On strongly rigid hyperfluctuating random measures

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Abstract

In contrast to previous belief, we provide examples of stationary ergodic random measures that are both hyperfluctuating and strongly rigid. Therefore, we study hyperplane intersection processes (HIPs) that are formed by the vertices of Poisson hyperplane tessellations. These HIPs are known to be hyperfluctuating, that is, the variance of the number of points in a bounded observation window grows faster than the size of the window. Here we show that the HIPs exhibit a particularly strong rigidity property. For any bounded Borel set B, an exponentially small (bounded) stopping set suffices to reconstruct the position of all points in B and, in fact, all hyperplanes intersecting B. Therefore, also the random measures supported by the hyperplane intersections of arbitrary (but fixed) dimension, are hyperfluctuating. Our examples aid the search for relations between correlations, density fluctuations, and rigidity properties.

Keywords: Strong rigidity, hyperfluctuation, hyperuniformity, Poisson hyperplane tessellations, hyperplane intersection processes

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1 Introduction

Let Φ be a *random measure* on the d-dimensional Euclidean space $\mathbb{R}^d$; see [10, 14]. In this note all random objects are defined over a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with associated expectation operator $\mathbb{E}$. Assume that Φ is *stationary*, that is distributionally invariant under translations. Assume also that Φ is *locally square integrable*, that is $\mathbb{E}[\Phi(B)^2] < \infty$ for all compact $B \subset \mathbb{R}^d$. Take a *convex body* W, that is a compact and convex subset of $\mathbb{R}^d$ and assume that W has positive volume $V_d(W)$. In many cases of interest one can define an *asymptotic variance* by the limit

$$\sigma^2 := \lim_{r\to\infty} \frac{\mathbb{V}\mathrm{ar}[\Phi(rW)]}{V_d(rW)}, \tag{1.1}$$

where the cases $\sigma^2 = 0$ and $\sigma^2 = \infty$ are allowed. This limit may depend on W; but we do not include this dependence into our notation. Quite often the asymptotic variance

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σ^2 is positive and finite. If, however, $\sigma^2 = 0$, then Φ is said to be *hyperuniform* [19, 20]. If $\sigma^2 = \infty$, then Φ is said to be *hyperfluctuating* [20]. In recent years hyperuniform random measures (in particular point processes) have attracted a great deal of attention. The local behavior of such processes can very much resemble that of a weakly correlated point process. Only on a global scale a regular geometric pattern might become visible. Large-scale density fluctuations remain anomalously suppressed similar to a lattice; see [19, 20, 5]. The concept of hyperuniformity connects a broad range of areas of research (in physics) [20], including unique effective properties of heterogeneous materials, Coulomb systems, avian photoreceptor cells, self-organization, and isotropic photonic band gaps.

A point process Φ on $\mathbb{R}^d$ is said to be *number rigid* if the number of points inside a given compact set is almost surely determined by the configuration of points outside [16, 2]. Examples of number rigid point processes include lattices independently perturbed by bounded random variables, Gibbs processes with certain long-range interactions [3], zeros of Gaussian entire functions [8], stable matchings from [11], and some determinantal processes with a projection kernel [4].

It was proved in [6] that in one and two dimensions a hyperuniform point process is number rigid, provided that the truncated pair-correlation function is decaying sufficiently fast. Quite remarkably, it was shown in [16] that in three and higher dimensions a Gaussian independent perturbation of a lattice (which is hyperuniform) is number rigid below a critical value of the variance but not number rigid above. It is believed [5] that a stationary number rigid point process is hyperuniform. In this note we show that this is not true. In fact we give examples of stationary and ergodic (in fact mixing) random measures that are both hyperfluctuating and rigid in a very strong sense. The authors are not aware of any previously known rigid and ergodic process that is non-hyperuniform in dimensions $d \geq 2$ (if W is the unit ball). An example for $d = 1$ has very recently been given in [12]. In this paper we will prove that the point process resulting from intersecting Poisson hyperplanes has very strong rigidity properties. This point process is hyperfluctuating [9] and, under an additional assumption on the directional distribution, mixing; see [18, Theorem 10.5.3] and Remark 2.1.

2 Poisson hyperplane processes

In this section we collect a few basic properties of Poisson hyperplane processes and the associated intersection processes. Let $\mathbb{H}^{d-1}$ denote the space of all hyperplanes in $\mathbb{R}^d$. Any such hyperplane H is of the form

$$H_{u,s} := \{y \in \mathbb{R}^d : \langle y, u\rangle = s\}, \tag{2.1}$$

where u is an element of the unit sphere $\mathbb{S}^{d-1}$, $s \in \mathbb{R}$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product. (Any hyperplane has two representations of this type.) We can make $\mathbb{H}^{d-1}$ a measurable space by introducing the smallest σ-field containing for each compact $K \subset \mathbb{R}^d$ the set

$$[K] := \{H \in \mathbb{H}^{d-1} : H \cap K \neq \emptyset\}. \tag{2.2}$$

In fact, $\mathbb{H}^{d-1} \cup \{\emptyset\}$ can be shown to be a closed subset of the space of all closed subsets of $\mathbb{R}^d$, equipped with the Fell topology. We refer to [18] for more details on this topology and related measurability issues; see also [14, Appendix A3].

We consider a (stationary) *Poisson hyperplane process*, that is a *Poisson process* η on $\mathbb{H}^{d-1}$ whose intensity measure is given by

$$\lambda = \gamma \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} \mathbf{1}\{H_{u,s} \in \cdot\} \, ds \, \mathbb{Q}(du), \tag{2.3}$$

where $\gamma > 0$ is an *intensity* parameter and $\mathbb{Q}$ (the *directional distribution* of η) is an even probability measure on $\mathbb{S}^{d-1}$. We assume that $\mathbb{Q}$ is not concentrated on a great subsphere. It would be helpful (even though not strictly necessary) if the reader is familiar with basic point process and random measure terminology; see e.g. [14]. For our purposes it is mostly enough to interpret η as a random discrete subset of $\mathbb{H}^{d-1}$. The number of points (hyperplanes) in a measurable set $A \subset \mathbb{H}^{d-1}$ is then given by $|\eta \cap A|$ and has a Poisson distribution with parameter $\lambda(A)$. Since λ is invariant under translations (we have for all $x \in \mathbb{R}^d$ that $\lambda(\cdot) = \lambda(\{H : H + x \in \cdot\})$), the Poisson process η is *stationary*, that is distributionally invariant under translations. Furthermore we can derive from *Campbell's theorem* (see e.g. [14, Proposition 2.7]) and (2.3) that

$$\mathbb{E}[|\eta \cap [K]|] < \infty, \quad K \subset \mathbb{R}^d \text{ compact}. \tag{2.4}$$

As usual we assume (without loss of generality) that $|\eta(\omega) \cap [K]| < \infty$ for all $\omega \in \Omega$ and all compact $K \subset \mathbb{R}^d$. More details on Poisson hyperplane processes can be found in [18, Section 4.4].

Let $m \in \{1, \dots, d\}$. We define a random measure Φ_m on $\mathbb{R}^d$ by

$$\Phi_m(B) := \frac{1}{m!} \sum_{H_1,\dots,H_m \in \eta}^{\neq} \mathcal{H}^{d-m}(B \cap H_1 \cap \dots \cap H_m) \tag{2.5}$$

for Borel sets $B \subset \mathbb{R}^d$, where $\sum^{\neq}$ denotes summation over pairwise distinct entries and where $\mathcal{H}^{d-m}$ is the Hausdorff measure of dimension $d - m$; see e.g. [14, Appendix A.3]. Using the arguments on p. 130 of [18] one can show that almost surely for all distinct $H_1, \dots, H_m \in \eta$ the intersection $H_1 \cap \dots \cap H_m$ is either empty or has dimension $d - m$. Combining this with (2.4), we see that the random measures $\Phi_1, \dots, \Phi_m$ are almost surely *locally finite*, that is finite on bounded Borel sets. The random variable $\Phi_m(B)$ is the volume contents (in the appropriate dimension) of all possible intersections of $d - m$ hyperplanes within B.

It can be shown that (almost surely) the intersection of $d + 1$ different hyperplanes from η is empty. Therefore the random measure Φ_d is almost surely a point process without multiplicities, so that $\Phi_d(B)$ is just the number of (intersection) points $x \in B$ with $\{x\} = H_1 \cap \dots \cap H_d$ for some $H_1, \dots, H_d \in \eta$. It is convenient to define a simple (and locally finite) point process Φ as the set of all points $x \in \mathbb{R}^d$ with $\{x\} = H_1 \cap \dots \cap H_d$ for some $H_1, \dots, H_d \in \eta$. When (as it is common) interpreting Φ as a random counting measure, we have that $\mathbb{P}(\Phi = \Phi_d) = 1$. Figure 1 shows two samples of η and Φ.

Among other things, Theorem 4.4.8 in [18] gives a formula for the *intensity* $\gamma_m := \mathbb{E}[\Phi_m([0, 1]^d)]$ of Φ_m. We only need to know that it is positive and finite. In the remaining part of this section we recall some second order properties of Φ_m. (At first reading some details could be skipped without too much loss.) Let A, B be bounded Borel subsets of $\mathbb{R}^d$. Using the theory of U-statistics [14, Section 12.3] it was shown in [13] that

$$\lim_{r \to \infty} r^{-(2d-1)} \, \mathbb{C}\text{ov}[\Phi_m(rA), \Phi_m(rB)] = C_m(A, B), \tag{2.6}$$

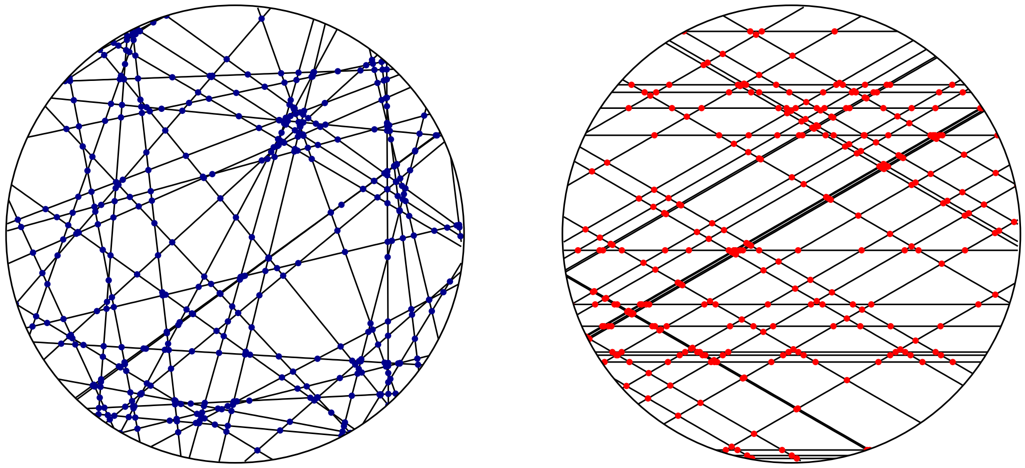


Figure 1: Samples of Poisson hyperplane processes η (lines) and the corresponding intersection processes Φ (solid circles) for two directional distributions: isotropic (left) and only three directions (right).

where

$$C_m(A,B) := \frac{1}{((m-1)!)^2} \int \bigg(\int \mathcal{H}^{d-m}(A \cap H_1 \cap \cdots \cap H_m)\, \lambda^{m-1}\big(d(H_2,\dots,H_m)\big) \bigg) \times \bigg(\int \mathcal{H}^{d-m}(B \cap H_1 \cap H_2' \cap \cdots \cap H_m')\, \lambda^{m-1}\big(d(H_2',\dots,H_m')\big) \bigg) \lambda(dH_1). \quad (2.7)$$

If $m=1$, this has to be read as

$$C_1(A,B) = \int \mathcal{H}^{d-1}(A \cap H_1)\mathcal{H}^{d-1}(B \cap H_1)\, \lambda(dH_1).$$

The asymptotic variance $C_m(A,A)$ was derived in [9]. We note that $C_m(A,A)$ is finite (this is implied by the form (2.3) of λ) and that $C_m(A,A)=0$ iff

$$\int \mathcal{H}^{d-m}(A \cap H_1 \cap \cdots \cap H_m)\, \lambda^m(d(H_1,\dots,H_m)) = 0.$$

Since $\mathbb{Q}$ is not concentrated on a great subsphere, this happens if and only if the Lebesgue measure of A vanishes; see the proof of [18, Theorem 4.4.8]. Therefore we obtain from (2.6) that the random measures $\Phi_1,\dots,\Phi_d$ are hyperfluctuating (if $d \geq 2$). The results in [9, 13] show that, for each finite collection $B_1,\dots,B_n$ of bounded Borel sets, the random vector $r^{-(d-1/2)}(\Phi_m(rB_1)-\mathbb{E}[\Phi_m(rB_1)],\dots,\Phi_m(rB_n)-\mathbb{E}[\Phi_m(rB_n)])$ converges in distribution to a multivariate normal distribution.

It is worth noting that the asymptotic covariances (2.7) are non-negative. If η is *isotropic* (meaning that $\mathbb{Q}$ is the uniform distribution on $\mathbb{S}^{d-1}$), there exist more detailed non-asymptotic second order results. In this case [9, p. 936] shows the *pair correlation function* ρ_2 (see e.g. [14, Section 8.2]) of the intersection point process $\Phi = \Phi_d$ is given by

$$\rho_2(x) = 1 + \sum_{i=1}^{d} a_i \gamma^{-i} \|x\|^{-i}, \quad x \in \mathbb{R}^d,\, x \neq 0, \quad (2.8)$$

where the coefficients $a_1, \dots, a_d$ are strictly poisitive and do only depend on the dimension. Hence, as $\|x\| \to \infty$, $\rho_2(x) - 1 \to 0$ only at speed $\|x\|^{-1}$. In particular, the *truncated* pair correlation function $\rho_2 - 1$ is not integrable outside of any neighborhood of the origin. Using the well-known formula [14, Exercise 8.9]

$$\mathbb{V}\mathrm{ar}[\Phi(B)] = \gamma_d V_d(B) + \gamma_d^2 \int V_d(B \cap (B + x))(\rho_2(x) - 1)\, dx,$$

(valid for all bounded Borel sets $B \subset \mathbb{R}^d$) and assuming that B is convex, it is not too hard to confirm (2.6) (using polar coordinates) for $A = B$ and a certain positive constant $C_d(B, B)$. The value of this constant can be found in [9].

Remark 2.1. Assume that $\mathbb{Q}$ vanishes on any great subsphere. Then the random measures $\Phi_1, \dots, \Phi_d$ have the following *mixing* property. Let $i \in \{1, \dots, d\}$. Then Φ_i can be interpreted as a random element in a suitably space $\mathbf{M}$ of measures on $\mathbb{R}^d$ equipped with a suitable σ-field [10, 14]. Let A, B be arbitrary measurable subsets of $\mathbf{M}$. Then

$$\lim_{\|x\| \to \infty} \mathbb{P}(\Phi_i \in A, \theta_x \Phi_i \in B) = \mathbb{P}(\Phi_i \in A)\mathbb{P}(\Phi_i \in B),$$

where the random measure $\theta_x \Phi_i$ is defined by $\theta_x \Phi_i(C) := \Phi_i(C + x)$ for Borel sets $C \subset \mathbb{R}^d$. This is a straightforward consequence of [18, Theorem 10.5.3] and the fact that Φ_i is derived from η in a translation invariant way. In particular Φ_i is *ergodic*, that is $\mathbb{P}(\Phi_i \in A) \in \{0, 1\}$ for each translation invariant measurable set $A \subset \mathbf{M}$.

3 A reconstruction algorithm

Let η be a Poisson hyperplane process as in Section 2. Let Φ be the intersection point process associated with η. (Recall from Section 2 that $\mathbb{P}(\Phi = \Phi_d) = 1$, where Φ_d is given by (2.5) for $m = d$.) Let $K \subset \mathbb{R}^d$ be a non-empty convex and compact set. In this section we describe an algorithm which reconstructs $\eta \cap [K]$ (see Algorithm 3.3 and Fig. 2) by observing the points of Φ in a (random) bounded domain in the complement of K. In the next section we shall show that this domain is exponentially small.

We say that $n \geq d$ points from $\mathbb{R}^d$ are in *general hyperplane position* if any d of them are affinely independent and span the same hyperplane. The straightforward idea of the algorithm comes from the following proposition of some independent interest. We have not been able to find this result in the literature.

Proposition 3.1. *Almost surely the following is true. Any distinct points $x_1, \dots, x_{2d-1} \in \Phi$ in general hyperplane position span a hyperplane $H \in \eta$.*

Proof. We start the proof with an auxiliary observation. Let $m \in \mathbb{N}$ and let f be a measurable function on $(\mathbb{H}^{d-1})^m$ taking values in the space of all non-empty closed subsets of $\mathbb{R}^d$. (We equip this space with the usual Fell–Matheron topology; see [18]). We assert that

$$\mathbb{P}(\text{there exist distinct } H_1, \dots, H_{m+1} \in \eta \text{ such that } f(H_1, \dots, H_m) \subset H_{m+1}) = 0. \quad (3.1)$$

Obviously the indicator function of the event in (3.1) can be bounded by

$$X := \sum_{H_1,\dots,H_{m+1}\in\eta}^{\neq} \mathbf{1}\{f(H_1,\dots,H_m) \subset H_{m+1}\}.$$

If $\mathbb{E}[X] = 0$, then (3.1) follows. By the multivariate Mecke formula [14, Theorem 4.5],

$$\mathbb{E}[X] = \int \mathbf{1}\{f(H_1,\dots,H_m) \subset H_{m+1}\}\, \lambda^{m+1}(d(H_1,\dots,H_{m+1})).$$

By Fubini's theorem it then enough to prove that

$$\int \mathbf{1}\{F \subset H\}\, \lambda(dH) = 0 \tag{3.2}$$

for any non-empty closed set $F \subset \mathbb{R}^d$. By monotonicity of integration it is sufficient to assume that $F = \{x\}$ for some $x \in \mathbb{R}^d$. But then (3.2) directly follows from (2.3) and $\int \mathbf{1}\{\langle x, u\rangle = r\}\, dr = 0$ for each $u \in \mathbb{S}^{d-1}$.

We now turn to the main part of the proof. Let $I_1,\dots,I_{2d-1} \subset \mathbb{N}$ be distinct with $|I_1| = \cdots = |I_{2d-1}| = d$. We shall refer to these sets as *blocks* and to subsets of blocks as *subblocks*. For convenience we assume that $I_1 = [d] := \{1,\dots,d\}$. Assume that $\cup_{i=1}^{2d-1} I_i = [n]$ for some $n \geq d$. Consider $(H_1,\dots,H_n) \in \eta^n$ with $H_i \neq H_j$ for $i \neq j$ and the following properties. For each $i \in \{1,\dots,2d-1\}$ we have that $\cap_{j\in I_i} H_j$ consists of a single point x_i and $x_1,\dots,x_{2d-1}$ are in general hyperplane position. Let H be affine hull of $\{x_1,\dots,x_{2d-1}\}$. We will show that almost surely $H \in \{H_1,\dots,H_{2d-1}\}$.

Let us assume on the contrary that $H \notin \{H_1,\dots,H_{2d-1}\}$. Then each $k \in [n]$ (for instance $k = 1$) belongs to at most $d-1$ of the blocks. Indeed, by the general hyperplane assumption we would otherwise have that $H_1 = H$. We will show that almost surely

$$\cap_{j\in I} H_j \subset H \tag{3.3}$$

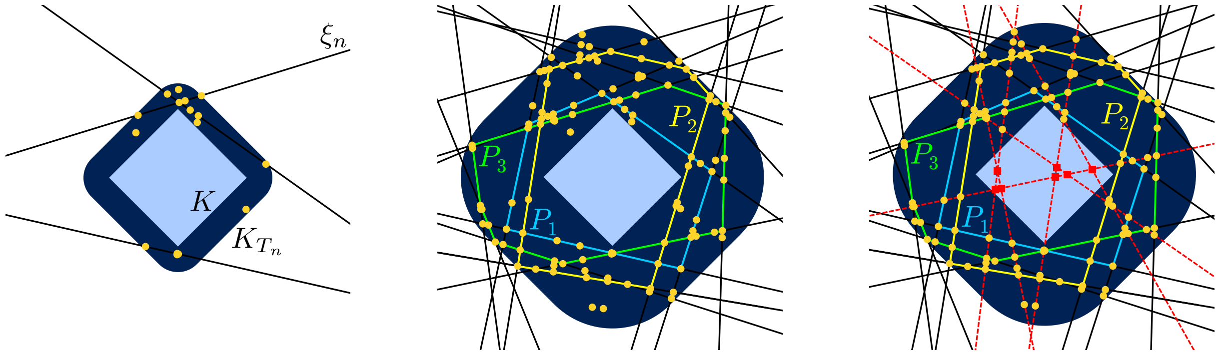
for all subblocks I.

We prove (3.3) by (descending) induction on the cardinality k of I. In the case $k = d$ (3.3) holds by definition of H. So assume that (3.3) holds for all subblocks of cardinality $k \in \{2,\dots,d\}$. We need to show that it holds for each subblock I of cardinality $k-1$. For notational convenience we take $I = [k-1]$. By induction hypothesis we have that

$$H_1 \cap \cdots \cap H_k \subset H. \tag{3.4}$$

Set $H' := H_1 \cap \cdots \cap H_{k-1} \cap H$. Since $H_1 \cap \cdots \cap H_{k-1} \neq \emptyset$ we have (almost surely) that $\dim H_1 \cap \cdots \cap H_{k-1} = (d-(k-1))$. Since $\dim H = d-1$ we therefore obtain that $\dim H' \in \{d-k, d-(k-1)\}$. Let us first assume that $\dim H' = d-k$. By (3.4) (and since $H_1 \cap \cdots \cap H_k \neq \emptyset$) we have that $\dim H_k \cap H' = d-k$. Therefore we obtain that $H_k \cap H' = H'$, that is $H' \subset H_k$. Since k is contained in at most $d-1$ of the subblocks, for instance in $I_1,\dots,I_{d-1}$, the blocks $I_d,\dots,I_{2d-1}$ still generate H, that is $H = \mathrm{aff}\{x_d,\dots,x_{2d-1}\}$. Therefore H' is "independent" of H_k, contradicting $H' \subset H_k$. More rigorously we can apply (3.1) to conclude that this case can almost surely not occur. Let us now assume that $\dim H' = d-(k-1)$. Then $\dim H' = \dim H_1 \cap \cdots \cap H_{k-1}$ and therefore $H' = H_1 \cap \cdots \cap H_{k-1}$. This means that $H_1 \cap \cdots \cap H_{k-1} \subset H$, as required to finish the induction.

Using (3.4) for subblocks of size 1, yields that $H_k = H$ for each $k \in [n]$. This contradiction finishes the proof of the lemma. □



$n = 16$ $\qquad$ $n = 112$ (i) $\qquad$ $n = 112$ (ii)

Figure 2: Reconstruction algorithm of $\eta \cap [K]$: Given a convex domain K, the algorithm recursively scans the points in $\Phi \cap K_{T_n}$ (solid circles). At step $n = 16$, three hyperplanes are reconstructed (left). At step $n = 112$, three polygons P_1, P_2, P_3 are reconstructed within the outer parallel set K_{T_n} (center). Hence, all hyperplanes in $\eta \cap [K]$ (dashed lines) can be reconstructed (right).

Remark 3.2. In general it is not possible to reduce the number $2d-1$ of points featuring in Proposition 3.1. To see this, we may consider the case $d = 3$ and a directional distribution which is concentrated on $\{e_1, e_2, e_3, -e_1, -e_2, -e_3\}$, where $\{e_1, e_2, e_3\}$ is an orthonormal system. In that case there exist infinitely many choices of four intersection points in general hyperplane position whose affine hull is not a hyperplane from η. Indeed, the hyperplanes tessellate space into cuboids and the four points can be chosen as endpoints of diametrically oposed edges of any cuboid.

Our algorithm requires some notation. Let

$$d(x, K) := \min\{\|y - x\| : y \in \mathbb{R}^d\}$$

denote the Euclidean distance between $x \in \mathbb{R}^d$ and K and let

$$K_r := \{x \in K^c : d(x, K) \leq r\} \tag{3.5}$$

denote the *outer parallel set* of K at distance $r \geq 0$. Note that $K_0 = \emptyset$. Define random times T_n, $n \geq 1$, inductively by setting

$$T_{n+1} := \min\{r > T_n : \Phi \cap (K_r \setminus K_{T_n}) \neq \emptyset\},$$

where $T_0 := 0$. We form a (random) set ξ_n of hyperplanes as follows. A hyperplane H belongs to ξ_n if it does not intersect K and if it contains $2d - 1$ different points from $\Phi \cap K_{T_n}$ in general hyperplane position. By Proposition 3.1 we have almost surely that $\xi_n \subset \eta$. For a hyperplane H with $H \cap K = \emptyset$ we let $H(K)$ denote the half-space bounded by H with $K \subset H(K)$.

Algorithm 3.3. The algorithm iterates over the random times T_n, $n \geq 1$, (recursively) scanning the points in $\Phi \cap K_{T_n}$. If the algorithm stops at time T_n, then it returns a set χ_n of hyperplanes that will be proved to coincide (almost surely) with $\eta \cap [K]$. Stage n of the algorithm is defined as follows (cf. Fig. 2):

(i) Determine ξ_n and check whether there are integers $k_1, \dots, k_{2d-1}$ and distinct $H_{i,j} \in \xi_n$ ($i \in [k_j]$, $j \in [2d-1]$) such that the boundary of

$$P_j := \bigcap_{i=1}^{k_j} H_{i,j}(K)$$

is contained in K_{T_n} for each $j \in [2d-1]$. If such hyperplanes do not exist, the algorithm continues with stage $n+1$. If they do exist, the algorithm continues with step (ii) and stops after it.

(ii) Find all collections of $2d-1$ points in $\Phi \cap K_{T_n}$ in general hyperplane position such that the generated hyperplane intersects K. If there are such points, χ_n is the set of all those hyperplanes. If there are no such points, then $\chi_n := \emptyset$.

Let $T := T_n$ if the algorithm stops at stage n. We set $T := \infty$ if it never stops. We can interpret T as the running time of the algorithm in continuous time. In the next section we will not only show that T is (almost surely) finite but does also have exponential moments. Here we wish to assure ourselves of the essentially geometric fact that the algorithm indeed determines $\eta \cap [K]$.

Proposition 3.4. *On the event $\{T < \infty\}$ we have almost surely that $\chi_T = \eta \cap [K]$.*

Proof. Assume that the algorithm stops at at stage n and let $P_1, \dots, P_{2d-1}$ be as in step (i) of the algorithm. These are bounded polytopes which contain K in their interior and which are made up of different hyperplanes from η. Assume that $H \in \eta$ intersects K. Then H intersects for each $i \in [2d-1]$ the boundary of the polytope P_i, and in fact, at least one of its edges. Therefore there exist distinct hyperplanes $H_1, \dots, H_{(2d-1)(d-1)} \in \eta \setminus \{H\}$ such that

$$H \cap \bigcap_{j \in I_i} H_j \neq \emptyset, \quad i \in [2d-1],$$

where $I_i := \{(i-1)(d-1)+1, \dots, i(d-1)\}$. Almost surely each of these intersections consists of only one point x_i, say. We assert that these points are in general hyperplane position. If they are not, then d among those points, $x_1, \dots, x_d$ say, are affinely dependent. Then one of those points, x_d say, must lie in $\operatorname{aff}\{x_1, \dots, x_{d-1}\}$. Therefore we need to show that the probability of finding distinct $H_0, \dots, H_{d(d-1)} \in \eta$ such that $\{x_i\} := H_0 \cap \cap_{j \in I_i} H_j$ is a singleton for each $i \in [d]$ and

$$x_d \in \operatorname{aff}\{x_1, \dots, x_{d-1}\}$$

is zero. Similarly as in the proof of (3.1) this probability can be bounded by

$$\iiint \mathbf{1}\{|H \cap \cap_{j \in I_1} H_j| = \cdots = |H \cap \cap_{j \in I_d} H_j| = 1\}$$
$$\mathbf{1}\{H \cap \cap_{j \in I_d} H_j \subset \operatorname{aff}\left(H \cap \cap_{j \in I_1} H_j, \dots, H \cap \cap_{j \in I_{d-1}} H_j\right)\}$$
$$\lambda(dH)\, \lambda^{d(d-1)}(d(H_1, \dots, H_{d(d-1)})).$$

Therefore it is enough to show that for λ-a.e. $H \in \mathbb{H}^{d-1}$ and each affine space $E \subset H$ of dimension at most $d-2$

$$\int \mathbf{1}\{|\cap_{j\in I_d} H_j| = 1, \cap_{j\in I_d} H_j \subset E\}\, \lambda_H^{d-1}(d(H_1,\dots,H_{d-1})) = 0, \tag{3.6}$$

where λ_H is the measure on the space of all affine subspaces of H given by

$$\lambda_H := \int \mathbf{1}\{H' \cap H \in \cdot\}\, \lambda(dH').$$

For λ-a.e. H, the measure λ_H is concentrated on the $(d-2)$-dimensional subspaces of H and invariant under translations in H. In fact, λ_H is the intensity measure of the Poisson process $\eta_H := \{H' \cap H : H' \in \eta\}$. Up to a constant multiple,

$$B \mapsto \int \mathbf{1}\{B \cap H_1 \cap \cdots \cap H_{d-1} \neq \emptyset\}\, \lambda_H^{d-1}(d(H_1,\dots,H_{d-1}))$$

is (as a function of the Borel set $B \subset H$) the intensity measure of the intersection process associated with η_H (see [18, p. 135]) and therefore proportional to Lebesgue measure on H. (It can also be checked more directly, that this function is a locally finite translation invariant measure.) Hence (3.6) follows. □

Remark 3.5. Assume that that the directional distribution $\mathbb{Q}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{H}^{d-1}$. Then the algorithm can be considerably simplified. In step (i) it is enough two find just two polytopes P_1, P_2 made up of distinct hyperplanes in ξ_n. Any hyperplane H from η that intersects K, intersects the boundary of the polytope P_1 in d affinely independent points from Φ and the same applies to P_2 (even without further assumptions of $\mathbb{Q}$). With some efforts it can be shown that the resulting $2d$ intersection points are almost surely in general hyperplane position. The forthcoming Theorem 4.1 remains valid. We do not go into the technical details.

Remark 3.6. The reconstruction algorithm 3.3 is not optimized for computational efficiency. For instance, the algorithm could already be stopped whenever there exists just one polytope which contains K in its interior but no points of Φ in the relative interior of its edges. (In this case $\eta \cap [K] = \emptyset$.) Moreover, it is not necessary that the boundaries of the polytopes P_j are completely contained in K_{T_n}. It would suffice to find polyhedral sets with sufficiently large parts of their boundaries contained in K_{T_n}.

4 Strong rigidity

In this section we shall exploit the algorithm from Section 3 to show that the intersection processes $\Phi_1, \dots, \Phi_m$ associated with a Poisson hyperplane process have very strong rigidity properties.

We start with giving a few definitions. Let Ψ be a random measure on $\mathbb{R}^d$ (for instance one of the $\Phi_1, \dots, \Phi_m$). For a Borel set $B \subset \mathbb{R}^d$ we denote by $\Psi_B := \Psi(\cdot \cap B)$ the restriction of Ψ to B. A mapping Z from Ω into the space of non-empty closed subsets of $\mathbb{R}^d$ is called Ψ-*stopping set* if $\{Z \subset F\} := \{\omega \in \Omega : Z(\omega) \subset F\}$ is for each closed set $F \subset \mathbb{R}^d$ an element of the σ-field $\sigma(\Psi_F)$ generated by Ψ_F. (In particular Z is then a *random closed set* [15].) By Ψ_Z we understand the restriction of Ψ to Z (that is the random measure $\omega \mapsto \Psi(\omega)_{Z(\omega)}$.) If Z is a Ψ-stopping set, then we say that $\eta \cap [K]$ is *almost surely determined by* Ψ_Z, if there exists a measurable mapping f (with suitable domain) such that $\eta \cap [K] = f(\Psi_Z)$ holds almost surely.

The following result shows that $\eta \cap [K]$ is almost surely determined by a Φ-stopping set $Z \subset K^c$ of exponentially small size. Here we quantify the size of a closed set $F \subset \mathbb{R}^d$ by the radius $R(F)$ of the smallest ball centred at the origin and containing F. (If F is not bounded then we set $R(F) := \infty$.)

Theorem 4.1. *Let $K \subset \mathbb{R}^d$ be convex and compact. Then there exists a Φ-stopping set Z with $Z \subset K^c$ and such that $\eta \cap [K]$ is almost surely determined by $\Phi \cap Z$. Moreover, there exist constants $c_1, c_2 > 0$ such that*

$$\mathbb{P}(R(Z) > s) \leq c_1 e^{-c_2 s}, \quad s \geq 1. \tag{4.1}$$

Proof. We consider the algorithm from Section 3 with running time T, defined after Algorithm 3.3. We assert that $Z := K_T$ has all desired properties, where $K_\infty := K^c$. The inclusion $Z \subset K^c$ is a direct consequence of the definitions. The stopping set property can be considered as pretty much obvious. The reader might wish to skip the following technical argument. Define $\mathbf{N}$ as the set of all locally finite subsets of $\mathbb{R}^d$. The algorithm from Section 3 can be used (in an obvious way) to define a measurable mapping $\tilde{Z}$ from $\mathbf{N}$ (equipped with the standard σ-field) to the space of all closed subsets of $\mathbb{R}^d$ such that $Z = \tilde{Z}(\Phi)$. We need to show that $\tilde{Z}$ is a stopping set, that is $\{\mu : \tilde{Z}(\mu) \subset F\}$ is for all closed sets $F \subset \mathbb{R}^d$ an element of the σ-field generated by the mapping $\mu \mapsto \mu \cap F$ from $\mathbf{N}$ to $\mathbf{N}$. To prove this we use [1, Proposition A.1]. According to this proposition it is sufficient to show that $\tilde{Z}((\psi \cap \tilde{Z}(\psi)) \cup \varphi) = \tilde{Z}(\psi)$ for all $\psi, \varphi \in \mathbf{N}$ with $\varphi \subset \tilde{Z}(\psi)^c$. But this follows from the definition of the algorithm. Indeed, suppose that $\psi \in \mathbf{N}$ is a realization of the intersection process and that the algorithm stops at time t. Restricting ψ to K_t and then adding a configuration φ in the complement of K_t does not change the running time t.

We show (4.1) by modifying the idea of the proof of Lemma 1 in [17]. Since $\mathbb{Q}$ is not concentrated on a great subsphere there exist linearly independent vectors $e_1, \ldots, e_d \in \mathbb{R}^d$ in the support of $\mathbb{Q}$. Since $\mathbb{Q}$ is even, the vectors $e_{d+1} := -e_1, \ldots, e_{2d} := -e_d$ are also in the support of $\mathbb{Q}$. We can then find a (large) constant $b > 0$ and (small) pairwise disjoint closed neighborhoods U_i of e_i, $i \in \{1, \ldots, 2d\}$, such that $U_{d+i} = \{-u : u \in U_i\}$ and each intersection

$$P = \bigcap_{i=1}^{2d} H^-(u_i, 1)$$

with $u_i \in U_i$, $i \in \{1, \ldots, 2d\}$, is a polytope with $R(P) \leq b$. Here we write, for given $u \in \mathbb{R}^d$ and $s \in \mathbb{R}$, $H^-(u, s) := \{y \in \mathbb{R}^d : \langle y, u \rangle \leq s\}$. Let $t \geq 0$. From linearity of the scalar product we then obtain that

$$R\left(\bigcap_{i=1}^{2d} H^-(u_i, t_i)\right) \leq b(R(K) + t), \tag{4.2}$$

whenever $R(K) \le t_i \le R(K) + t$ and $u_i \in U_i$ for $i \in \{1, \dots, 2d\}$.

We need a straightforward analytic fact. Since the determinant is a continuous function we can assume that there exists $a > 0$ such that

$$|\det(u_1, \dots, u_d)| \ge a, \quad (u_1, \dots, u_d) \in U_1 \times \cdots \times U_d. \tag{4.3}$$

For $i \in \{1, \dots, d\}$ let $u_i \in U_i \cup U_{d+i}$ and $s_i \in \mathbb{R}$. Then $H_{u_1,s_1} \cap \cdots \cap H_{u_d,s_d}$ consists of a single point x (by (4.3) and $U_{d+i} = -U_i$), whose Euclidean norm can be bounded as

$$\|x\| \le b' \max\{|s_i| : i = 1, \dots, d\}, \tag{4.4}$$

where $b' > 0$ is a constant that depends only on the dimension and the (fixed) sets $U_1, \dots, U_d$. To see this we note that x (now interpreted as a column vector) is the unique solution of the linear equation $Ax = s$, where A is the matrix with rows $u_1, \dots, u_d$ and s is the column vector with entries $s_1, \dots, s_d$. By (4.3) we have that $x = A^{-1}s$. It is well-known that

$$\|x\|_\infty \le \|A^{-1}\|_\infty \|s\|_\infty,$$

where $\|x\|_\infty := \max\{|x_i| : i = 1, \dots, d\}$ and $\|A^{-1}\|_\infty$ is the maximum absolute row sum of A^{-1}. In view of the explicit expression of A^{-1} in terms of $\det(A)^{-1}$ and the minors of A and the minimum principle for continuous functions we have that $\|A^{-1}\|_\infty$ is bounded from above by a positive constant. (Recall that $u_1, \dots, u_d$ are unit vectors.) Since $\|x\| \le c\|x\|_\infty$ for some $c > 0$ we obtain (4.4).

For notational simplicity we now assume that K is a ball with radius R centred at the origin. In fact, in view of the assertion this is no restriction of generality. Consider the following sets of hyperplanes:

$$A_i(t) := \{H(u, s) : u \in U_i, R < s \le R + t\}, \quad i \in [2d].$$

We assert the event inclusion

$$\bigcap_{i=1}^{2d} \{|\eta \cap A_i(t)| \ge 2d - 1\} \subset \{R(Z) \le b''(R + t)\}, \quad \mathbb{P}\text{-a.s.}, \tag{4.5}$$

where $b'' := \max\{b, b'\}$ with b' as in (4.4).

To show (4.5), we assume that $|\eta \cap A_i(t)| \ge 2d - 1$ for each $i \in [2d]$. Then we can find distinct hyperplanes $H_{i,j} \in \eta$ ($i \in [2d]$, $j \in [2d - 1]$) not intersecting K such that the polytopes

$$P_j := \bigcap_{i=1}^{2d} H_{i,j}(K), \quad j \in [2d - 1],$$

contain K in their interior and satisfy $R(P_j) \le b(R + t)$; see (4.2). Next we show that each $H_{i,j}$ is in ξ_n as soon as $T_n \ge b''(R + t) - R$. (Then our algorithm has identified these hyperplanes by time T_n.) Take $H_{1,1}$, for instance. Define $x_1, \dots, x_{2d-1} \in \Phi$ by $\{x_j\} := H_{1,1} \cap \cap_{i=2}^{d} H_{i,j}$. It can then be shown as in the proof of Proposition 3.4 that these points are in general hyperplane position. Therefore we obtain from (4.4) and the definition of $A_1(t)$ that $\|x_1\| \le b'(R+t)$ and in fact $\|x_j\| \le b'(R+t)$ for each $j \in [2d-1]$. Therefore $H_{i,j} \in \xi_n$, provided that $T_n \ge b'(R+t) - R$. We have already seen that $R(P_j) \le b(R+t)$, so that the boundary of P_j is contained in K_{T_n} if $T_n \ge b(R+t) - R$. (Note that K_{T_n} is a spherical shell with outer radius $R + T_n$ centred at the origin.) Altogether we obtain that $T \le b''(R+t) - R$ and hence $R(Z) = R(K_T) \le b''(R+t)$, proving (4.5).

Having established (4.5) we next note that

$$\begin{aligned}\mathbb{P}(R(Z) > b''(R+t)) &\le \mathbb{P}\bigg(\bigcup_{i=1}^{2d}\{|\eta \cap A_i(t)| \le 2d-2\}\bigg)\\ &\le \sum_{i=1}^{2d} \mathbb{P}(|\eta \cap A_i(t)| \le 2d-2)\\ &= \sum_{i=1}^{2d} \exp[-\lambda(A_i(t))] \sum_{j=0}^{2d-2} \frac{\lambda(A_i(t))^j}{j!},\end{aligned}$$

where we have used the defining properties of a Poisson process to obtain the final equality. By (2.3) we have that

$$\lambda(A_i(t)) = \gamma t\, \mathbb{Q}(U_i). \tag{4.6}$$

Setting $a := \min\{\mathbb{Q}(U_i) : i \in [2d]\}$ and using that $\mathbb{Q}(U_i) \le 1$ for each $i \in [2d]$, we obtain that

$$\mathbb{P}(R(Z) > b(R+t)) \le 2de^{-\gamma a t} \sum_{j=0}^{2d-2} \frac{\gamma^j}{j!} t^j.$$

This implies (4.1) for suitably chosen c_1, c_2. □

Remark 4.2. The stopping set Z in Theorem 4.1 depends measurably on $\Phi \cap K^c$ (is a measurable function of $\Phi \cap K^c$). This follows from the definition of the algorithm, but also from the following argument, which applies to general stopping sets Z with the property $Z \subset K^c$. By standard properties of random closed sets it suffices to check for each compact $F \subset \mathbb{R}^d$ that $\{Z \cap F = \emptyset\} \in \sigma(\eta \cap K^c)$. Since $Z \subset K^c$ we have that $\{Z \cap F = \emptyset\} = \{Z \cap (F \cup K) = \emptyset\}$. Since $F \cup K$ is compact, there is a decreasing sequence $(U_n)_{n\ge1}$ of open sets with intersection $F \cup K$ and such that

$$\{Z \cap (F \cup K) = \emptyset\} = \bigcup_{n=1}^{\infty}\{Z \cap U_n = \emptyset\} = \bigcup_{n=1}^{\infty}\{Z \subset U_n^c\}.$$

Since Z is a Φ-stopping set we have that the above right-hand side is contained in $\cup_{n=1}^{\infty}\sigma(\Phi_{U_n^c}) \subset \sigma(\Phi_{K^c})$, as asserted.

Theorem 4.1 implies the announced strong rigidity properties of the intersection processes.

Theorem 4.3. *Let $m \in \{1, \dots, d\}$ and let $B \subset \mathbb{R}^d$ be a bounded Borel set. Then there exists a Φ_m-stopping set Z with $Z \subset B^c$ and such that $(\Phi_m)_B$ is almost surely determined by $(\Phi_m)_Z$. Moreover, there exist constants $c_1, c_2 > 0$ such that* (4.1) *holds.*

Proof. Choose a convex and compact set $K \subset \mathbb{R}^d$ with $B \subset K$. Clearly, if the assertion holds in the case $B = K$, then we obtain it for all $B \subset K$. Hence we can assume that $B = K$. Let Z be as in Theorem 4.1. Since Φ_F is for each closed $F \subset \mathbb{R}^d$ a measurable function of $(\Phi_m)_F$ it follows that Z is a Φ_m-stopping set. Moreover, $(\Phi_m)_K$ is a (measurable) function of $\eta \cap [K]$. Hence Theorem 4.1 implies the assertions. □

The rigidity property in Theorem 4.3 is considerably stronger than the *strong rigidity* studied in [7]. The random measure $(\Phi_m)_B$ is not only determined by $(\Phi_m)_{B^c}$, but already by $(\Phi_m)_Z$ for an exponentially small stopping set $Z \subset B^c$.

5 Hyperfluctuating Cox processes and thinnings

The strong rigidity property of the random measures $\Phi_1, \dots, \Phi_m$ can easily be destroyed by additional randomization. For example we may consider, for $m \in \{1, \dots, d\}$, a *Cox process* Ψ_m directed by Φ_m [14, Chapter 13]. This means that the conditional distribution of Ψ_m given Φ_m is that of a Poisson process with intensity measure Φ_m. For $m = d$ this point process can be interpreted as a multiset (or a random measure). Each point of Φ_m gets (independently of the other points) a random multiplicity having a Poisson distribution of mean 1. Let $B \subset \mathbb{R}^d$ be a bounded Borel set. Then the well-known conditional variance formula (together with the stationarity of Φ_m) implies that

$$\mathbb{V}\mathrm{ar}[\Psi_m(B)] = \gamma_m V_d(B) + \mathbb{V}\mathrm{ar}[\Phi_m(B)],$$

where γ_m is the intensity of Φ_m; see [14, Proposition 13.6]. By (2.6), $\Psi_m(B)$ has the same variance asymptotics as $\Phi_m(B)$. In particular, Ψ_m is (for $d \geq 2$) hyperfluctuating. However, Ψ_m is not rigid. For example, given a Borel set B with positive volume, $\Psi_m(B)$ is not determined by the restriction of Ψ_m to the complement of B.

In the case of the intersection point process Φ there is an even simpler way of randomizing, namely to form a *p-thinning* Φ_p of Φ for some $p \in (0, 1)$. Formally, given Φ, the points of Φ are taken independently of each other as points of Φ_p with probability p [14, Section 5.3]. This point process is not rigid. A simple calculation (using the conditional variance formula for instance) shows that

$$\mathbb{V}\mathrm{ar}[\Phi_p(B)] = p^2\, \mathbb{V}\mathrm{ar}[\Phi(B)] + p(1-p)\mathbb{E}[\Phi(B)],$$

so that Φ_p inherits the variance asymptotics from Φ. It also not hard to see that the pair correlation function of Φ_p is the same as that of Φ and hence given by the slowly decaying function (2.8).

6 Concluding remarks

We have shown that the intersection point process Φ associated with a stationary Poisson hyperplane process is rigid in a very strong sense. This holds for any directional distribution which is not concentrated on a great subsphere. (Our arguments suggest that this might be true for more general stationary and mixing hyperplane processes with absolute continuous factorial moment measures.) On the other hand, Φ is hyperflunctating. Hence hyperuniformity is not necessary for rigidity as (weakly) conjectured in [6]. However, we completely agree with the authors of [6] that the precise relationships between rigidity and hyperuniformity constitute an interesting intriguing problem. We believe in the existence of generic point process assumptions that need to be added to rigidity to conclude hyperuniformity. Preferably these assumptions should be as minimal as possible.

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