

Forbidden Induced Subgraphs and the Łoś-Tarski Theorem

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Abstract

Let \mathcal{C} be a class of finite and infinite graphs that is closed under induced subgraphs. The well-known Łoś-Tarski Theorem from classical model theory implies that \mathcal{C} is definable in first-order logic (FO) by a sentence φ if and only if \mathcal{C} has a finite set of forbidden induced finite subgraphs. It provides a powerful tool to show nontrivial characterizations of graphs of small vertex cover, of bounded tree-depth, of bounded shrub-depth, etc. in terms of forbidden induced finite subgraphs. Furthermore, by the Completeness Theorem, we can compute from φ the corresponding forbidden induced subgraphs. We show that this machinery fails on finite graphs.

- There is a class \mathcal{C} of finite graphs which is definable in FO and closed under induced subgraphs but has no finite set of forbidden induced subgraphs.
- Even if we only consider classes \mathcal{C} of finite graphs which can be characterized by a finite set of forbidden induced subgraphs, such a characterization cannot be computed from an FO-sentence φ , which defines \mathcal{C} , and the size of the characterization cannot be bounded by $f(|\varphi|)$ for any computable function f .

Besides their importance in graph theory, the above results also significantly strengthen similar known results for arbitrary structures.

1. Introduction

Many classes of graphs can be defined by a finite set of forbidden induced finite subgraphs. One of the simplest examples is the class of graphs of bounded degree. Let $d \geq 1$ and \mathcal{F}_d consist of all graphs with vertex set $\{1, \dots, d+2\}$ and maximum degree exactly $d+1$. Then a graph G has degree at most d if and only if no graph in \mathcal{F}_d is isomorphic to an induced subgraph of G . Less trivial examples include graphs of small vertex cover (attributed to Lovász [9]), of bounded tree-depth [5], and of bounded shrub-depth [13]. As a matter of fact, understanding forbidden induced subgraphs for those graph classes is an important question in structural graph theory [7, 21, 12, 11]. However, a straightforward adaptation of a result in [10] shows that it is in general impossible to compute the forbidden induced subgraphs from a description of classes of graphs by Turing machines.

It is folklore [1, 17] that characterization by finitely many forbidden induced finite subgraphs is equivalent to definability by a universal sentence of first-order logic (FO). But only very recently, it was realized [2] that such a characterization can be further understood by the Łoś-Tarski theorem. Łoś [15] and Tarski [19] proved the first so-called preservation theorem of classical model theory. In its simplest form it says that the class $\text{GRAPH}(\varphi)$ of finite and infinite graphs that are models of a sentence φ of first-order logic is closed under induced subgraphs (or, that φ is preserved under induced subgraphs) if and only if there is a universal FO-sentence μ with $\text{GRAPH}(\varphi) = \text{GRAPH}(\mu)$. Recall that a universal sentence μ is a sentence of the form $\forall x_1 \dots \forall x_k \mu_0$, where μ_0 is quantifier-free.

Such a universal sentence $\mu = \forall x_1 \dots \forall x_k \mu_0$ expresses that certain patterns of induced subgraphs with at most k vertices are forbidden. In fact, let \mathcal{F} be a finite set of finite graphs and denote by $\text{FORB}(\mathcal{F})$ the class of (finite and infinite) graphs that do not contain an induced subgraph isomorphic to a graph in \mathcal{F} . Then for a universal sentence μ as above we have

$$\text{GRAPH}(\mu) = \text{FORB}(\mathcal{F}_k(\mu)). \quad (1)$$

Here for any FO-sentence φ and $k \geq 1$ by $\mathcal{F}_k(\varphi)$ we denote the class of graphs that are models of $\neg\varphi$ and whose universe is $\{1, \dots, \ell\}$ for some ℓ with $1 \leq \ell \leq k$. Clearly, $\mathcal{F}_k(\varphi)$ is finite.

We say that a class \mathcal{C} of finite and infinite graphs is *definable by a finite set of forbidden induced subgraphs* if there is a finite set \mathcal{F} of finite graphs such that $\mathcal{C} = \text{FORB}(\mathcal{F})$. Hence the graph-theoretic version of the Łoś-Tarski Theorem can be restated in the form:

- (I) Let \mathcal{C} be a class of finite and infinite graphs. The following are equivalent:
- (i) \mathcal{C} is closed under induced subgraphs and FO-axiomatizable.
 - (ii) \mathcal{C} is axiomatizable by a universal sentence.
 - (iii) \mathcal{C} is definable by a finite set of forbidden induced subgraphs.

This version of the Łoś-Tarski Theorem is already contained, at least implicitly, in the article [20] of Vaught published in 1954. In addition, it is easy to see that the equivalence between (ii) and (iii) holds for any class of finite graphs too.

Note that we have repeatedly mentioned that in the Łoś-Tarski Theorem graphs are allowed to be infinite. This is not merely a technicality. In [2], to obtain the forbidden induced subgraph characterization of graphs of bounded shrub-depth using the Łoś-Tarski Theorem, one simple but vital step is to extend the notion of shrub-depth to infinite graphs. Indeed, Tait [18] exhibited a class \mathcal{C} of finite structures (which might be understood as colored directed graphs) which is closed under induced substructures and FO-axiomatizable. Yet, \mathcal{C} is not definable by any universal sentence, thus cannot be characterized by a finite set of forbidden induced substructures. As the first result of this paper, we strengthen Tait's result to graphs.

Theorem 1.1. *There is a class \mathcal{C} of finite graphs with the following properties.*

- (i) \mathcal{C} is closed under induced subgraphs and FO-axiomatizable,
- (ii) \mathcal{C} is not definable by a finite set of forbidden induced subgraphs.

Even though we are interested in structural and algorithmic results for classes of finite graphs, we see that in order to apply the Łoś-Tarski Theorem for such purposes we have to consider classes of finite and infinite graphs. So in this paper “graph” means finite or infinite graph. As in the preceding result we mention it explicitly if we only consider finite graphs.

Complementing Theorem 1.1 we show that it is even undecidable whether a given FO-definable class of finite graphs which is closed under induced subgraphs can be characterized by a finite set of forbidden induced subgraphs. More precisely:

Theorem 1.2. *There is no algorithm that for any FO-sentence φ such that*

$$\text{GRAPH}_{\text{fin}}(\varphi) := \{G \mid G \text{ is a finite graph and a model of } \varphi\}$$

is closed under induced subgraphs decides whether φ is equivalent to a universal sentence on finite graphs.

As mentioned at the beginning, for a class of finite graphs definable by a finite set of forbidden induced subgraphs, it is preferable to have an explicit construction of those graphs. This however turns out to be difficult for many natural classes of graphs. For example, the forbidden induced subgraphs are only known for tree-depth at most 3 [7]. Let us consider the k -vertex cover problem for a constant $k \geq 1$. It asks whether a given graph has a vertex cover (i.e., a set of vertices that contains at least one endpoint of every edge) of size at most k . The class of all YES-instances of this problem, finite and infinite, is closed under induced subgraphs and FO-axiomatizable by the FO-sentence

$$\varphi_{\text{VC}}^k := \varphi_{\text{GRAPH}} \wedge \exists x_1 \dots \exists x_k \forall y \forall z \left(Eyz \rightarrow \bigvee_{1 \leq \ell \leq k} (x_\ell = y \vee x_\ell = z) \right),$$

where φ_{GRAPH} axiomatizes the class of graphs. Hence, by (I) the class of YES-instances can be defined by a finite set of forbidden induced subgraphs. As the reader will notice it is by no means trivial to find a universal sentence equivalent to φ_{VC}^k . But on the other hand, by the Completeness Theorem, we can search for such a universal sentence by enumerating all possible universal sentences μ and all possible proofs for $\vdash \varphi_{\text{VC}}^k \leftrightarrow \mu$, and then extract the corresponding forbidden induced subgraphs from μ as in (1).

To explain the hardness of constructing forbidden induced subgraphs, we prove two negative results.

Theorem 1.3. *There is no algorithm that for any FO-sentence φ which is equivalent to a universal sentence μ on finite graphs computes such a μ .*

Or equivalently, there is no algorithm that for any FO-sentence φ such that

$$\text{GRAPH}_{\text{fin}}(\varphi) = \text{FORB}_{\text{fin}}(\mathcal{F})$$

for a finite set \mathcal{F} of graphs computes such an \mathcal{F} . Here,

$$\text{FORB}_{\text{fin}}(\mathcal{F}) := \{G \mid G \text{ is a finite graph without induced subgraph isomorphic to a graph in } \mathcal{F}\}.$$

Theorem 1.4. *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a computable function. Then there is a class \mathcal{C} of finite graphs and an FO-sentence φ such that*

- (i) $\mathcal{C} = \text{GRAPH}_{\text{fin}}(\varphi)$.
- (ii) $\mathcal{C} = \text{GRAPH}_{\text{fin}}(\mu)$ for some universal sentence μ , in particular \mathcal{C} is closed under induced subgraphs.
- (iii) For every universal sentence μ with $\mathcal{C} = \text{GRAPH}_{\text{fin}}(\mu)$ we have $|\mu| \geq f(|\varphi|)$.

Theorem 1.3 significantly strengthens the aforementioned result of [10]. Even if a class \mathcal{C} of finite graphs definable by a finite set of forbidden induced subgraphs is given by an FO-sentence φ with $\mathcal{C} = \text{GRAPH}_{\text{fin}}(\varphi)$, instead of a much more powerful Turing machine, we still cannot compute an appropriate finite set of forbidden induced subgraphs for \mathcal{C} from φ . On top of it, Theorem 1.4 implies that the size of forbidden subgraphs for \mathcal{C} cannot be bounded by any computable function in terms of the size of φ .

There is an important precursor for Theorem 1.4,

Theorem 1.5 (Gurevich’s Theorem [14]). *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be computable. Then there is an FO-sentence φ such that the class $\text{MOD}(\varphi)$ of models of φ is closed under induced substructures but for every universal sentence μ with $\text{MOD}_{\text{fin}}(\mu) = \text{MOD}_{\text{fin}}(\varphi)$ we have $|\mu| \geq f(|\varphi|)$.*

Hence, Theorem 1.4 can be viewed as the graph-theoretic version of Theorem 1.5.

Besides its importance in graph theory, Theorem 1.4 is also relevant in the context of algorithmic model theory. For algorithmic applications, the Łoś-Tarski theorem provides a normal form (i.e., a universal sentence) for any FO-sentence preserved under induced substructures. In [3], it is shown that on *labelled trees* there is no *elementary bound* on the length of the equivalent universal sentence in terms of the original one. We should point out that Theorem 1.4 is not comparable to Theorem 6.1 in [3], since our lower bound is uncomputable (and thus, much higher than non-elementary) while the classes of graphs we construct in the proof are dense (thus very far from trees).

Our technical contributions. For every vocabulary it is well known that the class of structures of this vocabulary is FO-interpretable in graphs (see for example [8]). So one might expect that Theorem 1.1 and Theorem 1.4 can be derived easily from Tait’s Theorem and Gurevich’s Theorem using the standard FO-interpretations. However, an easy analysis shows that those interpretations result in classes of graphs that are not closed under induced subgraphs. So we introduce the notion of *strongly existential interpretation* which translates any class of structures preserved under induced substructures to a class of graphs closed under induced subgraphs. A lot of care is needed to construct strongly existential interpretations.

Related research. Let us briefly mention some further results related to the Łoś-Tarski Theorem. Essentially one could divide them into three categories: (a) The *positive results* showing that for certain classes \mathcal{C} of finite structures the analogue of the Łoś-Tarski Theorem holds if we restrict to structures in \mathcal{C} . For example, this is the case if \mathcal{C} is the class of all finite structures of tree-width k or less for some $k \in \mathbb{N}$ [1] or if \mathcal{C} is the class of all finite structures whose hypergraph satisfies certain properties [6]. (b) Both just mentioned papers contain also *negative results*, i.e., classes for which the analogue of the Łoś-Tarski Theorem fails: For example, in [1] this is shown for the class of finite planar graphs. (c) The third category contains generalizations of the Łoś-Tarski Theorem. For example, in [17] the authors, for $k \geq 1$ consider sentences of the form $\exists x_1 \dots \exists x_k \mu$, where μ is universal. Then the role of the closure under induced substructures is taken over by a semantic “core property $\text{PS}(k)$ ” which for $k = 0$ coincides with closure under induced substructures. Finally, we mention that in [4] the authors strengthen Tait’s result by showing that for every $n \geq 1$ there are first-order definable classes of finite structures closed under substructures which are not definable with n quantifier alternations.

Organization of this paper. In Section 2 we fix some notations and recall or derive some results about universal sentences we need in this paper. For the reader’s convenience, in Section 3 we include a proof of Tait’s result. Moreover, we prove a technical result, i.e., Proposition 3.11, which is an important tool in Gurevich’s Theorem. We introduce the concept of strongly existential interpretation in Section 4 and show that the results of the preceding section remain true under such interpretations. We present an appropriate strongly existential interpretation for graphs (in Section 5). Hence, we get the results of Section 3 for graphs. In Section 6 we first derive Gurevich’s Theorem and apply our interpretations to get the results for graphs. Finally, in Section 7, we prove that various problems related to our results are undecidable.

2. Preliminaries

We denote by \mathbb{N} the set of natural numbers greater or equal to 0. For $n \in \mathbb{N}$ let $[n] := \{1, 2, \dots, n\}$.

First-order logic FO. A *vocabulary* τ is a finite set of relation symbols. Each relation symbol has an *arity*. A *structure* \mathcal{A} of vocabulary τ , or τ -*structure*, consists of a (finite or infinite) nonempty set A , called the *universe* of \mathcal{A} and of an interpretation $R^{\mathcal{A}} \subseteq A^r$ of each r -ary relation symbol $R \in \tau$. If \mathcal{A} and \mathcal{B} are τ -structures, then \mathcal{A} is a *substructure* of \mathcal{B} , denoted by $\mathcal{A} \subseteq \mathcal{B}$, if $A \subseteq B$ and $R^{\mathcal{A}} \subseteq R^{\mathcal{B}}$, and \mathcal{A} is an *induced substructure* of \mathcal{B} , denoted by $\mathcal{A} \subseteq_{\text{ind}} \mathcal{B}$, if $A \subseteq B$ and $R^{\mathcal{A}} = R^{\mathcal{B}} \cap A^r$, where r is the arity of R . If, in addition, $A \subsetneq B$, then \mathcal{A} is a *proper induced substructure* of \mathcal{B} . By $\text{STR}[\tau]$ ($\text{STR}_{\text{fin}}[\tau]$) we denote the class of all (of all finite) τ -structures.

Formulas φ of *first-order logic* FO of vocabulary τ are built up from *atomic formulas* $x_1 = x_2$ and $Rx_1 \dots x_r$ (where $R \in \tau$ is of arity r and x_1, x_2, \dots, x_r are variables) using the boolean connectives \neg, \wedge, \vee and the universal \forall and existential \exists quantifiers. A relation symbol R is *positive* (*negative*) in φ if all atomic subformulas $R \dots$ in φ appear in the scope of an *even* (*odd*) number of negation symbols. By the notation $\varphi(\bar{x})$ with $\bar{x} = x_1, \dots, x_e$ we indicate that the variables free in φ are among x_1, \dots, x_e . If then \mathcal{A} is a τ -structure and $a_1, \dots, a_e \in A$, then $\mathcal{A} \models \varphi(a_1, \dots, a_e)$ means that $\varphi(\bar{x})$ holds in \mathcal{A} if x_i is interpreted by a_i for $i \in [e]$.

A *sentence* is a formula without free variables. For a sentence φ we denote by $\text{MOD}(\varphi)$ the class of models of φ and $\text{MOD}_{\text{fin}}(\varphi)$ is its subclass consisting of the finite models of φ . Sentences φ and ψ are *equivalent* if $\text{MOD}(\varphi) = \text{MOD}(\psi)$ and *finitely equivalent* if $\text{MOD}_{\text{fin}}(\varphi) = \text{MOD}_{\text{fin}}(\psi)$.

Graphs. Let $\tau_E := \{E\}$ with binary E . For all τ_E -structures we use the notation $G = (V(G), E(G))$ common in graph theory. Here $V(G)$, the universe of G , is the set of vertices, and $E(G)$, the interpretation of the relation symbol E , is the set of edges. The τ_E -structure $G = (V(G), E(G))$ is a *directed graph* if $E(G)$ does not contain self-loops, i.e., $(v, v) \notin E(G)$ for any $v \in V(G)$. If moreover $(u, v) \in E(G)$ implies $(v, u) \in E(G)$ for any pair (u, v) , then G is an (undirected) *graph*. The graph $H = (V(H), E(H))$ is an *induced subgraph* of G if

$$V(H) \subseteq V(G) \quad \text{and} \quad E(H) = E(G) \cap (V(H) \times V(H)).$$

We denote by GRAPH and $\text{GRAPH}_{\text{fin}}$ the class of all graphs and the class of finite graphs, respectively. Furthermore, for an $\text{FO}[\tau_E]$ -sentence φ by $\text{GRAPH}(\varphi)$ (and $\text{GRAPH}_{\text{fin}}(\varphi)$) we denote the class of graphs (and the class of finite graphs) that are models of φ .

Universal sentences and forbidden induced substructures. An FO-formula is *universal* if it is built up from atomic and negated atomic formulas by means of the connectives \wedge and \vee and the universal quantifier \forall . Often we say that a formula, say, containing the connective \rightarrow is universal if by replacing $\varphi \rightarrow \psi$ by $\neg\varphi \vee \psi$ (and “simple manipulations”) we get an equivalent universal sentence. Every universal sentence μ is equivalent to a sentence of the form $\forall x_1 \dots \forall x_k \mu_0$ for some $k \in \mathbb{N}$ and some quantifier-free μ_0 and moreover the length $|\mu|$ of μ is at most $|\varphi|$. If in the definition of universal formula we replace the universal quantifier by the existential one we get the definition of an *existential formula*.

One easily verifies that the class of models of a universal sentence is closed under induced substructures. As already mentioned in the Introduction for classes of graphs, Łoś [15] and Tarski [19] proved:

Theorem 2.1 (Łoś-Tarski Theorem). *Let τ be a vocabulary and φ an $\text{FO}[\tau]$ -sentence. Then $\text{MOD}(\varphi)$ is closed under induced substructures if and only if φ is equivalent to a universal sentence.*

We fix a vocabulary τ . Let \mathcal{F} be a finite set of finite τ -structures and denote by $\text{FORB}(\mathcal{F})$ (and $\text{FORB}_{\text{fin}}(\mathcal{F})$) the class of structures (of finite structures) that do not contain an induced substructure isomorphic to a structure in \mathcal{F} . Clearly for finite sets F and F' of finite τ -structures we have

$$\text{if } \mathcal{F} \subseteq \mathcal{F}', \text{ then } \text{FORB}(\mathcal{F}') \subseteq \text{FORB}(\mathcal{F}). \quad (2)$$

We say that a class \mathcal{C} of τ -structures (of finite τ -structures) is *definable by a finite set of forbidden induced substructures* if there is a finite set \mathcal{F} of finite structures such that $\mathcal{C} = \text{FORB}(\mathcal{F})$ ($\mathcal{C} = \text{FORB}_{\text{fin}}(\mathcal{F})$).

Recall that $\tau_E = \{E\}$ with binary E .

$$\varphi_{\text{DG}} := \forall x \neg Exx \quad \text{and} \quad \varphi_{\text{GRAPH}} := \forall x \neg Exx \wedge \forall x \forall y (Exy \rightarrow Eyx) \quad (3)$$

axiomatize the classes of directed graphs and of graphs, respectively. Let the τ_E -structures $H_0 = (V(H_0), E(H_0))$ and $H_1 = (V(H_1), E(H_1))$ be given by

$$V(H_0) := \{1\}, E(H_0) := \{(1, 1)\} \quad \text{and} \quad V(H_1) := \{1, 2\}, E(H_1) := \{(1, 2)\}.$$

Then $\text{FORB}(\{H_0\})$ and $\text{FORB}(\{H_0, H_1\})$ are the class of directed graphs and the class of graphs, respectively, i.e., $\text{MOD}(\varphi_{\text{DG}}) = \text{FORB}(\{H_0\})$ and $\text{MOD}(\varphi_{\text{GRAPH}}) = \text{FORB}(\{H_0, H_1\})$.

The following result generalizes this simple fact and establishes the equivalence between axiomatizability by a universal sentence and definability by a finite set of forbidden induced substructures. For an arbitrary vocabulary τ , an $\text{FO}[\tau]$ -sentence φ , and $k \geq 1$ let

$$\mathcal{F}_k(\varphi) := \{\mathcal{A} \in \text{STR}[\tau] \mid \mathcal{A} \models \neg\varphi \text{ and } A = [\ell] \text{ for some } \ell \in [k]\}. \quad (4)$$

Thus, $\mathcal{F}_k(\varphi)$ is, up to isomorphism, the class of structures with at most k elements which fail to be a model of φ . Note that $\mathcal{F}_1(\varphi_{\text{DG}}) = \{H_0\}$ and $\mathcal{F}_1(\varphi_{\text{GRAPH}}) = \{H_0, H_1\}$. Clearly, for a τ -sentence we have:

$$\begin{aligned} &\text{if } \text{MOD}(\varphi) \text{ is closed under induced substructures,} \\ &\text{then } \text{MOD}(\varphi) \subseteq \text{FORB}(\mathcal{F}_k(\varphi)) \text{ for all } k \geq 1. \end{aligned} \quad (5)$$

Proposition 2.2. *For a class \mathcal{C} of τ -structures and $k \geq 1$ the statements (i) and (ii) are equivalent.*

- (i) $\mathcal{C} = \text{MOD}(\mu)$ for some universal sentence $\mu := \forall x_1 \dots \forall x_k \mu_0$ with quantifier-free μ_0 .
- (ii) $\mathcal{C} = \text{FORB}(\mathcal{F})$ for some finite set \mathcal{F} of structures, all of at most k elements.

If (i) holds for μ , then $\mathcal{C} = \text{FORB}(\mathcal{F}_k(\mu))$.

Proof: (i) \Rightarrow (ii) Let $\mathcal{C} = \text{MOD}(\mu)$ for μ as in (i). Then $\text{MOD}(\mu)$ is closed under induced substructures and hence, $\mathcal{C} \subseteq \text{FORB}(\mathcal{F}_k(\mu))$ by (5).

Now assume that $\mathcal{A} \notin \mathcal{C}$. Then $\mathcal{A} \models \neg\mu$ and hence there are $a_1, \dots, a_k \in A$ with $\mathcal{A} \models \neg\mu_0(a_1, \dots, a_k)$. For $\mathcal{B} := [a_1, \dots, a_k]^{\mathcal{A}}$, the substructure of \mathcal{A} induced by a_1, \dots, a_k , we have $\mathcal{B} \models \neg\mu_0(a_1, \dots, a_k)$ (as μ_0 is quantifier-free) and thus, $\mathcal{B} \models \neg\mu$. Therefore, \mathcal{B} is isomorphic to a structure in $\mathcal{F}_k(\mu)$ and therefore, $\mathcal{A} \notin \text{FORB}(\mathcal{F}_k(\mu))$.

(ii) \Rightarrow (i) Let the τ -structure \mathcal{A} have at most k elements and let a_1, \dots, a_k be an enumeration of the elements of A (possibly with repetitions). Let $\delta(\mathcal{A}; a_1, \dots, a_k)$ be the conjunction of all literals (i.e., atomic or negated atomic formulas) $\lambda(x_1, \dots, x_k)$ such that $\mathcal{A} \models \lambda(a_1, \dots, a_k)$. Then for every τ -structure \mathcal{B} and $b_1, \dots, b_k \in B$ we have

$$\mathcal{B} \models \delta(\mathcal{A}; a_1, \dots, a_k)(b_1, \dots, b_k) \iff \text{the clauses } \pi(a_i) = b_i \text{ for } i \in [k] \\ \text{define an isomorphism from } \mathcal{A} \text{ onto } [b_1, \dots, b_k]^{\mathcal{B}}. \quad (6)$$

Now assume (ii), i.e., $\mathcal{C} = \text{FORB}(\mathcal{F})$ for some finite set \mathcal{F} of structures, all of at most k elements. If \mathcal{F} is empty, then $\mathcal{C} = \text{MOD}(\forall x x = x)$. Otherwise for every $\mathcal{A} \in \mathcal{F}$ we fix an enumeration $a_1^{\mathcal{A}}, \dots, a_k^{\mathcal{A}}$ of the elements of A . We set

$$\mu := \forall x_1 \dots \forall x_k \bigwedge_{\mathcal{A} \in \mathcal{F}} \neg \delta(\mathcal{A}; a_1^{\mathcal{A}}, \dots, a_k^{\mathcal{A}}).$$

Then $\text{FORB}(\mathcal{F}) = \text{MOD}(\mu)$. In fact, assume first that $\mathcal{B} \notin \text{MOD}(\mu)$. Then there are $b_1, \dots, b_k \in B$ and an $\mathcal{A} \in \mathcal{F}$ such that $\mathcal{B} \models \delta(\mathcal{A}; a_1^{\mathcal{A}}, \dots, a_k^{\mathcal{A}})(b_1, \dots, b_k)$. By (6), then \mathcal{A} is isomorphic to the induced substructure $[b_1, \dots, b_k]^{\mathcal{B}}$ of \mathcal{B} ; hence, $\mathcal{B} \in \text{FORB}(\mathcal{F})$.

Now assume $\mathcal{B} \notin \text{FORB}(\mathcal{F})$. Then there is an $\mathcal{A} \in \mathcal{F}$ and elements $b_1, \dots, b_k \in B$ such that the clauses $\pi(a_i^{\mathcal{A}}) = b_i$ for $i \in [k]$ define an isomorphism from \mathcal{A} onto $[b_1, \dots, b_k]^{\mathcal{B}}$. By (6), then $\mathcal{B} \models \delta(\mathcal{A}; a_1^{\mathcal{A}}, \dots, a_k^{\mathcal{A}})(b_1, \dots, b_k)$. Therefore, $\mathcal{B} \models \neg \mu$, i.e., $\mathcal{B} \notin \text{MOD}(\mu)$. \square

Corollary 2.3. *Let φ be a τ -sentence and $k \geq 1$. Then*

$$\text{MOD}(\varphi) = \text{FORB}(\mathcal{F}_k(\varphi)) \iff \varphi \text{ is equivalent to a universal sentence} \\ \text{of the form } \forall x_1 \dots \forall x_k \mu_0 \text{ with quantifier-free } \mu_0.$$

By (2) and (5) we get:

Corollary 2.4. *If $\text{MOD}(\mu) = \text{FORB}(\mathcal{F}_k(\mu))$ for some universal μ and some $k \in \mathbb{N}$, then $\text{MOD}(\mu) = \text{FORB}(\mathcal{F}_\ell(\mu))$ for all $\ell \geq k$.*

Corollary 2.5. *It is decidable whether two universal sentences are equivalent.*

Proof: Let μ and μ' be universal sentences. W.l.o.g. we may assume that $\mu = \forall x_1 \dots \forall x_k \mu_0$ and $\mu' = \forall x_1 \dots \forall x_\ell \mu'_0$ with $k \leq \ell$. By Corollary 2.3 and Corollary 2.4, we have

$$\text{MOD}(\mu) = \text{FORB}(\mathcal{F}_\ell(\mu)) \quad \text{and} \quad \text{MOD}(\mu') = \text{FORB}(\mathcal{F}_\ell(\mu')).$$

Thus μ and μ' are equivalent if and only if $\mathcal{F}_\ell(\mu) = \mathcal{F}_\ell(\mu')$. The right hand side of this equivalence is clearly decidable. \square

The last equivalence of this corollary shows:

Corollary 2.6. *For universal sentences μ and μ' we have*

$$\mu \text{ and } \mu' \text{ are equivalent} \iff \mu \text{ and } \mu' \text{ are finitely equivalent.}$$

The following consequence of Corollary 2.2 will be used in the next section.

Corollary 2.7. *Let $m, k \in \mathbb{N}$ with $m > k$ and let ψ_0 and ψ_1 be $\text{FO}[\tau]$ -sentences. Assume that \mathcal{A} is a finite model of $\psi_0 \wedge \psi_1$ with at least m elements and all its proper induced substructures with at most k elements are models of $\psi_0 \wedge \neg\psi_1$. Then $\psi_0 \wedge \neg\psi_1$ is not finitely equivalent to a universal sentence of the form $\mu := \forall x_1 \dots \forall x_k \mu_0$ with quantifier-free μ_0 .*

Proof: For a contradiction assume $\text{MOD}_{\text{fin}}(\psi_0 \wedge \neg\psi_1) = \text{MOD}_{\text{fin}}(\mu)$ for μ as above. As $\text{MOD}(\mu) = \text{FORB}(\mathcal{F}_k(\mu))$ by Proposition 2.2, we get (applying the finitely equivalence of $\psi_0 \wedge \neg\psi_1$ and μ to obtain the last equality)

$$\text{MOD}_{\text{fin}}(\psi_0 \wedge \neg\psi_1) = \text{MOD}_{\text{fin}}(\mu) = \text{FORB}_{\text{fin}}(\mathcal{F}_k(\mu)) = \text{FORB}_{\text{fin}}(\mathcal{F}_k(\psi_0 \wedge \neg\psi_1)).$$

However, by the assumptions the structure \mathcal{A} is contained in $\text{MOD}_{\text{fin}}(\psi_0 \wedge \neg\psi_1)$ but not in the class $\text{FORB}_{\text{fin}}(\mathcal{F}_k(\psi_0 \wedge \neg\psi_1))$. \square

Remark 2.8. Let \mathcal{C} be a class of τ -structures closed under induced substructures. For an $\text{FO}[\tau]$ -sentence φ we set $\text{MOD}_{\mathcal{C}}(\varphi) := \{\mathcal{A} \in \mathcal{C} \mid \mathcal{A} \models \varphi\}$. We say that the *Łoś-Tarski Theorem holds for \mathcal{C}* if for every $\text{FO}[\tau]$ -sentence φ such that the class $\text{MOD}_{\mathcal{C}}(\varphi)$ is closed under induced substructures there is a universal sentence μ such that

$$\text{MOD}_{\mathcal{C}}(\varphi) = \text{MOD}_{\mathcal{C}}(\mu).$$

The following holds:

Let \mathcal{C} and \mathcal{C}' be classes of τ -structures closed under induced substructures with $\mathcal{C}' \subseteq \mathcal{C}$. Furthermore assume that there is a universal sentence μ_0 such that $\mathcal{C}' = \text{MOD}_{\mathcal{C}}(\mu_0)$. If the analogue of the Łoś-Tarski Theorem holds for \mathcal{C} , then it holds for \mathcal{C}' , too

In fact, for every $\text{FO}[\tau]$ -sentence φ we have $\text{MOD}_{\mathcal{C}'}(\varphi) = \text{MOD}_{\mathcal{C}}(\mu_0 \wedge \varphi)$. Hence, if $\text{MOD}_{\mathcal{C}'}(\varphi)$ is closed under induced substructures, then by assumption there is a universal μ such that $\text{MOD}_{\mathcal{C}}(\mu_0 \wedge \varphi) = \text{MOD}_{\mathcal{C}}(\mu)$. Therefore, $\text{MOD}_{\mathcal{C}'}(\varphi) = \text{MOD}_{\mathcal{C}}(\mu) = \text{MOD}_{\mathcal{C}'}(\mu)$.

3. Basic ideas underlying the classical results

This section contains a proof of Tait's Theorem telling us that the analogue of the Łoś-Tarski-Theorem fails if we only consider finite structures. Afterwards we refine the argument to derive a generalization, namely Proposition 3.11, which is a key result to get Gurevich's Theorem.

We consider the vocabulary $\tau_0 := \{<, U_{\min}, U_{\max}, S\}$, where $<$ and S (the successor relation) are binary relation symbols and U_{\min} and U_{\max} are unary.

Let φ_0 be the conjunction of the universal sentences

- $\forall x \neg x < x, \quad \forall x \forall y (x < y \vee x = y \vee y < x), \quad \forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow x < z)$, i.e., “ $<$ is an ordering”
- $\forall x \forall y ((U_{\min} x \rightarrow (x = y \vee x < y))$ i.e., “every element in U_{\min} is a minimum w.r.t. $<$ ”
- $\forall x \forall y ((U_{\max} x \rightarrow (x = y \vee y < x))$ i.e., “every element in U_{\max} is a maximum w.r.t. $<$ ”
- $\forall xy (Sxy \rightarrow x < y)$
- $\forall x \forall y \forall z (x < y < z \rightarrow \neg Sxz)$.

Note that from the axioms it follows that there is at most one element in U_{\min} , at most one in U_{\max} , and that S is a subset of the successor relation w.r.t. $<$. We call τ_0 -orderings the models of φ_0 .

For τ_0 -structures \mathcal{A} and \mathcal{B} we write $\mathcal{B} \subseteq_{<} \mathcal{A}$ and say that \mathcal{B} is a $<$ -substructure of \mathcal{A} if \mathcal{A} is a substructure of \mathcal{B} with $<^{\mathcal{B}} = <^{\mathcal{A}} \cap (B \times B)$.

We remark that the relation symbols U_{\min} , U_{\max} , and S are negative in φ_0 . Therefore we have:

Lemma 3.1. *Let $\mathcal{B} \subseteq_{<} \mathcal{A}$. If $\mathcal{A} \models \varphi_0$, then $\mathcal{B} \models \varphi_0$.*

Let

$$\varphi_1 := \exists x U_{\min} x \wedge \exists x U_{\max} x \wedge \forall x \forall y (x < y \rightarrow \exists z Sxz). \quad (7)$$

We call models of $\varphi_0 \wedge \varphi_1$ *complete τ_0 -orderings*. Clearly, for every $k \geq 1$ there is a unique, up to isomorphism, complete τ_0 -ordering with exactly k elements. The next lemma shows that all its proper $<$ -substructures are models of $\varphi_0 \wedge \neg\varphi_1$.

Lemma 3.2. *Let \mathcal{A} and \mathcal{B} be τ_0 -structures. Assume that $\mathcal{A} \models \varphi_0$ and \mathcal{B} is a finite $<$ -substructure of \mathcal{A} that is a model of φ_1 . Then $\mathcal{B} = \mathcal{A}$ (in particular, $\mathcal{A} \models \varphi_1$).*

Proof: By the previous lemma we know that $\mathcal{B} \models \varphi_0$. Let $B := \{b_1, \dots, b_n\}$. As $<^{\mathcal{B}}$ is an ordering, we may assume that

$$b_1 <^{\mathcal{B}} b_2 <^{\mathcal{B}} \dots <^{\mathcal{B}} b_{n-1} <^{\mathcal{B}} b_n.$$

As $\mathcal{B} \models (\varphi_0 \wedge \varphi_1)$, we have $U_{\min}^{\mathcal{B}} b_1$, $U_{\max}^{\mathcal{B}} b_n$, and $S^{\mathcal{B}} b_i b_{i+1}$ for $i \in [n-1]$. As $\mathcal{B} \subseteq \mathcal{A}$, everywhere we can replace the upper index \mathcal{B} by \mathcal{A} .

We show $A = B$: Let $a \in A$. By $\mathcal{A} \models \varphi_0$, we have $b_1 \leq^{\mathcal{A}} a \leq^{\mathcal{A}} b_n$. Let $i \in [n]$ be maximal with $b_i \leq^{\mathcal{A}} a$. If $i = n$, then $b_n = a$. Otherwise $b_i \leq^{\mathcal{A}} a <^{\mathcal{A}} b_{i+1}$. As $S^{\mathcal{A}} b_i b_{i+1}$, we see that $b_i = a$ (by the last conjunct of φ_0). Now $A = B$ follows from $\mathcal{A} \models \varphi_0$. \square

Corollary 3.3. *Every proper $<$ -substructure of a finite model of $\varphi_0 \wedge \varphi_1$ is a model of $\varphi_0 \wedge \neg\varphi_1$.*

The class of finite τ_0 -orderings that are not complete is closed under $<$ -substructures but not axiomatizable by a universal sentence:

Theorem 3.4 (Tait's Theorem). *The class $\text{MOD}_{\text{fin}}(\varphi_0 \wedge \neg\varphi_1)$ is closed under $<$ -substructures (and hence, closed under induced substructures) but $\varphi_0 \wedge \neg\varphi_1$ is not finitely equivalent to a universal sentence.*

Proof: $\text{MOD}_{\text{fin}}(\varphi_0 \wedge \neg\varphi_1)$ is closed under $<$ -substructures: If $\mathcal{A} \models \varphi_0 \wedge \neg\varphi_1$ and \mathcal{B} is a finite $<$ -substructure of \mathcal{A} , then $\mathcal{B} \models \varphi_0$ (by Lemma 3.1). If $\mathcal{B} \models \neg\varphi_1$, we are done. If $\mathcal{B} \models \varphi_1$, then $\mathcal{A} \models \varphi_1$ by Lemma 3.2, which contradicts our assumption $\mathcal{A} \models \neg\varphi_1$.

Let $k \in \mathbb{N}$. It is clear that there is a finite model \mathcal{A} of $\varphi_0 \wedge \varphi_1$ with at least $k+1$ elements. By Corollary 3.3 every proper induced substructure of \mathcal{A} is a model of $\varphi_0 \wedge \neg\varphi_1$. Therefore, by Corollary 2.7, the sentence $\varphi_0 \wedge \neg\varphi_1$ is not finitely equivalent to a universal sentence of the form $\mu := \forall x_1 \dots \forall x_k \mu_0$ with quantifier-free μ_0 . As k was arbitrary, we get our claim. \square

Remark 3.5. A slight generalization of the previous proof shows that $\text{MOD}_{\text{fin}}(\varphi_0 \wedge \neg\varphi_1)$ is not even axiomatizable by a Π_2 -sentence, i.e., by a sentence χ of the form $\forall x_1 \dots \forall x_k \exists y_1 \dots \exists y_\ell \chi_0$ for some $k, \ell \geq 1$ and quantifier-free χ_0 . In fact, assume that $\text{MOD}_{\text{fin}}(\varphi_0 \wedge \neg\varphi_1) = \text{MOD}_{\text{fin}}(\chi)$. Again we choose a finite model \mathcal{A} of $\varphi_0 \wedge \varphi_1$ with at least $k+1$ elements. Then $\mathcal{A} \not\models \chi$. Hence there are $a_1, \dots, a_k \in A$ with $\mathcal{A} \models \neg \exists y_1 \dots \exists y_\ell \chi_0(a_1, \dots, a_k)$. Then $\mathcal{B} \models \neg \exists y_1 \dots \exists y_\ell \chi_0(a_1, \dots, a_k)$,

where $\mathcal{B} := [a_1, \dots, a_k]^{\mathcal{A}}$ is the substructure of \mathcal{A} induced by a_1, \dots, a_k . Hence, $\mathcal{B} \not\models \chi$ and therefore, $\mathcal{B} \not\models \varphi_0 \wedge \neg\varphi_1$. But this contradicts Corollary 3.3 as \mathcal{B} is a proper induced substructure of \mathcal{A} .

Note that $\varphi_0 \wedge \neg\varphi_1$ is (equivalent to) a Σ_2 -sentence, i.e., equivalent to the negation of a Π_2 -sentence.

We turn to a refinement of the previous statement that will be helpful to get Gurevich's Theorem.

Definition 3.6. (a) Let τ be obtained from the vocabulary τ_0 by adding finitely many relation symbols “in pairs,” the *standard* R together with its *complement* R^{comp} (intended as the complement of R). The symbols R and R^{comp} have the same arity and for our purposes we can restrict ourselves to unary or binary relation symbols (even though all results can be generalized to arbitrary arities). We briefly say that τ is obtained from τ_0 by adding pairs.

(b) Let τ be obtained from τ_0 by adding pairs. We say that $\varphi_{0\tau}$ is a τ -extension of φ_0 (where φ_0 is as above) if it is a universal sentence such that

- (i) the sentence φ_0 is a conjunct of $\varphi_{0\tau}$,
- (ii) the sentence $\bigwedge_{R \text{ standard}} \forall \bar{x} (\neg R\bar{x} \vee \neg R^{\text{comp}}\bar{x})$ is a conjunct of $\varphi_{0\tau}$,
- (iii) besides $<$ all relation symbols are negative in $\varphi_{0\tau}$ (if this is not the case for some new R or R^{comp} , the idea is to replace any positive occurrence of R or R^{comp} by $\neg R^{\text{comp}}$ and $\neg R$, respectively). For instance, we replace a subformula

$$x < y \wedge Rxy \quad \text{by} \quad x < y \wedge \neg R^{\text{comp}}xy.$$

(c) Let τ be obtained from τ_0 by adding pairs. Then we set

$$\varphi_{1\tau} := \varphi_1 \wedge \bigwedge_{R \text{ standard}} \forall \bar{x} (R\bar{x} \vee R^{\text{comp}}\bar{x}), \quad (8)$$

where φ_1 is as above (see (7)).

For a τ -structure \mathcal{B} with $\mathcal{B} \models \varphi_{0\tau} \wedge \varphi_{1\tau}$ we have

$$\mathcal{B} \models \bigwedge_{R \text{ standard}} \left(\forall \bar{x} (\neg R\bar{x} \vee \neg R^{\text{comp}}\bar{x}) \wedge \forall \bar{x} (R\bar{x} \vee R^{\text{comp}}\bar{x}) \right).$$

Hence,

$$\text{if } \mathcal{B} \models \varphi_{0\tau} \wedge \varphi_{1\tau}, \text{ then } (R^{\text{comp}})^{\mathcal{B}} \text{ is the complement of } R^{\mathcal{B}} \text{ for standard } R \in \tau. \quad (9)$$

Now we derive the analogues of Lemma 3.1–Theorem 3.4 essentially by the same proofs.

Lemma 3.7. *Let τ be obtained from τ_0 by adding pairs and let $\varphi_{0\tau}$ be an extension of φ_0 . If $\mathcal{B} \subseteq_{<} \mathcal{A}$ and $\mathcal{A} \models \varphi_{0\tau}$, then $\mathcal{B} \models \varphi_{0\tau}$.*

Proof: By Definition 3.6 (b) (iii) all relation symbols distinct from $<$ are negative in $\varphi_{0\tau}$. \square

Lemma 3.8. *Let τ be obtained from τ_0 by adding pairs and let $\varphi_{0\tau}$ be an extension of φ_0 . Assume that $\mathcal{A} \models \varphi_{0\tau}$ and that the finite $<$ -substructure \mathcal{B} of \mathcal{A} is a model of $\varphi_{1\tau}$. Then $\mathcal{B} = \mathcal{A}$ (in particular, $\mathcal{A} \models \varphi_{1\tau}$).*

Proof: Let $\mathcal{A} \upharpoonright \tau_0$ (and $\mathcal{B} \upharpoonright \tau_0$) be the τ_0 -structure obtained from \mathcal{A} (from \mathcal{B}) by removing all relations in $\tau \setminus \tau_0$.

By Lemma 3.2 we know that $\mathcal{B} \upharpoonright \tau_0 = \mathcal{A} \upharpoonright \tau_0$. Furthermore, $\mathcal{B} \models \varphi_{0\tau}$ by the previous lemma; thus, $\mathcal{B} \models \varphi_{0\tau} \wedge \varphi_{1\tau}$. Hence, by (9), $(R^{\text{comp}})^{\mathcal{B}}$ is the complement of $R^{\mathcal{B}}$ for standard R . Clearly, $R^{\mathcal{B}} \subseteq R^{\mathcal{A}}$ and $(R^{\text{comp}})^{\mathcal{B}} \subseteq (R^{\text{comp}})^{\mathcal{A}}$. As $A = B$ and \mathcal{A} is a model of the sentence $\bigwedge_{R \text{ standard}} \forall \bar{x} (\neg R\bar{x} \vee \neg R^{\text{comp}}\bar{x})$, we get $R^{\mathcal{B}} = R^{\mathcal{A}}$ and $(R^{\text{comp}})^{\mathcal{B}} = (R^{\text{comp}})^{\mathcal{A}}$. \square

Corollary 3.9. *Every proper $<$ -substructure of a finite model of $\varphi_{0\tau} \wedge \varphi_{1\tau}$ is a model of $\varphi_{0\tau} \wedge \neg\varphi_{1\tau}$.*

By replacing in the proof of Tait's Theorem the use of Lemma 3.1, Lemma 3.2, and Corollary 3.3 by Lemma 3.7, Lemma 3.8, and Corollary 3.9 respectively, we get:

Lemma 3.10. *Let τ be obtained from τ_0 by adding pairs and let $\varphi_{0\tau}$ be an extension of φ_0 . The class $\text{MOD}_{\text{fin}}(\varphi_{0\tau} \wedge \neg\varphi_{1\tau})$ is closed under $<$ -substructures (and hence, closed under induced substructures) but $\varphi_{0\tau} \wedge \neg\varphi_{1\tau}$ is not finitely equivalent to a universal sentence.*

Perhaps the reader will ask why we do not introduce for $<$ the ‘‘complement relation symbol’’ $<^{\text{comp}}$ and add the corresponding conjuncts to $\varphi_{0\tau}$ and $\varphi_{1\tau}$ (or, to φ_0 and φ_1) in order to get a result of the type of Lemma 3.8 (or already of the type of Lemma 3.2) where we can replace ‘‘ $<$ -substructure’’ by ‘‘substructure.’’ The reader will realize that corresponding proofs of $B = A$ break down.

The next proposition provides a uniform way to construct FO-sentences that are only equivalent to universal sentences of large size, which is the core of the proof of Gurevich's Theorem.

Proposition 3.11. *Again let τ be obtained from τ_0 by adding pairs and $\varphi_{0\tau}$ be an extension of φ_0 . Let $m \geq 1$ and γ be an FO[τ]-sentence such that*

$$\varphi_{0\tau} \wedge \varphi_{1\tau} \wedge \gamma \text{ has no infinite model but a finite model with at least } m \text{ elements.} \quad (10)$$

For

$$\chi := \varphi_{0\tau} \wedge (\varphi_{1\tau} \rightarrow \neg\gamma)$$

the statements (a) and (b) hold.

(a) *The class $\text{MOD}(\chi)$ is closed under $<$ -substructures.*

(b) *If $\mu := \forall x_1 \dots \forall x_k \mu_0$ with quantifier-free μ_0 is finitely equivalent to χ , then $k \geq m$.*

Proof: (a) Let $\mathcal{A} \models \chi$ and $\mathcal{B} \subseteq < \mathcal{A}$. Thus, $\mathcal{B} \models \varphi_{0\tau}$. If $\mathcal{B} \not\models \varphi_{1\tau}$, we are done. Assume $\mathcal{B} \models \varphi_{1\tau}$. In case B is infinite, we conclude by (10) that \mathcal{B} is a model of $\neg\gamma$ and hence of χ . Otherwise B is finite; then $\mathcal{B} = \mathcal{A}$ (by Lemma 3.8) and thus, $\mathcal{B} \models \chi$.

(b) According to (10) there is a finite model \mathcal{A} of $\varphi_{0\tau} \wedge \varphi_{1\tau} \wedge \gamma$, i.e., of $\varphi_{0\tau} \wedge \neg(\varphi_{1\tau} \rightarrow \neg\gamma)$, with at least m elements. By Corollary 3.9 every proper induced substructure of \mathcal{A} is not a model of $\varphi_{1\tau}$ and therefore, it is a model of $\varphi_{0\tau} \wedge (\varphi_{1\tau} \rightarrow \neg\gamma)$. Hence by Corollary 2.7, $\varphi_{0\tau} \wedge (\varphi_{1\tau} \rightarrow \neg\gamma)$ is not finitely equivalent to a universal sentence of the form $\mu := \forall x_1 \dots \forall x_k \mu_0$ with $k < m$ and quantifier-free μ_0 . \square

Remark 3.12. We can strengthen the statement (b) of the preceding proposition to:

If the Π_2 -sentence $\forall x_1 \dots \forall x_k \exists y_1 \dots \exists y_\ell \chi_0$ with quantifier-free χ_0 is finitely equivalent to χ , then $k \geq m$.

The proof is similar to that of the result in Remark 3.5 and is left to the reader.

4. The general machinery: strongly existential interpretations

We show that appropriate interpretations preserve the validity of Tait's theorem and of the statement of Proposition 3.11. Later on these interpretations will allow us to get versions of the results for graphs.

Let $\tau_E := \{E\}$ with binary E . As already remarked in the Preliminaries for all τ_E -structures we use the notation $G = (V(G), E(G))$ common in graph theory.

Let τ be obtained from τ_0 by adding pairs. Furthermore, let I be an interpretation of *width 2* (we only need this case) of τ -structures in τ_E -structures. This means that I assigns to every unary relation symbol $T \in \tau$ an $\text{FO}[\tau_E]$ -formula $\varphi_T(x_1, x_2)$ and to every binary relation symbol $T \in \tau$ an $\text{FO}[\tau_E]$ -formula $\varphi_T(x_1, x_2, y_1, y_2)$; moreover, I selects an $\text{FO}[\tau_E]$ -formula $\varphi_{\text{uni}}(x_1, x_2)$.

Then I assigns to every τ_E -structure G with $G \models \exists \bar{x} \varphi_{\text{uni}}(\bar{x})$ a τ -structure G_I , which we often denote by $\mathcal{O}_I(G)$, defined by

- $O_I(G) := \{\bar{a} \in V(G) \times V(G) \mid G \models \varphi_{\text{uni}}(\bar{a})\}$
- $T^{O_I(G)} := \{\bar{a} \in O_I(G) \mid G \models \varphi_T(\bar{a})\}$ for unary $T \in \tau$
- $T^{O_I(G)} := \{(\bar{a}, \bar{b}) \in O_I(G) \times O_I(G) \mid G \models \varphi_T(\bar{a}, \bar{b})\}$ for binary $T \in \tau$.

As the interpretation I is of width 2, we have

$$|O_I(G)| \leq |V(G)|^2. \quad (11)$$

Recall that for every sentence $\varphi \in \text{FO}[\tau]$ there is a sentence $\varphi^I \in \text{FO}[\tau_E]$ such that for all τ_E -structures G with $G \models \exists \bar{x} \varphi_{\text{uni}}(\bar{x})$ we have

$$(G_I \models) \mathcal{O}_I(G) \models \varphi \iff G \models \varphi^I. \quad (12)$$

For example, for the sentence $\varphi = \forall x \forall y Txy$ we have

$$\varphi^I = \forall \bar{x} \left(\varphi_{\text{uni}}(\bar{x}) \rightarrow \forall \bar{y} \left(\varphi_{\text{uni}}(\bar{y}) \rightarrow \varphi_T(\bar{x}, \bar{y}) \right) \right).$$

Furthermore there is a constant $c_I \in \mathbb{N}$ such that for all $\varphi \in \text{FO}[\tau]$,

$$|\varphi^I| \leq c_I \cdot |\varphi|. \quad (13)$$

Definition 4.1. Let τ be obtained from τ_0 by adding pairs and let I be an interpretation of τ_0 -structures in τ_E as just described. We say that I is *strongly existential* if all formulas of I are existential and $\varphi_{<}$ is even quantifier-free.

Lemma 4.2. Let τ be obtained from τ_0 by adding pairs and let $\varphi_{0\tau}$ be an extension of φ_0 . Then for every strongly existential interpretation I the sentence $\varphi_{0\tau}^I$ is (equivalent to) a universal sentence.

Proof: The claim holds as all relation symbols distinct from $<$ are negative in $\varphi_{0\tau}$. For example, for $\varphi := \forall x \forall y (U_{\min} x \rightarrow (x = y \vee x < y))$, we have

$$\varphi^I = \forall \bar{x} \left(\varphi_{\text{uni}}(\bar{x}) \rightarrow \forall \bar{y} \left(\varphi_{\text{uni}}(\bar{y}) \rightarrow (\varphi_{U_{\min}}(\bar{x}) \rightarrow ((x_1 = y_1 \wedge x_2 = y_2) \vee \varphi_{<}(\bar{x}, \bar{y}))) \right) \right). \quad \square$$

The following result shows that strongly existential interpretations preserve induced substructures in such a way that we can translate the results of the preceding section to the actual context.

Lemma 4.3. *Assume that I is strongly existential. Then for all τ_E -structures G and H with $H \subseteq_{\text{ind}} G$ and $O_I(H) \neq \emptyset$, we have $\mathcal{O}_I(H) \subseteq_{<} \mathcal{O}_I(G)$.*

Proof: As φ_{uni} is existential, we have $O_I(H) \subseteq O_I(G)$. Let $T \in \tau$ be distinct from $<$ and $\bar{b} \in T^{\mathcal{O}_I(H)}$. Then $H \models \varphi_T(\bar{b})$. As φ_T is existential, $G \models \varphi_T(\bar{b})$ and thus, $\bar{b} \in T^{\mathcal{O}_I(G)}$. Moreover, for $\bar{b}, \bar{b}' \in O_I(H)$ we have

$$\begin{aligned} \bar{b} <^{\mathcal{O}_I(H)} \bar{b}' &\iff H \models \varphi_{<}(\bar{b}, \bar{b}') \\ &\iff G \models \varphi_{<}(\bar{b}, \bar{b}') && \text{(as } H \subseteq_{\text{ind}} G \text{ and } \varphi_{<} \text{ is quantifier-free)} \\ &\iff \bar{b} <^{\mathcal{O}_I(G)} \bar{b}'. \end{aligned}$$

Putting all together we see that $\mathcal{O}_I(H) \subseteq_{<} \mathcal{O}_I(G)$. \square

We obtain from Lemma 3.8 the corresponding result in our framework.

Lemma 4.4. *Assume that I is strongly existential. Let $\varphi_{0\tau}$ be an extension of φ_0 . Let G be a τ_E -structure and $G \models \varphi_{0\tau}^I$. Let $H \subseteq_{\text{ind}} G$ with finite $O_I(H)$. If $H \models \varphi_{1\tau}^I$, then $\mathcal{O}_I(H) = \mathcal{O}_I(G)$ and $G \models \varphi_{1\tau}^I$.*

Proof: As $H \models \varphi_{1\tau}^I$, in particular $H \models (\exists x U_{\min} x)^I$; thus, $O_I(H) \neq \emptyset$. Therefore, $\mathcal{O}_I(H) \subseteq_{<} \mathcal{O}_I(G)$ by Lemma 4.3. By assumption and (12), $\mathcal{O}_I(G) \models \varphi_{0\tau}$ and $\mathcal{O}_I(H) \models \varphi_{1\tau}$. As $O_I(H)$ is finite, Lemma 3.8 implies $\mathcal{O}_I(H) = \mathcal{O}_I(G)$, and in particular $\mathcal{O}_I(G) \models \varphi_{1\tau}$. Hence, $G \models \varphi_{1\tau}^I$ by (12). \square

We now prove for strongly existential interpretations two results, Proposition 4.5 corresponds to Tait's Theorem (Theorem 3.4), and Proposition 4.6 corresponds to Proposition 3.11 (relevant to Gurevich's Theorem). In our application of these results to graphs in the next section the sentence ψ will be $\forall x \neg Exx \wedge \forall x \forall y (Exy \rightarrow Eyx)$, i.e., the sentence φ_{GRAPH} (cf. (3)) axiomatizing the class of graphs.

Proposition 4.5. *Let ψ be a universal τ_E -sentence. Assume that the interpretation I of τ_0 -structures in τ_E -structures is strongly existential. Furthermore, assume that for every sufficiently large finite complete τ_0 -ordering \mathcal{A} there is a finite τ_E -structure G with $\mathcal{O}_I(G) \cong \mathcal{A}$ and $G \models \psi$. Then there is an $\text{FO}[\tau_E]$ -sentence φ such that $\text{MOD}_{\text{fin}}(\psi \wedge \varphi)$ is closed under induced substructures, but $\psi \wedge \varphi$ is not finitely equivalent to a universal sentence.*

As φ we take the sentence

$$\varphi := \forall \bar{x} \neg \varphi_{\text{uni}}(\bar{x}) \vee (\varphi_0^I \wedge \neg \varphi_1^I)$$

(for the definition of φ_0 and φ_1 see page 8 and (7), respectively).

Proof: First we verify that the class $\text{MOD}_{\text{fin}}(\psi \wedge \varphi)$ is closed under induced substructures. Assume $G \models \psi \wedge \varphi$ and $H \subseteq_{\text{ind}} G$. Since ψ is universal, we have $H \models \psi$. If $G \models \forall \bar{x} \neg \varphi_{\text{uni}}(\bar{x})$, then $H \models \forall \bar{x} \neg \varphi_{\text{uni}}(\bar{x})$. Now assume that $G \models \varphi_0^I \wedge \neg \varphi_1^I$. Then $H \models \varphi_0^I$, as φ_0^I is universal by Lemma 4.2. If $H \models \forall \bar{x} \neg \varphi_{\text{uni}}(\bar{x})$ or $H \models \neg \varphi_1^I$, we are done. Otherwise $O_I(H) \neq \emptyset$ and $H \models \varphi_1^I$. Then $G \models \varphi_1^I$ (see Lemma 4.4), a contradiction.

Finally we show that for every $k \in \mathbb{N}$ the sentence $\psi \wedge \varphi$ is not finitely equivalent to a sentence of the form $\mu = \forall z_1 \dots \forall z_k \mu_0$ with quantifier-free μ_0 . Let

$$\mathcal{A} := (A, <^A, U_{\min}^A, U_{\max}^A, S^A)$$

be a complete τ_0 -ordering with at least $k^2 + 1$ elements. In particular, $\mathcal{A} \models \varphi_0 \wedge \varphi_1$. By assumption we can choose \mathcal{A} in such a way that there is a finite τ_E -structure G such that $\mathcal{O}_I(G) \cong \mathcal{A}$ and $G \models \psi$. Then $\mathcal{O}_I(G) \models \varphi_0 \wedge \varphi_1$, hence, $G \models \varphi_0^I \wedge \varphi_1^I$. Thus $G \models \psi \wedge \neg\varphi$. As $|\mathcal{O}_I(G)| = |\mathcal{A}| \geq k^2 + 1$, the graph G must contain more than k elements by (11).

We want to show that every induced substructure of G with at most k elements is a model of $\psi \wedge \varphi$. Then the result follows from Corollary 2.7. So let H be an induced substructure of G with at most k elements. Clearly, $H \models (\psi \wedge \varphi_0^I)$. If $H \models \forall \bar{x} \neg \varphi_{\text{uni}}(\bar{x})$ or $H \models \neg \varphi_1^I$, we are done. Otherwise $\mathcal{O}_I(H) \neq \emptyset$ and $H \models \varphi_1^I$. Then, Lemma 4.4 implies $\mathcal{O}_I(H) = \mathcal{O}_I(G)$. Recall $|V(H)| \leq k$, so $\mathcal{O}_I(H)$ has at most k^2 elements by (11), a contradiction as $|\mathcal{O}_I(G)| \geq k^2 + 1$. \square

Proposition 4.6. *Assume that ψ is a universal τ_E -sentence. Let τ be obtained from τ_0 by adding pairs and let $\varphi_{0\tau}$ be an extension of φ_0 . Let I be a strongly existential interpretation of τ -structures in τ_E -structures with the property that for every finite τ -structure \mathcal{A} , which is a model of $\varphi_{0\tau} \wedge \varphi_{1\tau}$, there is a finite τ_E -structure G with $\mathcal{O}_I(G) \cong \mathcal{A}$ and $G \models \psi$.*

Let $m \geq 1$ and γ be an $\text{FO}[\tau]$ -sentence such that

$$\varphi_{0\tau} \wedge \varphi_{1\tau} \wedge \gamma \text{ has no infinite model but a finite model with at least } m \text{ elements.} \quad (14)$$

For

$$\rho := \forall \bar{x} \neg \varphi_{\text{uni}}(\bar{x}) \vee (\varphi_{0\tau} \wedge (\varphi_{1\tau} \rightarrow \neg \gamma))^I \quad (15)$$

the statements (a) and (b) hold.

(a) *The class $\text{MOD}(\psi \wedge \rho)$ is closed under induced substructures.*

(b) *If $\mu := \forall x_1 \dots \forall x_k \mu_0$ with quantifier-free μ_0 is finitely equivalent to $\psi \wedge \rho$, then $k^2 \geq m$.*

Proof: (a) Assume that $G \models \psi \wedge \rho$ and $H \subseteq_{\text{ind}} G$. Clearly $H \models \psi$. If $H \models \forall \bar{x} \neg \varphi_{\text{uni}}(\bar{x})$, then we are done. Otherwise, the universe of $\mathcal{O}_I(H)$ and hence, that of $\mathcal{O}_I(G)$, are not empty. Then $G \models \varphi_{0\tau}^I$ and as $H \subseteq_{\text{ind}} G$, we have $H \models \varphi_{0\tau}^I$ by Lemma 4.2.

If $H \not\models \varphi_{1\tau}^I$, we are done. Otherwise, $H \models \varphi_{1\tau}^I$. If H_I is infinite, then $H_I \models \neg \gamma$ by (14) and we are again done. If H_I is finite, then $\mathcal{O}_I(H) = \mathcal{O}_I(G)$ by Lemma 4.4. Thus $\mathcal{O}_I(G) \models \varphi_{1\tau}$ and hence, $\mathcal{O}_I(G) \models \neg \gamma$ as $G \models \rho$. Therefore, $\mathcal{O}_I(H) \models \neg \gamma$ and thus, $H \models \rho$.

(b) By (14) there is a finite model \mathcal{A} of $\varphi_{0\tau} \wedge \varphi_{1\tau} \wedge \gamma$ with at least m elements. By assumption there is a finite τ_E -structure G with $\mathcal{O}_I(G) \cong \mathcal{A}$ and $G \models \psi$. Clearly, $G \models \neg \forall \bar{x} \neg \varphi_{\text{uni}}(\bar{x})$ and $G \models (\varphi_{0\tau} \wedge \varphi_{1\tau} \wedge \gamma)^I$. Hence, $G \models \psi \wedge \neg \rho$. Assume that $k^2 < m$. We want to show that every induced substructure of G with at most k elements is a model of $\psi \wedge \rho$. Then the claim (b) follows from Corollary 2.7.

So let H be an induced substructure of G with at most k elements. Clearly, $H \models (\psi \wedge \varphi_{0\tau}^I)$. If $H \models \forall \bar{x} \neg \varphi_{\text{uni}}(\bar{x})$ or $H \models \neg \varphi_{1\tau}^I$, we are done. Otherwise $\mathcal{O}_I(H) \neq \emptyset$ and $H \models \varphi_{1\tau}^I$. Then, $\mathcal{O}_I(H) = \mathcal{O}_I(G)$ by Lemma 4.4. This leads to a contradiction, as $\mathcal{O}_I(H)$ has at most k^2 elements by (12), while $\mathcal{O}_I(G)$ has m elements and we assumed $k^2 < m$. \square

Remark 4.7. The results corresponding to Remark 3.5 and Remark 3.12 are valid for Proposition 4.5 and Proposition 4.6 too. In particular, the sentence $\psi \wedge \varphi$ ($= \psi \wedge \forall \bar{x} \neg \varphi_{\text{uni}}(\bar{x}) \vee (\varphi_0^I \wedge \neg \varphi_1^I)$) is not equivalent to a Π_2 -sentence. Furthermore $\psi \wedge \varphi$ itself is equivalent to a Σ_2 -sentence. In fact, as all relation symbols besides $<$ are negative in φ_0 , the sentence φ_0^I is universal. Moreover, as U_{\min} , U_{\max} , and S are positive in φ_1 , the sentence φ_1^I (as φ_1) is equivalent to a Π_2 -sentence. Hence $\psi \wedge \varphi$ is equivalent to a Σ_2 -sentence.

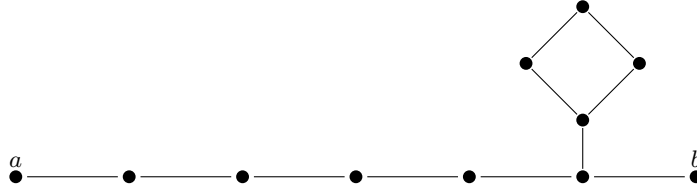


Figure 1. A path of length 6 with a 4-ear.

5. Tait's Theorem for finite graphs

In this section we introduce a strongly existential interpretation, which allows us to get Tait's Theorem for graphs. The corresponding result for Gurevich's Theorem will be derived in Section 6.

We first introduce a further concept. Let G be a graph and $a, b \in V(G)$. For $r, s \geq 3$ a *path from vertex a to vertex b of length r with an s -ear* is a path between a and b with a cycle of length s ; one vertex of this cycle is adjacent to the vertex adjacent to b on the path. Figure 1 is a path from a to b of length 6 with a 4-ear.

Lemma 5.1. *For $r, s \geq 3$ there are quantifier-free formulas $\varphi_{cr}(x, \bar{z})$ and $\varphi_{pe,r,s}(x, y, \bar{z}, \bar{w})$ such that for all graphs G we have*

- (a) $G \models \varphi_{cr}(a, \bar{u}) \iff \bar{u}$ is a cycle of length r containing a .
- (b) $G \models \varphi_{pe,r,s}(a, b, \bar{u}, \bar{v}) \iff \bar{u}$ is path from a to b of length r with the s -ear \bar{v} .

Proof: (a) We can take as $\varphi_{cr}(x, z_1, \dots, z_r)$ the formula

$$x = z_1 \wedge E z_r z_1 \wedge \bigwedge_{1 \leq i < r} E z_i z_{i+1} \wedge \bigwedge_{1 \leq i < j \leq r} \neg z_i = z_j.$$

(b) We can take as $\varphi_{pe,r,s}(x, y, z_0, \dots, z_r, w_1, \dots, w_s)$ the formula

$$x = z_0 \wedge y = z_r \wedge \bigwedge_{0 \leq i < r-1} E z_i z_{i+1} \wedge \bigwedge_{0 \leq i < j \leq r} \neg z_i = z_j \wedge \bigwedge_{0 \leq i \leq r, j \in [s]} \neg z_i = w_j \\ \wedge \varphi_{cs}(w_1, w_1, \dots, w_r) \wedge E z_{r-1} w_1. \quad \square$$

To understand better how we obtain the desired interpretation we first assign to every complete τ_0 -ordering \mathcal{A} , i.e., to every model of $\varphi_0 \wedge \varphi_1$, a τ_E -structure $G := G(\mathcal{A})$ which is a graph.

In a first step we extend \mathcal{A} to a τ_0^* -structure \mathcal{A}^* , where $\tau_0^* := \tau_0 \cup \{B, C, L, F\}$ in the following way. Here B, C are unary and L, F are binary relation symbols.

For every *original* (or, *basic*) element a , i.e., for every $a \in A$, we introduce a new element a' , the *companion* of a . We set

- $A^* := A \cup \{a' \mid a \in A\}$,
- $B^{A^*} := A, \quad C^{A^*} := \{a' \mid a \in A\}$,
- $L^{A^*} := \{(a, a') \mid a \in A\}, \quad F^{A^*} := \{(a', b), (b, a') \mid a, b \in A, a <^{\mathcal{A}} b\}$.



Figure 2. Turning an ordering to the relation F .

Note that the relation F is irreflexive and symmetric, i.e., (A^*, F^{A^*}) is already a graph, which is illustrated by Figure 2. Observe that F contains the whole information of the ordering $<^A$ up to isomorphism.

We use \mathcal{A}^* to define the desired graph $G = G(\mathcal{A})$. The vertex set $V(G)$ contains the elements of A^* , and the edge relation $E(G)$ contains F^{A^*} . Furthermore G contains just all the vertices and edges required by the following items:

- To $a \in U_{\min}^A$ we add a cycle of length 5 consisting of new vertices, i.e., not in A^* (besides a).
- To $a \in U_{\max}^A$ we add a cycle of length 7 consisting of new vertices (besides a).
- To $a \in B^{A^*}$ we add a cycle of length 9 consisting of new vertices (besides a).
- To $a \in C^{A^*}$ we add a cycle of length 11 consisting of new vertices (besides a).
- To $(a, b) \in S^A$ we add a path from a to b of length 17 with a 13-ear consisting of new vertices (besides a and b).
- To $(a, a') \in L^{A^*}$ we add a path from a to a' of length 17 with a 15-ear consisting of new vertices (besides a and a').

Hereby we meant by “add a cycle” or “add a path with an ear” that we only add the edges required by the corresponding formulas in Lemma 5.1.

To ease the discussion, we divide cycles in $G (= G(\mathcal{A}))$ into four categories.

[F -cycle] These are cycles in (A^*, F^{A^*}) , i.e., cycles using only edges of F^{A^*} .

[T -cycle] For every unary $T \in \{U_{\min}, U_{\max}, B, C\}$, a T -cycle is the cycle introduced for an $a \in T^A$.

[ear-cycle] These are the cycles constructed as ears on the gadgets for the relations S^{A^*} and L^{A^*} .

[mixed-cycle] All the other cycles are *mixed*.

For example, we get a mixed cycle if we start with a_2, a'_0, a_1 in Figure 2 and then add the path introduced for $(a_1, a_2) \in S^A$ (ignoring the ear).

A number of observations for these types of cycles are in order.

Lemma 5.2. (i) All the F -cycles are of even length.¹

(ii) Every U_{\min} -, U_{\max} -, B -, and C -cycle is of length 5, 7, 9, and 11, respectively.

(iii) Every ear-cycle is of length 13 or 15.

¹Moreover one can show that every *chordless* F -cycle has length 4.

(iv) Every mixed-cycle neither uses new vertices of any T -cycle for $T \in \{U_{\min}, U_{\max}, B, C\}$ nor any vertex of any ear-cycle.

(v) Every mixed-cycle has length at least 17.

Proof: (i) follows easily from the fact that (A^*, F^{A^*}) is a bipartite graph; (ii) and (iii) are trivial.

For (iv) assume that a mixed-cycle uses a *new* vertex b of a T -cycle \mathcal{C} introduced for some $a \in T^{A^*}$, where $T \in \{U_{\min}, U_{\max}, B, C\}$. As \mathcal{C} is mixed, it must contain a vertex $c \notin T^{A^*}$. To reach b from c the mixed cycle must pass through a and hence must contain one of the two segments of \mathcal{C} between b and a . As a consequence, in order for the mixed-cycle to go back from b to c , it must also use the other segment of \mathcal{C} between a and b . This means that it must be the T -cycle \mathcal{C} itself, instead of a mixed one. A similar argument shows that mixed cycles do not contain vertices of any ear-cycle.

To prove (v), let \mathcal{C} be a mixed-cycle. By (iv), \mathcal{C} must contain all vertices of a (at least one) path introduced for a pair $(a, a') \in L^{A^*}$ or $(a, b) \in S^{A^*}$ (ignoring the ear). As this path has length 17, we get our claim. \square

Conversely, given a τ_E -structure G , which is a graph, we construct a τ_0 -structure which we denote by $\mathcal{O}(G)$, possibly the empty structure. Recall the definitions of ‘‘cycle’’ and of ‘‘path with ear’’ given by Lemma 5.1.

- $O(G) := \{(a_1, a_2) \in V(G) \times V(G) \mid a_1 \text{ is a member of a cycle of length 9, } a_2 \text{ is a member of a cycle of length 11, and there is a path from } a_1 \text{ to } a_2 \text{ of length 17 with a 15-ear}\}$
- $<^{\mathcal{O}(G)} := \{((a_1, a_2), (b_1, b_2)) \in O(G) \times O(G) \mid \{a_2, b_1\} \in E(G)\}$
- $U_{\min}^{\mathcal{O}(G)} := \{(a_1, a_2) \in O(G) \mid a_1 \text{ is a member of a cycle of 5 elements}\}$
- $U_{\max}^{\mathcal{O}(G)} := \{(a_1, a_2) \in O(G) \mid a_1 \text{ is a member of a cycle of 7 elements}\}$
- $S^{\mathcal{O}(G)} := \{((a_1, a_2), (b_1, b_2)) \in O(G) \times O(G) \mid \text{there is a path from } a_1 \text{ to } b_1 \text{ of length 17 with a 13-ear}\}.$

Lemma 5.3. *For every complete τ_0 -ordering \mathcal{A} we have $\mathcal{O}(G(\mathcal{A})) \cong \mathcal{A}$.*

Proof: Let $G := G(\mathcal{A})$ and $\mathcal{A}^+ := \mathcal{O}(G)$. We claim that the mapping $h : \mathcal{A} \rightarrow \mathcal{A}^+$ defined by

$$h(a) := (a, a') \quad \text{for } a \in \mathcal{A}$$

is an isomorphism from \mathcal{A} to \mathcal{A}^+ . To that end, we first prove that

$$\mathcal{A}^+ = \{(a, a') \mid a \in \mathcal{A}\},$$

which implies that h is well defined and a bijection. For every $a \in \mathcal{A}$ it is easy to see that $(a, a') \in O(G)$ ($= \mathcal{A}^+$). For the converse, let $(a_1, a_2) \in O(G)$. In particular, a_1 is a member of a cycle of length 9. By Lemma 5.2, this must be a B -cycle which contains some $a \in \mathcal{A}$. Using the same argument, a_2 is a member of a C -cycle which contains a vertex b' being the companion of some $b \in \mathcal{A}$. Furthermore, there is a path from a_1 to a_2 of length 17 with a 15-ear. The 15-ear is a cycle of length 15. Again by Lemma 5.2 this cycle is an ear-cycle which belongs to the gadget we introduced for some $(c, c') \in L^{A^*}$ with $c \in \mathcal{A}$. Then it is easy to see that $a = c = b$. This finishes the proof that h is a bijection from \mathcal{A} to \mathcal{A}^+ .

Similarly, we can prove that h preserves all the relations. \square

We want to show that we can obtain $\mathcal{O}(G)$ from G by a strongly existential FO-interpretation. We set

$$\begin{aligned} \eta(x, x', \bar{x}, \bar{x}', \bar{z}, \bar{w}) &:= \text{“}\bar{x} \text{ is a cycle of length 9 containing } x, \bar{x}' \text{ is a cycle of length 11 containing } x', \\ &\text{and } \bar{z} \text{ is a path from } x \text{ to } x' \text{ of length 17 with the 15-ear } \bar{w}\text{”} \\ &= \varphi_{c9}(x, \bar{x}) \wedge \varphi_{c11}(x', \bar{x}') \wedge \varphi_{pe,17,15}(x, x', \bar{z}, \bar{w}). \end{aligned}$$

We define the desired interpretation I of width 2 of τ_0 -structures in graphs. We set

$$\varphi_{\text{uni}}(x, x') := \exists \bar{x} \exists \bar{x}' \exists \bar{z} \exists \bar{w} \eta(x, x', \bar{x}, \bar{x}', \bar{z}, \bar{w}).$$

Hence for every graph G ,

$$\mathcal{O}_I(G) = \{(a_1, a_2) \in V(G) \times V(G) \mid G \models \exists \bar{x} \exists \bar{x}' \exists \bar{z} \exists \bar{w} \eta(a_1, a_2, \bar{x}, \bar{x}', \bar{z}, \bar{w})\}.$$

Furthermore we define

- $\varphi_{U_{\min}}(x, x') := \exists \bar{z} \varphi_{c5}(x, \bar{z})$,
- $\varphi_{U_{\max}}(x, x') := \exists \bar{z} \varphi_{c7}(x, \bar{z})$,
- $\varphi_S(x, x', y, y') := \exists \bar{z} \exists \bar{w}$ “ \bar{z} is a path of length 17 from x to y with a 13-ear \bar{w} ”
 $= \exists \bar{z} \exists \bar{w} \varphi_{pe,17,13}(x, \bar{z}, \bar{w})$.

Then we have:

Lemma 5.4. *The interpretation I given by $(\varphi_{\text{uni}}, \varphi_{<}, \varphi_{U_{\min}}, \varphi_{U_{\max}}, \varphi_S)$ is strongly existential. For every complete τ_0 -ordering \mathcal{A} we have $\mathcal{O}_I(G(\mathcal{A})) = \mathcal{O}(G(\mathcal{A}))$ and hence, by Lemma 5.3,*

$$\mathcal{O}_I(G(\mathcal{A})) \cong \mathcal{A}.$$

Setting $\psi := \varphi_{\text{GRAPH}}$, the sentence axiomatizing the class of graphs, we get from Proposition 4.5:

Theorem 5.5 (Tait’s Theorem for graphs). *There is a τ_E -sentence φ such that $\text{GRAPH}_{\text{fin}}(\varphi)$, the class of finite graphs that are models of φ , is closed under induced subgraphs but φ is not equivalent to a universal sentence in finite graphs.*

In this section we presented a strongly existential interpretation of τ_0 -structures and applied it to finite complete τ_0 -orderings, i.e. to models of $\varphi_0 \wedge \varphi_1$. A straightforward generalization of the preceding proofs allows to show the following result for vocabularies obtained from τ_0 by adding pairs. We shall use it in Section 6.

Lemma 5.6. *Let τ be obtained from τ_0 by adding pairs. There is a strongly existential interpretation $I (= I_\tau)$ that for every extension $\varphi_{0\tau}$ of φ_0 assigns to every τ -structure \mathcal{A} that is a model of $\varphi_{0\tau} \wedge \varphi_{1\tau}$ a graph $G(\mathcal{A})$ with $\mathcal{O}_I(G(\mathcal{A})) \cong \mathcal{A}$. For finite \mathcal{A} the graph $G(\mathcal{A})$ is finite.*

Proof: We get the graph $G(\mathcal{A})$ as in the case $\tau := \tau_0$: For the elements of new unary relations we add cycles such that the lengths of the cycles are odd and distinct for distinct unary relations in τ . Let c be the maximal length of these cycles. Then we add paths with ears to the tuples of binary relations as above. For distinct binary relations the ears should have distinct length and again this length should be odd and greater than c . On the other hand, the length of added new paths can be the same for all binary relations but should be greater than the length of all the cycles. \square

Remark 5.7. (a) Let $\mathcal{C} := \text{MOD}_{\text{fin}}(\forall x \neg Exx)$ be the class of directed graphs. Then $\mathcal{C}' := \text{GRAPH}_{\text{fin}}$, the class of finite graphs, is a subclass of \mathcal{C} closed under induced substructures and definable in \mathcal{C} by the universal sentence $\forall x \forall y (Exy \rightarrow Eyx)$. As the Łoś-Tarski Theorem fails for the class of finite graphs, it fails for the class of directed graphs by Remark 2.8.

(b) Now let $\mathcal{C} := \text{GRAPH}_{\text{fin}}$ and $\mathcal{C}' := \text{PLANAR}_{\text{fin}}$ be the class of finite planar graphs, a subclass of $\text{GRAPH}_{\text{fin}}$ closed under induced subgraphs. As mentioned in the Introduction, in [1] it is shown that the Łoś-Tarski Theorem fails for $\text{PLANAR}_{\text{fin}}$. As $\text{PLANAR}_{\text{fin}}$ is not axiomatizable in $\text{GRAPH}_{\text{fin}}$ by a universal sentence, not even by a first-order sentence, we do not get the failure of the Łoś-Tarski Theorem for the class of finite graphs, i.e., Tait’s Theorem for graphs, by applying the result of Remark 2.8. We show that $\text{PLANAR}_{\text{fin}} = \text{FORB}_{\text{fin}}(\mathcal{F})$ for a finite set \mathcal{F} of finite graphs (or, equivalently, $\text{PLANAR}_{\text{fin}} = \text{MOD}_{\text{fin}}(\mu)$ for a universal μ) leads to a contradiction. Let k be the maximum size of the set of vertices of graphs in \mathcal{F} . Let G be the graph obtained from the clique K_5 of 5 vertices by subdividing each edge by $k + 1$. Clearly, $G \notin \text{PLANAR}_{\text{fin}}$. However, every subgraph of G induced on at most k elements is planar. Hence, $G \in \text{FORB}_{\text{fin}}(\mathcal{F})$.

(c) Let τ be any vocabulary with at least one at least binary relation T . Then the Łoś-Tarski Theorem fails for the class $\mathcal{C} := \text{STR}_{\text{fin}}[\tau]$, the class of all finite τ -structures. By Remark 2.8 it suffices to show the existence of a universally definable subclass \mathcal{C}' of \mathcal{C} which “essentially is the class of graphs.” We set

$$\mu := \forall x \forall \bar{u} \neg Txx\bar{u} \wedge \forall x \forall y \forall \bar{u} \forall \bar{v} (Txy\bar{u} \rightarrow Tyx\bar{v}) \wedge \bigwedge_{R \in \tau, R \neq T} \forall \bar{u} \neg R\bar{u}$$

and let \mathcal{C}' be $\text{MOD}_{\text{fin}}(\mu)$.

If τ only contains unary relation symbols, the Łoś-Tarski Theorem holds for $\text{STR}_{\text{fin}}[\tau]$. It is easy to see for an $\text{FO}(\tau)$ -sentence φ that the closure under induced substructures of $\text{MOD}_{\text{fin}}(\varphi)$ implies that of $\text{MOD}(\varphi)$.

6. Gurevich’s Theorem

The following discussion will eventually lead to a proof of Gurevich’s Theorem, i.e., Theorem 1.5. Our proof essentially follows Gurevich’s proof in [14], but it contains some elements of Rossman’s proof of the same result in [16].² Afterwards we show that it remains true if we restrict ourselves to graphs.

Our main tool is Proposition 3.11, and the goal is to construct a formula γ in (10) whose size is much smaller than the number m . Basically γ will describe a very long computation of a Turing machine on a short input. We fix a universal Turing machine M operating on an one-way infinite tape, the tape alphabet is $\{0, 1\}$, where 0 is also considered as blank, and Q is the set of states of M . The initial state is q_0 , and q_h is the halting state; thus $q_0, q_h \in Q$ and we assume that $q_0 \neq q_h$. An instruction of M has the form

$$qapbd,$$

where $q, p \in Q$, $a, b \in \{0, 1\}$ and $d \in \{-1, 0, 1\}$. It indicates that if M is in state q and the head of M reads an a , then the head replaces a by b and moves to the left (if $d = -1$), stays still (if $d = 0$), or moves to the right (if $d = 1$). In order to describe computations of M by FO-formulas we introduce binary predicates $H_q(x, t)$ for $q \in Q$ to indicate that at time t the machine is in state q and the head scans cell x , and a binary predicate $C_0(x, t)$ to indicate that the content of cell x at time t is 0.

²The reader of [14] will realize that the definition of φ^n on page 190 of [14] must be modified in order to ensure that the class of models of φ^n is closed under induced substructures.

The vocabulary τ_M is obtained from τ_0 by adding pairs (see Definition 3.6 (a)),

$$\tau_M := \tau_0 \cup \{H_q, H_q^{\text{comp}} \mid q \in Q\} \cup \{C_0, C_0^{\text{comp}}\}.$$

Intuitively, $H_q^{\text{comp}}(x, t)$ says that “at time t the machine is not in state q or the head is not in cell x ,” and $C_0^{\text{comp}}(x, t)$ says that “at time t the content of cell x is (not 0 and thus is) 1.” Sometimes we write C_1 instead of C_0^{comp} (e.g., below in φ_2 if $a = 1$ or $b = 0$).

Let φ_0 and φ_1 be the sentences already introduced in Section 3. For $w \in \{0, 1\}^*$ the sentence φ_{0w} will be an extension of φ_0 (compare Definition 3.6 (b)); hence, φ_{0w} will be a universal sentence and all relations symbols besides $<$ are negative in φ_{0w} ; in particular, it contains as conjuncts φ_0 and

$$\forall x \forall t (\neg C_0(x, t) \vee \neg C_0^{\text{comp}}(x, t)) \wedge \bigwedge_{q \in Q} \forall x \forall t (\neg H_q(x, t) \vee \neg H_q^{\text{comp}}(x, t)).$$

Finally, φ_{0w} will contain the following sentences φ_2 and φ_w as conjuncts. The sentence φ_2 describes one computation step. It contains for each instruction of M one conjunct. For example, the instruction $qapb1$ contributes the conjunct

$$\begin{aligned} & \forall x x' \forall t t' \forall y \left((H_q(x, t) \wedge C_a(x, t) \wedge S(x, x') \wedge S(t, t')) \right. \\ & \quad \rightarrow ((\neg C_{1-b}(x, t') \wedge \neg H_p^{\text{comp}}(x', t')) \\ & \quad \quad \wedge (y \neq x' \rightarrow \bigwedge_{r \in Q} \neg H_r(y, t')) \\ & \quad \quad \left. \wedge (y \neq x \rightarrow ((C_0(y, t) \rightarrow \neg C_0^{\text{comp}}(y, t)) \wedge (C_0^{\text{comp}}(y, t') \rightarrow \neg C_0(y, t')))) \right). \end{aligned}$$

For $w \in \{0, 1\}^*$ the sentence φ_w describes the initial configuration of M with input w (if $w = w_1 \dots w_{|w|}$, the first $|w|$ cells contain $w_1, \dots, w_{|w|}$, the remaining cells contain 0, and the head scans the first cell in the starting state q_0). Hence, as φ_w we can take the conjunction of

$$\begin{aligned} & - \forall x_1 \dots \forall x_{|w|} \left((U_{\min} x_1 \wedge \bigwedge_{i \in [|w|-1]} S x_i x_{i+1}) \right. \\ & \quad \left. \rightarrow (\bigwedge_{\substack{i \in [|w|] \\ w_i=0}} \neg C_0^{\text{comp}}(x_i, x_1) \wedge \bigwedge_{\substack{i \in [|w|] \\ w_i=1}} \neg C_0(x_i, x_1)) \right) \\ & - \forall x_1 \dots \forall x_{|w|} \forall x \left((U_{\min} x_1 \wedge \bigwedge_{i \in [|w|-1]} S x_i x_{i+1} \wedge x_{|w|} < x) \rightarrow \neg C_0^{\text{comp}}(x, x_1) \right) \\ & - \forall x \forall y \left(U_{\min} x \rightarrow (\neg H_{q_0}^{\text{comp}}(x, x) \wedge (y \neq x \rightarrow \bigwedge_{q \in Q} \neg H_q(y, x))) \right). \end{aligned}$$

Note that U_{\min} , U_{\max} , and S are negative in φ_{0w} . We set $\varphi_{1M} := \varphi_{1\tau_M}$; recall that by Definition 3.6 (c),

$$\varphi_{1M} = \varphi_1 \wedge \forall x \forall t (C_0(x, t) \vee C_0^{\text{comp}}(x, t)) \wedge \bigwedge_{q \in Q} \forall x \forall t (H_q(x, t) \vee H_q^{\text{comp}}(x, t)).$$

Let $w \in \{0, 1\}^*$ and $r \in \mathbb{N}$. Furthermore, let \mathcal{A} be a τ_M -structure where $<^{\mathcal{A}}$ is an ordering and $|A| \geq r+1$. Let a_0, \dots, a_r be the first $r+1$ elements of $<^{\mathcal{A}}$. Assume that M on the input $w \in \{0, 1\}^*$ runs at least r steps. We say that \mathcal{A} *correctly encodes r steps of the computation of M on w* if for i, j with $0 \leq i, j \leq r$,

$$(a_i, a_j) \in C_0^{\mathcal{A}} \iff \text{the content of cell } i \text{ after } j \text{ steps is } 0 \tag{16}$$

and for $q \in Q$,

$$(a_i, a_j) \in H_q^A \iff \text{after } j \text{ steps } M \text{ is in state } q \text{ and the head scans cell } j. \quad (17)$$

From the definitions of the sentences φ_{0w} and φ_{1M} , we see:

Lemma 6.1. *Let $w \in \{0, 1\}^*$ and \mathcal{A} be a model $\varphi_{0w} \wedge \varphi_{1M}$. If for $r \in \mathbb{N}$ we have $r + 1 \leq |A|$ (in particular, if A is infinite) and M on w runs at least r steps, then \mathcal{A} correctly encodes r steps of the computation of M on w .*

Finally, let γ_M be a sentence expressing that “the machine M reaches the halting state q_h in exactly ‘max’ steps,” more precisely,

$$\gamma_M := \exists t \exists x (U_{\max} t \wedge H_{q_h}(x, t) \wedge \forall t' \forall y (t' < t \rightarrow \neg H_{q_h}(y, t'))). \quad (18)$$

As a consequence of the preceding lemma, we obtain:

Corollary 6.2. *Let $w \in \{0, 1\}^*$ and assume that M on input w eventually halts, say in $h(w)$ steps, then*

$$\varphi_{0w} \wedge \varphi_{1M} \wedge \gamma_M$$

has no infinite model but a model with exactly $h(w) + 1$ elements (this model is unique up to isomorphism).

Proof : Let $\mathcal{A} \models \varphi_{0w} \wedge \varphi_{1M} \wedge \gamma_M$. Then $\mathcal{A} \upharpoonright \tau_0$ is a complete τ_0 -ordering and \mathcal{A} contains the description of the complete halting computation of M on the input w . As the machine M reaches the halting state in exactly $h(w)$ steps, we see that $|A| = h(w) + 1$; in particular, A is finite.

On the other hand, we can interpret (16) and (17) as defining relations C_0^A and H_q^A on the set $A := \{a_0, \dots, a_{h(w)}\}$ equipped with the “natural” ordering and its corresponding relations U_{\min} , U_{\max} , and S . If furthermore we let $(C_0^{\text{comp}})^A$ and $(H_q^{\text{comp}})^A$ be the complements in $A \times A$ of C_0^A and H_q^A , respectively, we get a model of $\varphi_{0w} \wedge \varphi_{1M} \wedge \gamma_M$ with exactly $h(w) + 1$ elements. \square

We set

$$\chi_w := \varphi_{0w} \wedge (\varphi_{1M} \rightarrow \neg \gamma_M). \quad (19)$$

By Proposition 3.11 and Corollary 6.2, we get:

Lemma 6.3. *Let M on input w eventually halt, say in $h(w)$ steps. Then:*

- (a) $\text{MOD}(\chi_w)$ is closed under $<$ -substructures.
- (b) If χ_w is finitely equivalent to a universal sentence μ , then $|\mu| \geq h(w) + 1$.

Now we show the following version of Gurevich’s Theorem.

Theorem 6.4. *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a computable function. Then there is a $w \in \{0, 1\}^*$ such that $\text{MOD}(\chi_w)$ is closed under $<$ -substructures but χ_w is not finitely equivalent to a universal sentence of length less than $f(|\chi_w|)$.*

Proof: By the previous lemma it suffices to find a $w \in \{0, 1\}^*$ such that M on input w halts in $h(w)$ steps with

$$h(w) \geq f(|\chi_w|).$$

W.l.o.g. we assume that f is increasing. An analysis of the formula χ_w shows that for some $c_M \in \mathbb{N}$ we have for all $w \in \{0, 1\}^*$,

$$|\chi_w| \leq c_M \cdot |w|. \quad (20)$$

We define $g : \mathbb{N} \rightarrow \mathbb{N}$ by

$$g(k) := f(5 \cdot c_M \cdot k). \quad (21)$$

Let M_0 be a Turing machine computing g , more precisely, the function $1^k \mapsto 1^{g(k)}$. We code M_0 and 1^k by a $\{0, 1\}$ -string $code(M_0, 1^k)$ such that M on $code(M_0, 1^k)$ simulates the computation of M_0 on 1^k .

Choose the least k such that for $w := code(M_0, 1^k)$ we have

$$|w| \leq 5k. \quad (22)$$

The universal Turing machine M on input w computes $1^{g(k)}$ and thus runs at least $g(k)$ steps, say, exactly $h(w)$ steps. By (20) – (22)

$$h(w) \geq g(k) = f(5 \cdot c_M \cdot k) \geq f(c_M \cdot |w|) \geq f(|\chi_w|). \quad \square$$

Finally we prove Gurevich's Theorem for graphs. For $\tau := \tau_M$ let I be an interpretation according to Lemma 5.6. For $w \in \{0, 1\}^*$ we consider the sentence

$$\rho_w := \forall \bar{x} \neg \varphi_{\text{uni}}(\bar{x}) \vee (\varphi_{0w} \wedge (\varphi_{1M} \rightarrow \neg \gamma_M))^I = \forall \bar{x} \neg \varphi_{\text{uni}}(\bar{x}) \vee \chi_w^I. \quad (23)$$

That is, for $G \models \rho_w$, either the graph G interprets an empty τ_M -structure, or a τ_M -structure which is a model of χ_w . If M halts in $h(w)$ steps on input w , then $\varphi_{0w} \wedge \varphi_{1M} \wedge \gamma_M$ has no infinite model but a finite model with $h(w) + 1$ elements by Corollary 6.2. Hence taking in Proposition 4.6 as ψ the sentence ψ_{GRAPH} axiomatizing the class of graphs we get the following analogue of Lemma 6.3.

Lemma 6.5. *Let M on input w halt in $h(w)$ steps. Then:*

- (a) $\text{GRAPH}(\rho_w)$, the class of graphs that are model of ρ_w , is closed under induced subgraphs (and hence equivalent in the class of graphs to a universal sentence).
- (b) If ρ_w is equivalent in the class of finite graphs to a universal sentence μ , then $|\mu|^2 \geq h(w)$.

Theorem 6.6 (Gurevich's Theorem for graphs). *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a computable function. Furthermore, let ρ_w be defined by (23), where I is an interpretation for $\tau := \tau_M$ according to Lemma 5.6. Then there is a $w \in \{0, 1\}^*$ such that $\text{GRAPH}(\rho_w)$ is closed under induced subgraphs but ρ_w is not equivalent in the class of finite graphs to a universal sentence of length less than $f(|\rho_w|)$.*

Proof: Again we assume that f is increasing. By the previous lemma it suffices to find a $w \in \{0, 1\}^*$ such that M on input w halts in $h(w)$ steps with

$$h(w) \geq f(|\rho_w|)^2.$$

There is a $c \in \mathbb{N}$, which depends on I but not on w , such that for c_I as in (13) and c_M as in (20) we have for $d_M := c + c_I \cdot c_M$,

$$|\rho_w| \leq c + c_I \cdot |\chi_w| \leq c + c_I \cdot c_M \cdot |w| \leq d_M \cdot |w|. \quad (24)$$

We define $g : \mathbb{N} \rightarrow \mathbb{N}$ by

$$g(k) := f(5 \cdot d_M \cdot k)^2 \quad (25)$$

and then proceed as in the proof of Theorem 6.4. Let M_0 be a Turing machine computing the function $1^k \mapsto 1^{g(k)}$. We code M_0 and 1^k by a $\{0, 1\}$ -string $code(M_0, 1^k)$ such that M on $code(M_0, 1^k)$ simulates the computation of M_0 on 1^k .

Choose the least k such that for $w := code(M_0, 1^k)$ we have

$$|w| \leq 5k. \quad (26)$$

The universal Turing machine M on input w computes $1^{g(k)}$ and thus runs at least $g(k)$ steps, say, exactly $h(w)$ steps. We have

$$h(w) \geq g(k) = f(5 \cdot d_M \cdot k)^2 \geq f(d_M \cdot |w|)^2 \geq f(|\rho_w|)^2$$

by (24) – (26). □

Remark 6.7. Using previous remarks (Remark 3.12 and Remark 4.7) one can even show that for every computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ the sentence χ_w is not finitely equivalent to a Π_2 -sentence of length less than $f(|\chi_w|)$ and the sentence ρ_w is not finitely equivalent in graphs to a Π_2 -sentence of length less than $f(|\chi_w|)$. Moreover, χ_w and ρ_w are equivalent to Σ_2 .

For this purpose note that in models of φ_{0w} the sentence γ_M is equivalent to

$$\exists t \exists x (U_{\max t} \wedge H_{q_h}(x, t)) \wedge \forall t_1 \forall t_2 \forall y (t_1 < t_2 \rightarrow \neg H_{q_h}(y, t_2)).$$

and hence equivalent to a Σ_2 and to a Π_2 -sentence. One easily verifies that the same holds for γ_M^I .

7. Some undecidable problems

In this section we show that various problems related to the results of the preceding sections are undecidable. Among others, these results explain why it might be hard, in fact impossible in general, to obtain forbidden induced subgraphs for various classes of graphs.

Proposition 7.1. *There is no algorithm that applied to any $\text{FO}[\tau_E]$ -sentence φ decides whether the class $\text{GRAPH}(\varphi)$ is closed under induced subgraphs.*

Proof: Assume \mathbb{A} is such an algorithm. By the Completeness Theorem there is an algorithm \mathbb{B} that assigns to every sentence φ with $\text{GRAPH}(\varphi)$ closed under induced subgraphs a universal sentence equivalent to φ in graphs. Define the function g by

$$g(\varphi) := \begin{cases} 0, & \text{if } \mathbb{A} \text{ rejects } \varphi \\ m, & \mathbb{B} \text{ needs } m \text{ steps to produce a universal sentence equivalent to } \varphi \end{cases}$$

and set $f(k) := \max\{g(\varphi) \mid |\varphi| \leq k\}$. Then f would contradict Gurevich's Theorem for graphs, i.e., Theorem 6.6. □

Corollary 7.2. *There is no algorithm that applied to any $\text{FO}[\tau_E]$ -sentence φ either reports that $\text{GRAPH}(\varphi)$ is not closed under induced subgraphs or it computes for $\text{GRAPH}(\varphi)$ a class of forbidden induced subgraphs.*

Proof: Otherwise we could use this algorithm as a decision algorithm for the previous result. \square

The following proposition is the analog of Proposition 7.1 for classes of finite graphs. We state it for $\text{FO}[\tau_E]$ -sentences and graphs even though we prove it for $\text{FO}[\tau_M]$ -sentences. One gets the version for graphs using the machinery we developed in previous sections similarly as we do it to get Corollary 7.5 from Proposition 7.4 below.

We write $M : w \mapsto \infty$ for the universal Turing machine M and a word $w \in \{0, 1\}^*$ if M on input w does not halt. We make use of the sentences φ_{0w} , φ_{1M} , and γ_M defined in the previous section.

Proposition 7.3. *There is no algorithm that applied to any $\text{FO}[\tau_E]$ -sentence φ decides whether the class $\text{GRAPH}_{\text{fin}}(\varphi)$ is closed under induced subgraphs.*

Proof: For the universal Turing machine M and a word $w \in \{0, 1\}^*$ consider the sentence

$$\pi_w := \varphi_{0w} \wedge \varphi_{1M} \wedge \gamma_M.$$

Then

$$\text{MOD}_{\text{fin}}(\pi_w) \text{ is closed under induced subgraphs} \iff M : w \mapsto \infty. \quad (27)$$

In fact, if $M : w \mapsto \infty$, then $\text{MOD}_{\text{fin}}(\pi_w) = \emptyset$, hence $\text{MOD}_{\text{fin}}(\pi_w)$ is trivially closed under induced subgraphs. If M on input w halts after $h(w)$ steps, then, up to isomorphism, there is a unique model \mathcal{A}_w of π_w and it has $h(w) + 1$ elements. By Lemma 3.8 every proper induced substructure of \mathcal{A}_w is not a model of π_w . Hence $\text{MOD}_{\text{fin}}(\pi_w)$ is not closed under induced subgraphs. As the halting problem for every universal Turing machine is not decidable, by (27) we get our claim. \square

Proposition 7.4. *There is no algorithm that applied to any $\text{FO}[\tau_M]$ -sentence, which is finitely equivalent to a universal sentence, computes such a universal sentence.*

Proof: Again we show that such an algorithm would allow us to decide for every $w \in \{0, 1\}^*$ whether the universal Turing machine M halts on input w . In (19) we defined χ_w by

$$\chi_w = \varphi_{0w} \wedge (\varphi_{1M} \rightarrow \neg\gamma_M).$$

If M halts on w , by Lemma 6.3 we know that $\text{MOD}(\chi_w)$ is closed under $<$ -substructures and thus equivalent to a universal sentence. The claimed algorithm (or, even the Completeness Theorem) will produce such a universal μ . Furthermore, by Corollary 6.2 we know that there is a finite model with $h(w) + 1$ elements, which is a model of $\varphi_{0w} \wedge \neg\chi_w$, hence it is a model of $\varphi_{0w} \wedge \neg\mu$.

If M does not halt on w , then we show that $\text{MOD}_{\text{fin}}(\chi_w) = \text{MOD}_{\text{fin}}(\varphi_{0w})$. Clearly $\text{MOD}_{\text{fin}}(\chi_w) \subseteq \text{MOD}_{\text{fin}}(\varphi_{0w})$. Now let \mathcal{A} be a finite model of φ_{0w} . If $\mathcal{A} \not\models \varphi_{1M}$, then $\mathcal{A} \models \chi_w$. Otherwise $\mathcal{A} \models \varphi_{1M}$, then \mathcal{A} correctly represents the first $|\mathcal{A}| - 1$ steps of the computation of M on w by Lemma 6.1. Thus \mathcal{A} is a model of $\neg\gamma_M$ as M does not halt on w . Therefore, \mathcal{A} is a model of χ_w .

Now we can see whether M does not halt on w by checking whether the universal sentence produced by the claimed algorithm is finitely equivalent to the universal sentence φ_{0w} . This can be checked effectively by Corollary 2.5 and Corollary 2.6. \square

Corollary 7.5. *There is no algorithm that applied to any $\text{FO}[\tau_E]$ -sentence φ such that $\text{GRAPH}_{\text{fin}}(\varphi)$ has a finite set of forbidden induced subgraphs computes such a set.*

Proof: Equivalently we show that there is no algorithm that applied to any $\text{FO}[\tau_E]$ -sentence φ such that $\text{GRAPH}_{\text{fin}}(\varphi) = \text{GRAPH}_{\text{fin}}(\mu)$ for some universal sentence μ computes such a μ .

For graphs let $I (= I_{\tau_M})$ be a strongly existential interpretation of τ_M -structures in graphs according to Lemma 5.6. We know that for every finite τ_M -structure \mathcal{A} there is a finite graph G such that $G_I \cong \mathcal{A}$.

For $w \in \{0, 1\}^*$ we consider the sentence ρ_w defined in (23) in the proof of Theorem 6.4,

$$\rho_w = \forall \bar{x} \neg \varphi_{\text{uni}}(\bar{x}) \vee (\varphi_{0w} \wedge (\varphi_{1M} \rightarrow \neg \gamma_M))^I = \forall \bar{x} \neg \varphi_{\text{uni}}(\bar{x}) \vee \chi_w^I.$$

We show that ρ_w is equivalent to a universal sentence μ on finite graphs. Moreover, M does not halt on input w if and only if μ is finitely equivalent to the universal sentence $\forall x \neg \varphi_{\text{uni}}(\bar{x}) \vee \varphi_{0w}^I$.

If M halts on w , then $\varphi_{0w} \wedge \varphi_{1M} \wedge \gamma_M$ has no infinite model but a finite model \mathcal{A} . Hence, by Proposition 4.6 we know that $\text{GRAPH}(\rho_w)$ is closed under induced subgraphs. Therefore, ρ_w is equivalent to a universal sentence μ in GRAPH . Let G be a finite graph with $G_I \cong \mathcal{A}$. Then $G \models (\varphi_{0w} \wedge \varphi_{1M} \wedge \gamma_M)^I$ and thus, $G \models \neg \rho_w$. Hence G is a finite graph which is a model of $\varphi_{0w}^I \wedge \neg \mu$. This means that μ is not equivalent to $\forall x \neg \varphi_{\text{uni}}(\bar{x}) \vee \varphi_{0w}^I$ on all finite graphs, as G is also a model of $\forall x \neg \varphi_{\text{uni}}(\bar{x}) \vee \varphi_{0w}^I$.

If $M : w \rightarrow \infty$, then we show that $\text{GRAPH}_{\text{fin}}(\rho_w) = \text{GRAPH}_{\text{fin}}(\forall x \neg \varphi_{\text{uni}}(\bar{x}) \vee \varphi_{0w}^I)$. Clearly $\text{GRAPH}_{\text{fin}}(\rho_w) \subseteq \text{GRAPH}_{\text{fin}}(\forall x \neg \varphi_{\text{uni}}(\bar{x}) \vee \varphi_{0w}^I)$. Now let the graph G be a model of $\forall x \neg \varphi_{\text{uni}}(\bar{x}) \vee \varphi_{0w}^I$. Further we can assume that $G \models \exists x \varphi_{\text{uni}}(\bar{x})$. In particular, $\mathcal{A} := G_I$ is well defined. If $G \not\models \varphi_{1M}^I$, then $G \models \chi_w^I$ and therefore, $G \models \rho_w$. If $G \models \varphi_{1M}^I$, then $\mathcal{A} \models \varphi_{0w} \wedge \varphi_{1M}^I$. As $M : w \rightarrow \infty$, by Lemma 6.1 the structure \mathcal{A} correctly represents the first $|\mathcal{A}| - 1$ steps of the computation of M on w . Thus, \mathcal{A} is a model of $\neg \gamma_M$, again as M does not halt on input w . It follows that G is a model of $\neg \gamma_M^I$, and then $G \models \rho_w$.

Now we can decide the halting problem for M . Given a word w , we use the claimed algorithm to get a universal sentence μ equivalent to ρ_w in the class of graphs. Finally we check whether μ is finitely equivalent to $\forall x \neg \varphi_{\text{uni}}(\bar{x}) \vee \varphi_{0w}^I$. This can be checked effectively again by Corollary 2.5 and Corollary 2.6. \square

Observe that Corollary 7.5 is precisely Theorem 1.3 as stated in the Introduction. Finally we prove Theorem 1.2, which is equivalent to the following result.

Theorem 7.6. *There is no algorithm that applied to an $\text{FO}[\tau_E]$ -sentence φ such that $\text{GRAPH}_{\text{fin}}(\varphi)$ is closed under induced subgraphs decides whether there is a finite set \mathcal{F} of graphs such that*

$$\text{GRAPH}_{\text{fin}}(\varphi) = \text{FORB}_{\text{fin}}(\mathcal{F}).$$

Proof: Again we prove the corresponding result for τ_M -sentences and τ_M -structures and leave it to the reader to translate it to graphs as in the previous proof. That is, we show:

There is no algorithm that applied to an $\text{FO}[\tau_M]$ -sentence φ such that $\text{MOD}_{\text{fin}}(\varphi)$ is closed under induced substructures decides whether there is a finite set F of finite τ_M -structures such that

$$\text{MOD}_{\text{fin}}(\varphi) = \text{FORB}_{\text{fin}}(F).$$

For $w \in \{0, 1\}^*$ let

$$\alpha_w := \varphi_{0w} \wedge (\varphi_{1M} \rightarrow \gamma_M).$$

We show that $\text{MOD}_{\text{fin}}(\alpha_w)$ is closed under induced subgraphs and that

$$M : w \rightarrow \infty \iff \alpha_w \text{ is not finitely equivalent to a universal sentence.}$$

Assume first that $M : w \rightarrow \infty$. Then $\varphi_{0w} \wedge \varphi_{1M} \wedge \gamma_M$ has no finite model by Lemma 6.1 and the definition (18) of γ_M . Therefore, $\text{MOD}_{\text{fin}}(\alpha_w) = \text{MOD}_{\text{fin}}(\varphi_{0w} \wedge \neg\varphi_{1M})$. By Lemma 3.10 we know that $\text{MOD}_{\text{fin}}(\varphi_{0w} \wedge \neg\varphi_{1M})$ is closed under induced substructures but not finitely equivalent to a universal sentence.

Now assume that M on input w halts in $h(w)$ steps. Then Corollary 6.2 guarantees that there is a unique model \mathcal{A}_w of $\varphi_{0w} \wedge \varphi_{1M} \wedge \gamma_M$ with $|A_w| = h(w) + 1$. We present a finite set \mathcal{F} of finite τ_M -structures such that

$$\text{MOD}_{\text{fin}}(\alpha_w) = \text{FORB}_{\text{fin}}(\mathcal{F}). \quad (28)$$

As φ_{0w} is universal, there is a finite set \mathcal{F}_0 of finite τ_M -structures such that

$$\text{MOD}_{\text{fin}}(\varphi_{0w}) = \text{FORB}_{\text{fin}}(\mathcal{F}_0).$$

Moreover, we set

$$\mathcal{F}_1 := \{\mathcal{B} \in \text{STR}[\tau_M] \mid \mathcal{B} \models \varphi_{0w} \wedge \varphi_{1M} \text{ and } B = [\ell] \text{ for some } \ell \leq h(w)\}$$

and

$$\mathcal{F}_2 := \{\mathcal{B} \in \text{STR}[\tau_M] \mid \mathcal{B} \models \varphi_{0w} \wedge \varphi_{1M}^* \wedge \forall t \forall t' (t < t' \rightarrow \forall y \neg H_{q_h}(y, t)) \\ \text{and } B = [h(w) + 2]\}.$$

Here φ_{1M}^* is obtained from φ_{1M} by replacing the conjunct φ_1 (see (7)) by

$$\varphi_1^* := \exists x U_{\min} x \wedge \forall x \forall y (x < y \rightarrow \exists z Sxz).$$

The difference is that φ_1^* does not require the set U_{\max} to be nonempty. Hence,

$$\varphi_{1M}^* = \varphi_1^* \wedge \forall x \forall t (C_0(x, t) \vee C_0^{\text{comp}}(x, t)) \wedge \bigwedge_{q \in Q} \forall x \forall t (H_q(x, t) \vee H_q^{\text{comp}}(x, t)).$$

Note that Lemma 6.1 remains true if in its statement we replace φ_{1M} by φ_{1M}^* .

For $\mathcal{F} := \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2$ we show (28). Assume first that a finite structure \mathcal{C} is a model of α_w . In particular, $\mathcal{C} \models \varphi_{0w}$ and therefore, \mathcal{C} has no induced substructure isomorphic to a structure in \mathcal{F}_0 .

Now, for a contradiction suppose that \mathcal{B} is an induced substructure of \mathcal{C} isomorphic to a structure in \mathcal{F}_1 . Then $\mathcal{B} \models \varphi_{1M}$ and thus, by Lemma 3.8, $\mathcal{C} = \mathcal{B}$. As $\mathcal{C} \models \alpha_w$, we get $\mathcal{C} \models \varphi_{0w} \wedge \varphi_{1M} \wedge \gamma_M$. Hence, $\mathcal{C} \cong \mathcal{A}_w$, a contradiction, as on the one hand $|\mathcal{C}| = |\mathcal{B}| \leq h(w)$ and on the other hand $|\mathcal{C}| = |\mathcal{A}_w| = h(w) + 1$.

Next we show that \mathcal{C} has no induced substructure \mathcal{B} isomorphic to a structure in \mathcal{F}_2 . As $\mathcal{B} \models \varphi_{0w} \wedge \varphi_{1M}^*$ and has $h(w) + 2$ elements, the first $h(w) + 1$ elements of \mathcal{B} correctly encode the first $h(w)$ steps of the computation of M on w , hence the full computation. As $|\mathcal{B}| = h(w) + 2$, this contradicts $\mathcal{B} \models \forall t \forall t' (t < t' \rightarrow \forall y \neg H_{q_h}(y, t))$.

As the final step let $\mathcal{C} \in \text{FORB}_{\text{fin}}(\mathcal{F})$. We show that $\mathcal{C} \models \alpha_w$. As \mathcal{C} omits the structures in \mathcal{F}_0 as induced substructures, we see that $\mathcal{C} \models \varphi_{0w}$. If $\mathcal{C} \not\models \varphi_{1M}$, we are done.

Recall that by Lemma 6.1 (more precisely, by the extension of Lemma 6.1 mentioned above) for finite structures \mathcal{B} of $\varphi_{0w} \wedge \varphi_{1M}^*$ we know:

- (a) if $|B| \leq h(w) + 1$, then \mathcal{B} encodes $|B| - 1$ steps of the computation of M on w ,
- (b) if $|B| > h(w) + 1$, then the first $h(w) + 1$ elements in the ordering $<^{\mathcal{B}}$ correctly encode the (full) computation of M on w .

Now assume that $\mathcal{C} \models \varphi_{1M}$, then (a) and (b) apply to \mathcal{C} . As no structure in \mathcal{F}_1 is isomorphic to an induced substructure of \mathcal{C} , we see that $|\mathcal{C}| \geq h(w) + 1$. But \mathcal{C} cannot have more than $h(w) + 1$ elements, as otherwise the substructure of \mathcal{C} induced on the first $h(w) + 2$ elements would be isomorphic to a structure \mathcal{B} in F_2 , a contradiction. \square

Remark 7.7. Mainly using Remark 6.7 one easily verifies that in all results but Proposition 7.3 of this section we can replace

There is no algorithm that applied to an $\text{FO}[\tau_E]$ -sentence $\varphi \dots$

by

There is no algorithm that applied to a Σ_2 -sentence $\varphi \dots$

In Proposition 7.3 we have to replace it by

There is no algorithm that applied to a Π_2 -sentence $\varphi \dots$

as φ_{1M} (and φ_{1M}^I) are Π_2 -sentences.

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