

Adaptive Augmented Lagrangian Methods: Algorithms and Practical Numerical Experience—Detailed Version

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In this paper, we consider augmented Lagrangian (AL) algorithms for solving large-scale nonlinear optimization problems that execute adaptive strategies for updating the penalty parameter. Our work is motivated by the recently proposed adaptive AL trust region method by Curtis, Jiang, and Robinson [Math. Prog., DOI: 10.1007/s10107-014-0784-y, 2013]. The first focal point of this paper is a new variant of the approach that employs a line search rather than a trust region strategy, where a critical algorithmic feature for the line search strategy is the use of convexified piecewise quadratic models of the AL function for computing the search directions. We prove global convergence guarantees for our line search algorithm that are on par with those for the previously proposed trust region method. A second focal point of this paper is the practical performance of the line search and trust region algorithm variants in MATLAB software, as well as that of an adaptive penalty parameter updating strategy incorporated into the LANCELOT software. We test these methods on problems from the CUTEst and COPS collections, as well as on challenging test problems related to optimal power flow. Our numerical experience suggests that the adaptive algorithms outperform traditional AL methods in terms of efficiency and reliability. As with traditional AL algorithms, the adaptive methods are matrix-free and thus represent a viable option for solving extreme-scale problems.

Keywords: nonlinear optimization, nonconvex optimization, large-scale optimization, augmented Lagrangians, matrix-free methods, steering methods

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1. Introduction

Augmented Lagrangian (AL) methods [25, 33] have recently regained popularity due to growing interests in solving extreme-scale nonlinear optimization problems. These methods are attractive in such settings as they can be implemented matrix-free [2, 3, 11, 28] and have global and local convergence guarantees under relatively weak assumptions [18, 26]. Furthermore, certain variants of AL methods [20, 21] have proved to be very efficient for solving certain structured problems [6, 34, 36].

A new AL trust region method was recently proposed and analyzed in [15]. The novel feature of that algorithm is an adaptive strategy for updating the penalty parameter inspired by techniques for performing such updates in the context of exact penalty methods [7, 8, 29]. This feature is designed to overcome a potentially serious drawback of traditional AL methods, which is that they may be ineffective during some (early) iterations due to poor choices of the penalty parameter and/or Lagrange multiplier estimates. In such situations, the poor choices of these quantities may lead to little or no improvement in the primal space and, in fact, the iterates may diverge from even a well-chosen initial

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iterate. The key idea for avoiding this behavior in the algorithm proposed in [15] is to adaptively update the penalty parameter *during* the step computation in order to ensure that the trial step yields a sufficiently large reduction in linearized constraint violation, thus *steering* the optimization process steadily toward constraint satisfaction.

The contributions of this paper are two-fold. First, we present an AL line search method based on the same framework employed for the trust region method in [15]. The main difference between our new approach and that in [15], besides the differences inherent in using line searches instead of a trust region strategy, is that we utilize a convexified piecewise quadratic model of the AL function to compute the search direction in each iteration. With this modification, we prove that our line search method achieves global convergence guarantees on par with those proved for the trust region method in [15]. The second contribution of this paper is that we perform extensive numerical experiments with a MATLAB implementation of the adaptive algorithms (i.e., both line search and trust region variants) and an implementation of an adaptive penalty parameter updating strategy in the LANCELOT software [12]. We test these implementations on problems from the CUTEEST [22] and COPS [5] collections, as well as on test problems related to optimal power flow [37]. Our results indicate that our adaptive algorithms outperform traditional AL methods in terms of efficiency and reliability.

The remainder of the paper is organized as follows. In §2, we present our adaptive AL line search method and state convergence results. Details and proofs of these results, which draw from those in [15], can be found in Appendices A and B. We then provide numerical results in §3 to illustrate the effectiveness of our implementations of our adaptive AL algorithms. We give conclusions in §4.

Notation. We often drop function arguments once a function is defined. We also use a subscript on a function name to denote its value corresponding to algorithmic quantities using the same subscript. For example, for a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, if x_k is the value for the variable x during iteration k of an algorithm, then $f_k := f(x_k)$. We also often use subscripts for constants to indicate the algorithmic quantity to which they correspond. For example, γ_μ denotes a parameter corresponding to the algorithmic quantity μ .

2. An Adaptive Augmented Lagrangian Line Search Algorithm

2.1 Preliminaries

We assume that all problems under our consideration are formulated as

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} f(x) \text{ subject to } c(x) = 0, \quad l \leq x \leq u. \quad (1)$$

Here, we assume that the objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and constraint function $c : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are twice continuously differentiable, and that the variable lower bound vector $l \in \mathbb{R}^n$ and upper bound vector $u \in \mathbb{R}^n$ satisfy $l \leq u$. Our goal is to design an algorithm that will compute a first-order primal-dual stationary point for problem (1). However, in order for the algorithm to be suitable as a general-purpose approach, it should have mechanisms for terminating and providing useful information when an instance of (1) is (locally) infeasible. In such cases, we have designed our algorithm so that it transitions to finding an infeasible first-order stationary point for the nonlinear feasibility problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} v(x) \text{ subject to } l \leq x \leq u, \quad (2)$$

where $v : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as $v(x) = \frac{1}{2} \|c(x)\|_2^2$.

As implied by the previous paragraph, our algorithm requires first-order stationarity conditions for problems (1) and (2), which can be stated in the following manner. First, introducing a Lagrange multiplier vector $y \in \mathbb{R}^m$, we define the Lagrangian for problem (1), call it $\ell : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, by

$$\ell(x, y) = f(x) - c(x)^T y.$$

Then, defining the gradient of the objective function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $g(x) = \nabla f(x)$, the Jacobian of the constraint functions $J : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times n}$ by $J(x) = \nabla c(x)$, and the projection operator onto the bounds $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$, component-wise for $i \in \{1, \dots, n\}$, by

$$[P(x)]_i = \begin{cases} l_i & \text{if } x_i \leq l_i \\ u_i & \text{if } x_i \geq u_i \\ x_i & \text{otherwise} \end{cases}$$

we may introduce the primal-dual stationarity measure $F_L : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ given by

$$F_L(x, y) = P(x - \nabla_x \ell(x, y)) - x = P(x - (g(x) - J(x)^T y)) - x.$$

First-order primal-dual stationary points for (1) can then be characterized as zeros of the primal-dual stationarity measure $F_{\text{OPT}} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{n+m}$ defined by stacking the stationarity measure F_L and the constraint function $-c$, i.e., a first-order primal-dual stationary point for (1) is any pair (x, y) with $l \leq x \leq u$ satisfying

$$0 = F_{\text{OPT}}(x, y) = \begin{pmatrix} F_L(x, y) \\ -c(x) \end{pmatrix} = \begin{pmatrix} P(x - \nabla_x \ell(x, y)) - x \\ \nabla_y \ell(x, y) \end{pmatrix}. \quad (3)$$

Similarly, a first-order primal stationary point for (2) is any x with $l \leq x \leq u$ satisfying

$$0 = F_{\text{FEAS}}(x), \quad (4)$$

where $F_{\text{FEAS}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined by

$$F_{\text{FEAS}}(x) = P(x - \nabla_x v(x)) - x = P(x - J(x)^T c(x)) - x.$$

In particular, if $l \leq x \leq u$, $v(x) > 0$, and (4) holds, then x is an infeasible stationary point for problem (1).

Over the past decades, a variety of effective numerical methods have been proposed for solving large-scale bound-constrained optimization problems. Hence, the critical issue in solving problem (1) is how to handle the presence of the equality constraints. As in the wide variety of penalty methods that have been proposed, the strategy adopted by AL methods is to remove these constraints, but push the algorithm to satisfy them through the addition of influential terms in the objective. In this manner, problem (1) (or at least (2)) can be solved via a sequence of bound-constrained subproblems—thus allowing AL methods to exploit the methods that are available for subproblems of this type. Specifically, AL methods consider a sequence of subproblems in which the objective is a weighted sum of the Lagrangian ℓ and the constraint violation measure v . By scaling ℓ by a penalty parameter $\mu \geq 0$, each subproblem involves the minimization of a function $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$, called the augmented Lagrangian (AL), defined by

$$\mathcal{L}(x, y, \mu) = \mu \ell(x, y) + v(x) = \mu(f(x) - c(x)^T y) + \frac{1}{2} \|c(x)\|_2^2.$$

Observe that the gradient of the AL with respect to x , evaluated at (x, y, μ) , is given by

$$\nabla_x \mathcal{L}(x, y, \mu) = \mu(g(x) - J(x)^T \pi(x, y, \mu)),$$

where we define the function $\pi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^m$ by

$$\pi(x, y, \mu) = y - \frac{1}{\mu} c(x). \quad (5)$$

Hence, each subproblem to be solved in an AL method has the form

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \mathcal{L}(x, y, \mu) \quad \text{subject to } l \leq x \leq u. \quad (6)$$

Given a pair (y, μ) , a first-order stationary point for problem (6) is any zero of the primal-dual stationarity measure $F_{\text{AL}} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^n$, defined similarly to F_{L} but with the Lagrangian replaced by the augmented Lagrangian; i.e., given (y, μ) , a first-order stationary point for (6) is any x satisfying

$$0 = F_{\text{AL}}(x, y, \mu) = P(x - \nabla_x \mathcal{L}(x, y, \mu)) - x. \quad (7)$$

Given a pair (y, μ) with $\mu > 0$, a traditional AL method proceeds by (approximately) solving (6), which is to say that it finds a point, call it $x(y, \mu)$, that (approximately) satisfies (7). If the resulting pair $(x(y, \mu), y)$ is not a first-order primal-dual stationary point for (1), then the method would modify the Lagrange multiplier y or penalty parameter μ so that, hopefully, the solution of the subsequent subproblem (of the form (6)) yields a better primal-dual solution estimate for (1). The function π plays a critical role in this procedure. In particular, observe that if $c(x(y, \mu)) = 0$, then $\pi(x(y, \mu), y, \mu) = y$ and (7) would imply $F_{\text{OPT}}(x(y, \mu), y) = 0$, i.e., that $(x(y, \mu), y)$ is a first-order primal-dual stationary point for (1). Hence, if the constraint violation at $x(y, \mu)$ is sufficiently small, then a traditional AL method would set the new value of y as $\pi(x, y, \mu)$. Otherwise, if the constraint violation is not sufficiently small, then the penalty parameter is decreased to place a higher priority on reducing it during subsequent iterations.

2.2 Algorithm Description

Our AL line search algorithm is similar to the AL trust region method proposed in [15], except for two key differences: it executes line searches rather than using a trust region framework, and it employs a convexified piecewise quadratic model of the AL function for computing the search direction in each iteration. The main motivation for utilizing a convexified model is to ensure that each computed search direction is a direction of strict descent for the AL function from the current iterate, which is necessary to ensure the well-posedness of the line search. However, it should be noted that, practically speaking, the convexification of the model does not necessarily add any computational difficulties when computing each direction; see §3.1.1. Similar to the trust region method proposed in [15], a critical component of our algorithm is the adaptive strategy for updating the penalty parameter μ during the search direction computation. This is used to ensure steady progress—i.e., steer the algorithm—toward solving (1) (or at least (2)) by monitoring predicted improvements in linearized feasibility.

The central component of each iteration of our algorithm is the search direction computation. In our approach, this computation is performed based on local models of the constraint violation measure v and the AL function \mathcal{L} at the current iterate, which at iteration k is given by (x_k, y_k, μ_k) . The local models that we employ for these functions

are, respectively, $q_v : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\tilde{q} : \mathbb{R}^n \rightarrow \mathbb{R}$, defined as follows:

$$\begin{aligned} q_v(s; x) &= \frac{1}{2} \|c(x) + J(x)s\|_2^2 \\ \tilde{q}(s; x, y, \mu) &= \mathcal{L}(x, y) + \nabla_x \mathcal{L}(x, y)^T s + \max\{\frac{1}{2} s^T (\mu \nabla_{xx}^2 \ell(x, y) + J(x)^T J(x)) s, 0\}. \end{aligned}$$

We note that q_v is a typical Gauss-Newton model of the constraint violation measure v , and \tilde{q} is a convexification of a second-order approximation of the augmented Lagrangian. (We use the notation \tilde{q} rather than simply q to distinguish between the model above and the second-order model—without the max—that appears extensively in [15].)

Our algorithm computes two types of steps during each iteration. The purpose of the first step, which we refer to as the steering step, is to gauge the progress towards linearized feasibility that may be achieved (locally) from the current iterate. This is done by (approximately) minimizing our model q_v of the constraint violation measure v within the bound constraints and a trust region. Then, a step of the second type is computed by (approximately) minimizing our model \tilde{q} of the AL function \mathcal{L} within the bound constraints and a trust region. If the reduction in the model q_v yielded by the latter step is sufficiently large—say, compared to that yielded by the steering step—then the algorithm proceeds using this step as the search direction. Otherwise, the penalty parameter may be reduced, in which case a step of the latter type is recomputed. This process repeats iteratively until a search direction is computed that yields a sufficiently large (or at least not too negative) reduction in q_v . As such, the iterate sequence is intended to make steady progress toward (or at least approximately maintain) constraint satisfaction throughout the optimization process, regardless of the initial penalty parameter value.

We now describe this process in more detail. During iteration k , the steering step r_k is computed via the optimization subproblem given by

$$\underset{r \in \mathbb{R}^n}{\text{minimize}} \quad q_v(r; x_k) \quad \text{subject to} \quad l \leq x_k + r \leq u, \quad \|r\|_2 \leq \theta_k, \quad (8)$$

where, for some constant $\delta > 0$, the trust region radius is defined to be

$$\theta_k := \delta \|F_{\text{FEAS}}(x_k)\|_2 \geq 0. \quad (9)$$

A consequence of this choice of trust region radius is that it forces the steering step to be smaller in norm as the iterates of the algorithm approach any stationary point of the constraint violation measure [35]. This prevents the steering step from being too large relative to the progress that can be made toward minimizing v . While (8) is a convex optimization problem for which there are efficient methods, in order to reduce computational expense our algorithm only requires r_k to be an approximate solution of (8). In particular, we merely require that r_k yields a reduction in q_v that is proportional to that yielded by the associated Cauchy step (see (16a) later on), which is defined to be

$$\bar{r}_k := \bar{r}(x_k, \theta_k) := P(x_k - \bar{\beta}_k J_k^T c_k) - x_k \quad (10)$$

for $\bar{\beta}_k := \bar{\beta}(x_k, \theta_k)$ such that, for some $\varepsilon_r \in (0, 1)$, the step \bar{r}_k satisfies

$$\Delta q_v(\bar{r}_k; x_k) := q_v(0; x_k) - q_v(\bar{r}_k; x_k) \geq -\varepsilon_r \bar{r}_k^T J_k^T c_k \quad \text{and} \quad \|\bar{r}_k\|_2 \leq \theta_k. \quad (11)$$

Appropriate values for $\bar{\beta}_k$ and \bar{r}_k —along with auxiliary nonnegative scalar quantities ε_k and Γ_k to be used in subsequent calculations in our method—are computed by Algorithm 1. The quantity $\Delta q_v(\bar{r}_k; x_k)$ representing the predicted reduction in constraint violation yielded by \bar{r}_k is guaranteed to be positive at any x_k that is not a first-order

stationary point for v subject to the bound constraints; see part (i) of Lemma A.4. We define a similar reduction $\Delta q_v(r_k; x_k)$ for the steering step r_k .

Algorithm 1 Cauchy step computation for the feasibility subproblem (8)

```

1: procedure CAUCHY_FEASIBILITY( $x_k, \theta_k$ )
2:   restrictions :  $\theta_k \geq 0$ .
3:   available constants :  $\{\varepsilon_r, \gamma\} \subset (0, 1)$ .
4:   Compute the smallest integer  $l_k \geq 0$  satisfying  $\|P(x_k - \gamma^{l_k} J_k^T c_k) - x_k\|_2 \leq \theta_k$ .
5:   if  $l_k > 0$  then
6:     Set  $\Gamma_k \leftarrow \min\{2, \frac{1}{2}(1 + \|P(x_k - \gamma^{l_k-1} J_k^T c_k) - x_k\|_2/\theta_k)\}$ .
7:   else
8:     Set  $\Gamma_k \leftarrow 2$ .
9:   end if
10:  Set  $\bar{\beta}_k \leftarrow \gamma^{l_k}$ ,  $\bar{r}_k \leftarrow P(x_k - \bar{\beta}_k J_k^T c_k) - x_k$ , and  $\varepsilon_k \leftarrow 0$ .
11:  while  $\bar{r}_k$  does not satisfy (11) do
12:    Set  $\varepsilon_k \leftarrow \max(\varepsilon_k, -\Delta q_v(\bar{r}_k; x_k)/\bar{r}_k^T J_k^T c_k)$ .
13:    Set  $\bar{\beta}_k \leftarrow \gamma \bar{\beta}_k$  and  $\bar{r}_k \leftarrow P(x_k - \bar{\beta}_k J_k^T c_k) - x_k$ .
14:  end while
15:  return :  $(\bar{\beta}_k, \bar{r}_k, \varepsilon_k, \Gamma_k)$ 
16: end procedure

```

After computing a steering step r_k , we proceed to compute a trial step s_k via

$$\underset{s \in \mathbb{R}^n}{\text{minimize}} \tilde{q}(s; x_k, y_k, \mu_k) \quad \text{subject to} \quad l \leq x_k + s \leq u, \|s\|_2 \leq \Theta_k, \quad (12)$$

where, given $\Gamma_k > 1$ from the output of Algorithm 1, we define the trust region radius

$$\Theta_k := \Theta(x_k, y_k, \mu_k, \Gamma_k) = \Gamma_k \delta \|F_{\text{AL}}(x_k, y_k, \mu_k)\|_2 \geq 0. \quad (13)$$

As for the steering step, we allow inexactness in the solution of (12) by only requiring the step s_k to satisfy a Cauchy decrease condition (see (16b) later on), where the Cauchy step for problem (12) is

$$\bar{s}_k := \bar{s}(x_k, y_k, \mu_k, \Theta_k, \varepsilon_k) := P(x_k - \bar{\alpha}_k \nabla_x \mathcal{L}(x_k, y_k, \mu_k)) - x_k \quad (14)$$

for $\bar{\alpha}_k = \bar{\alpha}(x_k, y_k, \mu_k, \Theta_k, \varepsilon_k)$ such that, for $\varepsilon_k \geq 0$ returned from Algorithm 1, \bar{s}_k yields

$$\begin{aligned} \Delta \tilde{q}(\bar{s}_k; x_k, y_k, \mu_k) &:= \tilde{q}(0; x_k, y_k, \mu_k) - \tilde{q}(\bar{s}_k; x_k, y_k, \mu_k) \\ &\geq -\frac{(\varepsilon_k + \varepsilon_r)}{2} \bar{s}_k^T \nabla_x \mathcal{L}(x_k, y_k, \mu_k) \quad \text{and} \quad \|\bar{s}_k\|_2 \leq \Theta_k. \end{aligned} \quad (15)$$

Algorithm 2 describes our procedure for computing $\bar{\alpha}_k$ and \bar{s}_k . (The importance of incorporating Γ_k in (13) and ε_k in (15) is revealed in the proofs of Lemmas A.2 and A.3.) The quantity $\Delta \tilde{q}(\bar{s}_k; x_k, y_k, \mu_k)$ representing the predicted reduction in $\mathcal{L}(\cdot, y_k, \mu_k)$ yielded by \bar{s}_k is guaranteed to be positive at any x_k that is not a first-order stationary point for $\mathcal{L}(\cdot, y_k, \mu_k)$ subject to the bound constraints; see part (ii) of Lemma A.4. A similar quantity $\Delta \tilde{q}(s_k; x_k, y_k, \mu_k)$ is also used for the search direction s_k .

Our complete algorithm is given as Algorithm 3 on page 8. In particular, the k th iteration proceeds as follows. Given the k th iterate tuple (x_k, y_k, μ_k) , the algorithm first determines whether the first-order primal-dual stationarity conditions for (1) or the first-order stationarity condition for (2) are satisfied. If either is the case, then the algorithm

Algorithm 2 Cauchy step computation for the Augmented Lagrangian subproblem (12).

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1: procedure CAUCHY_AL( $x_k, y_k, \mu_k, \Theta_k, \varepsilon_k$ )
2:   restrictions :  $\mu_k > 0, \Theta_k > 0,$  and  $\varepsilon_k \geq 0.$ 
3:   available constant :  $\gamma \in (0, 1).$ 
4:   Set  $\bar{\alpha}_k \leftarrow 1$  and  $\bar{s}_k \leftarrow P(x_k - \bar{\alpha}_k \nabla_x \mathcal{L}(x_k, y_k, \mu_k)) - x_k.$ 
5:   while (15) is not satisfied do
6:     Set  $\bar{\alpha}_k \leftarrow \gamma \bar{\alpha}_k$  and  $\bar{s}_k \leftarrow P(x_k - \bar{\alpha}_k \nabla_x \mathcal{L}(x_k, y_k, \mu_k)) - x_k.$ 
7:   end while
8:   return :  $(\bar{\alpha}_k, \bar{s}_k)$ 
9: end procedure

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terminates, but otherwise the method enters the **while** loop in line 11 to check for stationarity with respect to the AL function. This loop is guaranteed to terminate finitely; see Lemma A.1. Next, after computing appropriate trust region radii and Cauchy steps, the method enters a block for computing the steering step r_k and trial step s_k . Through the **while** loop on line 19, the overall goal of this block is to compute (approximate) solutions of subproblems (8) and (12) satisfying

$$\Delta \tilde{q}(s_k; x_k, y_k, \mu_k) \geq \kappa_1 \Delta \tilde{q}(\bar{s}_k; x_k, y_k, \mu_k) > 0, \quad l \leq x_k + s_k \leq u, \quad \|s_k\|_2 \leq \Theta_k, \quad (16a)$$

$$\Delta q_v(r_k; x_k) \geq \kappa_2 \Delta q_v(\bar{r}_k; x_k) \geq 0, \quad l \leq x_k + r_k \leq u, \quad \|r_k\|_2 \leq \theta_k, \quad (16b)$$

$$\text{and } \Delta q_v(s_k; x_k) \geq \min\{\kappa_3 \Delta q_v(r_k; x_k), v_k - \frac{1}{2}(\kappa_t t_j)^2\}. \quad (16c)$$

In these conditions, the method employs user-provided constants $\{\kappa_1, \kappa_2, \kappa_3, \kappa_t\} \subset (0, 1)$ and the algorithmic quantity $t_j > 0$ representing the j th constraint violation target. It should be noted that, for sufficiently small $\mu > 0$, many approximate solutions to (8) and (12) satisfy (16), but for our purposes (see Theorem 2.2) it is sufficient that, for sufficiently small $\mu > 0$, they are at least satisfied by $r_k = \bar{r}_k$ and $s_k = \bar{s}_k$. A complete description of the motivations underlying conditions (16) can be found in [15, Section 3]. In short, (16a) and (16b) are Cauchy decrease conditions while (16c) ensures that the trial step predicts progress toward constraint satisfaction, or at least predicts that any increase in constraint violation is limited (when the right-hand side is negative).

With the search direction s_k in hand, the method proceeds to perform a backtracking line search along the strict descent direction s_k for $\mathcal{L}(\cdot, y_k, \mu_k)$ at x_k . Specifically, for a given $\gamma_\alpha \in (0, 1)$, the method computes the smallest integer $l \geq 0$ such that

$$\mathcal{L}(x_k + \gamma_\alpha^l s_k, y_k, \mu_k) \leq \mathcal{L}(x_k, y_k, \mu_k) - \eta_s \gamma_\alpha^l \Delta \tilde{q}(s_k; x_k, y_k, \mu_k), \quad (17)$$

and then sets $\alpha_k \leftarrow \gamma_\alpha^l$ and $x_{k+1} \leftarrow x_k + \alpha_k s_k$. The remainder of the iteration is then composed of potential modifications of the Lagrange multiplier vector and target values for the accuracies in minimizing the constraint violation measure and AL function subject to the bound constraints. First, the method checks whether the constraint violation at the next primal iterate x_{k+1} is sufficiently small compared to the target $t_j > 0$. If this requirement is met, then a multiplier vector \hat{y}_{k+1} that satisfies

$$\|F_L(x_{k+1}, \hat{y}_{k+1})\|_2 \leq \min\{\|F_L(x_{k+1}, y_k)\|_2, \|F_L(x_{k+1}, \pi(x_{k+1}, y_k, \mu_k))\|_2\} \quad (18)$$

is computed. Two obvious potential choices for \hat{y}_{k+1} are y_k and $\pi(x_{k+1}, y_k, \mu_k)$, but another viable candidate would be an approximate least-squares multiplier estimate (which may be computed via a linearly constrained optimization subproblem). The method then checks if either $\|F_L(x_{k+1}, \hat{y}_{k+1})\|_2$ or $\|F_{AL}(x_{k+1}, y_k, \mu_k)\|_2$ is sufficiently small with respect

Algorithm 3 Adaptive Augmented Lagrangian Line Search Algorithm

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1: Choose  $\{\gamma, \gamma_\mu, \gamma_\alpha, \gamma_t, \gamma_T, \kappa_1, \kappa_2, \kappa_3, \varepsilon_r, \kappa_t, \eta_s, \eta_{vs}\} \subset (0, 1)$  and  $\{\delta, \epsilon, Y\} \subset (0, \infty)$  such
   that  $\eta_{vs} \geq \eta_s$ .
2: Choose initial primal-dual pair  $(x_0, y_0)$  and initialize  $\{\mu_0, \delta_0, t_0, t_1, T_1, Y_1\} \subset (0, \infty)$ 
   such that  $Y_1 \geq Y$  and  $\|y_0\|_2 \leq Y_1$ .
3: Set  $k \leftarrow 0$ ,  $k_0 \leftarrow 0$ , and  $j \leftarrow 1$ .
4: loop
5:   if  $F_{\text{OPT}}(x_k, y_k) = 0$ , then
6:     return the first-order stationary solution  $(x_k, y_k)$ .
7:   end if
8:   if  $\|c_k\|_2 > 0$  and  $F_{\text{FEAS}}(x_k) = 0$ , then
9:     return the infeasible stationary point  $x_k$ .
10:  end if
11:  while  $F_{\text{AL}}(x_k, y_k, \mu_k) = 0$ , do
12:    Set  $\mu_k \leftarrow \gamma_\mu \mu_k$ .
13:  end while
14:  Define  $\theta_k$  by (9).
15:  Use Algorithm 1 to compute  $(\bar{\beta}_k, \bar{r}_k, \varepsilon_k, \Gamma_k) = \text{CAUCHY\_FEASIBILITY}(x_k, \theta_k)$ .
16:  Define  $\Theta_k$  by (13).
17:  Use Algorithm 2 to compute  $(\bar{\alpha}_k, \bar{s}_k) = \text{CAUCHY\_AL}(x_k, y_k, \mu_k, \Theta_k, \varepsilon_k)$ .
18:  Compute approximate solutions  $r_k$  to (8) and  $s_k$  to (12) that satisfy (16a)–(16b).
19:  while (16c) is not satisfied or  $F_{\text{AL}}(x_k, y_k, \mu_k) = 0$ , do
20:    Set  $\mu_k \leftarrow \gamma_\mu \mu_k$  and define  $\Theta_k$  by (13).
21:    Use Algorithm 2 to compute  $(\bar{\alpha}_k, \bar{s}_k) = \text{CAUCHY\_AL}(x_k, y_k, \mu_k, \Theta_k, \varepsilon_k)$ .
22:    Compute an approximate solution  $s_k$  to (12) satisfying (16a).
23:  end while
24:  Set  $\alpha_k \leftarrow \gamma_\alpha^l$  where  $l \geq 0$  is the smallest integer satisfying (17).
25:  Set  $x_{k+1} \leftarrow x_k + \alpha_k s_k$ .
26:  if  $\|c_{k+1}\|_2 \leq t_j$ , then
27:    Compute any  $\hat{y}_{k+1}$  satisfying (18).
28:    if  $\min\{\|F_{\text{L}}(x_{k+1}, \hat{y}_{k+1})\|_2, \|F_{\text{AL}}(x_{k+1}, y_k, \mu_k)\|_2\} \leq T_j$ , then
29:      Set  $k_j \leftarrow k + 1$  and  $Y_{j+1} \leftarrow \max\{Y, t_{j-1}^{-\epsilon}\}$ .
30:      Set  $t_{j+1} \leftarrow \min\{\gamma_t t_j, t_j^{1+\epsilon}\}$  and  $T_{j+1} \leftarrow \gamma_T T_j$ .
31:      Set  $y_{k+1}$  from (19) where  $\alpha_y$  satisfies (20).
32:      Set  $j \leftarrow j + 1$ .
33:    else
34:      Set  $y_{k+1} \leftarrow y_k$ .
35:    end if
36:  else
37:    Set  $y_{k+1} \leftarrow y_k$ .
38:  end if
39:  Set  $\mu_{k+1} \leftarrow \mu_k$ .
40:  Set  $k \leftarrow k + 1$ .
41: end loop

```

to the target value $T_j > 0$. If so, then new target values $t_{j+1} < t_j$ and $T_{j+1} < T_j$ are set, $Y_{j+1} \geq Y_j$ is chosen, and a new Lagrange multiplier vector is set as

$$y_{k+1} \leftarrow (1 - \alpha_y)y_k + \alpha_y \hat{y}_{k+1}, \quad (19)$$

where α_y is the largest value in $[0, 1]$ such that

$$\|(1 - \alpha_y)y_k + \alpha_y \widehat{y}_{k+1}\|_2 \leq Y_{j+1}. \quad (20)$$

This updating procedure is well-defined since the choice $\alpha_y \leftarrow 0$ results in $y_{k+1} \leftarrow y_k$, for which (20) is satisfied since $\|y_k\|_2 \leq Y_j \leq Y_{j+1}$. If either line 26 or line 28 in Algorithm 3 tests false, then the method simply sets $y_{k+1} \leftarrow y_k$. We note that unlike more traditional augmented Lagrangian approaches [2, 11], the penalty parameter is not adjusted on the basis of a test like that on line 26, but instead relies on our steering procedure. Moreover, in our approach we decrease the target values at a linear rate for simplicity, but more sophisticated approaches may be used [11].

2.3 Well-posedness and global convergence

In this section, we state two vital results, namely that Algorithm 3 is well posed, and that limit points of the iterate sequence have desirable properties. Proofs of these results, which are similar to those in [15], are given in Appendices A and B. In order to show well-posedness of the algorithm, we make the following formal assumption.

ASSUMPTION 2.1 *At each given x_k , the objective function f and constraint function c are both twice-continuously differentiable.*

Under this assumption, we have the following theorem.

THEOREM 2.2 *Suppose that Assumption 2.1 holds. Then the k th iteration of Algorithm 3 is well posed. That is, either the algorithm will terminate in line 6 or 9, or it will compute $\mu_k > 0$ such that $F_{AL}(x_k, y_k, \mu_k) \neq 0$ and for the steps $s_k = \bar{s}_k$ and $r_k = \bar{r}_k$ the conditions in (16) will be satisfied, in which case $(x_{k+1}, y_{k+1}, \mu_{k+1})$ will be computed.*

According to Theorem 2.2, we have that Algorithm 3 will either terminate finitely or produce an infinite sequence of iterates. If it terminates finitely—which can only occur if line 6 or 9 is executed—then the algorithm has computed a first-order stationary solution or an infeasible stationary point and there is nothing else to prove about the algorithm’s performance in such cases. Therefore, it remains to focus on the global convergence properties of Algorithm 3 under the assumption that the sequence $\{(x_k, y_k, \mu_k)\}$ is infinite. For such cases, we make the following additional assumption.

ASSUMPTION 2.3 *The primal sequences $\{x_k\}$ and $\{x_k + s_k\}$ are contained in a convex compact set over which the objective function f and constraint function c are both twice-continuously differentiable.*

Our main global convergence result for Algorithm 3 is as follows.

THEOREM 2.4 *Suppose that Assumptions 2.2 and 2.3 hold. Then one of the following must hold:*

- (i) every limit point x_* of $\{x_k\}$ is an infeasible stationary point;
- (ii) $\mu_k \not\rightarrow 0$ and there exists an infinite ordered set $\mathcal{K} \subseteq \mathbb{N}$ such that every limit point of $\{(x_k, \widehat{y}_k)\}_{k \in \mathcal{K}}$ is first-order stationary for (1); or
- (iii) $\mu_k \rightarrow 0$, every limit point of $\{x_k\}$ is feasible, and if there exists a positive integer p such that $\mu_{k_j-1} \geq \gamma_\mu^p \mu_{k_j-1}$ for all sufficiently large j , then there exists an infinite ordered set $\mathcal{J} \subseteq \mathbb{N}$ such that any limit point of either $\{(x_{k_j}, \widehat{y}_{k_j})\}_{j \in \mathcal{J}}$ or $\{(x_{k_j}, y_{k_j-1})\}_{j \in \mathcal{J}}$ is

first-order stationary for (1).

Observe that the conclusions are exactly the same as in [15, Theorem 3.14]. We also call the readers attention to the comments following [15, Theorem 3.14], which discuss the consequences of these results. In particular, these comments suggest how Algorithm 3 may be modified to guarantee convergence to first-order stationary points, even in case (iii) of the theorem. However, as mentioned in [15], we do not consider these modifications to the algorithm to have practical benefits. This perspective is supported by the numerical tests presented in the following section.

3. Numerical Experiments

In this section, we provide evidence that steering can have a positive effect on the performance of AL algorithms. To best illustrate the influence of steering, we implemented and tested algorithms in two pieces of software. First, in MATLAB, we implemented our adaptive AL line search algorithm, i.e., Algorithm 3, and the adaptive AL trust region method given as [15, Algorithm 4]. Since these methods were implemented from scratch, we had control over every aspect of the code, which allowed us to implement all features described in this paper and in [15]. Second, we implemented a simple modification of the AL trust region algorithm in the LANCELOT software package [12]. Our only modification to LANCELOT was to incorporate a basic form of steering; i.e., we did not change other aspects of LANCELOT, such as the mechanisms for triggering a multiplier update. In this manner, we were also able to isolate the effect that steering had on numerical performance, though it should be noted that there were differences between Algorithm 3 and our implemented algorithm in LANCELOT in terms of, e.g., the multiplier updates.

While we provide an extensive amount of information about the results of our experiments in this section, further information can be found in Appendix C.

3.1 MATLAB *implementation*

3.1.1 *Implementation details*

Our MATLAB software was comprised of six algorithm variants. The algorithms were implemented as part of the same package so that most of the algorithmic components were exactly the same; the primary differences related to the step acceptance mechanisms and the manner in which the Lagrange multiplier estimates and penalty parameter were updated. First, for comparison against algorithms that utilized our steering mechanism, we implemented line search and trust region variants of a basic augmented Lagrangian method, given as [15, Algorithm 1]. We refer to these algorithms as **BAL-LS** (**basic augmented Lagrangian, line search**) and **BAL-TR** (**trust region**), respectively. These algorithms clearly differed in that one used a line search and the other used a trust region strategy for step acceptance, but the other difference was that, like Algorithm 3 in this paper, **BAL-LS** employed a convexified model of the AL function. (We discuss more details about the use of this convexified model below.) The other algorithms implemented in our software included two variants of Algorithm 3 and two variants of [15, Algorithm 4]. The first variants of each, which we refer to as **AAL-LS** and **AAL-TR** (**adaptive, as opposed to basic**), were straightforward implementations of these algorithms, whereas the latter variants, which we refer to as **AAL-LS-safe** and **AAL-TR-safe**, included an implementation of a safeguarding procedure for the steering mechanism. The safeguarding procedure will be described in detail shortly.

The main per-iteration computational expense for each algorithm variant can be at-

tributed to the search direction computations. For computing a search direction via an approximate solve of (12) or [15, Prob. (3.8)], all algorithms essentially used the same procedure. For simplicity, all algorithms considered variants of these subproblems in which the ℓ_2 -norm trust region was replaced by an ℓ_∞ -norm trust region so that the subproblems were bound-constrained. (The same modification was used in the Cauchy step calculations.) Then, starting with the Cauchy step as the initial solution estimate and defining the initial working set by the bounds identified as active by the Cauchy step, a projected conjugate gradient (PCG) method was used to compute an improved solution. During the PCG routine, if a new trial solution violated a bound constraint that was not already part of the working set, then this bound was added to the working set and the PCG routine was reinitialized. By contrast, if the reduced subproblem (corresponding to the current working set) was solved sufficiently accurately, then a check for termination was performed. In particular, multiplier estimates were computed for the working set elements. If these multiplier estimates were all nonnegative (or at least larger than a small negative number), then the subproblem was deemed to be solved and the routine terminated. Otherwise, an element corresponding to the most negative multiplier estimate was removed from the working set and the PCG routine was reinitialized. We do not claim that the precise manner in which we implemented this approach guaranteed convergence to an exact solution of the subproblem. However, the approach just described was based on well-established methods for solving bound-constrained quadratic optimization problems (QPs), and we found that it worked very well in our experiments. It should be noted that if, at any time, negative curvature was encountered in the PCG routine, then the solver terminated with the current PCG iterate. In this manner, the solutions were generally less accurate when negative curvature was encountered, but we claim that this did not have too adverse an effect on the performance of any of the algorithms.

A few additional comments are necessary to describe our search direction computation procedures. First, it should be noted that for the line search algorithms, the Cauchy step calculation in Algorithm 2 was performed with (15) as stated (i.e., with \tilde{q}), but the above PCG routine to compute the search direction was applied to (12) *without* the convexification for the quadratic term. However, we claim that this choice remains consistent with the stated algorithms since, for all algorithm variants, we performed a sanity check after the computation of the search direction. In particular, the reduction in the model of the AL function yielded by the search direction was compared against that yielded by the corresponding Cauchy step. If the Cauchy step actually provided a better reduction in the model, then the computed search direction was replaced by the Cauchy step. In this sanity check for the line search algorithms, we computed the model reductions *with* the convexification of the quadratic term (i.e., with \tilde{q}), which implies that, overall, our implemented algorithm guaranteed Cauchy decrease in the appropriate model for all algorithms. Second, we remark that for the algorithms that employed a steering mechanism, we did not employ the same procedure to approximately solve (8) or [15, Prob. (3.4)]. Instead, we simply used the Cauchy steps as approximate solutions of these subproblems. Finally, we note that in the steering mechanism, we checked condition (16c) with the Cauchy steps for each subproblem, despite the fact that the search direction was computed as a more accurate solution of (12) or [15, Prob. (3.8)]. This had the effect that the algorithms were able to modify the penalty parameter via the steering mechanism prior to computing the search direction; only Cauchy steps for the subproblems were needed for steering.

Most of the other algorithmic components were implemented similarly to the algorithm in [15]. As an example, for the computation of the estimates $\{\hat{y}_{k+1}\}$ (which are required to satisfy (18)), we checked whether $\|F_L(x_{k+1}, \pi(x_{k+1}, y_k, \mu_k))\|_2 \leq \|F_L(x_{k+1}, y_k)\|_2$; if so, then we set $\hat{y}_{k+1} \leftarrow \pi(x_{k+1}, y_k, \mu_k)$, and otherwise we set $\hat{y}_{k+1} \leftarrow y_k$. Furthermore,

for prescribed tolerances $\{\kappa_{\text{opt}}, \kappa_{\text{feas}}, \mu_{\text{min}}\} \subset (0, \infty)$, we terminated an algorithm with a declaration that a stationary point was found if

$$\|F_L(x_k, y_k)\|_\infty \leq \kappa_{\text{opt}} \quad \text{and} \quad \|c_k\|_\infty \leq \kappa_{\text{feas}}, \quad (21)$$

and terminated with a declaration that an infeasible stationary point was found if

$$\|F_{\text{FEAS}}(x_k)\|_\infty \leq \kappa_{\text{opt}}, \quad \|c_k\|_\infty > \kappa_{\text{feas}}, \quad \text{and} \quad \mu_k \leq \mu_{\text{min}}. \quad (22)$$

As in [15], this latter set of conditions shows that we did not declare that an infeasible stationary point was found unless the penalty parameter had already been reduced below a prescribed tolerance. This helps in avoiding premature termination when the algorithm could otherwise continue and potentially find a point satisfying (21), which was always the preferred outcome. Each algorithm terminated with a message of failure if neither (21) nor (22) was satisfied within k_{max} iterations. It should also be noted that the problems were pre-scaled so that the ℓ_∞ -norms of the gradients of the problem functions at the initial point would be less than or equal to a prescribed constant $G > 0$. The values for all of these parameters, as well as other input parameter required in the code, are summarized in Table 1. (Values for parameters related to updating the trust region radii required by [15, Algorithm 4] were set as in [15].)

Table 1. Input parameter values used in our MATLAB software.

Parameter	Value	Parameter	Value	Parameter	Value	Parameter	Value
γ	0.5	κ_2	1	η_{vs}	0.9	μ_{min}	10^{-8}
γ_μ	0.1	κ_3	10^{-4}	ϵ	0.5	k_{max}	10^4
γ_t	0.1	ϵ_r	10^{-4}	μ_0	1	G	10^2
γ_T	0.1	κ_t	0.9	κ_{opt}	10^{-5}		
κ_1	1	η_s	10^{-4}	κ_{feas}	10^{-5}		

We close this subsection with a discussion of some additional differences between the algorithms as stated in this paper and in [15] and those implemented in our software. We claim that none of these differences represents a significant departure from the stated algorithms; we merely made some adjustments to simplify the implementation and to incorporate features that we found to work well in our experiments. First, while all algorithms use the input parameter γ_μ given in Table 1 for decreasing the penalty parameter, we decrease the penalty parameter less significantly in the steering mechanism. In particular, in line 20 of Algorithm 3 and line 20 of [15, Algorithm 4], we replace γ_μ with 0.7. Second, in the line search algorithms, rather than set the trust region radii as in (9) and (13) where δ appears as a constant value, we defined a dynamic sequence, call it $\{\delta_k\}$, that depended on the step-size sequence $\{\alpha_k\}$. In this manner, δ_k replaced δ in (9) and (13) for all k . We initialized $\delta_0 \leftarrow 1$. Then, for all k , if $\alpha_k = 1$, then we set $\delta_{k+1} \leftarrow \frac{5}{3}\delta_k$, and if $\alpha_k < 1$, then we set $\delta_{k+1} \leftarrow \frac{1}{2}\delta_k$. Third, to simplify our implementation, we effectively ignored the imposed bounds on the multiplier estimates by setting $Y \leftarrow \infty$ and $Y_1 \leftarrow \infty$. This choice implies that we always chose $\alpha_y \leftarrow 1$ in (19). Fourth, we initialized the target values as

$$t_1 \leftarrow \max\{10^2, \min\{10^4, \|c_k\|_\infty\}\} \quad (23)$$

$$\text{and } T_1 \leftarrow \max\{10^0, \min\{10^2, \|F_L(x_k, y_k)\|_\infty\}\}. \quad (24)$$

Finally, in `AAL-LS-safe` and `AAL-TR-safe`, we safeguard the steering procedure by shutting it off whenever the penalty parameter was smaller than a prescribed tolerance. Specifically, we considered the `while` condition in line 19 of Algorithm 3 and line 19 of [15, Algorithm 4] to be satisfied whenever $\mu_k \leq 10^{-4}$.

3.1.2 Results on CUTESt test problems

We tested our MATLAB algorithms on the subset of problems from the CUTESt [24] collection that have at least one general constraint and at most 1000 variables and 1000 constraints. This set contains 383 test problems. However, the results that we present in this section are only for those problems for which at least one of our six solvers obtained a successful result, i.e., where (21) or (22) was satisfied. This led to a set of 323 problems that are represented in the numerical results in this section.

To illustrate the performance of our MATLAB software, we use performance profiles as introduced by Dolan and Moré [17] to provide a visual comparison of different measures of performance. Consider a performance profile that measures performance in terms of required iterations until termination. For such a profile, if the graph associated with an algorithm passes through the point $(\alpha, 0.\beta)$, then, on $\beta\%$ of the problems, the number of iterations required by the algorithm was less than 2^α times the number of iterations required by the algorithm that required the fewest number of iterations. At the extremes of the graph, an algorithm with a higher value on the vertical axis may be considered a more efficient algorithm, whereas an algorithm on top at the far right of the graph may be considered more reliable. Since, for most problems, comparing values in the performance profiles for large values of α is not enlightening, we truncated the horizontal axis at 16 and simply remark on the numbers of failures for each algorithm.

Figures 1 and 2 show the results for the three line search variants, namely `BAL-LS`, `AAL-LS`, and `AAL-LS-safe`. The numbers of failures for these algorithms were 25, 3, and 16, respectively. The same conclusion may be drawn from both profiles: the steering variants (with and without safeguarding) were both more efficient and more reliable than the basic algorithm, where efficiency is measured by either the number of iterations (Figure 1) or the number of function evaluations (Figure 2) required. We display the profile for the number of function evaluations required since, for a line search algorithm, this value is always at least as large as the number of iterations, and will be strictly greater whenever backtracking is required to satisfy (17) (yielding $\alpha_k < 1$). From these profiles, one may observe that unrestricted steering (in `AAL-LS`) yielded superior performance to restricted steering (in `AAL-LS-safe`) in terms of both efficiency and reliability; this suggests that safeguarding the steering mechanism may diminish its potential benefits.

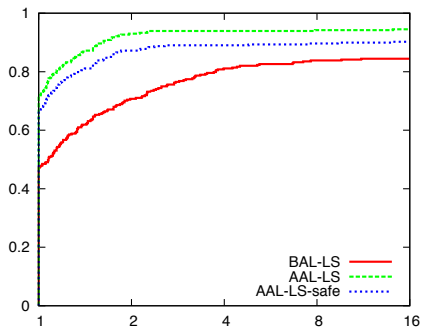


Figure 1. Performance profile for iterations: line search algorithms on the CUTESt set.

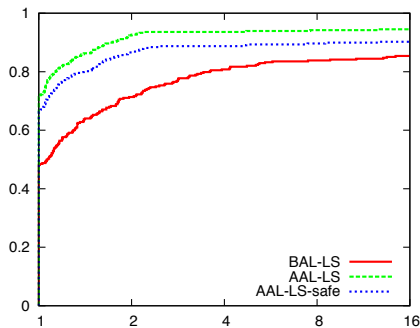


Figure 2. Performance profile for function evaluations: line search algorithms on the CUTESt set.

Figures 3 and 4 show the results for the three trust region variants, namely `BAL-TR`,

AAL-TR, and AAL-TR-safe, the numbers of failures for which were 30, 12, and 20, respectively. Again, as for the line search algorithms, the same conclusion may be drawn from both profiles: the steering variants (with and without safeguarding) are both more efficient and more reliable than the basic algorithm, where now we measure efficiency by either the number of iterations (Figure 3) or the number of gradient evaluations (Figure 4) required before termination. We observe the number of gradient evaluations here (as opposed to the number of function evaluations) since, for a trust region algorithm, this value is never larger than the number of iterations, and will be strictly smaller whenever a step is rejected and the trust-region radius is decreased because of insufficient decrease in the AL function. These profiles also support the other observation that was made by the results for our line search algorithms, i.e., that unrestricted steering may be superior to restricted steering in terms of efficiency and reliability.

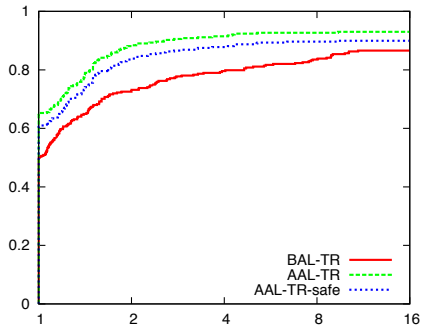


Figure 3. Performance profile for iterations: trust region algorithms on the CUTESt set.

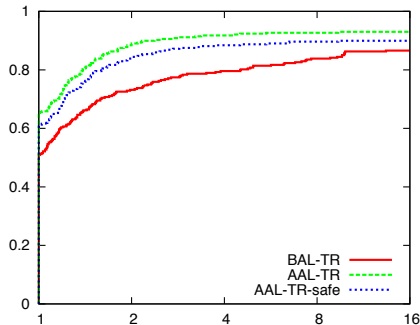


Figure 4. Performance profile for gradient evaluations: trust region algorithms on the CUTESt set.

The performance profiles in Figures 1–4 suggest that steering has practical benefits, and that safeguarding the procedure may limit its potential benefits. However, to be more confident in these claims, one should observe the final penalty parameter values typically produced by the algorithms. These observations are important since one may be concerned whether the algorithms that employ steering yield final penalty parameter values that are often significantly smaller than those yielded by basic AL algorithms. To investigate this possibility in our experiments, we collected the final penalty parameter values produced by all six algorithms; the results are in Table 2. The column titled μ_{final} gives a range for the final value of the penalty parameter. (For example, the value 27 in the BAL-LS column indicates that the final penalty parameter value computed by our basic line search AL algorithm fell in the range $[10^{-2}, 10^{-1})$ for 27 of the problems.)

Table 2. Numbers of CUTESt problems for which the final penalty parameter values were in the given ranges.

μ_{final}	BAL-LS	AAL-LS	AAL-LS-safe	BAL-TR	AAL-TR	AAL-TR-safe
1	139	87	87	156	90	90
$[10^{-1}, 1)$	43	33	33	35	46	46
$[10^{-2}, 10^{-1})$	27	37	37	28	29	29
$[10^{-3}, 10^{-2})$	17	42	42	19	49	49
$[10^{-4}, 10^{-3})$	22	36	36	18	29	29
$[10^{-5}, 10^{-4})$	19	28	42	19	25	39
$[10^{-6}, 10^{-5})$	15	19	11	9	11	9
$(0, 10^{-6})$	46	46	40	44	49	37

We remark on two observations about the data in Table 2. First, as may be expected, the algorithms that employ steering typically reduce the penalty parameter below its ini-

tial value on some problems on which the other algorithms do not reduce it at all. This, in itself, is not a major concern, since a reasonable reduction in the penalty parameter may cause an algorithm to locate a stationary point more quickly. Second, we remark that the number of problems for which the final penalty parameter was very small (say, less than 10^{-4}) was similar for all algorithms, even those that employed steering. This suggests that while steering was able to aid in guiding the algorithms toward constraint satisfaction, the algorithms did not reduce the value to such a small value that feasibility became the only priority. Overall, our conclusion from Table 2 is that steering typically decreases the penalty parameter more than does a traditional updating scheme, but one should not expect that the final penalty parameter value will be reduced unnecessarily small due to steering; rather, steering can have the intended benefit of improving efficiency and reliability by guiding a method toward constraint satisfaction more quickly.

3.1.3 Results on COPS test problems

We also tested our MATLAB software on the large-scale constrained problems available in the COPS [5] collection. This test set was designed to provide difficult test cases for nonlinear optimization software; the problems include examples from fluid dynamics, population dynamics, optimal design, mesh smoothing, and optimal control. For our purposes, we solved the smallest versions of the AMPL models [1, 19] provided in the collection. All of our solvers failed to solve the problems named *chain*, *dirichlet*, *henon*, *lane_emden*, and *robot1*, so these problems were excluded. The remaining set consisted of the following 17 problems: *bearing*, *camshape*, *catmix*, *channel*, *elec*, *gasoil*, *glider*, *marine*, *methanol*, *minsurf*, *pinene*, *polygon*, *rocket*, *steering*, *tetra*, *torsion*, and *triangle*. Since the size of this test set is relatively small, we have decided to display pair-wise comparisons of algorithms in the manner suggested in [30]. That is, for a performance measure of interest (e.g., number of iterations required until termination), we compare solvers, call them A and B , on problem j with the logarithmic *outperforming factor*

$$r_{AB}^j := -\log_2(m_A^j/m_B^j), \quad \text{where} \quad \begin{cases} m_A^j & \text{is the measure for } A \text{ on problem } j \\ m_B^j & \text{is the measure for } B \text{ on problem } j. \end{cases} \quad (25)$$

Therefore, if the measure of interest is iterations required, then $r_{AB}^j = p$ would indicate that solver A required 2^{-p} the iterations required by solver B . For all plots, we focus our attention on the range $p \in [-2, 2]$.

The results of our experiments are given in Figures 5–8. For the same reasons as discussed in §3.1.2, we display results for iterations and function evaluations for the line search algorithms, and display results for iterations and gradient evaluations for the trust region algorithms. In addition, here we ignore the results for **AAL-LS-safe** and **AAL-TR-safe** since, as in the results in §3.1.2, we did not see benefits in safeguarding the steering mechanism. In each figure, a positive (negative) bar indicates that the algorithm whose name appears above (below) the horizontal axis yielded a better value for the measure on a particular problem. The results are displayed according to the order of the problems listed in the previous paragraph. In Figures 5 and 6 for the line search algorithms, the red bars for problems *catmix* and *polygon* indicate that **AAL-LS** failed on the former and **BAL-LS** failed on the latter; similarly, in Figures 7 and 8 for the trust region algorithms, the red bar for *catmix* indicates that **AAL-TR** failed on it.

The results in Figures 5 and 6 indicate that **AAL-LS** more often outperforms **BAL-LS** in terms of iterations and functions evaluations, though the advantage is not overwhelming. On the other hand, it is clear from Figures 7 and 8 that, despite the one failure, **AAL-TR** is generally superior to **BAL-TR**. We conclude from these results that steering was beneficial

on this test set, especially in terms of the trust region methods.

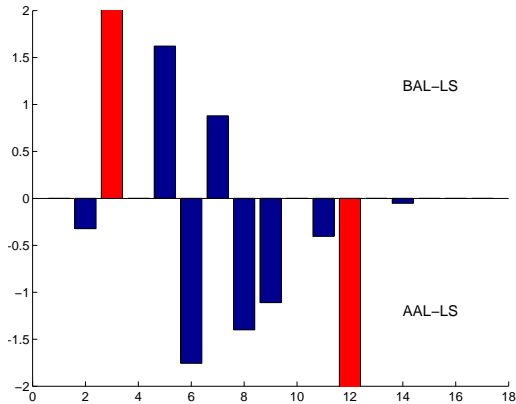


Figure 5. Outperforming factors for iterations: line search algorithms on the COPS set.

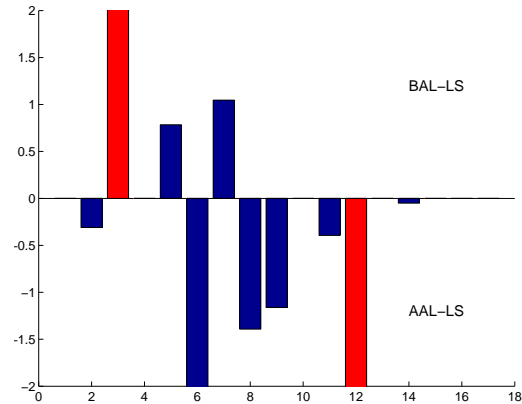


Figure 6. Outperforming factors for function evaluations: line search algorithms on the COPS set.

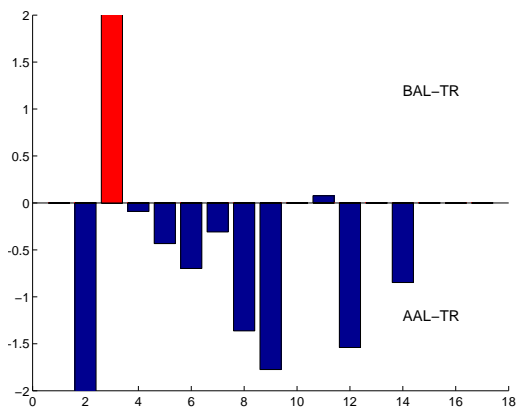


Figure 7. Outperforming factors for iterations: trust region algorithms on the COPS set.

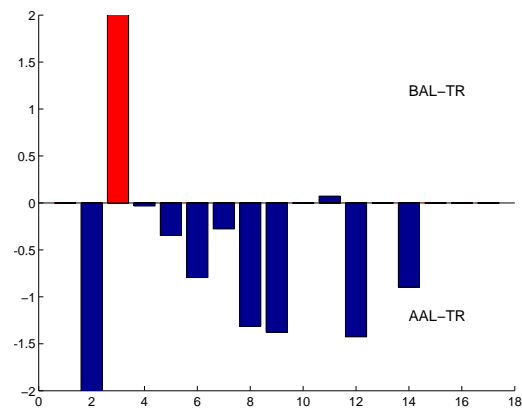


Figure 8. Outperforming factors for gradient evaluations: trust region algorithms on the COPS set.

3.1.4 Results on optimal power flow (OPF) test problems

As a third and final set of experiments for our MATLAB software, we tested our algorithms on a collection of optimal power flow (OPF) problems modeled in AMPL using data sets obtained from MATPOWER [37]. OPF problems represent a challenging set of nonconvex problems. The active and reactive power flow and the network balance equations give rise to equality constraints involving nonconvex functions while the inequality constraints are linear and result from placing operating limits on quantities such as flows, voltages, and various control variables. The control variables include the voltages at generator buses and the active-power output of the generating units. The state variables consist of the voltage magnitudes and angles at each node as well as reactive and active flows in each link. Our test set was comprised of 28 problems modeled on systems having 14 to 662 nodes from the IEEE test set. In particular, there are seven IEEE systems, each modeled in four different ways: (i) in Cartesian coordinates; (ii) in polar coordinates; (iii) with basic approximations to the sin and cos functions in the problem functions; and (iv) with linearized constraints based on DC power flow equations (in place of AC power flow). It should be noted that while linearizing the constraints in

formulation (iv) led to a set of linear optimization problems, we still find it interesting to investigate the possible effect that steering may have in this context. All of the test problems were solved by all of our algorithm variants.

We provide outperforming factors in the same manner as in §3.1.3. Figures 9 and 10 reveal that AAL-LS typically outperforms BAL-LS in terms of both iterations and function evaluations, and Figures 11 and 12 reveal that AAL-TR more often than not outperforms BAL-TR in terms of iterations and gradient evaluations. Interestingly, these results suggest more benefits for steering in the line search algorithm than in the trust region algorithm, which is the opposite of that suggested by the results in §3.1.3. However, in any case, we believe that we have presented convincing numerical evidence that steering often has an overall beneficial effect on the performance of our MATLAB solvers.

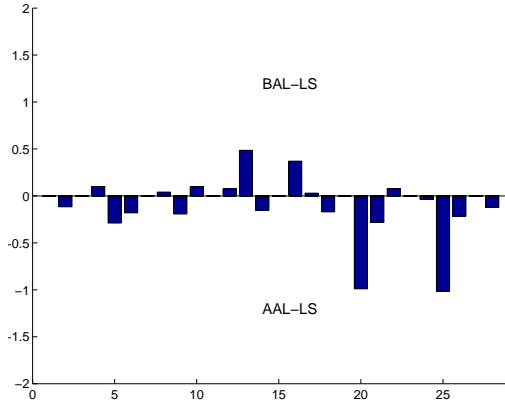


Figure 9. Outperforming factors for iterations: line search algorithms on OPF tests.

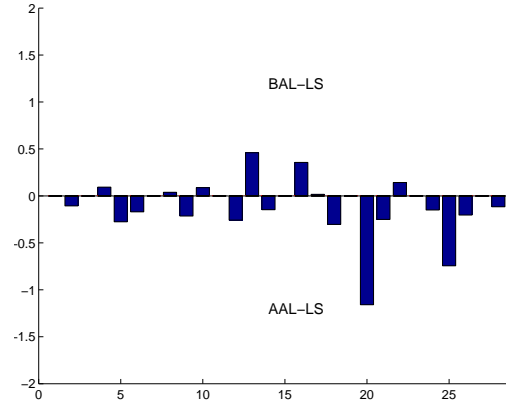


Figure 10. Outperforming factors for function evaluations: line search algorithms on OPF tests.

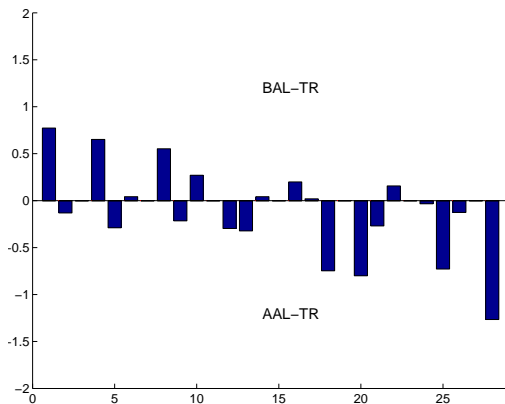


Figure 11. Outperforming factors for iterations: trust region algorithms on OPF tests.

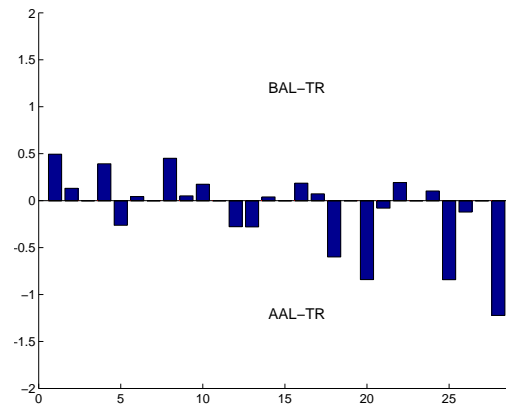


Figure 12. Outperforming factors for gradient evaluations: trust region algorithms on OPF tests.

3.2 An implementation of LANCELOT that uses steering

3.2.1 Implementation details

The results for our MATLAB software in the previous section illustrate that our adaptive line search AL algorithm and the adaptive trust region AL algorithm from [15] are often

more efficient and reliable than basic AL algorithms that employ traditional penalty parameter and Lagrange multiplier updates. Recall, however, that our adaptive methods are different from their basic counterparts in two key ways. First, the steering conditions (16) are used to dynamically decrease the penalty parameter during the optimization process for the AL function. Second, our mechanisms for updating the Lagrange multiplier estimate are different than the basic algorithm outlined in [15, Algorithm 1] since they use optimality measures for both the Lagrangian and the AL functions (see line 28 of Algorithm 3) rather than only that for the AL function. We believe this strategy is more adaptive since it allows for updates to the Lagrange multipliers when the primal estimate is still far from a first-order stationary point for the AL function subject to the bounds.

In this section, we isolate the effect of the first of these differences by incorporating a steering strategy in the LANCELOT [12, 13] package that is available in the GALAHAD library [23]. Specifically, we made three principle enhancements in LANCELOT. First, along the lines of the model q in [15] and the convexified model \tilde{q} defined in this paper, we defined the model $\hat{q} : \mathbb{R}^n \rightarrow \mathbb{R}$ of the AL function given by

$$\hat{q}(s; x, y, \mu) = s^T \nabla_x \ell(x, y + c(x)/\mu) + \frac{1}{2} s^T (\nabla_{xx} \ell(x, y) + J(x)^T J(x)/\mu) s$$

as an alternative to the Newton model $q_N : \mathbb{R}^n \rightarrow \mathbb{R}$, originally used in LANCELOT,

$$q_N(s; x, y, \mu) = s^T \nabla_x \ell(x, y + c(x)/\mu) + \frac{1}{2} s^T (\nabla_{xx} \ell(x, y + c(x)/\mu) + J(x)^T J(x)/\mu) s.$$

As in our adaptive algorithms, the purpose of employing such a model was to ensure that $\hat{q} \rightarrow q_v$ (pointwise) as $\mu \rightarrow 0$, which was required to ensure that our steering procedure was well-defined; see (A1a). Second, we added routines to compute generalized Cauchy points [9] for both the constraint violation measure model q_v and \hat{q} during the loop in which μ was decreased until the steering test (16c) was satisfied; recall the **while** loop starting on line 19 of Algorithm 3. Third, we used the value for μ determined in the steering procedure to compute a generalized Cauchy point for the Newton model q_N , which was the model employed to compute the search direction. For each of the models just discussed, the generalized Cauchy point was computed using either an efficient sequential search along the piece-wise Cauchy arc [10] or via a backtracking Armijo search along the same arc [31]. We remark that this third enhancement would not have been needed if the model \hat{q} were used to compute the search directions. However, in our experiments, it was revealed that using the Newton model typically led to better performance, so the results in this section were obtained using this third enhancement. In our implementation, the user was allowed to control which model was used via control parameters. We also added control parameters that allowed the user to restrict the number of times that the penalty parameter may be reduced in the steering procedure in a given iteration, and that disabled steering once the penalty parameter was reduced below a given tolerance (as in the safeguarding procedure implemented in our MATLAB software).

The new package was tested with three different control parameter settings. We refer to algorithm with the first setting, which did not allow any steering to occur, simply as **lancelot**. The second setting allowed steering to be used initially, but turned it off whenever $\mu \leq 10^{-4}$ (as in our safeguarded MATLAB algorithms). We refer to this variant as **lancelot-steering-safe**. The third setting allowed for steering to be used without any safeguards or restrictions; we refer to this variant as **lancelot-steering**. As in our MATLAB software, the penalty parameter was decreased by a factor of 0.7 until the steering test (16c) was satisfied. All other control parameters were set to their default **lancelot** values. The new package will be re-branded as LANCELOT in the next official release, GALAHAD 2.6.

GALAHAD was compiled with gfortran-4.7 with optimization -O and using Intel MKL

BLAS. The code was executed on a single core of an Intel Xeon E5620 (2.4GHz) CPU with 23.5 GiB of RAM.

3.2.2 Results on CUTESt test problems

We tested `lancelot`, `lancelot-steering`, and `lancelot-steering-safe` on the subset of CUTESt problems that have at least one general constraint and at most 10,000 variables and 10,000 constraints. This amounted to 457 test problems. The results are displayed as performance profiles in Figures 13 and 14, which were created from the 364 of these problems that were solved by at least one of the algorithms. As in the previous sections, since the algorithms are trust region methods, we use the number of iterations and gradient evaluations required as the performance measures of interest.

We can make two important observations from these profiles. First, it is clear that `lancelot-steering` and `lancelot-steering-safe` yielded similar performance in terms of iterations and gradient evaluations, which suggests that safeguarding the steering mechanism is not necessary in practice. Second, `lancelot-steering` and `lancelot-steering-safe` were both more efficient and reliable than `lancelot` on these tests, thus showing the positive influence that steering can have on performance.

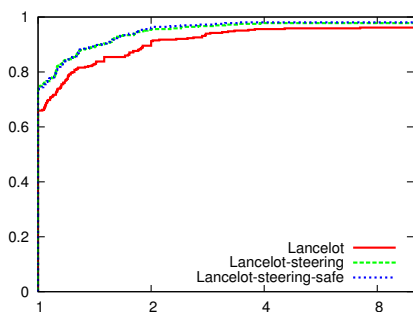


Figure 13. Performance profile for iterations: LANCELOT algorithms on the CUTESt set.

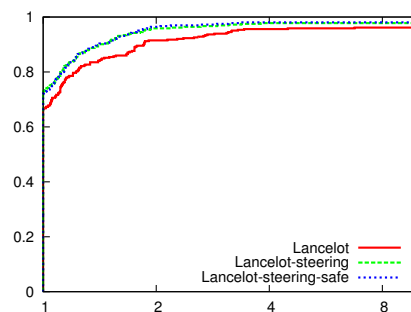


Figure 14. Performance profile for gradient evaluations: LANCELOT algorithms on the CUTESt set.

As in §3.1.2, it is important to observe the final penalty parameter values yielded by `lancelot-steering` and `lancelot-steering-safe` as opposed to those yielded by `lancelot`. For these experiments, we collected this information; see Table 3.

Table 3. Numbers of CUTESt problems for which the final penalty parameter values were in the given ranges.

μ_{final}	<code>lancelot</code>	<code>lancelot-steering</code>	<code>lancelot-steering-safe</code>
1	14	1	1
$[10^{-1}, 1)$	77	1	1
$[10^{-2}, 10^{-1})$	47	93	93
$[10^{-3}, 10^{-2})$	27	45	45
$[10^{-4}, 10^{-3})$	18	28	28
$[10^{-5}, 10^{-4})$	15	22	22
$[10^{-6}, 10^{-5})$	12	21	14
$(0, 10^{-6})$	19	18	25

We make a few remarks about the results in Table 3. First, as may have been expected, the `lancelot-steering` and `lancelot-steering-safe` algorithms typically reduced the penalty parameter below its initial value, even when `lancelot` did not reduce it at all throughout an entire run. Second, the number of problems for which the final penalty

parameter was less than 10^{-4} was 171 for `lancelot` and 168 for `lancelot-steering`. Combining this fact with the previous observation leads us to conclude that steering tended to reduce the penalty parameter from its initial value of 1, but, overall, it did not decrease it much more aggressively than `lancelot`. Third, it is interesting to compare the final penalty parameter values for `lancelot-steering` and `lancelot-steering-safe`. Of course, these values were equal in any run in which the final penalty parameter was greater than or equal to 10^{-4} , since this was the threshold value below which safeguarding was activated. Interestingly, however, `lancelot-steering-safe` actually produced *smaller* values of the penalty parameter compared to `lancelot-steering` when the final penalty parameter was smaller than 10^{-4} . We initially found this observation to be somewhat counterintuitive, but we believe that it can be explained by observing the penalty parameter updating strategy used by `lancelot`. (Recall that once safeguarding was activated in `lancelot-steering-safe`, the updating strategy became the same used in `lancelot`.) In particular, the decrease factor for the penalty parameter used in `lancelot` is 0.1, whereas the decrease factor used in steering the penalty parameter was 0.7. Thus, we believe that `lancelot-steering` reduced the penalty parameter more gradually once it was reduced below 10^{-4} while `lancelot-steering-safe` could only reduce it in the typical aggressive manner. (We remark that to (potentially) circumvent this inefficiency in `lancelot`, one could implement a different strategy in which the penalty parameter decrease factor is increased as the penalty parameter decreases, but in a manner that still ensures that the penalty parameter converges to zero when infinitely many decreases occur.) Overall, our conclusion from Table 3 is that steering typically decreases the penalty parameter more than a traditional updating scheme, but the difference is relatively small and we have implemented steering in a way that improves the overall efficiency and reliability of the method.

4. Conclusion

In this paper, we explored the numerical performance of adaptive updates to the Lagrange multiplier vector and penalty parameter in AL methods. Specific to the penalty parameter updating scheme is the use of steering conditions that guide the iterates toward the feasible region and toward dual feasibility in a balanced manner. Similar conditions were first introduced in [8] for exact penalty functions, but have been adapted in [15] and this paper to be appropriate for AL-based methods. Specifically, since AL methods are not exact (in that, in general, the trial steps do not satisfy linearized feasibility for any positive value of the penalty parameter), we allowed for a relaxation of the linearized constraints. This relaxation was based on obtaining a target level of infeasibility that is driven to zero at a modest, but acceptable, rate. This approach is in the spirit of AL algorithms since feasibility and linearized feasibility are only obtained in the limit. It should be noted that, like other AL algorithms, our adaptive methods can be implemented matrix-free, i.e., they only require matrix-vector products. This is of particular importance when solving large problems that have sparse derivative matrices.

As with steering strategies designed for exact penalty functions, our steering conditions proved to yield more efficient and reliable algorithms than a traditional updating strategy. This conclusion was made by performing a variety of numerical tests that involved our own MATLAB implementations and a simple modification of the well-known AL software LANCELOT. To test the potential for the penalty parameter to be reduced too quickly, we also implemented safeguarded variants of our steering algorithms. Across the board, our results indicate that safeguarding was not necessary and would typically degrade performance when compared to the unrestricted steering approach. We feel confident that these tests clearly show that although our theoretical global convergence guarantee

is weaker than some algorithms (i.e., we cannot prove that the penalty parameter will remain bounded under a suitable constraint qualification), this should not be a concern in practice. Finally, we suspect that the steering strategies described in this paper would also likely improve the performance of other AL-based methods such as [4, 27].

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Appendix A. Well-posedness

Our goal in this appendix is to prove that Algorithm 3 is well-posed under Assumption 2.1. Since this assumption is assumed to hold throughout the remainder of this appendix, we do not refer to it explicitly in the statement of each lemma and proof.

A.1 Preliminary results

Our proof of the well-posedness of Algorithm 3 relies on showing that it will either terminate finitely or will produce an infinite sequence of iterates $\{(x_k, y_k, \mu_k)\}$. In order

to show this, we first require that the **while** loop that begins at line 11 of Algorithm 3 terminates finitely. Since the same loop appears in the AL trust region method in [15] and the proof of the result in the case of that algorithm is the same as that for Algorithm 3, we need only refer to the result in [15] in order to state the following lemma for Algorithm 3.

LEMMA A.1 ([15, Lemma 3.2]) *If line 11 is reached, then $F_{AL}(x_k, y_k, \mu) \neq 0$ for all sufficiently small $\mu > 0$.*

Next, since the Cauchy steps employed in Algorithm 3 are similar to those employed in the method in [15], we may state the following lemma showing that Algorithms 1 and 2 are well defined when called in lines 15, 17, and 21 of Algorithm 3. It should be noted that a slight difference between Algorithm 2 and the similar procedure in [15] is the use of the convexified model \tilde{q} in (15). However, we claim that this difference does not affect the veracity of the result.

LEMMA A.2 ([15, Lemma 3.3]) *The following hold true:*

- (i) *The computation of $(\bar{\beta}_k, \bar{r}_k, \varepsilon_k, \Gamma_k)$ in line 15 is well defined and yields $\Gamma_k \in (1, 2]$ and $\varepsilon_k \in [0, \varepsilon_r)$.*
- (ii) *The computation of $(\bar{\alpha}_k, \bar{s}_k)$ in lines 17 and 21 is well defined.*

The next result, similar to [15, Lemma 3.4], highlights critical relationships between q_v and \tilde{q} as $\mu \rightarrow 0$. Indeed, much of the proof follows exactly the same logic as [15, Lemma 3.4], but we provide a complete proof to account for our present use of the convexified model \tilde{q} and the differences in the trust region radii for the subproblems employed in the algorithms. This result is crucial for showing that the steering condition (16c) is satisfied for all sufficient small μ (see Lemma A.5).

LEMMA A.3 *Let $(\bar{\beta}_k, \bar{r}_k, \varepsilon_k, \Gamma_k) \leftarrow \text{CAUCHY_FEASIBILITY}(x_k, \theta_k)$ with θ_k defined by (9) and, as quantities dependent on the penalty parameter $\mu > 0$, let $(\bar{\alpha}_k(\mu), \bar{s}_k(\mu)) \leftarrow \text{CAUCHY_AL}(x_k, y_k, \mu, \Theta_k(\mu), \varepsilon_k)$ with $\Theta_k(\mu) := \Gamma_k \delta \|F_{AL}(x_k, y_k, \mu)\|_2$ (see (13)). Then, the following hold true:*

$$\lim_{\mu \rightarrow 0} \left(\max_{\|s\|_2 \leq 2\theta_k} |\tilde{q}(s; x_k, y_k, \mu) - q_v(s; x_k)| \right) = 0, \quad (\text{A1a})$$

$$\lim_{\mu \rightarrow 0} \nabla_x \mathcal{L}(x_k, y_k, \mu) = J_k^T c_k, \quad (\text{A1b})$$

$$\lim_{\mu \rightarrow 0} \bar{s}_k(\mu) = \bar{r}_k, \quad (\text{A1c})$$

$$\text{and } \lim_{\mu \rightarrow 0} \Delta q_v(\bar{s}_k(\mu); x_k) = \Delta q_v(\bar{r}_k; x_k). \quad (\text{A1d})$$

Proof. Since x_k and y_k are fixed during iteration k , for ease of exposition we often drop these quantities from function dependencies for the purposes of this proof. From the definitions of q_v and \tilde{q} , it follows that for some constants $M_1 > 0$ and $M_2 > 0$ independent of μ we have

$$\begin{aligned} \max_{\|s\|_2 \leq 2\theta_k} |\tilde{q}(s; \mu) - q_v(s)| &= \max_{\|s\|_2 \leq 2\theta_k} |\mu \ell_k + \mu \nabla_x \ell_k^T s + \max\{\frac{\mu}{2} s^T \nabla_{xx} \ell_k s, -\frac{1}{2} \|J_k s\|_2^2\}| \\ &\leq \mu M_1 + \max\{\mu M_2, -\frac{1}{2} \|J_k s\|_2^2\}. \end{aligned}$$

Since the right-hand side of this expression vanishes as $\mu \rightarrow 0$, we have (A1a). Further,

$$\nabla_x \mathcal{L}(x_k, y_k, \mu) - J_k^T c_k = \mu(g_k - J_k^T y_k),$$

which implies that (A1b) holds.

We now show that (A1c) holds. Our proof considers two cases depending on the value $F_{\text{FEAS}}(x_k)$. Throughout consideration of these cases, it should be observed that all quantities in Algorithm 1 are unaffected by μ , so they can be considered as fixed quantities.

Case 1: If $F_{\text{FEAS}}(x_k) = 0$, then $\theta_k = \delta \|F_{\text{FEAS}}(x_k)\|_2 = 0$, from which it follows that $\bar{r}_k = 0$ and $\Delta q_v(\bar{r}_k) = 0$. Furthermore, from (A1b), we have $\Theta_k(\mu) \rightarrow 0$ as $\mu \rightarrow 0$, which means $\bar{s}_k(\mu) \rightarrow 0 = \bar{r}_k$, as desired.

Case 2: Now suppose that $F_{\text{FEAS}}(x_k) \neq 0$. In the following arguments, we define the following functions of a nonnegative integer l and positive scalar μ :

$$r_k(l) = P(x_k - \gamma^l J_k^T c_k) - x_k \quad \text{and} \quad s_k(l, \mu) = P(x_k - \gamma^l \nabla_x \mathcal{L}(\mu)) - x_k.$$

We also define $l_\beta \geq 0$ to be the integer such that $\bar{\beta}_k = \gamma^{l_\beta}$ (see Algorithm 1), which implies that

$$\bar{r}_k = r_k(l_\beta). \tag{A2}$$

We have as a consequence of (A1b) that

$$\lim_{\mu \rightarrow 0} s_k(l, \mu) = r_k(l) \quad \text{for any } l \geq 0.$$

In particular, this implies with (A2) that

$$\lim_{\mu \rightarrow 0} s_k(l_\beta, \mu) = r_k(l_\beta) = \bar{r}_k. \tag{A3}$$

Thus, (A1c) follows as long as

$$\bar{s}_k(\mu) = s_k(l_\beta, \mu) \quad \text{for all sufficiently small } \mu > 0. \tag{A4}$$

Since the computation of $\bar{s}_k(\mu)$ (via the CAUCHY_AL routine stated as Algorithm 2) computes a nonnegative integer $l_{\alpha, \mu}$ such that

$$\bar{s}_k(\mu) = P(x_k - \gamma^{l_{\alpha, \mu}} \nabla_x \mathcal{L}(\mu)) - x_k,$$

it follows that (A4) can be proved by showing that $l_{\alpha, \mu} = l_\beta$ for all sufficiently small $\mu > 0$. As a preliminary result in the proof of this fact, we first show that for l_k computed in Algorithm 1 we have

$$\min\{l_\beta, l_{\alpha, \mu}\} \geq l_k \quad \text{for all sufficiently small } \mu > 0. \tag{A5}$$

Indeed, if $l_k = 0$, then (A5) holds trivially. Thus, let us suppose that $l_k > 0$. According to the procedures in Algorithm 1, it is clear that $l_\beta \geq l_k$. Hence, we may turn our attention to $l_{\alpha, \mu}$. From the definition of $\Theta_k(\mu)$ (in the statement of this lemma), (A1b), the manner in which Γ_k is set in Algorithm 1, the fact that $\theta_k > 0$, and since $\|P(x_k - \gamma^{l_k-1} J_k^T c_k) - x_k\|_2 > \theta_k$ due to the manner in which l_k is set in Algorithm 1,

we have that

$$\begin{aligned}
\lim_{\mu \rightarrow 0} \Theta_k(\mu) &= \lim_{\mu \rightarrow 0} \Gamma_k \delta \|F_{\text{AL}}(x_k, y_k, \mu)\|_2 = \Gamma_k \delta \|F_{\text{FEAS}}(x_k)\|_2 = \Gamma_k \theta_k \\
&= \min \left\{ 2, \frac{1}{2} \left(1 + \frac{\|P(x_k - \gamma^{l_k-1} J_k^T c_k) - x_k\|_2}{\theta_k} \right) \right\} \theta_k \\
&= \min \left\{ 2\theta_k, \frac{1}{2} \left(\theta_k + \|P(x_k - \gamma^{l_k-1} J_k^T c_k) - x_k\|_2 \right) \right\} \\
&\in (\theta_k, \|P(x_k - \gamma^{l_k-1} J_k^T c_k) - x_k\|_2).
\end{aligned}$$

Along with (A1b), this implies that for all sufficiently small $\mu > 0$ we have

$$\lim_{\mu \rightarrow 0} \|P(x_k - \gamma^{l_k-1} \nabla_x \mathcal{L}(\mu)) - x_k\|_2 = \|P(x_k - \gamma^{l_k-1} J_k^T c_k) - x_k\|_2 > \Theta_k(\mu).$$

This shows that $l_{\alpha, \mu} \geq l_k$ holds for all sufficiently small $\mu > 0$. Consequently, we have (A5).

Having ensured that (A5) holds, we proceed to prove that $l_{\alpha, \mu} = l_\beta$ for all sufficiently small $\mu > 0$. It follows from the definition of l_β , (A2), the procedures of Algorithm 1 (e.g., the manner in which ε_k is set), and part (i) of Lemma A.2 that

$$\begin{aligned}
-\frac{\Delta q_v(\bar{r}_k)}{\bar{r}_k^T J_k^T c_k} &= -\frac{\Delta q_v(r_k(l_\beta))}{r_k(l_\beta)^T J_k^T c_k} \geq \varepsilon_r \\
\text{and } -\frac{\Delta q_v(r_k(l))}{r_k(l)^T J_k^T c_k} &\leq \varepsilon_k < \varepsilon_r \text{ for all integers } l_k \leq l < l_\beta.
\end{aligned} \tag{A6}$$

(Here, it is important to note that [14, Theorem 12.1.4] can be invoked to ensure that all denominators in (A6) are negative.) It follows from (A1b), (A3), (A1a), (A6), and part (i) of Lemma A.2 that

$$\lim_{\mu \rightarrow 0} -\frac{\Delta \tilde{q}(s_k(l_\beta, \mu))}{s_k(l_\beta, \mu)^T \nabla_x \mathcal{L}(\mu)} = -\frac{\Delta q_v(\bar{r}_k)}{\bar{r}_k^T J_k^T c_k} \geq \varepsilon_r > \frac{\varepsilon_k + \varepsilon_r}{2} \tag{A7}$$

and

$$\lim_{\mu \rightarrow 0} -\frac{\Delta \tilde{q}(s_k(l, \mu))}{s_k(l, \mu)^T \nabla_x \mathcal{L}(\mu)} = -\frac{\Delta q_v(r_k(l))}{r_k(l)^T J_k^T c_k} \leq \varepsilon_k < \frac{\varepsilon_k + \varepsilon_r}{2} \text{ for all integers } l_k \leq l < l_\beta. \tag{A8}$$

It now follows from (A5), (A7), (A8), and (15) that $l_{\alpha, \mu} = l_\beta$ for all sufficiently small $\mu > 0$. As previously mentioned, this proves (A1c).

Finally, we notice that (A1d) follows from (A1c) and continuity of the model q_v . \blacksquare

We also need the following lemma related to Cauchy decreases in the models q_v and \tilde{q} . The conclusions of the lemma are similar to [15, Lemma 3.5], but here we account for the convexified model \tilde{q} and other differences in the subproblems employed here as opposed to those in [15].

LEMMA A.4 *Let Ω be any scalar value such that*

$$\Omega \geq \max\{\|\mu_k \nabla_{xx}^2 \ell(x_k, y_k) + J_k^T J_k\|_2, \|J_k^T J_k\|_2\}. \tag{A9}$$

Then, the following hold true:

(i) For some $\kappa_4 \in (0, 1)$, the Cauchy step for subproblem (8) yields

$$\Delta q_v(\bar{r}_k; x_k) \geq \kappa_4 \|F_{\text{FEAS}}(x_k)\|_2^2 \min \left\{ \delta, \frac{1}{1 + \Omega} \right\}. \quad (\text{A10})$$

(ii) For some $\kappa_5 \in (0, 1)$, the Cauchy step for subproblem (12) yields

$$\Delta \tilde{q}(\bar{s}_k; x_k, y_k, \mu_k) \geq \kappa_5 \|F_{\text{AL}}(x_k, y_k, \mu_k)\|_2^2 \min \left\{ \delta, \frac{1}{1 + \Omega} \right\}. \quad (\text{A11})$$

Proof. Let $\Sigma_k := 1 + \sup\{|\omega_k(r)| : 0 < \|r\|_2 \leq \theta_k\}$, where

$$\omega_k(r) = \frac{-\Delta q_v(r; x_k) - r^T J_k^T c_k}{\|r\|_2^2} \text{ for all } r \in \mathbb{R}^n.$$

In fact, using (A9), the Cauchy-Schwartz inequality, and standard norm inequalities, we have that

$$\omega_k(r) = \frac{r^T J_k^T J_k r}{2\|r\|_2^2} \leq \Omega \text{ for all } r \in \mathbb{R}^n.$$

Hence, $\Sigma_k \leq 1 + \Omega$. The requirement (11) and [32, Theorem 4.4] then yield, for some $\bar{\kappa}_4 \in (0, 1)$, that

$$\Delta q_v(\bar{r}_k; x_k) \geq \epsilon_r \bar{\kappa}_4 \|F_{\text{FEAS}}(x_k)\|_2 \min \left\{ \theta_k, \frac{1}{\Sigma_k} \|F_{\text{FEAS}}(x_k)\|_2 \right\}.$$

which, with (9), implies that (A10) follows with $\kappa_4 := \epsilon_r \bar{\kappa}_4$.

We now show (A11) in a similar manner. Let $\bar{\Sigma}_k := 1 + \sup\{|\bar{\omega}_k(s)| : 0 < \|s\|_2 \leq \Theta_k\}$ where

$$\bar{\omega}_k(s) := \frac{-\Delta \tilde{q}(s; x_k, y_k, \mu_k) - s^T \nabla_x \mathcal{L}(x_k, y_k, \mu_k)}{\|s\|_2^2} \text{ for all } s \in \mathbb{R}^n.$$

Using (A9), we have in a similar manner as above that

$$\begin{aligned} \bar{\omega}_k(s) &= \frac{\max\{\mu_k s^T \nabla_{xx}^2 \ell(x_k, y_k, \mu_k) s + s^T J_k^T J_k s, 0\}}{2\|s\|_2^2} \\ &\leq \frac{\|\mu_k \nabla_{xx}^2 \ell(x_k, y_k, \mu_k) + J_k^T J_k\|_2 \|s\|_2^2}{2\|s\|_2^2} \leq \Omega. \end{aligned}$$

Thus, $\bar{\Sigma}_k \leq 1 + \Omega$. The requirement (15) and [32, Theorem 4.4] then yield, for some $\bar{\kappa}_5 \in (0, 1)$, that

$$\Delta \tilde{q}(\bar{s}_k; x_k, y_k, \mu_k) \geq \frac{\epsilon_k + \epsilon_r}{2} \bar{\kappa}_5 \|F_{\text{AL}}(x_k, y_k, \mu_k)\|_2 \min \left\{ \Theta_k, \frac{1}{\bar{\Sigma}_k} \|F_{\text{AL}}(x_k, y_k, \mu_k)\|_2 \right\},$$

which, with (13) and Lemma A.2(i), implies that (A11) follows with $\kappa_5 := \frac{1}{2} \epsilon_r \bar{\kappa}_5$. \blacksquare

The next lemma shows that the **while** loop at line 19, which is responsible for ensuring that our adaptive steering conditions in (16) are satisfied, terminates finitely. The proof of this is similar to the second part of [15, Theorem 3.6]

LEMMA A.5 *The **while** loop that begins at line 19 of Algorithm 3 terminates finitely.*

Proof. Since Lemma A.1 ensures that the latter condition in the **while** loop is satisfied for all sufficiently small $\mu_k > 0$, it suffices to show that $s_k = \bar{s}_k$ and $r_k = \bar{r}_k$ satisfy (16c) for all sufficiently small $\mu_k > 0$. To see this, we may borrow notation from Lemma A.3—i.e., to consider $s_k = \bar{s}_k$ as a quantity dependent on a parameter $\mu > 0$ —and observe that (A1d) implies

$$\lim_{\mu \rightarrow 0} \Delta q_v(s_k(\mu); x_k) = \lim_{\mu \rightarrow 0} \Delta q_v(\bar{s}_k(\mu); x_k) = \Delta q_v(\bar{r}_k; x_k) = \Delta q_v(r_k; x_k). \quad (\text{A12})$$

If $\Delta q_v(r_k; x_k) > 0$, then (A12) implies that (16c) is satisfied for sufficiently small $\mu_k > 0$. Otherwise,

$$\Delta q_v(r_k; x_k) = \Delta q_v(\bar{r}_k; x_k) = 0, \quad (\text{A13})$$

which along with (A10) implies that $F_{\text{FEAS}}(x_k) = 0$. We may now consider two cases depending on whether x_k is feasible for (1). If $c_k \neq 0$, then Algorithm 3 would have terminated in line 9, meaning that the **while** loop at line 19 would not have been reached. On the other hand, if $c_k = 0$, then (A13) implies

$$\min\{\kappa_3 \Delta q_v(r_k; x_k), v_k - \frac{1}{2}(\kappa_t t_j)^2\} = -\frac{1}{2}(\kappa_t t_j)^2 < 0. \quad (\text{A14})$$

This last strict inequality follows since $t_j > 0$ by construction and $\kappa_t \in (0, 1)$ by choice. Therefore, we can deduce that (16c) will be satisfied for sufficiently small $\mu_k > 0$ by observing (A12), (A13) and (A14). ■

The final lemma of this section shows that s_k is a strict descent direction for the AL function. The conclusion of this lemma is the primary motivation for our use of the convexified model \tilde{q} .

LEMMA A.6 *At line 24 of Algorithm 3, the search direction s_k is a strict descent direction for $\mathcal{L}(\cdot, y_k, \mu_k)$ from x_k . In particular,*

$$\nabla_x \mathcal{L}(x_k, y_k, \mu_k)^T s_k \leq -\Delta \tilde{q}(s_k; x_k, y_k, \mu_k) \leq -\kappa_1 \Delta \tilde{q}(\bar{s}_k; x_k, y_k, \mu_k) < 0. \quad (\text{A15})$$

Proof. From the definition of \tilde{q} , we find

$$\begin{aligned} \Delta \tilde{q}(s_k; x_k, y_k, \mu_k) &= \tilde{q}(0; x_k, y_k, \mu_k) - \tilde{q}(s_k; x_k, y_k, \mu_k) \\ &= -\nabla_x \mathcal{L}(x_k, y_k, \mu_k)^T s_k - \max\{\frac{1}{2} s_k^T (\mu_k \nabla_{xx}^2 \ell(x_k, y_k) + J_k^T J_k) s_k, 0\} \\ &\leq -\nabla_x \mathcal{L}(x_k, y_k, \mu_k)^T s_k. \end{aligned}$$

It follows from this inequality and (16a) that

$$\nabla_x \mathcal{L}(x_k, y_k, \mu_k)^T s_k \leq -\Delta \tilde{q}(s_k; x_k, y_k, \mu_k) \leq -\kappa_1 \Delta \tilde{q}(\bar{s}_k; x_k, y_k, \mu_k) < 0,$$

as desired. ■

A.2 Proof of well-posedness result

Proof of Theorem 2.2. If, during the k th iteration, Algorithm 3 terminates in line 6 or 9, then there is nothing to prove. Thus, to proceed in the proof, we may assume that line 11 is reached. Lemma A.1 then ensures that

$$F_{\text{AL}}(x_k, y_k, \mu) \neq 0 \text{ for all sufficiently small } \mu > 0. \quad (\text{A16})$$

Consequently, the **while** loop in line 11 will terminate for a sufficiently small $\mu_k > 0$. Next, by construction, conditions (16a) and (16b) are satisfied for any $\mu_k > 0$ by $s_k = \bar{s}_k$ and $r_k = \bar{r}_k$. Lemma A.5 then shows that for a sufficiently small $\mu_k > 0$, (16c) is also satisfied by $s_k = \bar{s}_k$ and $r_k = \bar{r}_k$. Therefore, line 24 will be reached. Finally, Lemma A.6 ensures that α_k in line 24 is well-defined. This completes the proof as all remaining lines in the k th iteration are explicit. \blacksquare

Appendix B. Global Convergence

We shall tacitly presume that Assumption 2.3 holds throughout this section, and not state it explicitly. This assumption and the bound on the multipliers enforced in line 31 of Algorithm 3 imply that there exists a positive monotonically increasing sequence $\{\Omega_j\}_{j \geq 1}$ such that for all $k_j \leq k < k_{j+1}$ we have

$$\|\nabla_{xx}^2 \mathcal{L}(\sigma, y_k, \mu_k)\|_2 \leq \Omega_j \text{ for all } \sigma \text{ on the segment } [x_k, x_k + s_k], \quad (\text{B1a})$$

$$\|\mu_k \nabla_{xx}^2 \ell(x_k, y_k) + J_k^T J_k\|_2 \leq \Omega_j, \quad (\text{B1b})$$

$$\text{and } \|J_k^T J_k\|_2 \leq \Omega_j. \quad (\text{B1c})$$

In the subsequent analysis, we make use of the subset of iterations for which line 29 of Algorithm 3 is reached. For this purpose, we define the iteration index set

$$\mathcal{Y} := \{k_j : \|c_{k_j}\|_2 \leq t_j, \min\{\|F_L(x_{k_j}, \hat{y}_{k_j})\|_2, \|F_{\text{AL}}(x_{k_j}, y_{k_j-1}, \mu_{k_j-1})\|_2\} \leq T_j\}. \quad (\text{B2})$$

B.1 Preliminary results

The following result provides critical bounds on differences in (components of) the augmented Lagrangian summed over sequences of iterations. We remark that the proof in [15] essentially relies on Assumption 2.3 and Dirichlet's Test [16, §3.4.10].

LEMMA B.1 ([15, Lemma 3.7].) *The following hold true.*

(i) *If $\mu_k = \mu$ for some $\mu > 0$ and all sufficiently large k , then there exist positive constants*

M_f , M_c , and $M_{\mathcal{L}}$ such that for all integers $p \geq 1$ we have

$$\sum_{k=0}^{p-1} \mu_k (f_k - f_{k+1}) < M_f, \quad (\text{B3})$$

$$\sum_{k=0}^{p-1} \mu_k y_k^T (c_{k+1} - c_k) < M_c, \quad (\text{B4})$$

$$\text{and } \sum_{k=0}^{p-1} (\mathcal{L}(x_k, y_k, \mu_k) - \mathcal{L}(x_{k+1}, y_k, \mu_k)) < M_{\mathcal{L}}. \quad (\text{B5})$$

(ii) If $\mu_k \rightarrow 0$, then the sums

$$\sum_{k=0}^{\infty} \mu_k (f_k - f_{k+1}), \quad (\text{B6})$$

$$\sum_{k=0}^{\infty} \mu_k y_k^T (c_{k+1} - c_k), \quad (\text{B7})$$

$$\text{and } \sum_{k=0}^{\infty} (\mathcal{L}(x_k, y_k, \mu_k) - \mathcal{L}(x_{k+1}, y_k, \mu_k)) \quad (\text{B8})$$

converge and are finite, and

$$\lim_{k \rightarrow \infty} \|c_k\|_2 = \bar{c} \text{ for some } \bar{c} \geq 0. \quad (\text{B9})$$

We also need the following lemma that bounds the step-size sequence $\{\alpha_k\}$ below.

LEMMA B.2 *There exists a positive monotonically decreasing sequence $\{C_j\}_{j \geq 1}$ such that, with the sequence $\{k_j\}$ computed in Algorithm 3, the step-size sequence $\{\alpha_k\}$ satisfies*

$$\alpha_k \geq C_j > 0 \text{ for all } k_j \leq k < k_{j+1}.$$

Proof. By Taylor's Theorem and Lemma A.6, it follows under Assumption 2.3 that there exists $\tau > 0$ such that for all sufficiently small $\alpha > 0$ we have

$$\mathcal{L}(x_k + \alpha s_k, y_k, \mu_k) - \mathcal{L}(x_k, y_k, \mu_k) \leq -\alpha \Delta \tilde{q}(s_k; x_k, y_k, \mu_k) + \tau \alpha^2 \|s_k\|_2^2. \quad (\text{B10})$$

On the other hand, during the line search implicit in line 24 of Algorithm 3, a step-size α is rejected if

$$\mathcal{L}(x_k + \alpha s_k, y_k, \mu_k) - \mathcal{L}(x_k, y_k, \mu_k) > -\eta_s \alpha \Delta \tilde{q}(s_k; x_k, y_k, \mu_k). \quad (\text{B11})$$

Combining (B10), (B11), and (16a) we have that a rejected step-size α satisfies

$$\alpha > \frac{(1 - \eta_s) \Delta \tilde{q}(s_k; x_k, y_k, \mu_k)}{\tau \|s_k\|_2^2} \geq \frac{(1 - \eta_s) \Delta \tilde{q}(s_k; x_k, y_k, \mu_k)}{\tau \Theta_k^2}.$$

From this bound, the fact that if the line search rejects a step-size it multiplies it by $\gamma_\alpha \in (0, 1)$, (16a), (A11), (B1b), (13), and $\Gamma_k \in (1, 2]$ (see Lemma A.2) it follows that,

for all $k \in [k_j, k_{j+1})$,

$$\begin{aligned} \alpha_k &\geq \frac{\gamma_\alpha(1 - \eta_s)\Delta\tilde{q}(s_k; x_k, y_k, \mu_k)}{\tau\Theta_k^2} \\ &\geq \frac{\gamma_\alpha(1 - \eta_s)\kappa_1\kappa_5\|F_{\text{AL}}(x_k, y_k, \mu_k)\|_2^2}{\tau\Gamma_k^2\delta^2\|F_{\text{AL}}(x_k, y_k, \mu_k)\|_2^2} \min\left\{\delta, \frac{1}{1 + \Omega_j}\right\} \\ &\geq \frac{\gamma_\alpha(1 - \eta_s)\kappa_1\kappa_5}{4\tau\delta^2} \min\left\{\delta, \frac{1}{1 + \Omega_j}\right\} =: C_j > 0, \end{aligned}$$

as desired. \blacksquare

We break the remainder of the analysis into two cases depending on whether there are a finite or an infinite number of modifications of the Lagrange multiplier estimate.

B.2 A finite number of multiplier updates

In this section, we suppose that the set \mathcal{Y} in (B2) is finite in that the counter j in Algorithm 3 satisfies

$$j \in \{1, 2, \dots, \bar{j}\} \text{ for some finite } \bar{j}. \quad (\text{B12})$$

This allows us to define, and consequently use in our analysis, the quantities

$$t := t_{\bar{j}} > 0 \text{ and } T := T_{\bar{j}} > 0. \quad (\text{B13})$$

We provide two lemmas in this subsection. The first considers cases when the penalty parameter converges to zero, and the second considers cases when the penalty parameter remains bounded away from zero. This first case—in which the multiplier estimate is only modified a finite number of times and the penalty parameter vanishes—may be expected to occur when (1) is infeasible. Indeed, in this case, we show that every limit point of the primal iterate sequence is an infeasible stationary point.

LEMMA B.3 *If $|\mathcal{Y}| < \infty$ and $\mu_k \rightarrow 0$, then there exist a vector y and integer $\bar{k} \geq 0$ such that*

$$y_k = y \text{ for all } k \geq \bar{k}, \quad (\text{B14})$$

and for some constant $\bar{c} > 0$, we have the limits

$$\lim_{k \rightarrow \infty} \|c_k\|_2 = \bar{c} > 0 \text{ and } \lim_{k \rightarrow \infty} F_{\text{FEAS}}(x_k) = 0. \quad (\text{B15})$$

Therefore, every limit point of $\{x_k\}_{k \geq 0}$ is an infeasible stationary point.

Proof. It follows from (B12), (B13), and the manner in which the multiplier estimates are updated in Algorithm 3 that there exists y and a scalar $\bar{k} \geq k_{\bar{j}}$ such that (B14) holds. Thus, all that remains is to prove that (B15) holds for some $\bar{c} > 0$.

From (B9) and the supposition that $\mu_k \rightarrow 0$, it follows that $\|c_k\|_2 \rightarrow \bar{c}$ for some $\bar{c} \geq 0$. If $\bar{c} = 0$, then by Assumption 2.3, (B14), and the fact that $\mu_k \rightarrow 0$ it follows that $\lim_{k \rightarrow \infty} \nabla_x \mathcal{L}(x_k, y, \mu_k) = \lim_{k \rightarrow \infty} J_k^T c_k = 0$, which implies that $\lim_{k \rightarrow \infty} F_{\text{AL}}(x_k, y, \mu_k) = \lim_{k \rightarrow \infty} F_{\text{FEAS}}(x_k) = 0$. This would imply that for some $k \geq \bar{k}$ the algorithm would set

$j \leftarrow \bar{j} + 1$, thus violating (B12). Consequently, we may conclude that $\bar{c} > 0$, which proves the first limit in (B15). Now, to reach a contradiction to the second limit in (B15), suppose that $F_{\text{FEAS}}(x_k) \rightarrow 0$. This, together with Assumption 2.3, (B14), and the supposition that $\mu_k \rightarrow 0$, implies that there exist a positive constant ϵ and an infinite index set \mathcal{K} such that

$$\|F_{\text{AL}}(x_k, y_k, \mu_k)\|_2 \geq \epsilon \text{ for all } k \in \mathcal{K}. \quad (\text{B16})$$

It follows from (17), (16a), (A11), (B16), and Lemma B.2 that for all $k \in \mathcal{K}$ we have

$$\begin{aligned} \mathcal{L}(x_{k+1}, y_k, \mu_k) &= \mathcal{L}(x_k + \alpha_k s_k, y_k, \mu_k) \\ &\leq \mathcal{L}(x_k, y_k, \mu_k) - \eta_s \alpha_k \Delta \tilde{q}(s_k; x_k, y_k, \mu_k) \\ &\leq \mathcal{L}(x_k, y_k, \mu_k) - \eta_s \alpha_k \kappa_1 \Delta \tilde{q}(\bar{s}_k; x_k, y_k, \mu_k) \\ &\leq \mathcal{L}(x_k, y_k, \mu_k) - \eta_s \alpha_k \kappa_1 \kappa_5 \|F_{\text{AL}}(x_k, y_k, \mu_k)\|_2^2 \min \left\{ \delta, \frac{1}{1 + \Omega_{\bar{j}}} \right\} \\ &\leq \mathcal{L}(x_k, y_k, \mu_k) - \eta_s C_{\bar{j}} \kappa_1 \kappa_5 \epsilon^2 \min \left\{ \delta, \frac{1}{1 + \Omega_{\bar{j}}} \right\}. \end{aligned}$$

This implies that, for all $k \in \mathcal{K}$, the reduction $\mathcal{L}(x_k, y_k, \mu_k) - \mathcal{L}(x_{k+1}, y_k, \mu_k)$ is greater than or equal to a positive constant. In the meantime, we know from Lemma A.6 and the way we update x_k at each iteration that $\mathcal{L}(x_k, y_k, \mu_k) - \mathcal{L}(x_{k+1}, y_k, \mu_k) \geq 0$ for all k . Therefore, we have reached a contradiction to (B8). This implies that our supposition that $F_{\text{FEAS}}(x_k) \rightarrow 0$ cannot be true, so we have (B15). \blacksquare

The next lemma considers the case when μ stays bounded away from zero. This is possible, for example, if the algorithm converges to an infeasible stationary point that is stationary for the AL function for the final Lagrange multiplier estimate and penalty parameter computed in the algorithm.

LEMMA B.4 *If $|\mathcal{Y}| < \infty$ and $\mu_k = \mu$ for some $\mu > 0$ for all sufficiently large k , then with t defined in (B13) there exist a vector y and integer $\bar{k} \geq 0$ such that*

$$y_k = y \text{ and } \|c_k\|_2 \geq t \text{ for all } k \geq \bar{k}, \quad (\text{B17})$$

and we have the limit

$$\lim_{k \rightarrow \infty} F_{\text{FEAS}}(x_k) = 0. \quad (\text{B18})$$

Therefore, every limit point of $\{x_k\}_{k \geq 0}$ is an infeasible stationary point.

Proof. Since $|\mathcal{Y}| < \infty$, we know that (B12) and (B13) hold for some $\bar{j} \geq 0$, and since we suppose that $\mu_k = \mu > 0$ for all sufficiently large k , it follows by the mechanisms for updating the Lagrange multiplier estimates in Algorithm 3 that there exists y and a scalar $k' \geq k_{\bar{j}}$ such that

$$\mu_k = \mu \text{ and } y_k = y \text{ for all } k \geq k'. \quad (\text{B19})$$

Our next goal is to prove that

$$\lim_{k \rightarrow \infty} \|F_{\text{AL}}(x_k, y, \mu)\|_2 = 0. \quad (\text{B20})$$

Indeed, to reach a contradiction, suppose that (B20) does not hold. It then follows that there exist a positive number ζ and an infinite index set \mathcal{K}' with all elements greater than or equal to k' such that

$$\|F_{\text{AL}}(x_k, y, \mu)\|_2 \geq \zeta \text{ for all } k \in \mathcal{K}'. \quad (\text{B21})$$

Similar to the proof of Lemma B.3, it then follows from (B21), (17), (16a), (A11), and Lemma B.2 that for all $k \in \mathcal{K}'$ we have

$$\begin{aligned} \mathcal{L}(x_{k+1}, y, \mu) &= \mathcal{L}(x_k + \alpha_k s_k, y, \mu) \\ &\leq \mathcal{L}(x_k, y, \mu) - \eta_s \alpha_k \Delta \tilde{q}(s_k; x_k, y, \mu) \\ &\leq \mathcal{L}(x_k, y, \mu) - \eta_s \alpha_k \kappa_1 \Delta \tilde{q}(\bar{s}_k; x_k, y, \mu) \\ &\leq \mathcal{L}(x_k, y, \mu) - \eta_s \alpha_k \kappa_1 \kappa_5 \|F_{\text{AL}}(x_k, y, \mu)\|_2^2 \min \left\{ \delta, \frac{1}{1 + \Omega_{\bar{j}}} \right\} \\ &\leq \mathcal{L}(x_k, y, \mu) - \eta_s C_{\bar{j}} \kappa_1 \kappa_5 \zeta^2 \min \left\{ \delta, \frac{1}{1 + \Omega_{\bar{j}}} \right\}. \end{aligned}$$

This implies that, for all $k \in \mathcal{K}'$, the reduction $\mathcal{L}(x_k, y, \mu) - \mathcal{L}(x_{k+1}, y, \mu)$ is greater than or equal to a positive constant. However, we know from Assumption 2.3 that $\mathcal{L}(x_k, y, \mu)$ is bounded below. Therefore, we have reached a contradiction, so (B20) must hold.

The first consequence of (B20) is that it allows us to prove (B17). Indeed, it follows that there exists $\bar{k} \geq k'$ such that $\|c_k\|_2 \geq t$ for all $k \geq \bar{k}$, since otherwise (B20) would imply that, for some $k \geq \bar{k}$, Algorithm 3 would set $j \leftarrow \bar{j} + 1$, which violates (B12). Thus, along with (B19), we have proved (B17).

The second consequence of (B20) is that it allows us to prove (B18), which is all that remains to complete the proof of the lemma. It follows from (16a), (B20), and part (i) of Lemma A.2 that

$$\lim_{k \rightarrow \infty} \|s_k\|_2 \leq \lim_{k \rightarrow \infty} \Theta_k = \lim_{k \rightarrow \infty} \Gamma_k \delta \|F_{\text{AL}}(x_k, y, \mu)\|_2 = 0. \quad (\text{B22})$$

Furthermore, from (B22) and Assumption 2.3, we have

$$\lim_{k \rightarrow \infty} \Delta q_v(s_k; x_k) = 0, \quad (\text{B23})$$

and, along with (B13) and (B17), we have

$$v_k - \frac{1}{2}(\kappa_t t_{\bar{j}})^2 \geq \frac{1}{2}t^2 - \frac{1}{2}(\kappa_t t)^2 = \frac{1}{2}(1 - \kappa_t^2)t^2 > 0 \text{ for all } k \geq \bar{k}. \quad (\text{B24})$$

We may use these facts to prove $F_{\text{FEAS}}(x_k) \rightarrow 0$. In particular, in order to derive a contradiction, suppose that $F_{\text{FEAS}}(x_k) \not\rightarrow 0$. Then, there exist a positive number ξ and an infinite index set \mathcal{K}'' such that

$$\|F_{\text{FEAS}}(x_k)\|_2 \geq \xi \text{ for all } k \in \mathcal{K}''. \quad (\text{B25})$$

Using (16b), (A10), (B1c), and (B12), we then find for $k \in \mathcal{K}''$ that

$$\Delta q_v(r_k; x_k) \geq \kappa_2 \Delta q_v(\bar{r}_k; x_k) \geq \kappa_2 \kappa_4 \xi^2 \min \left\{ \frac{1}{1 + \Omega_{\bar{j}}}, \delta \right\} =: \zeta' > 0. \quad (\text{B26})$$

We may now combine (B26), (B24), and (B23) to state that (16c) must be violated for sufficiently large $k \in \mathcal{K}''$ and, consequently, the penalty parameter will be decreased. However, this is a contradiction to (B19), so we conclude that $F_{\text{FEAS}}(x_k) \rightarrow 0$. The fact that every limit point of $\{x_k\}_{k \geq 0}$ is an infeasible stationary point follows since $\|c_k\|_2 \geq t$ for all $k \geq \bar{k}$ from (B17) and $F_{\text{FEAS}}(x_k) \rightarrow 0$. ■

This completes the analysis for the case that the set \mathcal{Y} is finite.

B.3 An infinite number of multiplier updates

We now suppose that $|\mathcal{Y}| = \infty$. In this case, it follows from the procedures for updating the Lagrange multiplier estimate and target values in Algorithm 3 that

$$\lim_{j \rightarrow \infty} t_j = \lim_{j \rightarrow \infty} T_j = 0. \quad (\text{B27})$$

As in the previous subsection, we split the analysis in this subsection into two results. This time, we begin by considering the case when the penalty parameter remains bounded below and away from zero. In this scenario, we state the following result that a subsequence of the iterates converges to a first-order stationary point. The proof of the corresponding result in [15] applies for Algorithm 3, so we do not provide it here for the sake of brevity; the proof is relatively straightforward, essentially relying on the fact that (B27) squeezes the constraint violation and stationarity measure error to zero to yield (B28).

LEMMA B.5 ([15, Lemma 3.10].) *If $|\mathcal{Y}| = \infty$ and $\mu_k = \mu$ for some $\mu > 0$ for all sufficiently large k , then*

$$\lim_{j \rightarrow \infty} c_{k_j} = 0 \quad (\text{B28a})$$

$$\text{and } \lim_{j \rightarrow \infty} F_L(x_{k_j}, \hat{y}_{k_j}) = 0. \quad (\text{B28b})$$

Thus, any limit point (x_, y_*) of $\{(x_{k_j}, \hat{y}_{k_j})\}_{j \geq 0}$ is first-order stationary for (1).*

Finally, we consider the case when the penalty parameter converges to zero. Again, we do not provide a proof of the following lemma since that of the corresponding proof in [15] suffices here as well.

LEMMA B.6 ([15, Lemma 3.13]) *If $|\mathcal{Y}| = \infty$ and $\mu_k \rightarrow 0$, then*

$$\lim_{k \rightarrow \infty} c_k = 0. \quad (\text{B29})$$

If, in addition, there exists a positive integer p such that $\mu_{k_j-1} \geq \gamma_\mu^p \mu_{k_j-1-1}$ for all sufficiently large j , then there exists an infinite ordered set $\mathcal{J} \subseteq \mathbb{N}$ such that

$$\lim_{j \in \mathcal{J}, j \rightarrow \infty} \|F_L(x_{k_j}, \hat{y}_{k_j})\|_2 = 0 \quad \text{or} \quad \lim_{j \in \mathcal{J}, j \rightarrow \infty} \|F_L(x_{k_j}, \pi(x_{k_j}, y_{k_j-1}, \mu_{k_j-1}))\|_2 = 0. \quad (\text{B30})$$

In such cases, if the first (respectively, second) limit in (B30) holds, then along with (B29) it follows that any limit point of $\{(x_{k_j}, \hat{y}_{k_j})\}_{j \in \mathcal{J}}$ (respectively, $\{(x_{k_j}, y_{k_j-1})\}_{j \in \mathcal{J}}$) is a first-order stationary point for (1).

B.4 Proof of global convergence result

Proof of Theorem 2.4. Lemmas B.3, B.4, B.5 and B.6 cover the only four possible outcomes of Algorithm 3; the result follows from those described in these lemmas. ■

Appendix C. Numerical Results

In this appendix, we provide detailed results of our experiments described in §3. The problems listed in the tables in this appendix are only those that were used in the performance data provided in §3. Also, note that we provide problem size information for the reformulated model, i.e., that resulting after transformations were employed to create a model of the form (1). For example, a general inequality constraint with lower and upper bounds would have been augmented with two auxiliary variables to form two equality constraints with lower bounds on the auxiliary variables.

In Tables C1–C6 for our MATLAB software, we indicate the name (Name) along with the numbers of variables (n), equality constraints (m_e), and bound constraints (m_b) of each problem solved. Then, for each algorithm, we indicate the termination flag (Flag) along with the numbers of iterations (Iter.), function evaluations (Func.), and gradient evaluations (Grad.) required before termination. The flags indicate whether a first-order stationary point was found (Opt.), an infeasible stationary point was found (Inf.), the iteration limit was reached (Itr.), or the time limit was reached (Time).

Table C1.: MATLAB line search algorithms, results on CUTESt problems

Name	n	m_e	m_b	BAL-LS			AAL-LS				AAL-LS-safe				
				Flag	Iter.	Func.	Grad.	Flag	Iter.	Func.	Grad.	Flag	Iter.	Func.	Grad.
ACOPP30	154	142	166	Itr.	10000	10032	10002	Itr.	10000	10011	10002	Itr.	10000	10009	10002
ACOPR14	106	96	88	Opt.	5976	5998	5978	Opt.	72	78	74	Opt.	67	74	69
AIRCRAFT	5	5	0	Opt.	151	152	153	Opt.	151	152	153	Opt.	151	152	153
AIRPORT	126	42	210	Opt.	176	226	178	Opt.	64	80	66	Opt.	113	129	115
ALSOAME	2	1	4	Opt.	10	11	12	Opt.	9	10	11	Opt.	9	10	11
ANTWERP	29	10	53	Opt.	215	216	217	Opt.	54	55	56	Opt.	54	55	56
ARGAUSS	3	15	0	Inf.	158	159	160	Inf.	158	159	160	Inf.	158	159	160
AVGASA	18	10	26	Opt.	190	191	192	Opt.	163	164	165	Opt.	163	164	165
AVGASB	18	10	26	Opt.	104	105	106	Opt.	189	190	191	Opt.	189	190	191
BATCH	109	73	155	Opt.	116	138	118	Opt.	101	111	103	Opt.	101	111	103
BIGGSC4	17	13	21	Opt.	136	137	138	Opt.	32	33	34	Opt.	32	33	34
BOOTH	2	2	0	Opt.	1	2	3	Opt.	1	2	3	Opt.	1	2	3
BT1	2	1	0	Opt.	689	690	691	Opt.	29	30	31	Opt.	29	30	31
BT10	2	2	0	Opt.	140	142	142	Opt.	49	50	51	Opt.	49	50	51
BT11	5	3	0	Opt.	103	104	105	Opt.	37	38	39	Opt.	37	38	39
BT12	5	3	0	Opt.	11	12	13	Opt.	8	9	10	Opt.	8	9	10
BT13	5	1	1	Opt.	9	10	11	Opt.	774	1758	776	Itr.	10000	10984	10002
BT2	3	1	0	Opt.	96	97	98	Opt.	10	11	12	Opt.	10	11	12
BT3	5	3	0	Opt.	19	20	21	Opt.	20	21	22	Opt.	20	21	22
BT4	3	2	0	Opt.	26	27	28	Opt.	12	13	14	Opt.	12	13	14
BT5	3	2	0	Opt.	6	10	8	Opt.	5	9	7	Opt.	5	9	7
BT6	5	2	0	Opt.	11	13	13	Opt.	19	21	21	Opt.	19	21	21
BT7	5	3	0	Opt.	138	157	140	Opt.	53	71	55	Opt.	56	74	58
BT8	5	2	0	Itr.	10000	10002	10002	Opt.	23	27	25	Opt.	23	27	25
BT9	4	2	0	Opt.	62	85	64	Opt.	63	86	65	Opt.	63	86	65
BURKEHAN	2	1	2	Inf.	0	1	2	Inf.	0	1	2	Inf.	0	1	2
BYRDSPHR	3	2	0	Opt.	28	78	30	Opt.	34	84	36	Opt.	34	84	36
C-RELOAD	426	284	684	Opt.	110	111	112	Opt.	40	41	42	Opt.	40	41	42
CANTILVR	6	1	6	Opt.	201	207	203	Opt.	372	379	374	Opt.	372	379	374
CB2	6	3	3	Opt.	30	42	32	Opt.	20	32	22	Opt.	20	32	22
CB3	6	3	3	Opt.	26	27	28	Opt.	24	25	26	Opt.	24	25	26
CHACONN1	6	3	3	Opt.	22	27	24	Opt.	22	27	24	Opt.	22	27	24
CHACONN2	6	3	3	Opt.	33	34	35	Opt.	24	25	26	Opt.	24	25	26
CLUSTER	2	2	0	Opt.	3125	3126	3127	Opt.	3125	3126	3127	Opt.	3125	3126	3127
CONGIGMZ	8	5	5	Opt.	5330	5342	5332	Opt.	133	146	135	Opt.	133	146	135
COOLHANS	9	9	0	Itr.	10000	10001	10002	Itr.	10000	10001	10002	Itr.	10000	10001	10002
CRESC100	206	200	205	Inf.	36	56	38	Inf.	24	25	26	Inf.	24	25	26
CRESC4	14	8	13	Opt.	88	96	90	Opt.	20	23	22	Opt.	8186	8193	8188
CRESC50	106	100	105	Inf.	41	57	43	Inf.	31	32	33	Inf.	31	32	33
CSF1	8	5	9	Opt.	1173	1233	1175	Opt.	59	94	61	Opt.	59	94	61
CSF2	8	5	8	Opt.	1165	1216	1167	Opt.	9179	9234	9181	Itr.	10000	10055	10002
CUBENE	2	2	0	Opt.	8	14	10	Opt.	8	14	10	Opt.	8	14	10

STATIC3	434	96	144	Itr.	10000	10001	10002	Opt.	21	22	23	Itr.	10000	10001	10002
STEENBRA	432	108	432	Opt.	14	15	16	Opt.	11	12	13	Opt.	11	12	13
STEENBRB	468	108	468	Opt.	160	161	63	Opt.	18	19	14	Opt.	18	19	14
STEENBRC	540	126	540	Opt.	284	285	164	Opt.	32	33	17	Opt.	32	33	17
STEENBRD	468	108	468	Opt.	297	298	148	Opt.	33	34	20	Opt.	33	34	20
STEENBRE	540	126	540	Opt.	270	271	186	Opt.	67	68	19	Opt.	67	68	19
STEENBRF	468	108	468	Opt.	163	164	97	Opt.	15	16	11	Opt.	15	16	11
STEENBRG	540	126	540	Opt.	329	330	261	Opt.	20	21	16	Opt.	20	21	16
SUPERSIM	2	2	1	Opt.	2	3	4	Opt.	2	3	4	Opt.	2	3	4
SWOPF	97	92	34	Opt.	440	441	406	Opt.	292	293	228	Opt.	292	293	228
SYNTHES1	12	6	18	Opt.	54	55	56	Opt.	36	37	38	Opt.	36	37	38
SYNTHES2	25	15	34	Opt.	371	372	373	Opt.	416	417	418	Opt.	432	433	434
SYNTHES3	38	23	55	Opt.	6669	6670	6671	Opt.	7314	7315	7316	Opt.	7314	7315	7316
TABLE7	624	230	1108	Itr.	10000	10001	1028	Opt.	33	34	35	Opt.	33	34	35
TAME	2	1	2	Opt.	1	2	3	Opt.	1	2	3	Opt.	1	2	3
TARGUS	162	63	277	Itr.	10000	10001	10002	Opt.	35	36	37	Opt.	35	36	37
TENBARS1	19	9	15	Itr.	10000	10001	9937	Opt.	186	187	143	Opt.	186	187	143
TENBARS3	18	8	12	Opt.	107	108	73	Itr.	10000	10001	76	Itr.	10000	10001	76
TENBARS4	19	9	11	Itr.	10000	10001	961	Opt.	642	643	494	Itr.	10000	10001	9852
TRIGGER	6	6	0	Opt.	9	10	9	Opt.	8	9	8	Opt.	8	9	8
TRIMLOSS	197	75	319	Opt.	151	152	112	Opt.	89	90	75	Opt.	89	90	75
TRO6X2	46	21	26	Itr.	10000	10001	9751	Inf.	147	148	92	Itr.	10000	10001	9945
TRUSPYR1	12	4	9	Itr.	10000	10001	9961	Opt.	1005	1006	934	Opt.	1013	1014	942
TRY-B	2	1	2	Opt.	10	11	12	Opt.	18	19	17	Opt.	18	19	17
TWOBARS	4	2	6	Opt.	54	55	50	Opt.	50	51	47	Opt.	50	51	47
WACHBIEG	3	2	2	Opt.	24	25	26	Opt.	22	23	22	Opt.	22	23	22
WATER	31	10	62	Opt.	9	10	11	Opt.	8	9	10	Opt.	8	9	10
WOMFLET	6	3	3	Opt.	8	9	9	Opt.	41	42	29	Opt.	41	42	29
YFITNE	3	17	0	Opt.	58	59	56	Itr.	10000	10001	9996	Itr.	10000	10001	9996
YORKNET	312	256	288	Opt.	149	150	123	Opt.	135	136	106	Inf.	305	306	276
ZAMB2-10	264	96	528	Opt.	47	48	44	Opt.	43	44	45	Opt.	43	44	45
ZAMB2-11	264	96	528	Opt.	17	18	19	Opt.	17	18	19	Opt.	17	18	19
ZAMB2-8	132	48	264	Opt.	17	18	19	Opt.	26	27	28	Opt.	26	27	28
ZAMB2-9	132	48	264	Opt.	2297	2298	2299	Itr.	10000	10001	10002	Itr.	10000	10001	10002
ZANGWIL3	3	3	0	Opt.	4	5	6	Opt.	5	6	7	Opt.	5	6	7
ZECEVIC2	4	2	6	Opt.	5	6	7	Opt.	5	6	7	Opt.	5	6	7
ZECEVIC3	4	2	6	Opt.	69	70	64	Opt.	47	48	44	Opt.	47	48	44
ZECEVIC4	4	2	6	Opt.	12	13	14	Opt.	15	16	17	Opt.	15	16	17
ZY2	5	2	6	Opt.	8	9	10	Opt.	8	9	10	Opt.	8	9	10

Table C3.: MATLAB line search algorithms, results on COPS problems

Name	n	m_e	m_b	BAL-LS			AAL-LS				
				Flag	Iter.	Func.	Grad.	Flag	Iter.	Func.	Grad.
bearing1	2500	0	2500	Opt.	4	5	6	Opt.	4	5	6
camshape1	3198	2398	3998	Opt.	30	31	32	Opt.	24	25	26
catmix1	1098	798	600	Opt.	366	672	368	Itr.	10000	10201	10002
channell	1598	1598	0	Opt.	357	690	359	Opt.	357	690	359
elec1	150	50	0	Opt.	65	119	67	Opt.	200	205	202
gasoil1	1001	998	3	Opt.	223	441	225	Opt.	66	86	68
glider1	499	400	402	Opt.	75	107	77	Opt.	138	221	140
marinel	1615	1592	15	Opt.	124	126	126	Opt.	47	48	49
methanol1	1202	1197	5	Opt.	123	226	125	Opt.	57	101	59
minsurl	2500	0	2500	Opt.	7	8	9	Opt.	7	8	9
pinene1	2000	1995	5	Inf.	45	46	47	Inf.	34	35	36
polygon1	1371	1273	1469	Time	627	691	629	Opt.	54	83	56
rocket1	1601	1200	2401	Opt.	37	39	39	Opt.	37	39	39
steering1	999	800	403	Opt.	28	29	30	Opt.	27	28	29
tetral	3693	2826	2927	Opt.	5	6	7	Opt.	5	6	7
torsion1	2500	0	5000	Opt.	4	5	6	Opt.	4	5	6
triangle1	3317	1797	1802	Opt.	5	6	7	Opt.	5	6	7

Table C4.: MATLAB trust region algorithms, results on COPS problems

Name	n	m_e	m_b	BAL-TR			AAL-TR				
				Flag	Iter.	Func.	Grad.	Flag	Iter.	Func.	Grad.
bearing1	2500	0	2500	Opt.	3	4	5	Opt.	3	4	5
camshape1	3198	2398	3998	Inf.	4726	4727	4728	Opt.	16	17	18
catmix1	1098	798	600	Inf.	745	746	496	Time	8021	8022	7908
channell	1598	1598	0	Opt.	610	611	402	Opt.	573	574	393
elec1	150	50	0	Opt.	108	109	75	Opt.	80	81	59
gasoil1	1001	998	3	Opt.	193	194	163	Opt.	119	120	94
glider1	499	400	402	Opt.	140	141	103	Opt.	113	114	85
marinel	1615	1592	15	Opt.	144	145	144	Opt.	56	57	58
methanol1	1202	1197	5	Opt.	366	367	174	Opt.	107	108	67
minsurl	2500	0	2500	Opt.	14	15	11	Opt.	14	15	11
pinene1	2000	1995	5	Inf.	37	38	39	Inf.	39	40	41
polygon1	1371	1273	1469	Opt.	125	126	86	Opt.	43	44	32
rocket1	1601	1200	2401	Opt.	49	50	45	Opt.	49	50	45
steering1	999	800	403	Opt.	99	100	80	Opt.	55	56	43
tetral	3693	2826	2927	Opt.	5	6	7	Opt.	5	6	7
torsion1	2500	0	5000	Opt.	2	3	4	Opt.	2	3	4
triangle1	3317	1797	1802	Opt.	5	6	7	Opt.	5	6	7

Table C5.: MATLAB line search algorithms, results on OPF problems

Name	n	m_e	m_b	BAL-LS				AAL-LS			
				Flag	Iter.	Func.	Grad.	Flag	Iter.	Func.	Grad.
OPFapproxIEEE014	76	68	84	Opt.	16	19	18	Opt.	16	19	18
OPFcartIEEE014	108	100	98	Opt.	13	14	15	Opt.	12	13	14
OPFdcIEEE014	49	49	40	Opt.	2	3	4	Opt.	2	3	4
OPFpolarIEEE014	76	68	84	Opt.	14	15	16	Opt.	15	16	17
OPFapproxIEEE030	151	142	160	Opt.	22	23	24	Opt.	18	19	20
OPFcartIEEE030	215	206	190	Opt.	34	36	36	Opt.	30	32	32
OPFdcIEEE030	107	107	82	Opt.	4	5	6	Opt.	4	5	6
OPFpolarIEEE030	151	142	160	Opt.	35	37	37	Opt.	36	38	38
OPFapproxIEEE057	288	267	312	Opt.	24	29	26	Opt.	21	25	23
OPFcartIEEE057	410	389	369	Opt.	29	32	31	Opt.	31	34	33
OPFdcIEEE057	204	204	156	Opt.	6	7	8	Opt.	6	7	8
OPFpolarIEEE057	288	267	312	Opt.	38	55	40	Opt.	40	46	42
OPFapproxIEEE118	696	634	718	Opt.	15	16	17	Opt.	21	22	23
OPFcartIEEE118	944	882	836	Opt.	30	31	32	Opt.	27	28	29
OPFdcIEEE118	463	463	358	Opt.	3	4	5	Opt.	3	4	5
OPFpolarIEEE118	696	634	718	Opt.	24	25	26	Opt.	31	32	33
OPFapproxIEEE162	959	901	1008	Opt.	50	85	52	Opt.	51	86	53
OPFcartIEEE162	1285	1227	1170	Opt.	99	116	101	Opt.	88	94	90
OPFdcIEEE162	727	727	568	Opt.	4	5	6	Opt.	4	5	6
OPFpolarIEEE162	959	901	1008	Opt.	143	201	145	Opt.	72	90	74
OPFapproxIEEE300	1661	1487	1770	Opt.	73	94	75	Opt.	60	79	62
OPFcartIEEE300	2263	2089	2070	Opt.	91	125	93	Opt.	96	138	98
OPFdcIEEE300	1119	1119	822	Opt.	19	20	21	Opt.	19	20	21
OPFpolarIEEE300	1661	1487	1770	Opt.	78	101	80	Opt.	76	91	78
OPFapproxIEEE662	3665	3427	3834	Opt.	105	134	107	Opt.	52	80	54
OPFcartIEEE662	4991	4753	4496	Opt.	72	76	74	Opt.	62	66	64
OPFdcIEEE662	2691	2691	2032	Opt.	10	11	12	Opt.	10	11	12
OPFpolarIEEE662	3665	3427	3834	Opt.	37	39	39	Opt.	34	36	36

Table C6.: MATLAB trust region algorithms, results on OPF problems

Name	n	m_e	m_b	BAL-TR				AAL-TR			
				Flag	Iter.	Func.	Grad.	Flag	Iter.	Func.	Grad.
OPFapproxIEEE014	76	68	84	Opt.	24	25	22	Opt.	41	42	31
OPFcartIEEE014	108	100	98	Opt.	23	24	21	Opt.	21	22	23
OPFdcIEEE014	49	49	40	Opt.	2	3	4	Opt.	2	3	4
OPFpolarIEEE014	76	68	84	Opt.	14	15	16	Opt.	22	23	21
OPFapproxIEEE030	151	142	160	Opt.	22	23	24	Opt.	18	19	20
OPFcartIEEE030	215	206	190	Opt.	34	35	33	Opt.	35	36	34
OPFdcIEEE030	107	107	82	Opt.	3	4	5	Opt.	3	4	5
OPFpolarIEEE030	151	142	160	Opt.	30	31	30	Opt.	44	45	41
OPFapproxIEEE057	288	267	312	Opt.	36	37	28	Opt.	31	32	29
OPFcartIEEE057	410	389	369	Opt.	34	35	31	Opt.	41	42	35
OPFdcIEEE057	204	204	156	Opt.	2	3	4	Opt.	2	3	4
OPFpolarIEEE057	288	267	312	Opt.	43	44	40	Opt.	35	36	33
OPFapproxIEEE118	696	634	718	Opt.	15	16	17	Opt.	12	13	14
OPFcartIEEE118	944	882	836	Opt.	34	35	36	Opt.	35	36	37
OPFdcIEEE118	463	463	358	Opt.	2	3	4	Opt.	2	3	4
OPFpolarIEEE118	696	634	718	Opt.	27	28	29	Opt.	31	32	33
OPFapproxIEEE162	959	901	1008	Opt.	75	76	58	Opt.	76	77	61
OPFcartIEEE162	1285	1227	1170	Opt.	183	184	144	Opt.	109	110	95
OPFdcIEEE162	727	727	568	Opt.	2	3	4	Opt.	2	3	4
OPFpolarIEEE162	959	901	1008	Opt.	162	163	120	Opt.	93	94	67
OPFapproxIEEE300	1661	1487	1770	Opt.	94	95	74	Opt.	78	79	70
OPFcartIEEE300	2263	2089	2070	Opt.	131	132	91	Opt.	146	147	104
OPFdcIEEE300	1119	1119	822	Opt.	14	15	16	Opt.	14	15	16
OPFpolarIEEE300	1661	1487	1770	Opt.	87	88	68	Opt.	85	86	73
OPFapproxIEEE662	3665	3427	3834	Opt.	106	107	95	Opt.	64	65	53
OPFcartIEEE662	4991	4753	4496	Opt.	60	61	62	Opt.	55	56	57
OPFdcIEEE662	2691	2691	2032	Opt.	5	6	7	Opt.	5	6	7
OPFpolarIEEE662	3665	3427	3834	Opt.	113	114	98	Opt.	47	48	42

In Table C7 for our LANCELOT software, we indicate the name (Name) along with the numbers of variables (n), equality constraints (m_e), and bound constraints (m_b) of each problem solved. Then, for each algorithm, we indicate the termination flag (Flag) along with the numbers of iterations (Iter.), gradient evaluations (Grad.), and time (Time) required before termination. The flags indicate whether a first-order stationary point was found (Opt.), the problem was suspected to be infeasible (Inf.), the iteration limit was reached (Itr.), or the algorithm determined that no more progress could be made. In this last situation, we verified whether the final point was approximately stationary based on a loosened threshold criteria (Thr.) or not (Ter.).

