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**Testing for Unit Root against LSTAR Model:
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Keywords:	Unit root Test, Dickey-Fuller test, STAR model, GARCH (1, 1), Wavelet method, MODWT
Abstract:	In this paper, we propose a Nonlinear Dickey-Fuller (NDF) test for unit root against LSTAR (1) model with time as the transition variable. The asymptotic distribution of the test is analytically derived while the small sample distributions are investigated by Monte Carlo experiment. The results have shown that there is a serious size distortion for the NDF test under GARCH errors which lead to an over-rejection of the unit root null hypothesis. Wavelet technique has been used to improve the size property and an empirical example is used to compare our test with the traditional Dickey-Fuller test.



Testing for Unit Root against LSTAR Model: Wavelet Improvement under GARCH Distortion

By

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Abstract

In this paper, we propose a Nonlinear Dickey-Fuller F test for unit root against first order Logistic Smooth Transition Autoregressive LSTAR (1) model with time as the transition variable. The Nonlinear Dickey-Fuller F test statistic is established under the null hypothesis of random walk without drift and the alternative model is a nonlinear LSTAR (1) model. The asymptotic distribution of the test is analytically derived while the small sample distributions are investigated by Monte Carlo experiment. The size and power properties of the test have been investigated using Monte Carlo experiment. The results have shown that there is a serious size distortion for the test when GARCH errors appear in the Data Generating Process (DGP), which lead to an over-rejection of the unit root null hypothesis. To solve this problem, we use the Wavelet technique to count off the GARCH distortion and to improve the size property of the test under GARCH error. We also discuss the asymptotic distributions of the test statistics in GARCH and wavelet environments.

Keywords: Unit root Test, Dickey-Fuller F test, STAR model, GARCH (1, 1), Wavelet method, MODWT

I. Introduction

Empirical studies show that many economic variables display nonlinear features where the economic behaviors change when certain variables lie in different regions. To capture such features, several nonlinear models have been introduced. Among them, STAR models allow nonlinear structures between the data regimes to be described with a smooth regime transition function. They are of particular interest in

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4 macroeconomics which always contains mass of economic agents, where even if the
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6 decisions are made discretely, the aggregated behaviors will show smooth regime
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8 changes (see Teräsvirta, 1994). However, before applying nonlinear models such as
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10 STAR models, testing linearity against nonlinearity is essential, and unit root tests
11
12 against nonlinear model need further cautious consideration. In STAR models, Eklund
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14 (2003) proposed unit root tests against LSTAR with transition variables being the
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16 lagged dependent variables. Later, He and Sandberg (2006) proposed the nonlinear
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18 Dickey-Fuller ρ and t test statistics with time as the transition variable. In this paper
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20 we will first derive Nonlinear Dickey-Fuller F test of unit root against LSTAR models
21
22 with time as the transition variable.
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26 We next investigate the size property of the Nonlinear Dickey-Fuller F test when the
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28 error in the DGP shows conditional heteroskedasticity as a GARCH(1,1) model. As
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30 ARCH/GARCH models are always employed to model the conditional variance
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32 without paying enough attention to the specification of the conditional mean, and that
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34 the misspecification may lead to further inconsistent estimates, specification tests
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36 including unit root test under ARCH/GARCH error has attracted much attention. Peter
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38 and Veloce (1988), Kim and Schmidt (1993), Cook (2006) showed that Dickey-Fuller
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40 test is generally not robust in the near integrated situation in GARCH error, especially
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42 when the GARCH process exhibits a high degree of volatility. Therefore, to improve
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44 the test property, numerous studies pay attention to deriving unit root test based on
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46 Maximum Likelihood Estimation (MLE), which jointly estimates the parameters of
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48 unit root model and the GARCH error model. (see Seo,1999; Ling and Li, 1998).
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50 However, the MLE is not a perfect solution to the GARCH error problem. Charles
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52 and Darné (2008) re-examined Seo's test which is based on MLE and showed that the
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54 empirical size and power of the Dickey-Fuller test is generally better than Seo's test
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56 when the GARCH paramter β is superior to ARCH parameter α (The definition of
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58 β and α please refer to Section IV). Moreover, in our LSTAR model, if we use MLE
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60 method, the estimated dimensional parameter space is larger than the linear case and it

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4 will be numerically quite complicated to obtain. Thus in this paper, we consider an
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6 alternative to MLE method to improve the unit root test under GARCH error.
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10 We apply the wavelet method, which has been widely used after its theoretic
11 foundation in 1980s (see Grossmann and Morelet, 1984 and Mallat, 1989). In
12 economics, Schleicher (2002) mentioned that since economic behaviors take place at
13 different frequencies, the wavelet method can catch landscape characteristics in
14 addition to the microscopic detail in economic areas. In this paper, we use the wavelet
15 method to count off the finest local behavior of the series in the form of conditional
16 heteroskedasticity in GARCH errors, whose information is caught by the highest scale
17 in wavelet coefficients. The same logic can be found in Schleicher (2002), who
18 pointed out that lower scales hold most of the energy of the unit root process and that
19 non-lasting disturbances are captured by the higher scale coefficients. This logic is
20 also reflected in Fan and Gençay (2006), who stated that the spectrum of a unit root
21 process is infinite at frequency 0. They proposed a unit root test on the perspective of
22 the frequency domain as the test is the ratio of the energy of the low frequency scale
23 to the total energy of the time series. Here our Nonlinear Dickey-Fuller F test statistic
24 is in a time domain where we use the scaling coefficient directly in the test statistics;
25 in this way, the asymptotic distribution of the test statistics will not be influenced
26 under the wavelet environment. We use Maximal Overlap Discrete Wavelet
27 Transform (MODWT) as it has no restriction on the sample size and LA (8) wavelet
28 filter as it has better band pass character. For more information about the MODWT
29 methods and LA filter, we refer to Vidakovic (1998), Percival and Walden (2000), and
30 to Gençay Selcuk and Whicher (2001b).
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52
53 The paper is organized as follows. Section II presents the LSTAR model, the
54 procedure for testing unit root against the LSTAR alternatives, the asymptotic
55 properties and the finite sample distribution of the test statistics. Section III
56 investigates the size and power property of the test, and offers an empirical example.
57
58 Section IV shows the size distortion of the test statistics under GARCH (1, 1) error.
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Section V presents the wavelet size improvement of the small samples and the asymptotical distribution. Concluding remarks can be found in the final section. All proofs of theorems in this paper are omitted to save place, but available from the authors up on request.

II. Model, Test procedure, The Nonlinear Dickey-Fuller F test

In STAR models, the LSTAR model can catch the asymmetric feature of a process in two extreme states: when the economic contractions are always more violent, and when expansions are more stationary and persistent. This paper only considers the nonlinear LSTAR models with the following structure:

$$y_t = \pi_{10} + \pi_{11}y_{t-1} + (\pi_{20} + \pi_{21}y_{t-1})F(t, \gamma, c) + u_t \quad (1)$$

$$F(t; \gamma, c) = \frac{1}{(1 + \exp\{-\gamma(t - c)\})} - \frac{1}{2}.$$

Here the transition variable is defined as time t , which implies that the equilibrium regimes switch as the time evolves. Parameter γ determines the speed of transition from one extreme regime to another at time c : the larger the γ is, the steeper the transition function will be. Meanwhile, set c fixed, as $\gamma \rightarrow \infty$, the function turns into a step function of t and the model becomes a two regimes threshold autoregressive model (TAR). When setting t and c fixed, the situation when $\gamma \rightarrow 0$ leads the resulting model to be linear. Therefore, the linear hypothesis is equivalent to the hypothesis: $\gamma = 0$. Our goal is to test the null hypothesis of a random walk without drift against the nonlinear LSTAR (1) model. The null hypothesis can be expressed as the following parameter restrictions:

$$H_0 : \gamma = 0, \pi_{10} = 0, \pi_{11} = 1.$$

Since $\gamma = 0$ will lead to an identification problem under the null hypotheses (see Teräsvirta, 1994), we follow the approach used by Luukkonen, Saikkonen and

Teräsvirta (1988), by applying a Taylor expansion of the $F(t, \gamma, c)$ with γ around 0.

As the first-order expansion will lead to low power if the transition takes place only in the drift, we show also the third-order Taylor expansion which is more robust:

$$F_1(t; \gamma, c) = \frac{\gamma(t-c)}{4} + o(\gamma). \quad (2)$$

$$F_3(t; \gamma, c) = \frac{\gamma(t-c)}{4} + \frac{\gamma^3(t-c)^3}{48} + o(\gamma^3). \quad (3)$$

Substituting equations (2) and (3) into equations (1), we obtain the following auxiliary regressions:

$$\text{Order1: } y_t = s_{1t}' \lambda_1 + (y_{t-1} s_{1t})' \varphi_1 + u_{1t}.$$

$$\text{Order3: } y_t = s_{3t}' \lambda_3 + (y_{t-1} s_{3t})' \varphi_3 + u_{3t}.$$

$$s_{1t} = \begin{pmatrix} 1 \\ t \end{pmatrix}, \lambda_1 = \begin{pmatrix} \lambda_{10} \\ \lambda_{11} \end{pmatrix}, \varphi_1 = \begin{pmatrix} \varphi_{10} \\ \varphi_{11} \end{pmatrix}; s_{3t} = \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}, \lambda_3 = \begin{pmatrix} \lambda_{30} \\ \lambda_{31} \\ \lambda_{32} \\ \lambda_{33} \end{pmatrix}, \varphi_3 = \begin{pmatrix} \varphi_{30} \\ \varphi_{31} \\ \varphi_{32} \\ \varphi_{33} \end{pmatrix}$$

The unit root test in the nonlinear time series model is a joint test of both unit root and linearity. The corresponding auxiliary null hypotheses are:

$$H_{0m} : \lambda_{mi} = 0, \forall i; \varphi_{m0} = 1, \varphi_{mj} = 0, j \geq 1; m = 1, 3.$$

Following the above auxiliary regressions and null hypothesis, we now derive the unit root test statistics and investigate their distribution properties. Here we assume that the error terms in equations (1) are independent identically distributed (*i.i.d.*). In the distribution form, the $W(\cdot)$ represents a standard Brownian motion on $[0, 1]$.

Assumption 1: Let $\{u_t\}$ is *i.i.d.* with $E(u_t) = 0, \text{Var}(u_t) = \sigma_u^2$, and $E(u_t^4) < \infty$.

Theorem2: Assume that the following models $y_t = s_{mt}' \lambda_m + (y_{t-1} s_{mt})' \varphi_m + u_{mt}$ hold,

and assume that $(u_{mt})_{t=1}^{\infty}$ fulfills Assumption 1, then for $m = 1, 3$, we have:

$$\hat{\psi}_m - \psi_m \xrightarrow{p} 0, \quad \Upsilon_m (\hat{\psi}_m - \psi_m) \xrightarrow{L} \Psi_m^{-1} \Pi_m, \quad s_m^2 \Upsilon_m (\sum x_{mt} x_{mt}')^{-1} \Upsilon_m \xrightarrow{L} \sigma_u^2 \Psi_m^{-1}.$$

Where the parameters are defined as follows:

$$\Upsilon_1 = \text{diag}\{T_1\}, \quad T_1 = [T^{1/2} \quad T^{3/2} \quad T \quad T^2],$$

$$\Upsilon_3 = \text{diag}\{T_3\}, T_3 = [T^{1/2} T^{3/2} T^{5/2} T^{7/2} T T^2 T^3 T^4],$$

$$\hat{\psi}_m = \begin{pmatrix} \hat{\lambda}_m \\ \hat{\phi}_m \end{pmatrix}, \psi_m = \begin{pmatrix} \lambda_m \\ \phi_m \end{pmatrix}, x_{mt} = \begin{bmatrix} s_{mt} \\ y_{t-1}s_{mt} \end{bmatrix}, \Psi_m = \begin{bmatrix} A_m & \sigma_u B_m \\ \sigma_u B_m' & \sigma_u^2 C_m \end{bmatrix}, \Pi_m = \begin{bmatrix} \sigma_u D_m \\ \sigma_u^2 E_m \end{bmatrix},$$

$$A_m = [a_{ij}]_{(m+1) \times (m+1)}, B_m = [b_{ij}]_{(m+1) \times (m+1)}, C_m = [c_{ij}]_{(m+1) \times (m+1)}, D_m = [d_i]_{(m+1) \times 1}, E_m = [e_i]_{(m+1) \times 1},$$

$$a_{ij} = T^{-(i+j-1)} \sum_{t=1}^T t^{i+j-2}, b_{ij} = \int_0^1 r^{i+j-2} W(r) dr, c_{ij} = \int_0^1 r^{i+j-2} W(r)^2 dr,$$

$$d_i = W(1) - (i-1) \int_0^1 r^{i-2} W(r) dr, e_i = \frac{(W(1)^2 - (i-1) \int_0^1 r^{i-2} W(r)^2 dr - 1/i)}{2}.$$

Based on *Theorem 1*, under the null hypothesis $H_0: R_m \psi_m = r_m$, we have the

Nonlinear Dickey-Fuller F test statistic as follows:

$$\begin{aligned} F_m &= (\hat{\psi}_m - \psi_m)' R_m' \{s_m^2 R_m (\sum x_{mt} x_{mt}')^{-1} R_m'\}^{-1} R_m (\hat{\psi}_m - \psi_m) / 2 \\ &= (\hat{\psi}_m - \psi_m)' R_m' Y_m * \{s_m^2 Y_m R_m (\sum x_{mt} x_{mt}')^{-1} R_m' Y_m'\}^{-1} * Y_m R_m (\hat{\psi}_m - \psi_m) / 2 \\ &\xrightarrow{L} (\Psi_m^{-1} \Pi_m)' \{\sigma_u^2 \Psi_m^{-1}\}^{-1} (\Psi_m^{-1} \Pi_m) / 2 = \Pi_m' \Psi_m^{-1} \Pi_m / 2 \sigma_u^2. \end{aligned}$$

Where: $R_m = I_{2 \times (m+1)}$, $r_1' = [0 \ 0 \ 1 \ 0]$, $r_3' = [0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0]$.

To find out the finite-sample distributions of the test, we generate data from the model $y_t = y_{t-1} + u_t$ where $u_t \sim n.i.d.(0,1)$ with desired sample sizes. The number of Monte Carlo replication is 20000. To avoid high parameter dimension which shows in third-order Taylor expansion, we only report the critical value table with Taylor expansion of Order 1, and we also only use this case in the following part of the paper.

Table 1. Critical values for the Nonlinear D-F F test

T	99%	97.5%	95%	90%	10%	5%	2.5%	1%
50	1.1282	1.3710	1.6378	2.0181	6.7710	7.7315	8.6741	9.7927
100	1.2205	1.4894	1.7665	2.1512	7.0376	8.1185	9.2100	10.4860
250	1.2500	1.5324	1.8055	2.1816	7.2652	8.4135	9.4990	10.7145
500	1.2850	1.5682	1.8660	2.2380	7.2687	8.3699	9.3918	10.6378

III. Size and power property of the test, Empirical Example

We again use the Monte Carlo method to investigate the size and power properties of our test statistics. We ignore the size table as it is unbiased due to that we use same Monte Carlo experiment when simulating the critical value table. Thus we only examine the power property of the test. As we are more interested in the variation of the dynamic parameters, we set the drift parameter stable with $\pi_{10}=0$, $\pi_{20}=1$. We also set the transition speed parameter $\gamma=1$, and the transition time $c = T/2$. Thus the changing parameters are only dynamical parameters π_{11} , π_{21} . We also impose the Lagrange stability condition $\pi_{11} + \pi_{21} \in (0,1)$ with $\pi_{11} + \pi_{21} = 0.9$ to ensure the stable trajectories (Tong (1990)). To observe the power changes with the nonlinear dynamics impact measured by π_{21} , we set: $\pi_{11} \in (0.1, 0.3, 0.4, 0.5, 0.6)$, $\pi_{21} \in (0.8, 0.6, 0.5, 0.4, 0.3)$.

Table 2. Power property for the test

T	$\pi_{11}=0.6$	$\pi_{11}=0.5$	$\pi_{11}=0.4$	$\pi_{11}=0.3$	$\pi_{11}=0.1$
	$\pi_{21}=0.3$	$\pi_{21}=0.4$	$\pi_{21}=0.5$	$\pi_{21}=0.6$	$\pi_{21}=0.8$
50	0.1925	0.3869	0.6056	0.8108	0.9820
100	0.7024	0.9329	0.9925	0.9995	1
250	0.9400	1	1	1	1
500	1	1	1	1	1

Table 2 shows that for small sample 50, the power depends mainly on the proportions of the nonlinear parts; the higher the nonlinear part, the better power is. With sample size increases to 250 and 500, the power of the test reaches 1.

We now show an empirical example to compare our Nonlinear Dickey-Fuller F test to the traditional Dickey-Fuller F test. For the unemployment rates in 10 OECD countries¹ from 1955 to 1999 and by using the Dickey-Fuller F test, the unit root reject none of them, while using the Nonlinear F test, the unit root was rejected in 3 series: Germany, Japan, France. We use the data from France for more detailed procedure.

¹ Austria, Denmark, Finland, France, Germany, Japan, Netherland, Norway, Sweden, Belgium

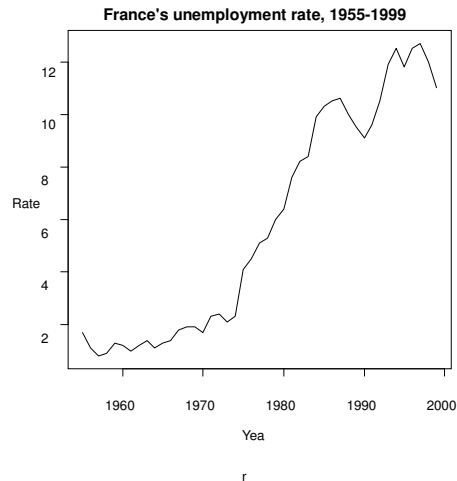


Figure 1. France's unemployment rate, 1955- 1999

In Figure 1, we can see that the time series shows an obvious data break around 1975, and the whole series is divided into two data periods with a smooth region changes. Thus we suppose that a STAR model should be a good choice. For linear Dickey-Fuller F , we first fit the data with an order one autoregressive model and the AR(1) process is: $y_t = 0.2274 + 0.9971y_{t-1} + u_t$ with sum of residue squares 15.04 and F statistic 2.7538, after we compare it to the critical value 4.86 in Table B.7 in Hamilton, J.D. (1994), the unit root is not rejected at the 5% level. However, from the graph we can see that for the unemployment rate data, the series are initially at an equilibrium state but after around 1975 there shows another state, and the whole series shows a nonlinear structure. Therefore, the Dickey-Fuller F test may be not valid with its linearity assumption. *Chow test* is used to test the series linearity against a single break at 1976 and we obtain the F_{Chow} statistic: $F_{Chow} = 0.2369$. At the 20% critical value, we reject the null hypothesis. Next we apply our Nonlinear test from the auxiliary model $y_t = -0.5518 + 0.0595t + 0.0834y_{t-1} - 0.06ty_{t-1} + u_t$. With sum of residue squares 11.58 and F_{NL} statistic 9.3687, after comparing it to 7.7315 in Table 1, the null hypothesis of a unit root is rejected at the 5% level. Then we are building a LSTAR model by Nonlinear Least Square (NLS) regression with t from 2 to 45 (correspond to the year from 1956 to 1999):

$$y_t = 2.3243 + 0.3797y_{t-1} + (2.5870 + 0.6153y_{t-1}) \left[\frac{1}{(1 + \exp\{-0.3408(t - 20.6567)\})} - \frac{1}{2} \right] + u_t$$

The estimation of c is 20.6567, which shows data that the break occurs around year 1975. The economical explanation for this break may related to the OPEC energy price rising in 1975 when the oil price raised 10% , which brought a huge shock to the economic field, including the job market. Therefore there are two different states of the unemployment rates: before and after the oil price change. Moreover, the sum of residual squares for LSTAR model is 9.53, which is the lowest of the three models.

IV. Size property under GARCH (1, 1) error

We turn here to the question of how the size property will be affected when GARCH errors appear. The GARCH (1, 1) is the most frequently used due to its simplicity and robustness. Thus we only concentrate the test property when the error of the DGP exhibits GARCH (1, 1). The unit root process with GARCH (1,1) error is as follows:

$$y_t = y_{t-1} + u_t,$$

$$u_t = \eta_t \sqrt{h_t}, h_t = w + \alpha u_{t-1}^2 + \beta h_{t-1}; \eta_t \sim i.i.d.(0,1), w = 1 - \alpha - \beta.$$

For the unit root based on the LSE in the linear case, Cook (2006) observed that size distortions of GARCH error are mainly caused by the volatility parameter α . Hence, our experiment design will include both the cases where $\alpha \geq \beta$ and $\alpha < \beta$ in the following situations: medium and high GARCH effect: $\alpha + \beta = 0.75$; $\alpha + \beta = 0.95$. To judge the reasonability of the size property at 5% nominal size, an unbiased estimated should lay between the approximate 95% confidence intervals of the actual size 5%. With replication number equal to 10000, the approximate 95% confidence interval for the estimated size is: $0.05 \pm 1.96 \sqrt{\frac{0.05(1-0.05)}{10000}} = (0.0457, 0.0543)$. The size table of the test under the medium GARCH error is now:

Table 3. Size property for the test under GARCH (1, 1) error, $\alpha + \beta = 0.75$

	$\alpha = 0.3$	$\alpha = 0.35$	$\alpha = 0.4$	$\alpha = 0.45$	$\alpha = 0.5$
T	$\beta = 0.45$	$\beta = 0.4$	$\beta = 0.35$	$\beta = 0.3$	$\beta = 0.25$
50	0.0897	0.0951	0.1074	0.1088	0.1102
100	0.0849	0.0891	0.0962	0.1038	0.1032
250	0.0734	0.0746	0.088	0.0935	0.0994
500	0.0752	0.0769	0.0756	0.0853	0.0885

Table 3 shows that when $\alpha + \beta = 0.75$, there exists obvious size distortion which is more serious as α increases and less serious in larger samples. This can be interpreted as follows: when GARCH effect is not high, the size distortion is mainly due to the volatility parameter α and it will be milder when the sample size grows. For the situation of high GARCH effect where $\alpha + \beta = 0.95$, the size is over biased as well and we will present it later together with the wavelet improved size in Table 6.

V. Wavelet improvement of size distortion under GARCH error

In this section we use the wavelet method to solve the problem of over-rejection under GARCH error. The process is simple: first we generate a new table of critical values where the DGP is the first level boundary wavelet scale coefficients get by Maximal Overlap Discrete Wavelet Transform (MODWT). Next, we apply the test using these scale coefficients instead of the original series. The logic behind this method is that, after wavelet decomposition, the wavelet's high frequency coefficients which contain short time volatility information brought by GARCH (1, 1) error are counted off. Those scale coefficients contain all the non stationary information when the original time series follows a unit root process, while the scale coefficients are still stationary when the original time series is stationary. Thus when conducting the unit root test,

we use the scale coefficients $V_t = \sum_{l=0}^{L-1} g_l y_{t-l \bmod T}$ instead of the original data, while g_l is

the scaling filter satisfying: $\sum_l g_l = 1$, $\sum_l g_l^2 = 1/2$, $\sum_l g_l g_{l+2^n} = 0$

Assumption 2: $\{w_t\}$ is a linear process which can be defined as follows:

$$w_t = \psi(L)u_t = \sum_{j=0}^{\infty} \psi_j u_{t-j}, \psi(1) \neq 0, \text{ and } \sum_{j=0}^{\infty} j |\psi_j| < \infty.$$

Theorem 2: Under Assumptions 1 and 2, the asymptotical distributions of the Nonlinear Dickey-Fuller F test statistics will not be influenced when we use wavelet scale coefficients $V_t = \sum_{l=0}^{L-1} g_l y_{t-l \bmod T}$ instead of original series y_t with $y_t = y_{t-1} + u_t$, where u_t fulfills Assumption 1.

For small sample, Monte Carlo experiment gives following critical value table:

Table 4. Critical values for the wavelet improved Nonlinear D-F F test

T	99%	97.5%	95%	90%	10%	5%	2.5%	1%
50	0.1699	0.2600	0.3797	0.5626	3.9150	4.7469	5.5354	6.6924
100	0.2607	0.4087	0.5792	0.8282	4.5163	5.3862	6.2805	7.2699
250	0.6007	0.8122	1.0262	1.3412	5.4247	6.3390	7.2056	8.3035
500	0.9216	1.1318	1.3484	1.6354	5.8999	6.8178	7.7707	8.9749

We can see that as the sample size increase, the critical values will approach the one we obtain from Table 1, which also implies that the distribution will not be influenced asymptotically. The size of the test using the wavelet scale coefficient is still unbiased under $u \sim n.i.d.(0,1)$. We are more interested to see the size property of the wavelet improved test when the original DGP suffered from GARCH (1, 1) error, when $\alpha + \beta = 0.75$ and 0.95, see Table 5 below.

When comparing Table 5 to Table 3, where no wavelet method is applied, we see that although there are only a few unbiased size in Table 5, the over-rejection problem get improved in each grid.

Table 5. Size property for the test in wavelet under GARCH (1, 1), $\alpha + \beta = 0.75$

	$\alpha = 0.3$	$\alpha = 0.35$	$\alpha = 0.4$	$\alpha = 0.45$	$\alpha = 0.5$
T	$\beta = 0.45$	$\beta = 0.4$	$\beta = 0.35$	$\beta = 0.3$	$\beta = 0.25$
50	0.0597	0.0625	0.0681	0.0628	0.0670
100	0.0710	0.0638	0.0691	0.0726	0.0700
250	0.0644	0.0690	0.0711	0.0720	0.0700
500	0.0669	0.0691	0.0674	0.0740	0.0737

For the case when, $\alpha + \beta = 0.95$, the wavelet improved size property and the original size which is in () are as follows:

Table 6. Size property for the test under GARCH (1, 1) error, $\alpha + \beta = 0.95$

	$\alpha = 0.4$	$\alpha = 0.45$	$\alpha = 0.5$	$\alpha = 0.55$	$\alpha = 0.6$
T	$\beta = 0.55$	$\beta = 0.5$	$\beta = 0.45$	$\beta = 0.4$	$\beta = 0.35$
50	0.0500(0.0762)	0.0521(0.0745)	0.0514(0.0831)	0.0487(0.0785)	0.0472(0.0754)
100	0.0609(0.0905)	0.0608(0.0891)	0.0543(0.0875)	0.0543(0.0879)	0.0513(0.0853)
250	0.0741(0.1013)	0.0732(0.0962)	0.0699(0.0909)	0.0580(0.0901)	0.0629(0.0901)
500	0.0799(0.1055)	0.0795(0.1116)	0.0718(0.1064)	0.0681(0.0950)	0.0626(0.0908)

Table 6 shows over-rejection problem of the size is obviously improved, especially when the sample size is small, and where the wavelet improved size is almost unbiased and lies between the 95% confidence interval. However, when the sample size increases, there is still some size distortion, which may due to the aggregate influence of the GARCH effect in large sample sizes for the near integrated GARCH error when $\alpha + \beta = 0.95$.

VI. Conclusions

In this paper we first propose a nonlinear Dickey-Fuller F test against LSTAR (1) model with time as the transition variable. The asymptotic distribution of the Nonlinear Dickey-Fuller F test statistic is derived while distributions of finite samples

are obtained by Monte Carlo simulations. The size of the test statistics is unbiased and the power shows good property in larger samples. We also use an empirical example to compare the nonlinear Dickey-Fuller F with traditional Dickey-Fuller F test.

Technically speaking, the Nonlinear Dickey-Fuller F test is not very innovative as it is mainly an addition of the nonlinear Dickey-Fuller ρ and t test proposed by He and Sandberg (2006). The main point of this paper is to show that our test suffered from serious size distortion under medium and high GARCH (1, 1) error. To resolve the problem, we use wavelet method as the wavelet scale coefficient can maintain the unit root information while count the GARCH effort off. We show that by using the wavelet method, the asymptotic distribution is not influenced, while the over-rejection problem in small sample size is improved.

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