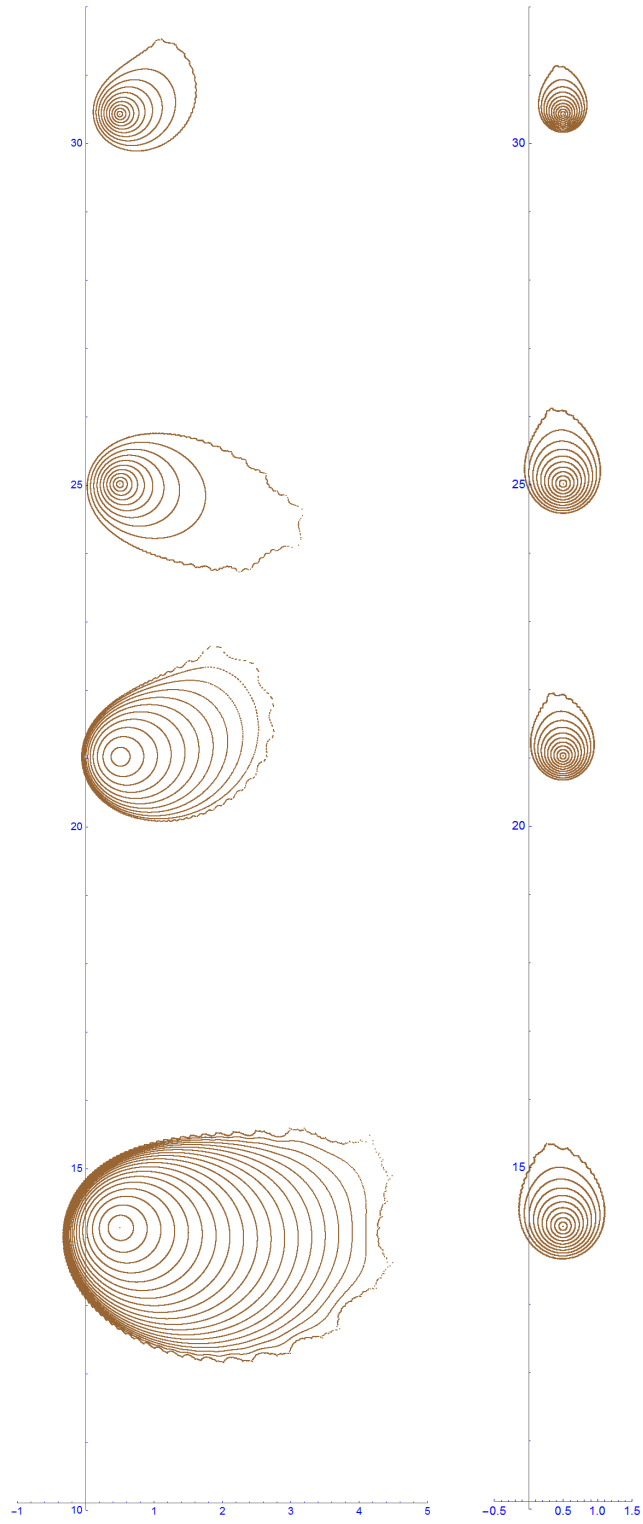


Figure 4: The first four Siegel disks (?) of ν_ζ (left) and ν_ξ (right).

n	$\text{Im } \gamma_n$	θ_n	$[a_1, a_2, \dots, a_{25}, a_{26}, \dots, a_{50}]$
1	14.1347	0.0112552	[88,1,5,1,1,2,2,5,2,15,2,4,2,4,1,9,1,1,5,2,10,1,1,5,1,2,7,100,9,2,2,3,2,5,1,8,179,23,1,1,35,1,3,1,2,8,7,34,4,1]
2	21.0220	0.00756943	[132,9,14,1,1,1,2,1,52,1,9,3,4,1,1,1,1,2,2,3,2,1,10,1,1,1,9,1,1,6,5,1,5,3,1,5,2,6,1,135,1,1,5,1,2,3,2,9,1,3]
3	25.0109	0.00636259	[157,5,1,12,3,1,1,1,1,2,11,1,29,5,1,4,1,1,3,5,14,1,3,1,2,3,6,14,4,1,41,4,1,1,7,4,1,3,21,1,4,3,1,2,2,1,16,1,2,3]
4	30.4249	0.00523061	[191,5,2,15,3,2,2,7,2,1,2,46,2,1,1,6,1,4,2,2,4,1,6,1,1,2,5,1,8,1,2,2,5,1,4,39,3,19,5,2,9,1,1876,2,12,1,4,4,1,6]
10	49.7738	0.00319746	[312,1,2,1,48,1,1,4,1,3,1,2,5,1,21,1,4,1,3,2,1,1,8,1,6,9,1,2,3,2,1,2,1,1,1,4,1,1,1,5,2,1,3,126,1,24,3,2,29,5]
10^2	236.524	0.00067289	[1486, 7,1,4,1,2,1,53,2,8,1,4,6,3,1,3,1,13,3,1,7,2,18,3,1,1,1,1,1,2,1,210,3,4,1,3,1,2,2,8,1,1,7,2,2,2,1,12,2,2]
10^3	1419.42	0.000112127	[8918,2,48,220,1,15,1,1,6,1,1,6,2,4,149,1,15,3,6,1,4,1,1,64,11,1,1,13,11,2,4,2,3,2,4,1,2,1,1,2,3,46,1,11,3,1,18,1,6,4]
10^4	9877.78	0.0000161124	[62063,1,15,2,1,2,1,8,1,2,6,2,1,2,2,1,5,1,1,186,3,5,4,14,9,1,12,1,12,1,6,4,22,1,3,15,1,2,1,14,1,2,2,12,3,1,1,1,45,2]

Table 1: Continued fraction expansions for some zeros of ζ .

Proposition 13 (Fixed points of N_g) *The point $\alpha \in \mathbb{C}$ is a fixed point of N_g if and only if α is a zero or a pole of g . Moreover,*

- *if α is a zero of g of order $m \geq 1$, then α is an attracting fixed point of multiplier $N'_g(\alpha) = 1 - \kappa/m$ and its index is $\iota(N_g, \alpha) = m/\kappa$.*
- *if α is a pole of g of order $m \geq 1$, then α is a repelling fixed point of multiplier $N'_g(\alpha) = 1 + \kappa/m$ and its index is $\iota(N_g, \alpha) = -m/\kappa$.*

In particular, N_g has no fixed point of multiplier 1.

The proof is similar to that of Proposition 5 and left to the readers.

Newton's method. The idea of (relaxed) Newton's method is to use the attracting fixed points of N_g to detect the location of the zeros of g . More precisely, by taking an initial value z_0 sufficiently close to the zero α , the sequence $\{N_g^n(z_0)\}_{n \geq 0}$ converges rapidly to the attracting fixed point α . It is practical to use the traditional value $\kappa = 1$, since the convergence to the simple zero is quadratic. That is, we have $N_g(z) - \alpha = O((z - \alpha)^2)$ near α .

Holomorphic index and the argument principle. Let us restrict our attention

to the traditional case $\kappa = 1$. For the fixed point α of N_g its holomorphic index is

$$\iota(N_g, \alpha) = \frac{1}{2\pi i} \int_C \frac{1}{z - N_g(z)} dz = \frac{1}{2\pi i} \int_C \frac{g'(z)}{g(z)} dz$$

where C is a small circle around α . This is exactly the argument principle applied to g .

The Riemann hypothesis. When we apply Newton's method to the Riemann zeta, all zeros of ζ become attracting fixed points of $N_\zeta(z) = z - \zeta(z)/\zeta'(z)$. By Proposition 2, we have:

Theorem 14 *The Riemann hypothesis is affirmative if and only if there is no topological disk D contained in the stripe $\mathcal{S}' = \{z \in \mathbb{C} : 1/2 < \operatorname{Re} z < 1\}$ satisfying $\overline{N_\zeta(D)} \subset D$.*

Proof. The Riemann hypothesis holds if and only if there is no zero of ζ in the stripe $\mathcal{S}' \subset \mathcal{S}$. Equivalently, by Proposition 13, there is no attracting fixed point of N_ζ in \mathcal{S}' . Now the theorem is a simple application of Proposition 2. ■

Some pictures. It is easier to draw pictures of the Julia set of Newton's map N_ζ than those of ν_ζ . After the list of references, we present some pictures of N_ζ , N_η , N_ξ and N_{\cosh} with comments.

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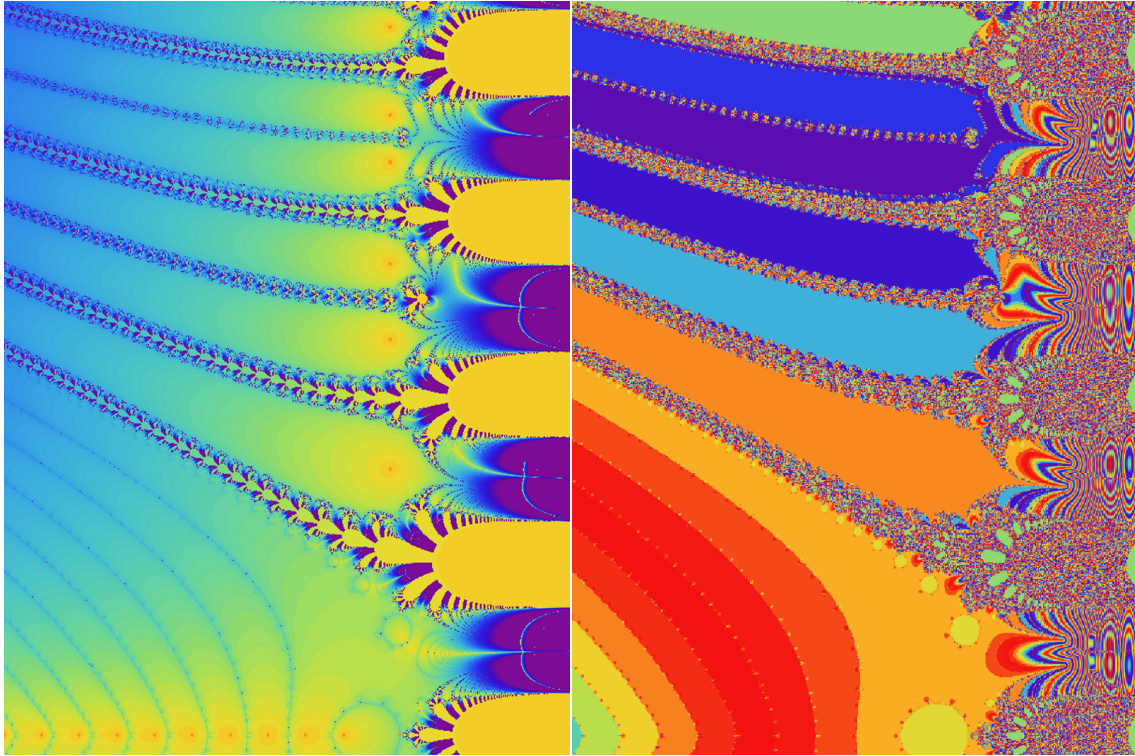


Figure 5: Left: The Julia set of N_ζ in $\{\operatorname{Re} z \in [-20, 10], \operatorname{Im} z \in [-1, 39]\}$. The orange dots are disks that are close to the zeros of ζ . Right: The same region in different colors. Colors distinguish the zeros to converge.

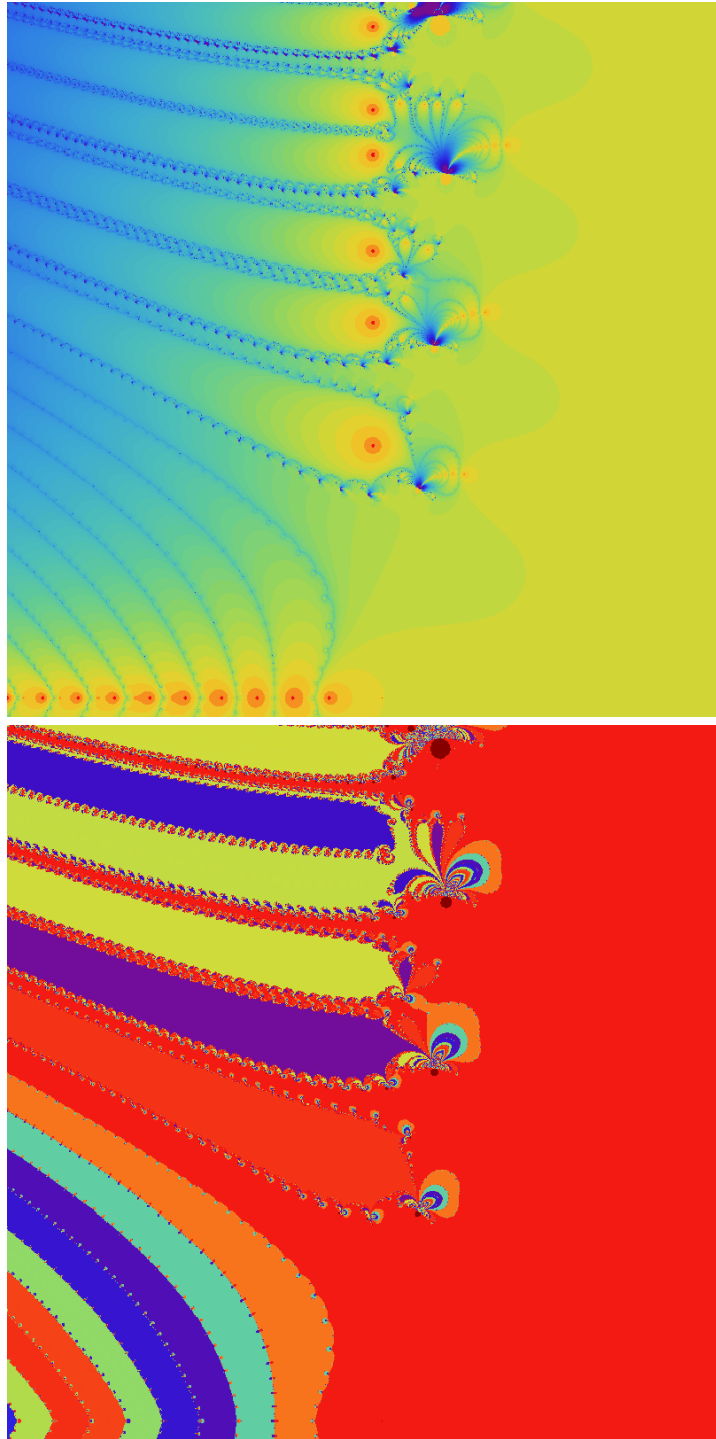


Figure 6: The Julia set of N_η in $\{\operatorname{Re} z \in [-20, 20], \operatorname{Im} z \in [-1, 39]\}$, drawn in the same colors as Figure 5. Probably because η is entire, the dynamics of N_η is simpler than that of N_ζ . It is known that for any entire function g and its zeros, their immediate basins (the connected components of the Fatou set of N_g which contain the zeros) are simply connected and unbounded ([MS]).

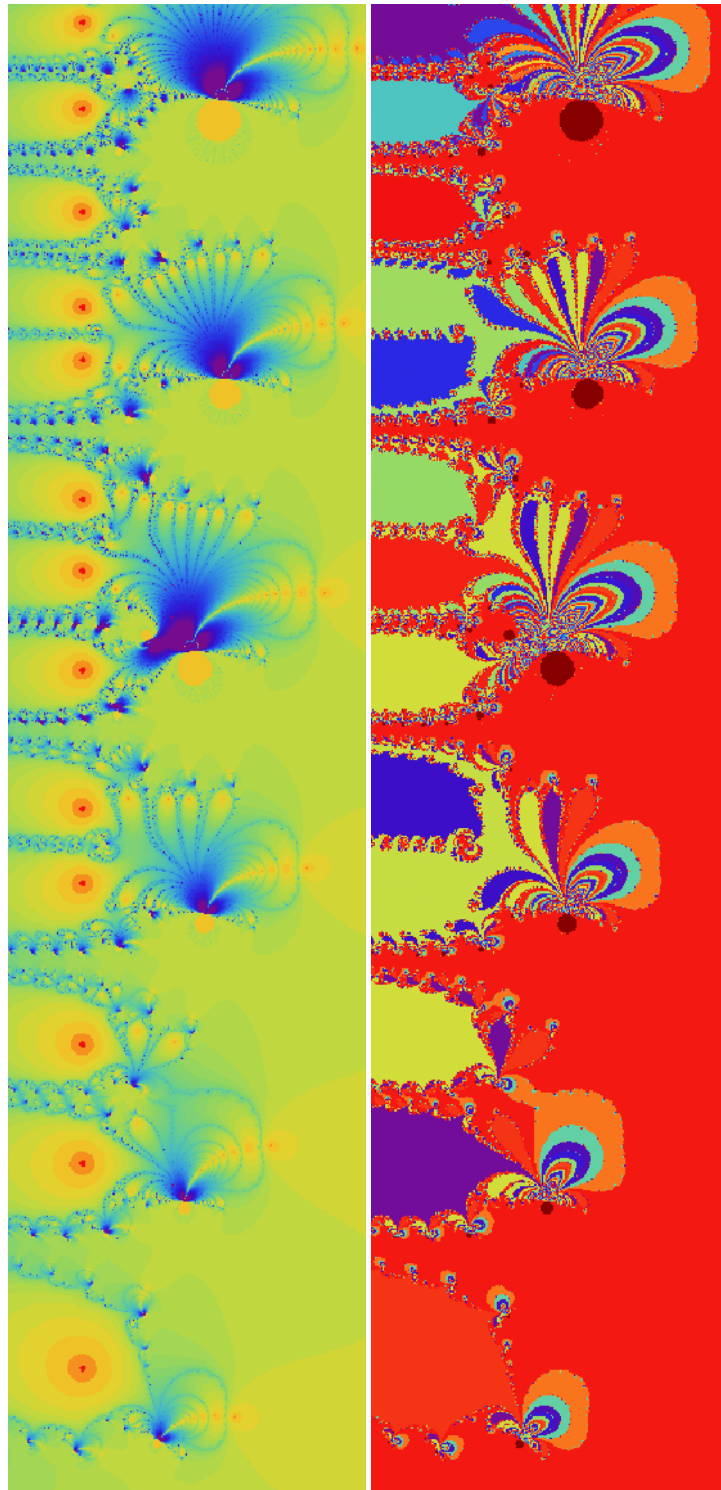


Figure 7: Details of the Julia set of N_η in $\{\text{Re } z \in [-2, 10], \text{Im } z \in [10, 60]\}$ (“Heads of Chickens”).

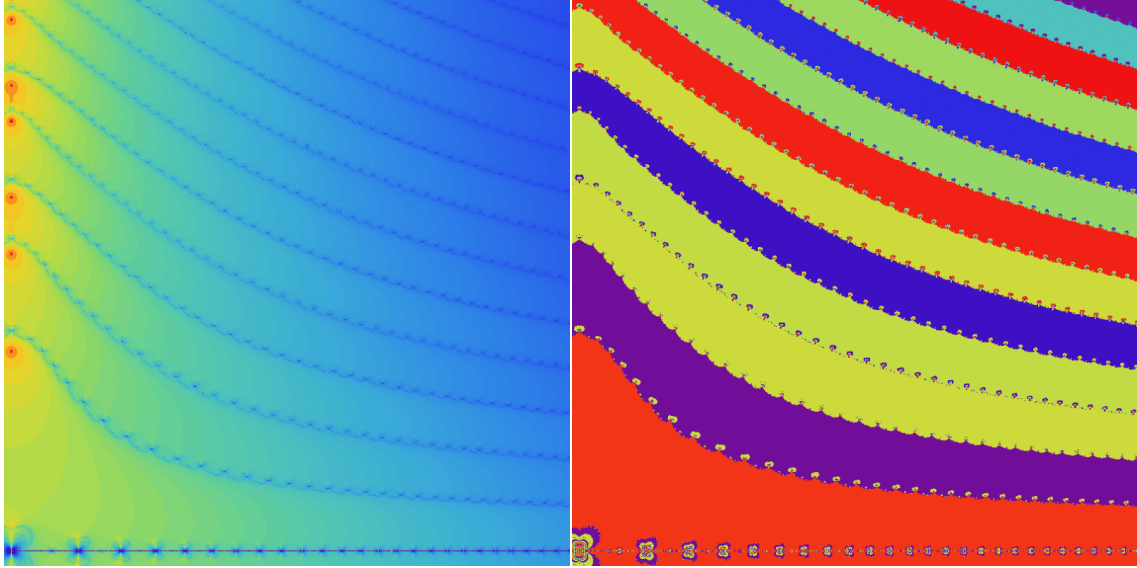


Figure 8: Details of the Julia set of N_ξ in $\{\operatorname{Re} z \in [0, 40], \operatorname{Im} z \in [-1, 39]\}$. The dynamics seems surprisingly simple. Compare with the case of hyperbolic cosine in Figure 10.

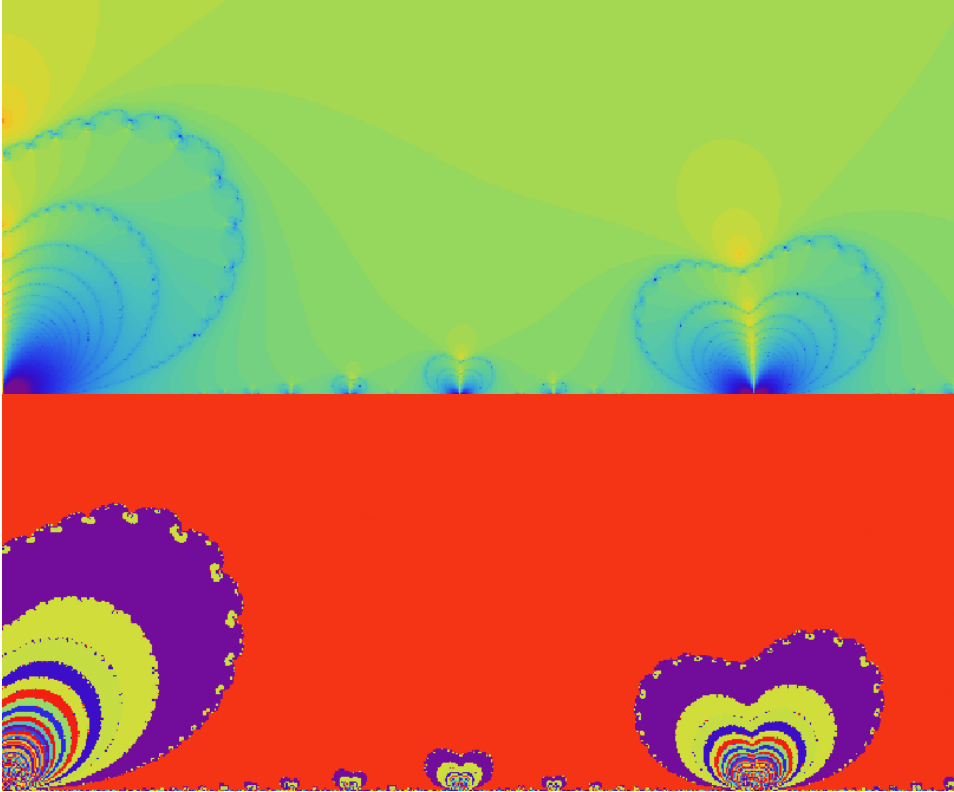


Figure 9: Details of the Julia set of N_ξ in $\{\operatorname{Re} z \in [0.5, 6.5], \operatorname{Im} z \in [0, 2.5]\}$.

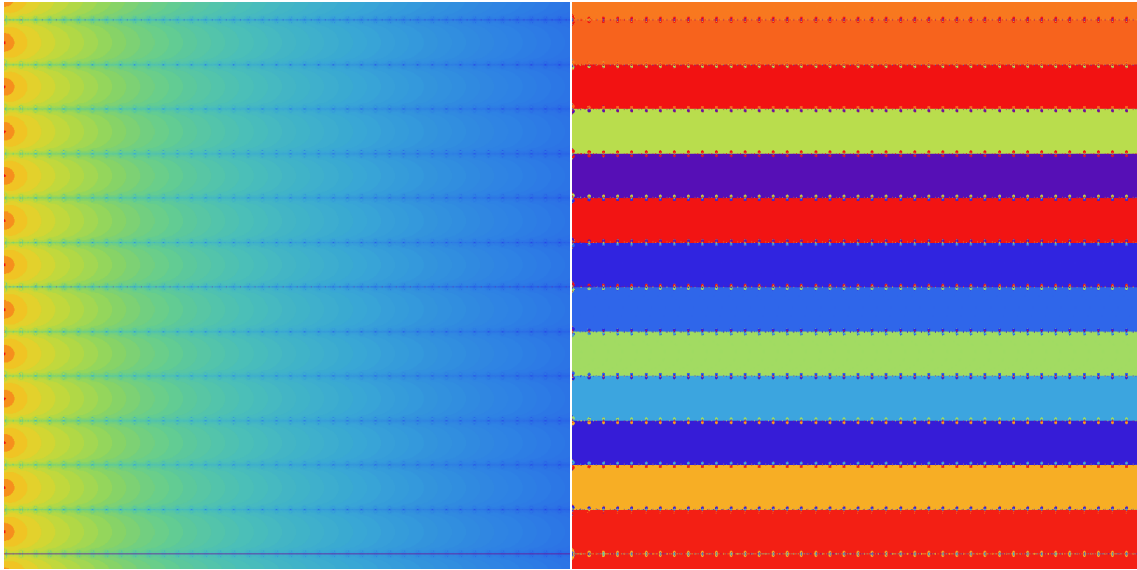


Figure 10: The Julia set of the Newton map N_{\cosh} of the hyperbolic cosine $\cosh z$ in $\{\operatorname{Re} z \in [0, 40], \operatorname{Im} z \in [-1, 39]\}$. The Julia set seems topologically the same as that of N_{ξ} . Do they belong to the same deformation space?

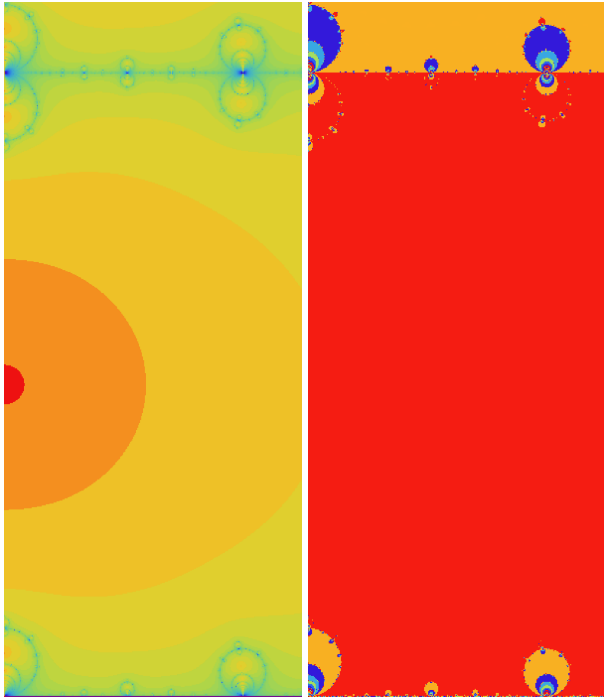


Figure 11: Details of the Julia set of the Newton map N_{\cosh} in $\{\operatorname{Re} z \in [0, 1.5], \operatorname{Im} z \in [0, 3.5]\}$.