Optimal Advertising Outsourcing Strategy with Different Effort Levels and Uncertain Demand

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Abstract

This paper studies the issue of advertising outsourcing and production planning for a manufacturer facing asymmetric advertising cost and uncertain market demand. To improve product sales, a manufacturer would hire an advertising agency to provide professional service on product advertising before the production takes place. A contract taking into account both advertising effort level and payment is introduced to incentivize the advertising agency to report the exact cost to the manufacturer. Furthermore, a model with the goal of maximizing the manufacturer's net-profit is proposed, in which both product demand and payment to the advertising agency are affected by the advertising effort level. Analytical solutions of the optimal strategies including the optimal advertising effort level and the optimal payment to the advertising agency are derived. Optimal retail price and the optimal production quantity are also obtained for the manufacturer in making managerial decisions.

Keywords: Production; Advertising Outsourcing; Principal-agent Problem; Revelation Principle; Uncertain Demand.

1 Introduction

Due to the globalization of economics and the consideration of the core competencies and cost issues, manufacturers are inclined to outsource their product advertising to other experienced and professional agent companies. This is made possible by carefully monitoring the tastes of consumers and industry for unexpected fads and increasing immediate sales at the retailer level. There are three major strategies for companies adopting outsourcing: business improvement, business impact and commercial exploitation, see for instance, Diromualdo and Gurbaxani [6]. Holcomb and Hitt [8] introduced two main perspectives, transaction cost theory and resource based view, which are available for application in strategic source. A lot of studies focusing on outsourcing strategies can be found in the literature, see for instance, Cachon and Zhang [4], Hsieh and Kuo [9], Shen et al. [21], Zhen [26] and Kaya [13]. In most outsourcing models, one commonly used assumption is that both the principal and the agent know the information during the whole process. However, the principal may not react to market changes as fast as the agent, since the agent is more professional in some areas. In this paper, the issues of outsourcing strategy with asymmetric information and the coordination between the manufacturer (he) and the advertising agency (she) are investigated.

A direct motivation of our paper comes from a problem faced by manufacturers that they may not be as experienced in "trendy" or "attractive" product advertising design as a professional agent company. In this case, the manufacturer needs to put extra effort in the advertising design in order to achieve the same design level as a professional agent company can provide. A significant investment in a higher level of advertising effort can develop trendier products and therefore results in products of greater value to consumers and hence elicit a greater willingness to pay. Enhancing effort level is costly, nevertheless, it brings an increase in market demand for the product. For example, Cachon et al. [5] found that although the price of the fashion product is higher with the enhanced design system, it is beneficial to the firm if the increase in cost for enhanced design is not too high. Thus, in order to reduce his cost and increase the product competitiveness, the manufacturer may hire a professional agent company to advertise his product.

For the conciseness of this paper, we adopt the advertising agency as the agent company who is employed for advertising the product. Before the production starts, the manufacturer has to decide a proper strategy on the advertising effort level and the payment to the agent company based on the reported advertising cost. There are typically fixed costs such as wages for a large design staff and trend spotters. However, costs could be uncertain due to labor-intensive production processes and costly local labor [5]. Hence the manufacturer may not know the exact cost of the advertising in practice. In other words, the information between the manufacturer and the advertising agency is asymmetric. The advertising agency may tend to report a cost higher than the actual cost for making more profit, which is obviously not favorable to the manufacturer. This would affect the manufacturer's decisions on the production planning and also his profit. Although with the current technology, more and more information can be shared by the participants in a supply chain, information asymmetry remains a key issue in the supply chain relationships. Interested readers can refer to Koulamas [14], Löffler [16], Myerson [17], Xiao and Yang [23], Zhang et al. [24] and Zhang et al. [25]. Our model presented here tries to address the production planning problems in such environments.

Recently, considerable attentions have been paid to incentive models which induce truthful information using the revelation principle [17]. If asymmetric information occurs regarding the agent's cost, then the principal can design a mechanism or a contract under which the agent would get her maximum profit when she reports the true cost. For example, Zhang et al. [25] proposed a supplier switching model with asymmetric information. A quantity-price contract was presented to demonstrate the advantage of the contract based on the principal-agent theory. Xiao and Yang [23] designed a wholesale price-order quantity contract to induce the retailer to report her risk sensitivity information truthfully based on the information revelation mechanism. In practice, a properly designed contract could be a useful and effective method to achieve supply chain coordination. It enhances the revenues of participants or the total revenue of a supply chain, such as returns policy [15] and revenue sharing contract [14]. The effect of cooperative advertising between a manufacturer and a retailer has been investigated, see for instance, Ahmadi Javid and Hoseinpour [1], Berger [3] and Jørgensen [?]. However, little is known for the coordination between a manufacturer and an advertising agency which is also common in modern manufacturing industry.

In this paper, we develop a manufacturer-led advertising outsourcing model to maximize the manufacturer's profit based on the principal-agent framework. Here the manufacturer is the principal and the advertising agency is the agent. In order to ensure that the advertising cost parameter reported by the advertising agency is true, a contract taking into account the effort level and the payment to the advertising agency is introduced. The model is also applicable to the situation when a manufacturer plans to outsource part of his business, such as product packaging and collecting production information from the market. Here we investigate the optimal production strategies for the manufacturer to maximize his profit. The demand for the product is assumed to be dependent on the retail price and the effort level. The parameter which represents market sensitivity in responding to the advertising of the product would be affected by the economic state, and this uncertainty would be only resolved at the end of the period. Hence, the situation of demand as a stochastic function with stochastic parameter is discussed, for example, with the Bernoulli distributed parameter and the uniformly distributed parameter. This stochastic property is usually neglected in earlier research, see for instance, Lau and Lau [15] and Mussa and Rosen [18]. Recently, a lot of literature such as Song et al. [20] and Perdikaki et al. [19] assumed that the market demand depends on the economic state. Our work extends the above assumption by considering an environment where the market sensitivity parameter in responding to the product advertising is stochastic in the demand function. The analytical optimal solutions, including four decision variables: (i) retail price, (ii) production size, (iii) advertising effort level, and (iv) payment to the advertising agency, are derived under different demand assumptions.

The rest of this paper is structured as follows. In Section 2, we propose a manufacturer-led advertising outsourcing model. Section 3 provides analytical solutions for the optimization problem under different market demand assumptions. Section 4 presents some numerical examples to illustrate the effectiveness of the methods. It describes prospects for outsourcing contract to achieve coordination and examines the effect of information asymmetry on the profit of the manufacturer. Finally, Section 5 concludes the paper.

2 The Basic Model

In this section, we present our manufacturer-led advertising outsourcing model. It is assumed that a manufacturer hires another company (the advertising agency) to advertise the product before production takes place. Here we assume that the advertising agency can invite teams with different professional levels or popularities for the projects with different payments. She can also employ different teams to study the market trend, collect reliable market information and adopt multiple forms of advertising to build a brand image of the product. Hence the advertising effort level provided by the advertising agency is defined as the professional level of the employed design team and the forms of advertising. A higher level of advertising effort can develop trendier product and therefore results in products of greater value to consumers and hence elicit a greater market demand. Furthermore, the manufacturer can adopt the optimal strategy on the retail price, production size and the advertising effort level. In reality, the manufacturer faces an unknown demand that is sensitive to both the retail price and the level of advertising effort. He also does not have complete knowledge about the actual cost of the advertising effort level.

Let m be the advertising effort level adopted by the manufacturer. The cost of advertising effort is supposed to be quadratic and is given by

$$C(x, m) = xm^2 \tag{1}$$

where x is the advertising cost parameter. This function has been widely used to describe the cost of advertising effort, see for instance, [**?**]. We adopt the quadratic function in our discussion to describe the cost of advertising effort because it is the simplest function that satisfies two important features. Firstly, it is strictly increasing because higher effort level means higher costs in human resources, materials and time. Secondly, with the convexity of the function, one is able to show that the marginal cost of advertising effort is increasing, see for instance, Huang et al. [10]. In practice, the relationship between an advertising effort level and the cost of advertising effort might be more complex. Other types of function $C(x, m)$ can be considered and analyzed using a similar method, which would be investigated in our future research.

Most of the literature considers a constant advertising cost parameter x [12]. In this work, we assume that x is a variable parameter depending on the market, which is affected by the material cost, labor cost, and other direct expense, etc. To simplify the problem considered here, we assume that the true value of the advertising cost parameter x for each advertising agency is the same. In reality, the manufacturer usually does not know the value of the agency's advertising cost parameter x exactly. It is assumed that the manufacturer only has the knowledge about the probability distribution of x that defined in the interval $[\underline{b}, \bar{b}]$ with the probability density function $f(\cdot)$ and cumulative distribution $F(\cdot)$. The inverse hazard rate is denoted by $h(x)$, i.e., $h(x) = F(x)/f(x)$, and $h(x)$ is an increasing function in x, which is a usual regularity condition [25].

The sequence of the events is given as follows:

1. The manufacturer offers a contract to the advertising agency, which includes different advertising effort levels and the corresponding payments.

2. The advertising agency reports an advertising cost parameter according to the operational cost and the contract.

3. The manufacturer decides the advertising effort level and payment to the advertising agency based on the reported value of the advertising cost parameter according to the

contract.

4. The manufacturer decides the production size and carries out the production.

5. The manufacturer decides the retail price based on the economic environment of the selling season.

To facilitate our discussion, we introduce the following notation that will be used in this paper.

(i) p, the unit retail price (a decision variable);

(ii) Q, the production size (a decision variable);

(iii) c, the unit manufacturing cost;

(iv) x, the true value of the advertising cost parameter;

(v) y, the reported value of the advertising cost parameter.

In general, when the reported advertising cost parameter is high, the manufacturer would lower his requirement of the effort level to control the cost while still improving the market demand of the product. Therefore the manufacturer would determine the advertising effort level of his product based on the advertising cost parameter reported by the advertising agency. Then the advertising effort level $m(\cdot)$ is assumed to depend on the reported value of the advertising cost parameter y. It satisfies:

$$\underline{m} \leq m(y) \leq \overline{m}. \tag{2}$$

The cost of advertising effort is rewritten as $xm(y)^2$.

In Huang et al. [10], they showed that the false returns in a reverse supply chain can be reduced by exerting costly effort, and the effort level is assumed to be continuous. Competition is fierce in the market, in order to keep her company competitive, here we also suppose that the advertising agency is able to give a continuous effort.

We assume that the leftovers have no salvage value. The demand $\Lambda(p, m(\cdot))$ satisfies:

$$\Lambda(p, m(x)) = A - \lambda p + km(x) \tag{3}$$

where A, λ, and k are positive quantities. It implies that the manufacturer can develop trendier and more attractive product by investing in advertising efforts, which would result in products of greater value to consumers and more demand of the product. This linear demand function has been widely used in literature, see for instance, He et al. [7] and Perdikaki et al. [19]. When $p = c$, the sales of the products usually should be good, so it is natural to assume that $A \gg \lambda c$. Moreover, $A \gg km(x)$ implies that the effect of the promotion on product sales is limited in the market. Here λ and k are measures for market sensitivity in response to the change of the retail price and the advertising effort, respectively.

Due to the revelation principle, the manufacturer can design a mechanism in which the advertising agency receives her maximum profit when the actual advertising cost parameter x is reported. In practice, the manufacturer requests the advertising agency to submit a budget for the advertising cost parameter y. Then the manufacturer decides the advertising effort level $m(y)$ and pays the advertising cost $t(y)$ to the advertising agency based on the reported value y. It is an important problem for the manufacturer to decide his optimal advertising effort level based on the reported value y. This is because if the reported value y is higher than the actual value x, the manufacturer may lower the advertising effort level to control his cost, and this would cause the reduction of both market demand of product and the expected profit of the manufacturer. Here the setting of advertising effort level and advertising cost depending on y aims to ensure that the reported value of the advertising cost parameter is true. This will be further discussed in Eqs. (4) - (6).

Denote by $\beta(x, y)$ the advertising agency's profit when she reports her value of the advertising cost parameter as y with the actual value x, which can be described as

$$\beta(x, y) = t(y) - xm(y)^2. \tag{4}$$

In order to maximize the manufacturer's profit, a contract between the manufacturer and the advertising agency is designed based on the revelation principle to induce the advertising agency to report the exact value of the advertising cost parameter. By the revelation principle, the manufacturer has to ensure that the advertising agency getting her maximum profit when she reports the actual advertising cost parameter, i.e., the Incentive Compatibility (IC) constraint, namely,

$$\beta(x, y) = t(y) - xm(y)^2 \leq \beta(x, x), \quad \text{for all } y, x \in [\underline{b}, \bar{b}]. \tag{5}$$

Without loss of generality, we assume that the advertising agency's reservation profit is zero. The Participation Constraint (PC) is needed to ensure the advertising agency at least gets the reservation profit zero when she reports the true cost, i.e.,

$$\beta(x,x) = t(x) - xm(x)^2 \geq 0, \quad \text{for all } x \in [\underline{b}, \bar{b}]. \tag{6}$$

With these constraints, the advertising agency will report the true value, thus the expected net profit of the manufacturer is given as follows:

$$\begin{aligned} E(\pi) &= \int_{\underline{b}}^{\bar{b}} (p \cdot \min\{Q, \Lambda(p, m(x))\} - c \cdot Q - t(x)) f(x) dx \\ &= \int_{\underline{b}}^{\bar{b}} [(p-c)Q - p[Q - (A - \lambda p + km(x))]^+ - t(x)] f(x) dx. \end{aligned} \tag{7}$$

The manufacturer's task is to determine the values of p, Q, $m(\cdot)$, and $t(\cdot)$ so as to maximize his total expected profit $E(\pi)$ under the constraints of IC, PC and DC (Decision Constraint). The following is the optimization problem:

$$\begin{cases} & \max\limits_{p,Q,m(\cdot),t(\cdot)} \ E(\pi) \\ \text{subject to} & \\ & (IC): \ \beta(x,x) \geq \beta(x,y), \quad \forall x, y \in [\underline{b}, \bar{b}], \\ & (PC): \ \beta(x,x) \geq 0, \quad \forall x \in [\underline{b}, \bar{b}], \\ & (DC): \ \underline{m} \leq m(x) \leq \overline{m}, \quad \forall x \in [\underline{b}, \bar{b}]. \end{cases} \tag{8}$$

Similar constraints of IC, PC and DC can be found in Zhang et al. [25]. In [25], a supplier switching model was proposed where the buying firm has to ensure the entrant supplier can get the maximum profit when the entrant supplier reports the true cost.

3 Model Analysis

In this section, we consider solving the optimization problem given in Eq. (8) under different assumptions on k, (see Eq.(3)), the market sensitivity in responding to the change of product advertising. We first present an equivalent optimization problem in the following proposition.

Proposition 1. *Optimization problem (8) can be converted to the following optimization prob-*

lem:

$$\begin{cases} \max_{p,Q,m(x)} \int_{\underline{b}}^{\bar{b}} [(p-c)Q - p[Q-(A-\lambda p + km(x))]^+ - sm(x)^2] f(x)dx \\ \text{subject to} \\ m'(x) \leq 0, \quad \forall x \in [\underline{b}, \bar{b}], \\ m(\underline{b}) = \overline{m}, \quad m(\bar{b}) \geq \underline{m}, \end{cases} \tag{9}$$

and the optimal value of $t(\cdot)$ in optimization problem (8) can be calculated as follows:

$$t^*(x) = xm^*(x)^2 + \int_x^{\bar{b}} m^*(\tau)^2 d\tau, \tag{10}$$

with $s = x + h(x)$. Here $h(x)$ is the inverse hazard rate i.e., $h(x) = F(x)/f(x)$.

Proof. See Appendix 6.1.

Eq. (10) implies that the manufacturer has to give an additional payment to the advertising agency with the aim of inducing the advertising agency to report the true value of the advertising cost parameter. The integral term in Eq. (10) is the additional payment that can be considered as the "information rent". Without this "information rent", the advertising agency may tend to report a higher value of the advertising cost parameter with the purpose of getting more profit. In that case, the manufacturer would choose a lower advertising effort level than that required under the actual value. Then the market demand of the product would be affected and tends to decline, which would reduce the expected profit of the manufacturer. However, if the decrease in his profit due to a false advertising cost parameter is less than his loss due to an additional payment, the manufacturer would abandon the strategy of paying "information rent". Hence comparing to the case with "information rent", there must be an upper bound on the reported value in responding to each true value of the advertising cost parameter, under which both of advertising agency and manufacturer's profits are maximized. An interesting future research issue is to consider collaboration of the advertising agency and the manufacturer under this circumstance.

Proposition 1 suggests that the payment $t(\cdot)$ can be calculated by the advertising effort level $m(\cdot)$, thus the optimization problem is simplified. Therefore, we only need to decide p, Q and $m(\cdot)$. In the following sections, we will solve the optimization problem in Eq. (9) by considering different cases of k. The manufacturer may face a market with a demand embedded with uncertainty which is only resolved at the end of the period. Hence we first consider k as a constant, and then extend it to the situation that k is a random variable following a general distribution.

3.1 Case I: k is Constant

When the economic environment is steady and the production time is short, then k in Eq. (3) is predictable. In this case, the demand depends linearly on retail price and the advertising effort level of the product. To solve optimization problem (9) when k is a constant, for the sake of convenience, we suppose that the manufacturer chooses an arbitrary $m(x)$. Then we have the following lemma.

Lemma 1. *Suppose that k is a constant. For any given $m(x)$, the optimal solution of optimization problem (9) is given by*

$$\begin{cases} p^* = \dfrac{A + km(x)}{2\lambda} + \dfrac{c}{2}, \\ Q^* = \dfrac{A + km(x)}{2} - \dfrac{\lambda c}{2}, \end{cases} \tag{11}$$

where p^*, Q^* are the optimal retail price and the optimal production size, respectively. The manufacturer's profit with $p = p^*$ and $Q = Q^*$ is given by

$$\pi = \frac{\left(A + km(x) - \lambda c\right)^2}{4\lambda} - sm(x)^2.$$

Proof. See Appendix 6.2.

In Lemma 1, we notice that both the production size Q^* and the retail price p^* depend linearly on the advertising effort level $m(\cdot)$ with slope $k/2$ and $k/(2\lambda)$, respectively. This implies that the manufacturer should aim at a larger production size and a higher retail price if they decide to choose a higher advertising effort level. Higher advertising effort level costs more, so it is reasonable for the manufacturer to increase the production size and raise the retail price. The slope depends on k, which means such increase is constrained by the demand to avoid loss from leftovers. Moreover, notice that p^*, Q^* and $t^*(\cdot)$ depend on $m(\cdot)$, we only have to decide $m(\cdot)$ to achieve the optimal solution of problem (9). From Lemma 1, we have

$$\pi = \frac{\left(A + km(x) - \lambda c\right)^2}{4\lambda} - sm(x)^2. \tag{12}$$

To attain the optimal $m(x)$, we denote by m_0 the solution of $\frac{\partial \pi}{\partial m(x)} = 0$, then we have

$$m_0(x) = \frac{(A - c\lambda)k}{4\lambda s - k^2}. \tag{13}$$

If $4\lambda \underline{b} - k^2 > 0$, then $m_0(\cdot)$ will only take positive values (We recall that $A - c\lambda \gg 0$ and

$s = x + h(x) > \underline{b}$). Note that from the constraints of optimization problem (9), we have

$$m^*(x) = \begin{cases} \underline{m}, & \text{if } m_0(x) \leq \underline{m}, \\ m_0(x), & \text{if } \underline{m} < m_0(x) \leq \overline{m}, \\ \overline{m}, & \text{if } m_0(x) > \overline{m}. \end{cases} \tag{14}$$

Based on Lemma 1 and Eqs. (12) - (14), the optimal solution is presented in the following proposition.

Proposition 2. *If k is constant and $4\lambda\underline{b} - k^2 > 0$, then the optimal solution of optimization problem (9) is given by*

$$\begin{cases} p^* &= \dfrac{A + km^*(x)}{2\lambda} + \dfrac{c}{2}, \\ Q^* &= \dfrac{A + km^*(x)}{2} - \dfrac{\lambda c}{2}, \end{cases} \tag{15}$$

and

$$m^*(x) = \begin{cases} \underline{m}, & \text{if } m_0(x) \leq \underline{m}, \\ m_0(x), & \text{if } \underline{m} < m_0(x) \leq \overline{m}, \\ \overline{m}, & \text{if } m_0(x) > \overline{m}, \end{cases} \tag{16}$$

where $m_0(x) = \frac{(A-c\lambda)k}{4\lambda s - k^2}$.

Proposition 2 suggests that when parameters λ, k and $\underline{b}$ satisfy $4\lambda\underline{b} - k^2 > 0$, one can determine the optimal solution for $m(\cdot)$. From Eq. (10) and Eq. (16), once the advertising agency reports her advertising cost parameter x, the manufacturer can decide the advertising effort level and the payment to the advertising agency accordingly. Furthermore, when the advertising agency reports a higher value of the advertising cost parameter, the manufacturer will decrease the advertising effort level and reduce the payment to the advertising agency. Hence, the advertising agency would incline to report her true advertising cost parameter.

3.2 Case II: k is a Random Variable

In this section, we assume that k is a random variable which is defined in the interval $[k_L, k_H]$, and $g(\cdot)$ is the density function. This corresponds to the case when k is sensitive to the economic environment (a blooming economy or a downturn economy). The value of k depends on the economic states and it also reflects the uncertainty of the effect of the advertising effort level on the market demand. The rationale behind this assumption is that customers would pay more attention to the product advertisement and be willing to purchase a product with a fancy and attractive advertisement when the economy is blooming. On the contrary, if there is an economic recession, customers might focus on the practicality of a product rather than its advertisement.

During the production period, there are three decision variables need to be determined: production size, advertising effort level and corresponding payment to the advertising agency. These variables should be decided before k is known. But the retail price p can be decided just before the selling season, for the accuracy of our model, we may assume that p depends on k, which will be certain when k is known. We then try to obtain the optimal solution of problem (9) under this "new" demand, and a lemma under a fixed advertising effort level $m(\cdot)$ is given.

Lemma 2. *Suppose k is a random variable with probability density function $g(\cdot)$, $k \in [k_L, k_H]$. Let*

$$k_a = \int_{k_L}^{k_H} kg(k)dk$$

be the mean (average value) of k. Denote

$$\begin{aligned} \pi_{r1} &= \frac{(A + k_a m(x) - \lambda c)^2}{4\lambda} - sm(x)^2, \\ \pi_{r2} &= \int_{k_q}^{k_H} \frac{A + km(x)}{\lambda} Qg(k)dk - \int_{k_q}^{k_H} \frac{Q^2}{\lambda} g(k)dk - sm(x)^2 - cQ \\ &\quad + \int_{k_L}^{k_q} \frac{(A + km(x))^2}{4\lambda} g(k)dk, \end{aligned}$$

where $k_q = \frac{2Q-A}{m(x)}$. For any given $m(\cdot)$, there are two cases for the optimal solution of optimization problem (9).

(i) If $(k_a - k_L)m(x) \leq c\lambda$, then

$$p^* = \frac{A + km(x) - Q}{\lambda}, \tag{17}$$

$$Q^* = \frac{A + k_a m(x) - \lambda c}{2}, \tag{18}$$

$$\pi = \pi_{r1}(Q^*). \tag{19}$$

(ii) If $(k_a - k_L)m(x) > c\lambda$, *then*

$$p^* = \begin{cases} \dfrac{A + km(x) - Q}{\lambda}, & \textit{if } k \in [k_q, k_H], \\ \dfrac{A + km(x)}{2\lambda}, & \textit{if } k \in [k_L, k_q], \end{cases} \tag{20}$$

$$Q^* = \operatorname{argmax}_Q \{\pi_{r2}\}, \ \textit{where } Q \in \left[\frac{A + k_L m(x)}{2}, \frac{A + k_H m(x)}{2}\right], \tag{21}$$

$$\pi = \pi_{r2}(Q^*). \tag{22}$$

Proof. See Appendix 6.3.

For Case (i) in Lemma 2, we suppose that $\underline{m} \leq \frac{c\lambda}{k_a - k_L}$. Typically, the optimal solution of maximizing π_{r1} is

$$m_{r1}(x) = \frac{(A - c\lambda)k_a}{4s\lambda - k_a^2},$$

and it is assumed that $4\underline{b}\lambda > k_a^2$ which ensures that $m_{r1}(x)$ is positive. Then Lemma 3 can be obtained easily.

Lemma 3. *Suppose that* $\underline{m} \leq \frac{c\lambda}{k_a - k_L}$ *and* $4\underline{b}\lambda > k_a^2$. *Denote by* π_{r1}^* *the maximum value of* π_{r1} *under* $m(x) \in \left[\underline{m}, \frac{c\lambda}{k_a - k_L}\right]$. *We have*

$$\pi_{r1}^* = \begin{cases} \pi_{r1}\left(\dfrac{c\lambda}{k_a - k_L}\right), \ \textit{if } m_{r1}(x) \geq \dfrac{c\lambda}{k_a - k_L}, \\ \pi_{r1}(m_{r1}(x)), \ \textit{if } \underline{m} \leq m_{r1}(x) < \dfrac{c\lambda}{k_a - k_L}, \\ \pi_{r1}(\underline{m}), \ \textit{if } m_{r1}(x) < \underline{m}. \end{cases} \tag{23}$$

According to Lemmas 2 and 3, an algorithm is provided to find the optimal solution of the problem when the density function $g(k)$ is complicated.

Algorithm 1 Solving Problem (9) Based on Lemmas 2 and 3

Step 1 We employed grid search method
Set $m(x) \in [\frac{c\lambda}{k_a - k_L}, \overline{m}]$ and $Q \in [\frac{A + k_L m(x)}{2}, \frac{A + k_H m(x)}{2}]$. For each $m(x)$, using discrete methods to find the optimal Q such that π_{r2} is maximized;

Step 2 Find the optimal $m(x)$ and the corresponding optimal Q such that π_{r2} is maximized. The maximum value is denoted as π_{r2}^*. The optimal p is given in Eq. (20);

Step 3 Find π_{r1}^* and the optimal $m(x)$ according to Lemma 3. The corresponding optimal values of Q and p are given by Eq. (17);

Step 4 The optimal profit of the manufacturer is $\pi^* = \max\{\pi_{r1}^*, \pi_{r2}^*\}$, and find the corresponding optimal $m(x)$, Q and p.

Furthermore, we consider two particular scenarios: k follows the *Bernoulli distribution* and k follows the *uniform distribution*.

3.2.1 The Bernoulli Distribution Case

In this section, the change rate of the market demand in responding to the advertising effort level is assumed to follow a Bernoulli distribution which allows us to derive the closed-form solution of the optimization problem. Based on the existing market information, one can obtain the characteristics of the demand. For the ease of discussion, we assume that there are two values of k, corresponding to different states of the economy, namely,

$$k = \begin{cases} k_H, & \text{if the economy is in High (H) state,} \\ k_L, & \text{if the economy is in Low (L) state.} \end{cases} \tag{24}$$

Let θ be the probability that the economy is in State L, $0 < \theta < 1$. The probability that the economy is in State H is then given by $(1-\theta)$. In this case, the average value of k is $k_a = \theta k_L + (1-\theta)k_H$. A similar assumption can be found in Burnetas and Ritchken [2].

Based on Lemma 2, one can obtain the optimal solution here. First, we give a lemma under a fixed advertising effort level $m(\cdot)$.

Lemma 4. *Suppose that k follows the distribution that is described in Eq. (24). For any fixed $m(\cdot)$, there are two cases for the optimal solution of optimization problem (9):*
(i) If $(1-\theta)(k_H - k_L)m(x) \leq c\lambda$, then

$$\begin{cases} p^* = \dfrac{A + km(x) - Q}{\lambda}, \\ Q^* = \dfrac{A + k_a m(x) - \lambda c}{2}, \end{cases} \tag{25}$$

$$\pi = \frac{(A + k_a m(x) - \lambda c)^2}{4\lambda} - sm(x)^2.$$

(ii) If $(1-\theta)(k_H - k_L)m(x) > c\lambda$, then

$$p^* = \begin{cases} \dfrac{A + k_H m(x) - Q}{\lambda}, & \text{if } k = k_H, \\ \dfrac{A + k_L m(x)}{2\lambda}, & \text{if } k = k_L, \end{cases} \tag{26}$$

$$Q^* = \frac{A + k_H m(x)}{2} - \frac{c\lambda}{2(1-\theta)}, \tag{27}$$

$$\pi = \theta\frac{(A+k_Lm(x))^2}{4\lambda} + (1-\theta)\frac{(A+k_Hm(x)^2)}{4\lambda} + \frac{c^2\lambda}{4(1-\theta)} - c\frac{A+k_Hm(x)}{2} - sm(x)^2. \tag{28}$$

Proof. See Appendix 6.4.

It is shown in Lemma 4 that both Q^* and p^* are piecewise functions depending on $m(\cdot)$. More specifically, Q^* is linearly dependent on the advertising effort level $m(\cdot)$ with a linear scale factor relating to the expectation of k, which is similar to the result in Lemma 1. In addition, p^* not only depends on $m(\cdot)$, but also on k, which is not decided during the production period. The distribution of p^* can be obtained from the distribution of k (see Eq. (26)). It is not necessary to decide the retail price p^* before the selling season when k is deterministic. The manufacturer can also approximate p^* by its expectation before the selling season. We still need to solve the optimal solution for $m(\cdot)$. It follows from Lemma 4 that we can obtain the optimal solution in the following proposition.

Proposition 3. *If k is Bernoulli distributed as described in Eq. (24) and $4\lambda\underline{b} - E(k^2) > 0$, then the optimal solution of $m(x)$ in optimization problem (9) is given by $m^*(x)$:*

$$m^*(x) = \begin{cases} \underline{m}, & \text{if } m_3(x) \leq \underline{m}, \\ m_3(x), & \text{if } \underline{m} < m_3(x) \leq \overline{m}, \\ \overline{m}, & \text{if } m_3(x) > \overline{m}, \end{cases} \tag{29}$$

where

$$m_3(x) = \begin{cases} m_{34}(x) = \dfrac{Ak_a - c\lambda k_H}{4s\lambda - (\theta k_L^2 + (1-\theta)k_H^2)}, & \text{if } s < \dfrac{D}{4c\lambda^2}, \\ m_{31}(x) = \dfrac{(A-c\lambda)k_a}{4s\lambda - k_a^2}, & \text{if } s \geq \dfrac{D}{4c\lambda^2}, \end{cases} \tag{30}$$

with

$$D = k_a[Ak_a - (A-c\lambda)k_L].$$

Furthermore, $m_{34}(x)$ and $m_{31}(x)$ satisfy the following conditions:
(i) If $s < \frac{D}{4c\lambda^2}$, then $(1-\theta)(k_H - k_L)m_{34}(x) > c\lambda$;
(ii) If $s \geq \frac{D}{4c\lambda^2}$, then $(1-\theta)(k_H - k_L)m_{31}(x) \leq c\lambda$.

Proof. See Appendix 6.5.

From Proposition 3, we notice that once the advertising agency reports her advertising cost parameter x, we can decide the optimal solution for $m(\cdot)$. And the optimal solutions of p and Q can be calculated by Lemma 4 based on Conditions (i) and (ii). Moreover, notice that

$s = x + h(x)$, so both $m_{34}(x)$ and $m_{31}(x)$ are decreasing functions. By observing Conditions (i) and (ii), we have

$$m_{34}(x_0) \geq \frac{c\lambda}{(1-\theta)(k_H - k_L)} > m_{31}(x_0),$$

where $x_0 + h(x_0) = \frac{D}{4c\lambda^2}$. Thus, the advertising effort level $m(\cdot)$ decreases when the report value of the parameter x increases, and the payment to the advertising agency also decreases accordingly. In addition, we have $Ak_a - c\lambda k_H \leq (A - c\lambda)k_a$, which implies that $m(\cdot)$ would decrease faster after the threshold x_0.

3.2.2 The Uniform Distribution Case

The impact of the change of the product advertising effort on the market demand is uncertain, thus it is necessary to simulate the model under different situations of k. Here k is a measure for market sensitivity to the product advertising effort. In particular, for a "new" product, no relevant information is available, one may assume that k follows a uniform distribution. Then the manufacturer would face a uniform market demand. This assumption can refer to Taylor [22], where the author considered a properly designed target rebate and returns contract under a uniform demand. In this section, we solve optimization problem (9) under the assumption that k is uniformly distributed, i.e., k follows $U(k_L, k_H)$. In this case, the average value of k is $k_a = \frac{k_L + k_H}{2}$. We present a lemma under the assumption that $m(\cdot)$ is given.

Lemma 5. *If k follows $U(k_L, k_H)$ and $m(\cdot)$ is given, then the optimal solution of optimization problem (9) is given by the following two cases:*
Case I: If $(k_H - k_L)m(x) \leq 2c\lambda$, then

$$\begin{cases} p^* &= \frac{A+km(x)-Q}{\lambda}, \\ Q^* &= \frac{A+k_a m(x)-\lambda c}{2}, \end{cases} \tag{31}$$

and

$$\pi = \frac{(A + k_a m(x) - \lambda c)^2}{4\lambda} - sm(x)^2,$$

Case II: If $(k_H - k_L)m(x) > 2c\lambda$, then

$$p^* = \begin{cases} \dfrac{A + km(x) - Q}{\lambda}, & \text{if } \dfrac{2Q - A}{m(x)} < k \leq k_H, \\ \dfrac{A + km(x)}{2\lambda}, & \text{if } k_L \leq k \leq \dfrac{2Q - A}{m(x)}, \end{cases} \tag{32}$$

$$Q^* = \frac{A + k_H m(x) - \sqrt{2\lambda c m(x)(k_H - k_L)}}{2}, \tag{33}$$

and

$$\begin{aligned}\pi = &\frac{1}{12\lambda}[(A + k_H m(x))^2 + (A + k_H m(x))(A + k_L m(x)) + (A + k_L m(x))^2] \\ &+ \frac{c\sqrt{2\lambda c m(x)(k_H - k_L)}}{3} - c\frac{A + k_H m(x)}{2} - s m(x)^2.\end{aligned} \tag{34}$$

Proof. *See Appendix 6.6.*

When we compare the results in Lemma 5 with those in Lemma 4, there is a remarkable resemblance between them. We note that $\frac{2\lambda c}{k_H - k_L}$ is the threshold in Lemma 5, which is the same as in Lemma 4 with $\theta = \frac{1}{2}$. But in this case, both p^* and Q^* do not linearly depend on $m(\cdot)$. And follows from Lemma 5, the optimal solution for $m(\cdot)$ is obtained in the following proposition.

Proposition 4. *If k follows $U(k_L, k_H)$ and $4\lambda\underline{b} - E(k^2) > 0$, then the optimal solution of $m(x)$ in optimization problem (9) is given by $m^*(x)$:*

$$m^*(x) = \begin{cases} \underline{m}, & \text{if } m_4(x) \le \underline{m}, \\ m_4(x), & \text{if } \underline{m} < m_4(x) \le \overline{m}, \\ \overline{m}, & \text{if } m_4(x) > \overline{m}, \end{cases} \tag{35}$$

where

$$D = k_a[Ak_a - (A - \lambda c)k_L],$$

and

$$m_4(x) = \begin{cases} m_{44}(x) \text{ (see appendix)} & , \text{ if } s < \dfrac{D}{4c\lambda^2}, \\ m_{41}(x) = \dfrac{(A - c\lambda)k_a}{4s\lambda - k_a^2} & , \text{ if } s \ge \dfrac{D}{4c\lambda^2}. \end{cases} \tag{36}$$

Furthermore, $m_{44}(x)$ and $m_{41}(x)$ satisfy the following conditions:

(i) If $s < \frac{D}{4c\lambda^2}$, then $(k_H - k_L)m_{44}(x) > 2c\lambda$;

(ii) If $s \ge \frac{D}{4c\lambda^2}$, then $(k_H - k_L)m_{41}(x) \le 2c\lambda$.

Proof. See Appendix 6.7.

With Conditions (i) and (ii) in Proposition 4, one can find the optimal solutions for p and Q through Lemma 5. Furthermore, it is shown that $m^*(x)$ decreases in x in the interval $[\underline{b}, \bar{b}]$, which means when the advertising agency reports a higher value of the parameter, the manufacturer will decide a lower advertising effort level. This agrees with the results that we obtained in the previous two cases.

4 Numerical Examples

This section gives some numerical examples to show the effectiveness and feasibility of the models in the presence of the contract. The true value of the advertising cost parameter x is assumed to be uniformly distributed over $[30, 80]$, which indicates that the advertising cost depends on the market and is uncertain. Other parameters are $A = 500$ units, $\lambda = 10$, $c = 5$ dollars, $\underline{m} = 1$, $\overline{m} = 20$. Five different cases of k are given, which represent different cases of the impacts of advertising on the market demand respectively.

Table 1: Five cases of k

Case (i)	Case (ii)	Case (iii)	Case (iv)	Case (v)
Constant $k = 15$	Constant $k = 20$	Bernoulli distributed $k_L = 10$, $k_H = 30$ $\theta = \frac{1}{3}$	Bernoulli distributed $k_L = 10$, $k_H = 30$ $\theta = \frac{1}{2}$	Uniformly distributed $k_L = 10$, $k_H = 30$

4.1 Feasibility Analysis

This part discusses the outsourcing contract designed above from the aspect of implementation as well as its impact on the profits of the manufacturer and the advertising agency. Case (v) is used as an example and other cases can be analyzed using the same method. We consider two different situations taking into account the development levels of advertising market: one is with a continuous advertising effort level and the other one is with a discrete advertising effort level.

4.1.1 Outsourcing Contract with a Continuous Advertising Effort Level

Table 2 presents the information of the contract when the advertising agency is able to provide a continuous advertising effort level. It includes the advertising effort level $m(y)$ and the payment $t(y)$ with respect to the reported value of the advertising cost parameter y. It shows that increase in y leads to decrease in both $m(y)$ and $t(y)$. Furthermore, Figures 1, 2 and 3 indicate that the advertising agency's net profit $\beta(x, y)$ reaches the maximum value when $y = x$ for any given x, which implies that the manufacturer is able to induce the advertising agency to report the actual advertising cost parameter under the outsourcing contract. They also show that a lower x results in a larger $\beta(x, x)$. However, the manufacturer fully bears the financial burden of increasing the advertising agency's revenue. This suggests that it would be costly for the manufacturer to establish such a contract when the actual advertising cost parameter is very low. Therefore, how to design an outsourcing contract to achieve coordination and win-win between a manufacturer and an advertising agency when the advertising cost parameter decreases would be an interesting research topic.

Table 2: Outsourcing contract.

y	30	35	40	45	50	55	60	65	70	75	80
$m(y)$	11.4	7.5	5.6	4.5	3.8	3.2	2.8	2.5	2.3	2.0	1.9
$t(y)$	4942.6	2611.9	1688.0	1206.6	914.1	718.9	580.1	476.6	396.6	333.0	281.3

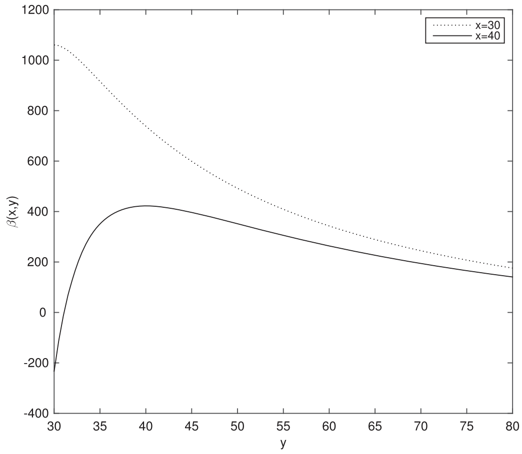

Figure 1: Advertising agency's net profit $\beta(x, y)$ when $x = 30$ and $x = 40$.

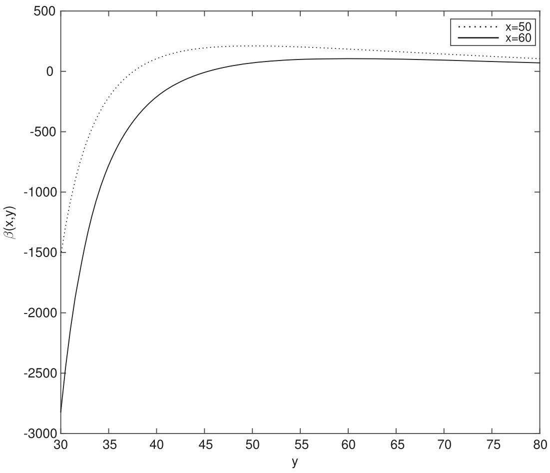

Figure 2: Advertising agency's net profit $\beta(x, y)$ when $x = 50$ and $x = 60$.

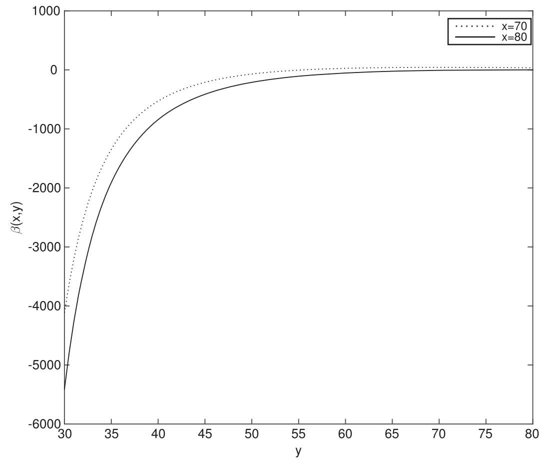

Figure 3: Advertising agency's net profit $\beta(x, y)$ when $x = 70$ and $x = 80$.

4.1.2 Outsourcing Contract with a Discrete Advertising Effort Level

The outsourcing contract with a discrete advertising effort level is analyzed using Algorithm 1. Assume that the range of the advertising effort level m is from 1 to 11, and for the conciseness of this section, only the first few values of y will be considered. Numerical results in Table 3 indicate that the payment $t(y)$ is sensitive to the change of the advertising effort level $m(y)$. It shows that as y increases, $m(y)$ tends to decrease and the decline in $t(y)$ is large only if $m(y)$ changes.

Table 3: Outsourcing contract: m is discrete.

y	30	30.5	31	31.5	32	32.5	33	33.5	34	34.5
$m(y)$	11	11	10	10	9	9	9	8	8	8
$t(y)$	4690.7	4689.7	4043.4	4042.6	3438.1	3435.9	3437.1	2871.7	2869.9	2870.6

Table 4 provides the optimal production plan for the manufacturer. Numerical analysis indicates that both the optimal production size Q and the optimal retail price p decline as y increases. The reason behind this is that the manufacturer would lower the advertising effort level to control the cost when y increases. As a result, the effect of the advertising on the market demand would be reduced. Then in order to improve the market demand of the product, the manufacturer would reduce the retail price p. However, the decline in the optimal production size Q indicates that the total demand would not be enhanced at the optimal retail price.

Table 4: Manufacturer's production plan: m is discrete.

y	30	30.5	31	31.5	32	32.5	33	33.5	34	34.5
$E(p)$	38.5	38.5	37.5	37.5	36.5	36.5	36.5	35.5	35.5	35.5
Q	341	341	329	329	318	318	318	307	307	307

Table 5 presents the advertising agency's net profit. It shows that the advertising agency can achieve the maximum profit for any given x when she reports the actual advertising cost parameter, i.e., $y = x$ or y is slightly less than the actual advertising cost parameter. This result indicates that the "information rent" can induce the advertising agency to report the true advertising cost parameter. Thus the outsourcing contract in our model can also be applied to handle the case with a discrete level of advertising effort. However, Table 5 also implies that the "information rent" is a little higher than it should be, making the advertising agency may even tend to report a lower value of the advertising cost parameter in this case. It suggests that the "information rent" should be scaled down in this discrete advertising effort level case.

Table 5: Advertising agency's net profit $\beta(x, y)$: m is discrete.

y \ x	30	30.5	31	31.5	32	32.5	33	33.5	34
30	1060.7	1000.2	939.7	879.2	818.7	758.2	697.7	637.2	576.7
30.5	1059.7	999.2	938.7	878.2	817.7	757.2	696.7	636.2	575.7
31	1043.4	993.4	943.4	893.4	843.4	793.4	743.4	693.4	643.4
31.5	1042.6	992.6	942.6	892.6	842.6	792.6	742.6	692.6	642.6
32	1008.1	967.6	927.1	886.6	846.1	805.6	765.1	724.6	684.1
32.5	1005.9	965.4	924.9	884.4	843.9	803.4	762.9	722.4	681.9
33	1007.1	966.6	926.1	885.6	845.1	804.6	764.1	723.6	683.1
33.5	951.7	919.7	887.7	855.7	823.7	791.7	759.7	727.7	695.7
34	949.9	917.9	885.9	853.9	821.9	789.9	757.9	725.9	693.9

4.2 Sensitivity Analysis

The previous section demonstrates that the outsourcing contract is able to induce the advertising agency to report the actual value of the advertising cost parameter. This section provides the sensitivity analysis of the proposed models under the true advertising cost parameter x.

Figures 4 and 5 depict the optimal advertising effort level $m(x)$ of the product and the optimal payment $t(x)$ to the advertising agency with respect to x. It shows that when x increases, the manufacturer would decrease his advertising effort level accordingly so that he would pay less to the advertising agency. Figure 6 shows that there is a positive correlation between $m(x)$ and $t(x)$ in all five cases. By comparing Cases (i) and (ii), (iii) and (iv), the correlation coefficient is related to the expectation of the demand factor k: the payment $t(\cdot)$ grows faster with a larger $E(k)$. It indicates that when the sensitivity of the market demand to the advertising effort level increases, the manufacturer is suggested to pay more attention to promote the coordination with the advertising agency.

Figures 7 and 8 present the optimal production size Q and the expectation of the optimal retail price $E(p)$. It shows that both of them go down as x goes up. The intuition is that a higher x makes it more likely for the manufacturer to lower his advertising effort level and retail price, which makes the market demand of the product to decrease slowly.

From Figures 4, 5, 7 and 8, it is observed that as x increases, the rates of decline in other factors including $m(x)$, $t(x)$, Q and $E(p)$ would slow down. It also shows that these factors drop much faster with a larger expectation (average) of k, meaning that the state of economic has a great influence on the outsourcing strategy and the production plan. Furthermore, by comparing cases (ii), (iv) and (v), which are different in k but with the same $E(k)$, it is found that when $s < \frac{D}{4\lambda^2 c}$, i.e., $x < 42.5$, $m(\cdot)$, $t(\cdot)$, Q and $E(p)$ are different in different cases. However, they are exactly the same if $x \geq 42.5$, which can also be observed by comparing Propositions 2, 3 and 4. Thus it suggests that if the advertising agency reports the true value of parameter x higher than $\frac{D}{4\lambda^2 c} - h(x)$, then the manufacturer can decide $m(\cdot)$, $t(\cdot)$ and Q based on the value of $E(k)$.

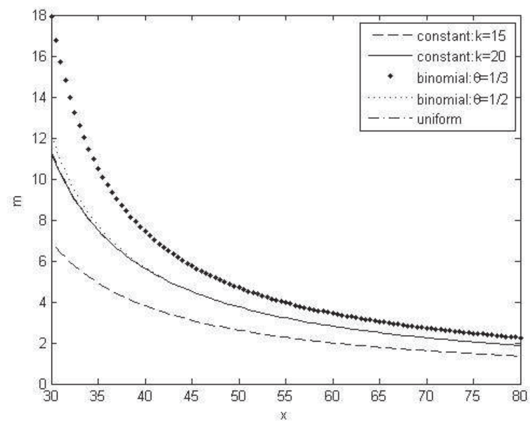

Figure 4: Advertising effort level of the product.

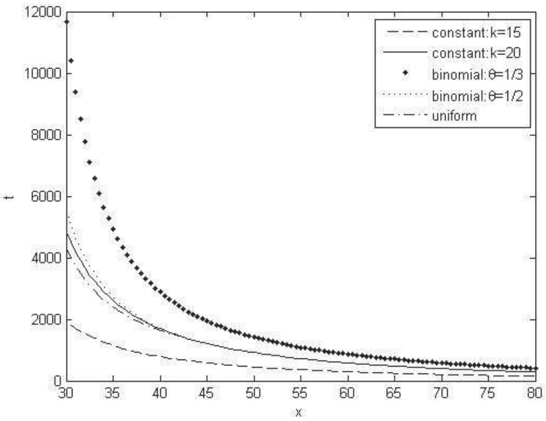

Figure 5: Payment to the advertising agency.

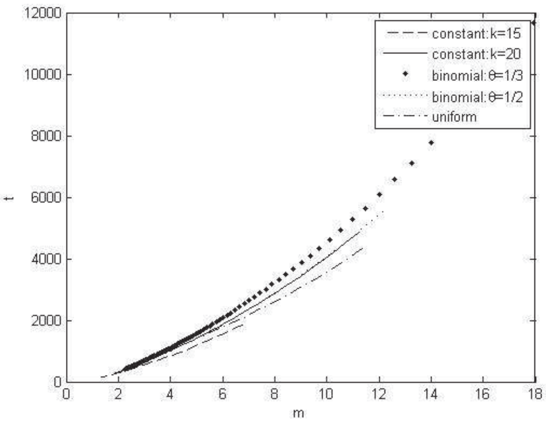

Figure 6: Relation between advertising effort level and payment to the advertising agency.

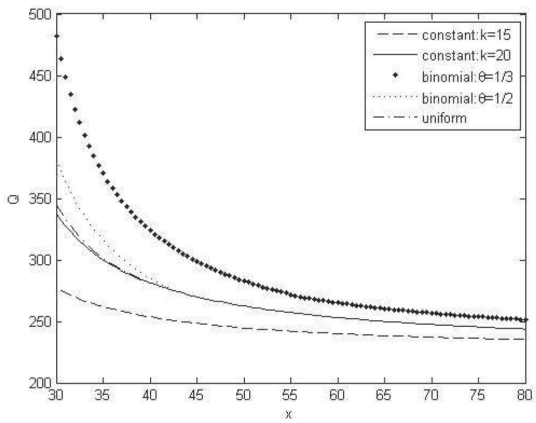

Figure 7: Production size.

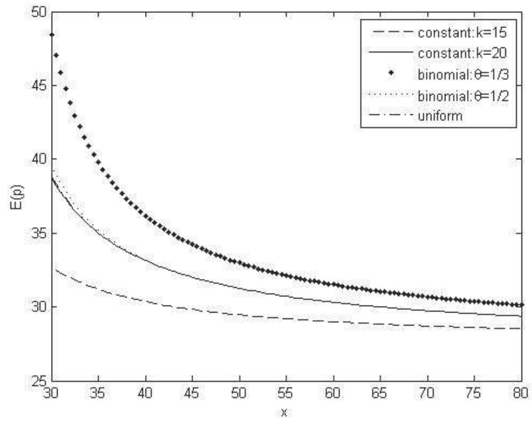

Figure 8: Expectation of the retail price.

5 Conclusions

This paper studies outsourcing strategies between one manufacturer and one advertising agency with an advertising effort-level-dependent and retail-price-dependent demand. We focus much of our discussion on one of the key aspects that affect the demand of product: advertising effort. As the advertising agency is the only one who knows the actual value of the advertising cost parameter of her advertising service, so asymmetric information problem exists in this system. The goal of the manufacturer is to maximize his profit. The contribution of this paper is to propose a mechanism by designing a contract so as to induce the advertising agency to report her actual advertising cost parameter. Here we solve this problem based on the revelation principle under the principal-agent framework. Another problem is the demand uncertainty. We consider different cases of the demand: deterministic and random. Through computing the partial derivatives of the objective function, we obtain analytical solutions of the optimization problem, taking into account the advertising effort level, payment to the advertising agency, production size and retail price. Based on our model, once the advertising agency reports her cost, the manufacturer can decide the above decision variables. Numerical examples are presented to illustrate the effectiveness and feasibility of the proposed models. Some useful management insights are obtained for the decision makers. This work adopts the linear function to describe the product demand when the advertising effort level is m. In practice, the relationship between an advertising effort level and the demand might be more complicated, such as non-linear demand in Huang and Li [11]. For future research, one may also consider a more complex supply chain, for example, by adding more outside competitors from the market, like multiple manufacturers or advertising agencies.

6 Appendix

6.1 Proof of Proposition 1

Proof. From (IC), we know that

$$\beta(x,y) = t(y) - xm(y)^2 \leq \beta(x,x), \quad \text{for} \quad y \in [\underline{b}, \bar{b}],$$

which implies

$$\frac{\partial \beta}{\partial y} \mid_{y=x} = t'(y) \mid_{y=x} -2xm(y)m'(y) \mid_{y=x} = 0, \tag{37}$$

$$\begin{aligned} \frac{\partial^2 \beta}{\partial y^2} \mid_{y=x} &= t''(y) - 2x[m'(y)^2 + m(y)m''(y)] \mid_{y=x} \\ &= t''(x) - 2x[m'(x)^2 + m(x)m''(x)] \leq 0. \end{aligned} \tag{38}$$

From Eq. (37), we have $t'(x) = 2xm(x)m'(x)$. Then we differentiate it with respect to x and obtain

$$t''(x) - 2[m(x)m'(x) + xm'(x)^2 + xm(x)m''(x)] = 0. \tag{39}$$

It follows from Eq. (38) and Eq. (39) that $m'(x) \leq 0$. So from Eq. (37) and Eq. (38), we have

$$\begin{cases} t'(x) = 2xm(x)m'(x), \\ m'(x) \leq 0. \end{cases} \tag{40}$$

On the other hand, Eq. (37) and Eq. (38) can be deduced from Eq. (40). Thus (IC) in Eq. (8) is equivalent to Eq. (40).

We note that $\beta(x,x) = t(x) - xm(x)^2$, so $\beta(\bar{b},\bar{b}) = t(\bar{b}) - \bar{b}m(\bar{b})^2$. Since

$$\frac{\partial \beta(x,x)}{\partial x} = -m(y)^2 \leq 0,$$

$\beta(x,x)$ decreases when x increases. Consequently, to maximize the profit of the manufacturer, (PC) can be written as $\beta(\bar{b},\bar{b}) = 0$, i.e., $t(\bar{b}) = \bar{b}m(\bar{b})^2$. Because $m'(x) \leq 0$, (DC) can be described as $m(\underline{b}) = \overline{m}$, $m(\bar{b}) \geq 1$.

By integrating both sides of the equation $t'(x) = 2xm(x)m'(x)$, we obtain

$$t(x) = xm(x)^2 + \int_x^{\bar{b}} m(s)^2 ds. \tag{41}$$

By substituting Eq. (41) into the profit function (7), one can obtain

$$E(\pi) = \int_{\underline{b}}^{\bar{b}} [(p-c)Q - p[Q-(A-\lambda p + km(x))]^{+} - (x+h(x))m(x)^2]f(x)dx.$$

Here $h(x)$ is the inverse hazard rate i.e., $h(x) = F(x)/f(x)$. The proof is complete.

6.2 Proof of Lemma 1

Proof. As we assume that $m(x)$ is given, our optimization problem is the following:

$$\max_{p,Q}\{pQ - p[Q-(A-\lambda p + km(x))]^{+} - sm(x)^2 - cQ\}. \tag{42}$$

In this case, once the unit retail price p is decided, the demand is deterministic as k here is constant. Thus the production quantity Q should be equal to the demand in this optimization problem, i.e., $Q = (A - \lambda p + km(x))$. Rewrite the problem as follows:

$$\max_{p}\{(p-c)(A-\lambda p + km(x)) - sm(x)^2\},$$

which is concave with respect to p. The optimal unit retail price p can be calculated as Eq. (11), the optimal production quantity Q can be obtained. The proof is complete.

6.3 Proof of Lemma 2

Proof. If k is a random variable, then our optimization problem would be Eq. (42) with $k \in [k_L, k_H]$ and $g(\cdot)$ is the density function. Now we are going to solve the problem assuming $m(\cdot)$ is already known. First we divide the problem into five situations to eliminate $[\cdot]^{+}$ in the objective function, and then by calculating the partial derivatives of the objective function, we can get the following results:

Case I. If $A + k_L m(x) < A + k_H m(x) \le Q$, by considering the partial derivatives and the boundary constraints, we have

$$\begin{cases} p^* &= \dfrac{A+km(x)}{2\lambda}, \\ Q^* &= A + k_H m(x), \\ \pi_1 &= \displaystyle\int_{k_L}^{k_H} \frac{(A+km(x))^2}{4\lambda} g(k)dk - c(A+k_H m(x)) - sm(x)^2. \end{cases} \tag{43}$$

Case II. If $A + k_L m(x) \leq Q < A + k_H m(x) < 2Q$, then

$$\begin{cases} p^* &= \dfrac{A + km(x)}{2\lambda}, \\ Q^* &= A + k_L m(x), \\ \pi_2 &= \displaystyle\int_{k_L}^{k_H} \frac{\theta(A + km(x))^2}{4\lambda} g(k)dk - c(A + k_L m(x)) - sm(x)^2. \end{cases} \tag{44}$$

Case III. If $Q < 2Q \leq A + k_L m(x) < A + k_H m(x)$, then we have the solution

$$\begin{cases} p^* &= \frac{A+km(x)-Q}{\lambda}, \\ \pi &= \displaystyle\int_{k_L}^{k_H} \frac{1}{\lambda}(A + km(x) - Q)Qg(k)dk - cQ - sm(x)^2 \\ &= \dfrac{A + k_a m(x)}{\lambda} Q - \dfrac{Q^2}{\lambda} - cQ - sm(x)^2, \end{cases} \tag{45}$$

where

$$k_a = \int_{k_L}^{k_H} kg(k)dk.$$

If we set $\frac{\partial \pi}{\partial Q} = 0$, then we have

$$Q_0 = \frac{A + k_a m(x) - \lambda c}{2}.$$

If $Q_0 \leq \frac{A+k_L m(x)}{2}$, i.e.,

$$(k_a - k_L)m(x) \leq c\lambda, \tag{46}$$

then we have the solution

$$\begin{cases} Q^* &= \frac{A+k_a m(x)-\lambda c}{2}, \\ \pi_{31} &= \frac{(A+k_a m(x)-\lambda c)^2}{4\lambda} - sm(x)^2. \end{cases} \tag{47}$$

And if $(k_a - k_L)m(x) > c\lambda$, the solution would be

$$\begin{cases} Q^* &= \frac{A+k_L m(x)}{2}, \\ \pi_{32} &= \frac{(A+k_a m(x))(A+k_L m(x))}{2\lambda} - \frac{(A+k_L m(x))^2}{4\lambda} - c\frac{A+k_L m(x)}{2} - sm(x)^2. \end{cases} \tag{48}$$

Case IV. If $Q < A + k_L m(x) \leq 2Q \leq A + k_H m(x)$, there exists $k_q \in [k_L, k_H]$ such that $A + k_q m(x) = 2Q$, then it has

$$p^* = \begin{cases} \dfrac{A + km(x) - Q}{\lambda}, \text{ if } k \in [k_q, k_H], \\ \dfrac{A + km(x)}{2\lambda}, \text{ if } k = [k_L, k_q], \end{cases} \tag{49}$$

$$\pi_4 = Q\int_{k_q}^{k_H}\frac{A+km(x)}{\lambda}g(k)dk - \frac{Q^2}{\lambda}\int_{k_q}^{k_H}g(k)dk - sm(x)^2 - cQ + \int_{k_L}^{k_q}\frac{(A+km(x))^2}{4\lambda}g(k)dk. \tag{50}$$

Case V. If $Q < A + k_L m(x) < A + k_H m(x) \leq 2Q$, then we have

$$\begin{cases} p^* = \dfrac{A+km(x)}{2\lambda}, \\ Q^* = \dfrac{A+k_H m(x)}{2}, \\ \pi_5 = \displaystyle\int_{k_L}^{k_H}\frac{(A+km(x))^2}{4\lambda}g(k)dk - sm(x)^2 - cQ. \end{cases} \tag{51}$$

By direct verification, we have $\pi_4 \geq \pi_5 > \pi_2 > \pi_1$ Now we are going to divide the problem into two different situations,

$$(k_a - k_L)m(x) \leq c\lambda \quad \text{and} \quad (k_a - k_L)m(x) > c\lambda.$$

(1) If $(k_a - k_L)m(x) \leq c\lambda$, by computation, $\pi_{31} \geq \pi_4$, then the maximum value of the manufacturer's profit would be π_{31};
(2) If $(k_a - k_L)m(x) > c\lambda$, then by direct verification, we have $\pi_4 \geq \pi_{32}$. Hence the maximum value of the manufacturer's profit would be π_4. The proof is complete.

6.4 Proof of Lemma 4

Proof. If k is Bernoulli distributed, then our optimization problem would be Eq. (42) with k satisfying Eq. (24). $k_a = \theta k_L + (1-\theta)k_H$ is the average value of k. Now we are going to solve the problem assuming $m(\cdot)$ is already known. Based on Lemma 2, we can get the following results and cases I, II, V have been omitted:

Case III. If $Q < 2Q \leq A + k_L m(x) < A + k_H m(x)$, then we have the solution

$$\begin{cases} p^* &= \frac{A+km(x)-Q}{\lambda}, \\ \pi &= \frac{A+k_a m(x)}{\lambda}Q - \frac{1}{\lambda}Q^2 - cQ - sm(x)^2. \end{cases} \tag{52}$$

If we set $\frac{\partial \pi}{\partial Q} = 0$, then we have

$$Q_0 = \frac{A + k_a m(x) - \lambda c}{2}.$$

If $Q_0 \le \frac{A+k_L m(x)}{2}$, i.e.,

$$(1-\theta)(k_H - k_L)m(x) \le c\lambda, \tag{53}$$

then we have the solution

$$\begin{cases} Q^* &= \frac{A+k_a m(x)-\lambda c}{2}, \\ \pi_{31} &= \frac{(A+k_a m(x)-\lambda c)^2}{4\lambda} - sm(x)^2. \end{cases} \tag{54}$$

Case IV. If $Q < A + k_L m(x) \le 2Q \le A + k_H m(x)$, the computational method of Case IV would be similar to Case III, and by direct verification, Condition (53) also holds in Case IV. Thus we have

$$p^* = \begin{cases} \dfrac{A + k_H m(x) - Q}{\lambda}, \text{ if } k = k_H, \\ \dfrac{A + k_L m(x)}{2\lambda}, \text{ if } k = k_L. \end{cases} \tag{55}$$

There are two situations.

(1) If $(1-\theta)(k_H - k_L)m(x) \le c\lambda$, then the optimal solution would be

$$\begin{aligned} Q^* &= \frac{A + k_L m(x)}{2}, \\ \pi_{41} &= \theta \frac{(A + k_L m(x))^2}{4\lambda} + (1-\theta)\frac{(A + k_L m(x))(A + (2k_H - k_L)m(x))}{4\lambda} \\ &\quad - 2c\lambda \frac{A + k_L m(x)}{4\lambda} - sm(x)^2. \end{aligned} \tag{56}$$

(2) If $(1-\theta)(k_H - k_L)m(x) > c\lambda$, then the optimal solution would be

$$\begin{aligned} Q^* &= \frac{A + k_H m(x)}{2} - \frac{c\lambda}{2(1-\theta)}, \\ \pi_{42} &= \theta \frac{(A + k_L m(x))^2}{4\lambda} + (1-\theta)\frac{(A + k_H m(x))^2}{4\lambda} + \frac{c^2\lambda}{4(1-\theta)} - c\frac{A + k_H m(x)}{2} - sm(x)^2. \end{aligned} \tag{57}$$

Now we are going to divide the problem into two different situations,

$$(1-\theta)(k_H - k_L)m(x) \le c\lambda \quad \text{and} \quad (1-\theta)(k_H - k_L)m(x) > c\lambda.$$

(1) If $(1-\theta)(k_H - k_L)m(x) \le c\lambda$, then by direct verification, we have $\pi_{31} > \pi_4$. Hence the optimal solution would be Eq. (54);
(2) If $(1-\theta)(k_H - k_L)m(x) > c\lambda$, the optimal solution is given in Eq. (57).

The proof is complete.

6.5 Proof of Proposition 3

Proof. From Lemma 4, we have the following two situations:

(S1) if $(1-\theta)(k_H - k_L)m(s) \le c\lambda$, then our optimization problem would be

$$\max_{m(\cdot)} \left\{ \frac{(A + k_a m(x))^2}{4\lambda} - \frac{c(A + k_a m(x))}{2} + \frac{\lambda c^2}{4} - sm(x)^2 \right\}.$$

By computing the partial derivative of the objective function, we have the optimal solution $m_0 = \frac{(A-c\lambda)k_a}{4s\lambda - k_a^2}$ without considering the boundaries. Then we are going to compare m_0 with the boundary $\frac{c\lambda}{(1-\theta)(k_H-k_L)}$:

$$m_0 - \frac{c\lambda}{(1-\theta)(k_H - k_L)} = \frac{k_a[Ak_a - (A - c\lambda)k_L] - 4c\lambda^2 s}{(4s\lambda - k_a^2)(1-\theta)(k_H - k_L)}.$$

Set

$$D = k_a[Ak_a - (A - c\lambda)k_L].$$

From the assumption above, we know that $4s\lambda > k_a^2$, so we have two cases:

Case I: If $4c\lambda^2 s \ge D$, then we have

$$m_{31}(x) = m_0 = \frac{(A - c\lambda)k_a}{4s\lambda - k_a^2},$$

denoting m_{31} as the optimal solution;

Case II: If $4c\lambda^2 s < D$, then we have

$$m_{32}(x) = \frac{c\lambda}{(1-\theta)(k_H - k_L)},$$

denoting m_{32} as the optimal solution.

(S2) If $(1-\theta)(k_H - k_L)m(x) > c\lambda$, then our optimization problem would be

$$\max_{m(\cdot)} \left\{ \theta \frac{(A + k_L m(x))^2}{4\lambda} + (1-\theta)\frac{(A + k_H m(x))^2}{4\lambda} + \frac{c^2\lambda}{4(1-\theta)} - c\frac{A + k_H m(s)}{2} - sm(x)^2 \right\}. \tag{58}$$

Then we will conduct similar computations as in (S1). It is very interesting that we can get the same boundary D, namely, the optimal solution will be divided by D in (S2) too. And the optimal solutions are:

Case I. If $4c\lambda^2 s \ge D$, then we have $m_{33}(x) = \frac{c\lambda}{(1-\theta)(k_H-k_L)}$, denoting m_{33} as the optimal solution;

Case II. If $4c\lambda^2 s < D$, then we have $m_{34}(x) = \frac{Ak_a - c\lambda k_H}{4s\lambda - (\theta k_L^2 + (1-\theta)k_H^2)}$, denoting m_{34} as the optimal solution.

By direct computation, if $4c\lambda^2 x \geq D$, then we have $\pi_{31}(m_{31}) > \pi_{42}(m_{33})$, so $m^* = m_{31}$. If $4c\lambda^2 x < D$, then we have $\pi_{42}(m_{34}) > \pi_{31}(m_{32})$, so $m^* = m_{34}$. And then we can find the corresponding p^* and Q^*. The proof is then complete.

6.6 Proof of Lemma 5

Proof. First, we eliminate $[\cdot]^+$ in the objective function, we divide the problem into five situations exactly as we do in the proof of Lemma 2. We omit the detailed calculations here and we use the same notations as we used in the proof of Lemma 2. By direct calculation and denote the average value of k as $k_a = \frac{k_H + k_L}{2}$, we have the following results.

(i) If $(k_H - k_L)m \leq 2\lambda c$, the optimal solution would be in Case III, which is

$$p^* = \frac{A + km(x) - Q}{\lambda}, \tag{59}$$

$$Q^* = \frac{A + k_a m(x) - \lambda c}{2}, \tag{60}$$

$$\pi = \frac{(A + k_a m - \lambda c)^2}{4\lambda} - sm(x)^2. \tag{61}$$

(ii) If $(k_H - k_L)m > 2\lambda c$, the optimal solution would be in Case IV, which is

$$p^* = \begin{cases} \dfrac{A + km(x) - Q}{\lambda}, & \text{if } \dfrac{2Q - A}{m(x)} < k \leq k_H, \\ \dfrac{A + km(x)}{2\lambda}, & \text{if } k_L \leq k \leq \dfrac{2Q - A}{m(x)}, \end{cases} \tag{62}$$

$$Q* = \frac{A + k_H m(x) - \sqrt{2\lambda c m(x)(k_H - k_L)}}{2}, \tag{63}$$

$$\begin{aligned} \pi = &\frac{1}{12\lambda}[(A + k_H m(x))^2 + (A + k_H m(x))(A + k_L m(x)) + (A + k_L m(x))^2] \\ &+ \frac{c\sqrt{2\lambda c m(x)(k_H - k_L)}}{3} - c\frac{A + k_H m(x)}{2} - sm(x)^2. \end{aligned} \tag{64}$$

The proof is complete.

6.7 Proof of Proposition 4

Proof. Using the results in Lemma 5, with the similar arguments in the proof of Proposition 3, we have the following two situations.

(S1) If $(k_H - k_L)m(x) \leq 2c\lambda$, then our optimization problem is given by

$$\max_{m(\cdot)} \left\{ \frac{(A + k_a m(x) - \lambda c)^2}{4\lambda} - sm(x)^2 \right\}. \tag{65}$$

Denote

$$D = k_a[Ak_a - (A - \lambda c)k_L],$$

and the solution is given by

Case I: If $4c\lambda^2 s \geq D$, then we have $m_{41}(x) = \frac{k_a(A-\lambda c)}{4\lambda s - {k_a}^2}$, denoting m_{41} as the optimal solution;

Case II: If $4c\lambda^2 s < D$, then we have $m_{42}(x) = \frac{2\lambda c}{k_H - k_L}$, denoting m_{42} as the optimal solution.

(S2) If $(k_H - k_L)m(x) > 2c\lambda$, then the optimization problem is

$$\max_{m(\cdot)} \left\{ \begin{array}{l} \frac{1}{12\lambda}[(A + k_H m(x))^2 + (A + k_H m(x))(A + k_L m(x)) + (A + k_L m(x))^2] \\ \qquad\qquad + \frac{c\sqrt{2\lambda c m(x)(k_H - k_L)}}{3} - c\frac{A + k_H m(x)}{2} - sm(x)^2 \end{array} \right\}. \tag{66}$$

Without considering the boundary $(k_H - k_L)m(x) > 2c\lambda$, we denote by $m_{44}(x)$ the optimal solution, which is the solution of the following equation:

$$\begin{aligned} &\frac{1}{12\lambda}[2k_H(A + k_H m(x)) + k_H(A + k_L m(x)) + 2k_L(A + k_L m(x)) + k_L(A + k_H m(x))] \\ &\qquad - \frac{ck_H}{2} + \frac{c}{6}\sqrt{\frac{2\lambda c(k_H - k_L)}{m(x)}} - 2sm(x) = 0. \end{aligned} \tag{67}$$

Using the assumption and the fact that $A \gg \lambda c$, Eq. (67) has three real roots and only one of them is positive, and the positive one is $m_{44}(x)$. We can get the solution $m_{44}(x) = M/N$ where

$$\begin{aligned} M \;=\; & [-Ak_L^2 - 2Ak_L^2 k_H - 2Ak_H k_L^2 - Ak_L^3 + 2ck_H^3 r + 2ck_H^2 k_L \lambda + 2ck_H k_L^2 \lambda \\ & +12Ak_H\lambda s + 12Ak_L\lambda s - 24ck_H\lambda^2 s + (-2c\lambda\sqrt{c(k_H - k_L)\lambda}(k_H^2 + k_H k_L + k_L^2 - 12\lambda s)^2 \\ & +(((k_1^2 + k_1 k_2 + k_2^2 - 12\lambda s)^3(A^3(k_H + k_L)^3 - 6A^2 ck_H(k_H + k_L)^2\lambda + 12Ac^2 k_H^2(k_H + k_L)\lambda^2 \\ & -4c^3\lambda^3(k_H^3 + k_L^3 + 12k_H\lambda s - 12k_L\lambda s))))^{2/3}]^2, \end{aligned}$$

and

$$
\begin{aligned}
N &= [2(k_H^2 + k_H k_L + k_L^2 - 12\lambda s)^2(-2c\lambda\sqrt{c(k_H - k_L)\lambda}(k_H^2 + k_H k_L + k_L^2 - 12\lambda s)^2 \\
&\quad +(((k_H^2 + k_H A k_L + k_L^2 - 12\lambda s)^3(A^3(k_H + k_L)^3 - 6A^2 c k_H (k_H + k_L)^2\lambda \\
&\quad +12Ac^2 k_H^2 (k_H + k_L)\lambda^2 - 4c^3\lambda^3(k_H^3 + k_L^3 + 12k_H\lambda s - 12k_L\lambda s))))^{2/3}].
\end{aligned}
$$

By substituting the boundary $\frac{2c\lambda}{k_H - k_L}$ into Eq. (67), we can check that D is the same as in (S1). Then by direct computation, we have

Case I: If $4c\lambda^2 s \geq D$, then we have $m_{43}(x) = \frac{2\lambda c}{k_H - k_L}$, denoting m_{43} as the optimal solution;

Case II: If $4c\lambda^2 s < D$, then we have $m_{44}(x)$ as the optimal solution.

By direct verification, if $4c\lambda^2 x \geq D$, then we have $\pi_3(m_{41}) > \pi_4(m_{43})$, so $m^* = m_{41}$. If $4c\lambda^2 x < D$, then we have $\pi_4(m_{44}) > \pi_3(m_{42})$, so $m^* = m_{44}$. And then we can find the corresponding p^* and Q^*. The proof is complete.

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