Mechanism design for aggregating energy consumption and quality of service in speed scaling scheduling*

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Abstract

We consider a strategic game, where players submit jobs to a machine that executes all jobs in a way that minimizes energy while respecting the given deadlines. The energy consumption is then charged to the players in some way. Each player wants to minimize the sum of that charge and of their job's deadline multiplied by a priority weight. Two charging schemes are studied, the *proportional cost share* which does not always admit pure Nash equilibria, and the *marginal cost share*, which does always admit pure Nash equilibria, at the price of overcharging by a constant factor.

keywords scheduling, energy management, quality of service, optimization, mechanism design.

1 Introduction

In many computing systems, maximizing quality of service comes generally at the price of a high energy consumption. This is also the case for the speed scaling scheduling model considered in this paper. It has been introduced in [16], and triggered a lot of work on offline and online algorithms; see [1] for an overview.

The online and offline optimization problem for minimizing flow time while respecting a maximum energy consumption has been studied for the single machine setting in [15, 2, 6, 9] and for the parallel machines setting in [3]. For the variant where an aggregation of energy and flow time is considered, polynomial approximation algorithms have been presented in [8, 4, 12].

In this paper we propose to study this problem from a different perspective, namely as a strategic game. In society many ecological problems are either addressed in a centralized manner, like forcing citizens to sort household waste, or in a decentralized manner, like tax incentives to enforce ecological behavior. This paper proposes incentives for a scheduling game, in form of an energy cost charging scheme.

Consider a scheduling problem for a single processor, that can run at variable speed, such as the modern microprocessors Intel SpeedStep, AMD PowerNow! or IBM EnergyScale.

*A preliminary version of this article appeared in *Proc. of the 9th Conf. on Web and Internet Economics* (WINE 2013).

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Higher speed means that jobs finish earlier at the price of a higher energy consumption. Each job has some workload, representing a number of instructions to execute, and a release time before which it cannot be scheduled. Every user submits a single job to a common processor, declaring the job parameters, together with a deadline, that the player chooses at his convenience.

The processor will schedule the submitted jobs preemptively, so that all release times and deadlines are respected and the overall energy usage is minimized. The energy consumed by the schedule needs to be charged to the users. The individual goal of each user is to minimize the sum of the energy cost share and of the requested weighted deadline. The weight is a private priority factor representing the individual importance of a small deadline. This factor includes implicitly a conversion factor that allows for an aggregation of the deadline and energy consumption into a single individual penalty.

In a companion paper [10] we study this game from the point of view of the game regulator, in a different setting. The players announce with their job their priority factors, and the regulator gets to decide on the completion time of the jobs. The usual questions one asks for such a game, is the existence of a cost sharing mechanism that would be truthful on the priority factors and which charge to the players amounts that sum up to a value comparable within a constant factor to the actual energy consumption of the schedule. This contrasts with the setting considered in this paper, where the player's strategies are the job's deadlines.

2 The model

Formally, we consider a non-cooperative game with n players and a regulator. The regulator manages the machine where the jobs are executed. Each player has a job i with a workload w_i, a release time r_i and a priority p_i, representing a quality of service coefficient. The player submits its job together with a deadline $d_i > r_i$ to the regulator. Workloads, release times and deadlines are public information known to all players, while quality of service coefficients can be private.

The regulator implements some *cost sharing mechanism*, which is known to all users. This mechanism defines a cost share function b_i specifying how much player i is charged. The penalty of player i is the sum of two values: his *energy cost share* $b_i(w, r, d)$ defined by the mechanism, where $w = (w_1, \ldots, w_n), r = (r_1, \ldots, r_n), d = (d_1, \ldots, d_n)$ are the input values, and his *waiting cost*, which can be either $p_i d_i$ or $p_i(d_i - r_i)$; we use the former waiting cost throughout the article but all our results apply to both settings. The sum of all player's penalties, i.e., energy cost shares and waiting costs will be called the *utilitarian social cost*.

The regulator computes a minimum energy schedule for a single machine in the speed scaling model. In this model at any point in time t the processor can run at some speed $s(t) \geq 0$. As a result, for any time interval I, the workload executed in I is $\int_{t \in I} s(t)\,\mathrm{d}t$ at the price of an energy consumption valuated at $\int_{t \in I} s(t)^\alpha\,\mathrm{d}t$ for some fixed physical constant $\alpha \in [2, 3]$ which is device dependent [7].

The sum of the energy used by this optimum schedule and of all the players' waiting costs will be called the *effective social cost*.

The minimum energy schedule can be computed in time $O(n^2 \log n)$ [11] and has (among others) the following properties [16]. The jobs in the schedule are executed by preemptive earliest deadline first order (EDF), and the speed $s(t)$ at which they are processed is piecewise constant. Preemptive EDF means that at every time point among all jobs which are already released and not yet completed, the job with the smallest deadline is executed, using job indexes to break ties.

The cost sharing mechanism defines the game completely. Ideally, we would like the game and the mechanism to have the following properties.

existence of pure Nash equilibria This means that there is always a strategy profile vector d such that no player can unilaterally deviate from his strategy d_i while strictly decreasing his penalty.

budget balance The mechanism is c-budged balanced, when the sum of the cost shares is no smaller than the total energy consumption and no larger than c times the energy consumption. Ideally we would like c to be close to 1.

In the sequel we introduce and study two different cost sharing mechanisms, namely Proportional Cost Sharing where every player pays exactly the cost generated during the execution of his job, and Marginal Cost Sharing where every player pays the increase of energy cost generated by adding this player to the game.

3 Proportional cost sharing

The proportional cost sharing is the simplest budget balanced cost sharing scheme one can think of. Every player i is charged exactly the energy consumed during the execution of his job. Unfortunately this mechanism does not behave well as we show in Theorem 1.

Fact 1. *In a single player game, the player's penalty is minimized by the deadline*

$$r_1 + w_1(\alpha - 1)^{1/\alpha} p_1^{-1/\alpha}.$$

Proof. If player 1 chooses deadline $d_1 = r_1 + x$, then his job is processed in the time interval $[r_1,\ r_1 + x]$ at speed w_1/x. Therefore his penalty is

$$p_1(r_1 + x) + x^{1-\alpha} w_1^{\alpha}.$$

Differentiating this expression in x, and using the fact that the penalty is concave in x for any $x > 0$ and $\alpha > 0$, we have that the optimal x for the player will set the derivative to zero. This implies the claimed deadline. □

If there are at least two players however, the game does not have nice properties as we show now.

Theorem 1. *The* Proportional Cost Sharing *does not always admit a pure Nash equilibrium.*

The proof consists of an example consisting of two identical players with identical jobs, say $w_1 = w_2 = 1$, $r_1 = r_2 = 0$ and $p_1 = p_2 = 1$. First we determine the best response of player 1 as a function of player 2. Then we conclude that there is no pure Nash equilibrium.

argument	value	applicable range	
$d_1^1 = d_2 + (\alpha-1)^{1/\alpha}$	$g_1(d_2) = d_2 + \alpha\,(\alpha-1)^{1/\alpha-1}$	$d_2 \le \tau_3$	$\tau_3 = (\alpha-1)^{1/\alpha}$
$d_1^2 = 2\left(\frac{\alpha-1}{2}\right)^{1/\alpha}$	$g_2(d_2) = \alpha\left(\frac{\alpha-1}{2}\right)^{1/\alpha-1}$	$\tau_1 \le d_2 \le \tau_5$	$\tau_1 = \left(\frac{\alpha-1}{2}\right)^{1/\alpha}$, $\tau_5 = 2\tau_1$
$d_1^3 = \frac{d_2}{2}$	$g_3(d_2) = d_2/2 + (d_2/2)^{1-\alpha}$	$d_2 \le \tau_6$	
$d_1^4 = (\alpha-1)^{1/\alpha}$	$g_4(d_2) = \alpha(\alpha-1)^{1/\alpha-1}$	$d_2 \ge \tau_6$	$\tau_6 = 2(\alpha-1)^{1/\alpha}$

Table 1: The local minimum in the range of f corresponding to f_i is a function of α and d_2, which we denote by d_1^i. The value at such local minimum is again a function of α and d_2, which we denote by $g_i(d_2)$. These are only conditional minima: they exist if and only if the condition given in the last column is satisfied.

Lemma 1. *Given the second player's choice d_2, the penalty of the first player as a function of his choice d_1 is given by*

$$f(d_1) = \begin{cases} f_1(d_1) = d_1 + (d_1 - d_2)^{1-\alpha} & \text{if } d_1 \ge 2d_2 \\ f_2(d_1) = d_1 + (\frac{d_1}{2})^{1-\alpha} & \text{if } d_2 \le d_1 \le 2d_2 \\ f_3(d_1) = d_1 + (\frac{d_2}{2})^{1-\alpha} & \text{if } \frac{d_2}{2} \le d_1 \le d_2 \\ f_4(d_1) = d_1 + d_1^{1-\alpha} & \text{if } d_1 \le \frac{d_2}{2}. \end{cases} \tag{1}$$

The local minima of $f(d_1)$ are summarized in Table 1, and the penalties corresponding to player 1 picking these minima are illustrated in Figure 1.

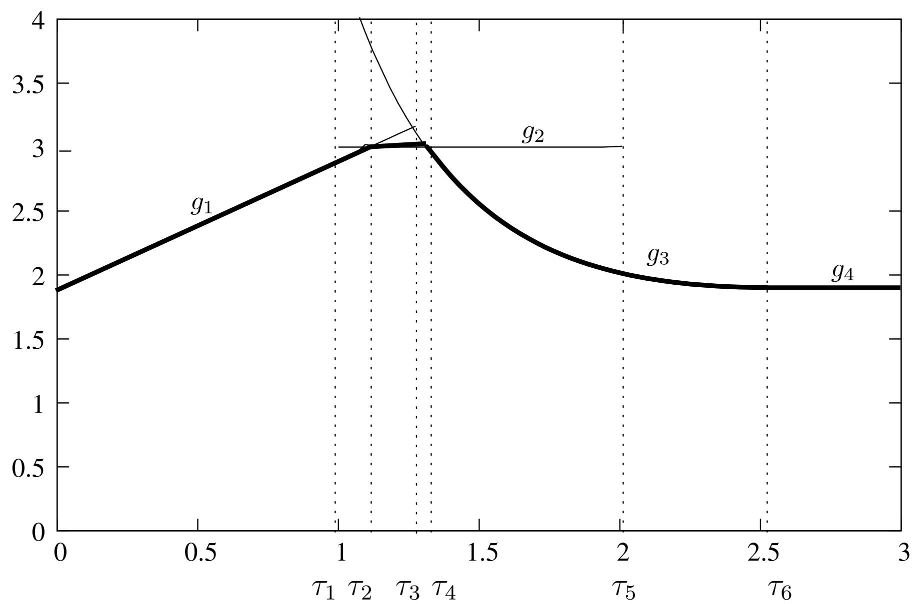

Figure 1: First player's penalty (in bold) when choosing his best response as a function of second player's strategy d_2, here for $\alpha = 3$. As seen on the plot, τ_2 (τ_4), formally defined in Lemma 2, is the argument for which g_1 and g_2 (g_2 and g_3) attain the same value.

Proof. Formula (1) follows by a straightforward case inspection. Then, to find all the local minima of f, we first look at the behavior of each of f_i, finding its local minima in their respective intervals, and afterwards we inspect the border points of these intervals.

Range of f_1: The derivative of f_1 is

$$f_1'(d_1) = 1 - (\alpha - 1)(d_1 - d_2)^{-\alpha}, \tag{2}$$

whose derivative in turn is positive for $\alpha > 1$. Hence, f_1 has a local minimum at d_1^1 as specified. The existence of this local minimum requires $d_1^2 \geq 2d_2$.

Range of f_2: The derivative of f_2 is

$$f_2'(d_1) = 1 - \frac{\alpha - 1}{2}(d_1/2)^{-\alpha}, \tag{3}$$

whose derivative in turn is positive for $\alpha > 1$. Hence, f_2 has a local minimum at d_1^2 as specified. The existence of this local minimum requires $d_2 \leq d_1^2 \leq 2d_2$, which is equivalent to $d_1^3/2 \leq d_2 \leq d_1^2$.

Range of f_3: f_3 is an increasing function, and therefore it attains a minimum value only at the lower end of its range, d_1^2. However, if d_1^3 is to be a local minimum of f, there can be no local minimum of f in the range of f_4 (immediately to the left), so the applicable range of d_1^3 is the complement of that of d_1^4.

Range of f_4: The derivative of f_4 is

$$f_4'(d_1) = 1 - (\alpha - 1)d_1^{-\alpha},$$

whose derivative in turn is positive for $\alpha > 1$. Therefore, f_4 has a local minimum at d_1^4 as specified. Since we require that this local minimum is within the range where f coincides with f_4, the necessary and sufficient condition is $d_1^4 \leq d_2/2$.

Now let us consider the boundaries of the ranges of each f_i. Since f_3 is strictly increasing, the border point of the ranges of f_3 and f_2 is not a local minimum of f. This leaves only the border point $d_1^3 = 2d_2$ of the ranges of f_2 and f_1 to consider. Clearly, d_1^3 is a local minimum of f if and only if $f_2'(d_1^3) \leq 0$ and $f_1'(d_1^3) \geq 0$. However, by (3), $f_2'(d_1^3) = 2 - (\alpha - 1)d_2^{-\alpha}$, and by (2), $f_1'(d_1^3) = 2 - 2(\alpha - 1)d_2^{-\alpha} < f_2'(d_1^3)$, so d_1^3 is not a local minimum of f either. □

Note that the range of g_4 is disjoint with the ranges of g_2 and g_1, and with the exception of the shared border value $2(\alpha - 1)^{1/\alpha}$, also with the range of g_3. However, the ranges of g_3, g_2 and g_1 are not disjoint. Therefore, we now focus on their shared range, and determine which of the functions gives rise to the true local minimum.

Lemma 2. *The functions $g_2(d_2)$ and $g_4(d_2)$ are constant, the function $g_1(d_2)$ is increasing and linear, and the function $g_3(d_2)$ is decreasing for $d_2 \leq \tau_6$. Moreover, there exist two unique values τ_2 and τ_4 with*

$$g_1(\tau_2) = g_2(\tau_2) \text{ and } g_2(\tau_4) = g_3(\tau_4).$$

In addition we have

$$\tau_1 < \tau_2 < \tau_3 < \tau_5 < \tau_6$$

and

$$\tau_2 \leq \tau_4 < \tau_5.$$

Proof. It follows from their definitions in Table 1 that g_2 and g_4 are constant and g_1 strictly increasing. In order to show that the $g_3(d_2)$ is decreasing in the range $0 \le d_2 \le \tau_6$, we show that its derivative in d_2 is non-positive, namely

$$\begin{aligned}
\frac{1}{2} - 2^{\alpha-1}(\alpha-1)d_2^{-\alpha} \le 0 & \qquad \equiv \\
1 \le 2^{\alpha}(\alpha-1)d_2^{-\alpha} & \qquad \equiv \\
d_2^{\alpha} \le 2^{\alpha}(\alpha-1) & \qquad \equiv \\
d_2 \le 2(\alpha-1)^{1/\alpha} = \tau_6. &
\end{aligned}$$

We define τ_2 as the unique root of $g_1(d_2) = g_2(d_2)$, namely

$$\tau_2 = \alpha(\alpha-1)^{1/\alpha-1}(2^{1-1/\alpha}-1).$$

Now we show that there is value τ_4 such that $g_3(\tau_4) = g_2$. This follows from the fact that g_3 is continuous and decreasing, that its limit at $d_2 \to 0$ is ∞ and that $g_3(\tau_6) = g_4 < g_2$.

For the bounds on τ_4 we need to show

$$g_3(\tau_5) < g_2 \le g_3(\tau_2).$$

We start with the left inequality:

$$\begin{aligned}
g_3(\tau_5) &= g_3\left(2\left(\frac{\alpha-1}{2}\right)^{1/\alpha}\right) \\
&= \left(\frac{\alpha-1}{2}\right)^{1/\alpha}\left(1+\frac{2}{\alpha-1}\right) \\
&= \left(\frac{\alpha-1}{2}\right)^{1/\alpha-1}\frac{\alpha+1}{2} \\
&< \left(\frac{\alpha-1}{2}\right)^{1/\alpha-1}\alpha.
\end{aligned}$$

For the right inequality, we first note that

$$\begin{aligned}
g_3(\tau_2) &= \tau_2/2 + (\tau_2/2)^{1-\alpha} \\
&= \tau_2(1/2 + (\tau_2)^{-\alpha}2^{\alpha-1}) \\
&= \alpha(\alpha-1)^{1/\alpha-1}(2^{1-1/\alpha}-1)\left(\frac{2^{\alpha-1}}{(\alpha(\alpha-1)^{1/\alpha-1}(2^{1-1/\alpha}-1))^{\alpha}}+\frac{1}{2}\right),
\end{aligned}$$

hence $g_3(\tau_2) \geq \alpha \left(\frac{\alpha-1}{2}\right)^{1/\alpha-1}$ is equivalent to

$$(2^{1-1/\alpha}-1)\left(\frac{2^{\alpha-1}}{\alpha^\alpha(\alpha-1)^{1-\alpha}(2^{1-1/\alpha}-1)^\alpha}+\frac{1}{2}\right) \geq 2^{1-1/\alpha} \qquad \equiv$$

$$(2^{1-1/\alpha}-1)\left(\frac{2^{\alpha-1}}{\alpha\alpha^{\alpha-1}(\alpha-1)^{1-\alpha}(2^{1-1/\alpha}-1)^\alpha}-\frac{1}{2}\right) \geq 1 \qquad \equiv$$

$$\frac{2^{\alpha-1}(\alpha-1)^{\alpha-1}\alpha^{1-\alpha}}{\alpha(2^{1-1/\alpha}-1)^\alpha}-\frac{1}{2} \geq \frac{1}{2^{1-1/\alpha}-1} \qquad \equiv$$

$$\frac{1}{\alpha}\frac{\left(2-\frac{2}{\alpha}\right)^{\alpha-1}}{(2^{1-1/\alpha}-1)^\alpha} \geq \frac{1}{2^{1-1/\alpha}-1}+\frac{1}{2} \qquad \equiv$$

$$\frac{1}{\alpha}\frac{\left(2-\frac{2}{\alpha}\right)^{\alpha-1}}{\left(\frac{2-2^{1/\alpha}}{2^{1/\alpha}}\right)^\alpha} \geq \frac{2^{1-1/\alpha}+1}{2(2^{1-1/\alpha}-1)} \qquad \equiv$$

$$\frac{1}{\alpha}\frac{\left(2-\frac{2}{\alpha}\right)^{\alpha-1}}{(2-2^{1/\alpha})^{\alpha-1}(2-2^{1/\alpha})}2 \geq \frac{(2+2^{1/\alpha})2^{-1/\alpha}}{2(2-2^{1/\alpha})2^{-1/\alpha}} \qquad \equiv$$

$$\frac{1}{\alpha}\left(\frac{2-\frac{2}{\alpha}}{2-2^{1/\alpha}}\right)^{\alpha-1} \geq \frac{2+2^{1/\alpha}}{4}.$$

We claim that for $\alpha \geq 2$ we have

$$\frac{2-\frac{2}{\alpha}}{2-2^{1/\alpha}} \geq \frac{1}{2-\sqrt{2}}. \tag{4}$$

Since we have equality at $\alpha = 2$, it suffices to prove that the left hand side is increasing. To this end, we consider its derivative

$$\frac{2\left(2-2^{\frac{1}{\alpha}}-\ln 2\cdot(1-\frac{1}{\alpha})\cdot 2^{\frac{1}{\alpha}}\right)}{\left(\alpha(2-2^{\frac{1}{\alpha}})\right)^2} = \frac{4-2^{1+\frac{1}{\alpha}}\cdot\left(1+(1-\frac{1}{\alpha})\ln 2\right)}{\left(\alpha(2-2^{\frac{1}{\alpha}})\right)^2},$$

and note that its enumerator is increasing in α and equals 0 for $\alpha = 1$. Thus (4) holds.

This permits us to define

$$z(\alpha) := (2-\sqrt{2})^{1-\alpha}/\alpha,$$

and to upper bound

$$\frac{1}{\alpha}\left(\frac{2-\frac{2}{\alpha}}{2-2^{1/\alpha}}\right)^{\alpha-1} \geq z(\alpha).$$

In order to lower bound

$$z(\alpha) \geq \frac{2+2^{1/\alpha}}{4}$$

for $\alpha \geq 2$, it suffices to show that z is increasing with α, since the right hand side is decreasing with α. Its derivative is

$$z'(\alpha) = -\frac{(2-\sqrt{2})^{1-\alpha}(1+\alpha\ln(2-\sqrt{2}))}{\alpha^2}.$$

Observe that $\alpha \ln(2-\sqrt{2}) < -1$ for every $\alpha \geq 2$, and therefore z' is positive and z is increasing as required. This concludes the existence of τ_4 with the required properties.

It remains to prove the remaining relations among τ's. We begin with

$$\begin{aligned}
\tau_1 &< \tau_2 & \equiv \\
\left(\frac{\alpha-1}{2}\right)^{1/\alpha} &< \alpha(\alpha-1)^{1/\alpha-1}(2^{1-1/\alpha}-1) & \equiv \\
2^{-1/\alpha} &< \frac{\alpha}{\alpha-1}(2^{1-1/\alpha}-1) & \equiv \\
1 &< \frac{\alpha}{\alpha-1}(2-2^{1/\alpha}) & \equiv \\
\frac{\alpha-1}{\alpha} &< 2-2^{1/\alpha} & \equiv \\
2^{1/\alpha}-1/\alpha &< 1.
\end{aligned}$$

To prove this inequality, we note that it holds as an equality in the limit $\alpha \to \infty$, and that the left hand side is increasing in α, since its derivative is

$$\frac{1-2^{1/\alpha}\ln(2)}{\alpha^2},$$

which is positive for $\alpha \geq 2$.

Now we show

$$\begin{aligned}
\tau_2 &< \tau_3 & \equiv \\
\alpha(\alpha-1)^{1/\alpha-1}(2^{1-1/\alpha}-1) &< (\alpha-1)^{1/\alpha} & \equiv \\
\alpha(2^{1-1/\alpha}-1) &< \alpha-1 & \equiv \\
2^{1-1/\alpha}-1 &< 1-1/\alpha & \equiv \\
2^{1-1/\alpha}+1/\alpha &< 2.
\end{aligned}$$

We observe that the left hand side has value 2 both at $\alpha = 1$ and in the limit $\alpha \to \infty$. To conclude that this holds for all $\alpha \in [1,\ \infty)$, we inspect the derivative of the left hand side with respect to α, which is

$$\frac{2^{1-1/\alpha}\ln(2)-1}{\alpha^2}.$$

Since $2^{1/\alpha-1}$ is monotone, there is a unique value $\alpha_0 = \frac{\ln 2}{\ln 2+\ln\ln 2} \approx 2.1221$ such that

$$2^{1/\alpha_0-1} = \ln(2).$$

In conclusion, the derivative is negative for $1 \leq \alpha < \alpha_0$ and then positive for $\alpha > \alpha_0$, so inside the interval the function never exceeds 2.

The inequality $\tau_3 < \tau_5$ follows from the equality $\tau_5 = 2^{1-1/\alpha}\tau_3$, while the inequality $\tau_5 < \tau_6$ follows from the equality $\tau_6 = 2^{1/\alpha}\tau_5$. This concludes the proof of the lemma. □

With Lemma 1 and Lemma 2, whose statements are summarized in Table 1 and Figure 1, we can finally determine what is the best response of the first player as a function of d_2. The following corollary follows from the definitions of τ_2, τ_4 and the inequalities on $\tau_1, \ldots, \tau_6$.

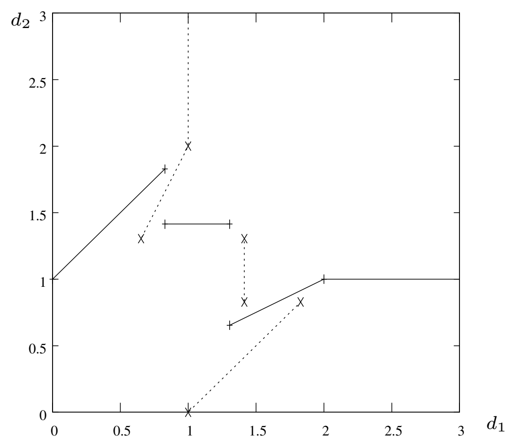

Figure 2: Best response of player 1 as function of d_2 (dashed lines), and best response of player 2 as function of d_1 (solid lines). Here for $\alpha = 3$.

Corollary 1. *The best response for player 1 as function of d_2 is*

$$\begin{aligned} d_1^1 &= d_2 + (\alpha-1)^{1/\alpha} && \text{if} \quad 0 < d_2 \le \tau_2, \\ d_1^2 &= 2\left(\frac{\alpha-1}{2}\right)^{1/\alpha} && \text{if} \quad \tau_2 < d_2 \le \tau_4, \\ d_1^3 &= \frac{d_2}{2} && \text{if} \quad \tau_4 < d_2 \le \tau_6, \\ d_1^4 &= (\alpha-1)^{1/\alpha} && \text{if} \quad \tau_6 < d_2. \end{aligned}$$

By the symmetry of the players, the second player's best response is in fact an identical function of d_1 as the one stated in Corollary 1. By straightforward inspection it follows that there is no fix point (d_1, d_2) to this game, which concludes the proof of Theorem 1, see Figure 2 for illustration.

4 Marginal cost sharing

In this section we propose a different cost sharing scheme, that improves on the previous one in the sense that it admits pure Nash equilibria, but does so at the price of overcharging by a constant factor.

Before we give the formal definition we need to introduce some notations. Let OPT(d) be the energy minimizing schedule for the given instance, and OPT(d_{-i}) be the energy minimizing schedule for the instance where job i is removed. We denote by $E(S)$ the energy cost of schedule S.

In the marginal cost sharing scheme, player i pays the penalty function

$$p_i d_i + E(\mathrm{OPT}(d)) - E(\mathrm{OPT}(d_{-i})).$$

This scheme defines an exact potential game by construction [13]. Formally, let n be the number of players, $D = \{d | \forall j : d_j > r_j\}$ be the set of action profiles (deadlines) over the action sets D_i of each player.

Let us denote the effective social cost corresponding to a strategy profile d by $\Phi(d)$. Then

$$\Phi(d) = \sum_{i=1}^{n} p_i d_i + E(\mathrm{OPT}(d)).$$

Clearly, if a player i changes its strategy d_i and his penalty decreases by some amount Δ, then the effective social cost decreases by the same amount Δ, because $E(\mathrm{OPT}(d_{-i}))$ remains unchanged.

4.1 Existence of Equilibria

While the best response function is not continuous in the strategy profile, precluding the use of Brouwer's fixed-point theorem, existence of pure Nash equilibria can nevertheless be easily established.

To this end, note that the global minimum of the effective social cost, if it exists, is a pure Nash equilibrium. Its existence follows from (1) compactness of a non-empty sub-space of strategies with bounded social cost and (2) continuity of Φ.

For (2), note that $\sum_i p_i d_i$ is clearly continuous in d, and hence $\Phi(d)$ is continuous if $E(\mathrm{OPT}(d))$ is. The continuity of the latter follows from the fact that $E(\mathrm{OPT}(d))$ is the solution to a linear program with a convex objective function, with an optimum being continuous in d [6].

For (1), let d' be any (feasible) strategy profile such that $d'_i > r_i$ for each player i. The subspace of strategy profiles d such that $\Phi(d) \leq \Phi(d')$ is clearly closed, and bounded due to the $p_i d_i$ terms. Thus it is a compact subspace that contains the global minimum of Φ.

4.2 Convergence can take forever

In this game the strategy set is infinite. Moreover, the convergence time can be infinite as we demonstrate below in Theorem 2. Note that this also proves that in general there are no dominant strategies in this game.

Theorem 2. *For the game with the marginal cost sharing mechanism, the convergence time to reach a pure Nash equilibrium can be unbounded.*

Proof. The proof is by exhibiting again the same small example, with 2 players, release times 0, unit weights, unit penalty factors, and $\alpha > 2$. From the previous section, we know that the game admits a pure Nash equilibrium, and by symmetry of the players, in fact two pure Nash equilibria. Following [10], the first one is

$$d_1 = \left(\frac{\alpha - 1}{2}\right)^{1/\alpha}, \; d_2 = d_1 + (\alpha - 1)^{1/\alpha},$$

while the second one is symmetric for players 1 and 2.

In the remainder of the proof, we assume that player 1 chooses a deadline which is close to the pure Nash equilibrium above. By analyzing the best responses of the players, we conclude that after a best response of player 2, and then of player 1 again, he chooses a deadline which is even closer to the pure Nash equilibrium above but still different from it, leading to an infinite convergence sequence of best responses.

Now suppose that $d_1 = \delta \left(\frac{\alpha-1}{2}\right)^{1/\alpha}$ for some $1 < \delta < 2^{1/\alpha}$. What is the best response for player 2?

Lemma 3. *Given the first player's choice d_1, the penalty of the second player as a function of his choice d_2 is given by*

$$h(d_2, d_1) = \begin{cases} h_1(d_2, d_1) = d_2 + d_2^{1-\alpha} + (d_1 - d_2)^{1-\alpha} - d_1^{1-\alpha} & \text{if } d_2 \le \frac{d_1}{2} \\ h_2(d_2, d_1) = d_2 + (2^\alpha - 1)d_1^{1-\alpha} & \text{if } \frac{d_1}{2} \le d_2 \le d_1 \\ h_3(d_2, d_1) = d_2 + 2^\alpha d_2^{1-\alpha} - d_1^{1-\alpha} & \text{if } d_1 \le d_2 \le 2d_1 \\ h_4(d_2, d_1) = d_2 + (d_2 - d_1)^{1-\alpha} & \text{if } d_2 \ge 2d_1, \end{cases}$$

and the best response for player 2 as function of d_1 is

$$d_1 + (\alpha - 1)^{1/\alpha} = (\alpha - 1)^{1/\alpha}(1 + 2^{-1/\alpha}\delta) \tag{5}$$

Proof. We first analyze the behavior of each of h_i, finding their local minima in the respective intervals, and afterwards we show that the equation (5) defines the best response for player 2 as function of d_1. For convenience we omit parameter d_1 in each function h_i. Figure 3 illustrate the best response for player 2 when player 1 chooses $d_1 = (\frac{\alpha-1}{2})^{1/\alpha}\delta$ for some $\delta > 1$.

Range of h_4: The derivative of h_4 in d_2 is

$$1 - (\alpha - 1)(d_2 - d_1)^{-\alpha},$$

which is zero exactly for the value

$$d_1 + (\alpha - 1)^{1/\alpha}.$$

As the second derivative $h_1''(d_2) = \alpha(\alpha - 1)(d_2 - d_1)^{-\alpha-1}$ is positive, the choice $d_2 = d_1 + (\alpha - 1)^{1/\alpha}$ minimizes the penalty among $d_2 \ge 2d_1$. Therefore, h_1 has a local minimum if

$$(\alpha - 1)^{1/\alpha} \ge d_1 = \delta \left(\frac{\alpha - 1}{2}\right)^{1/\alpha},$$

which holds by assumption $\delta < 2^{1/\alpha}$.

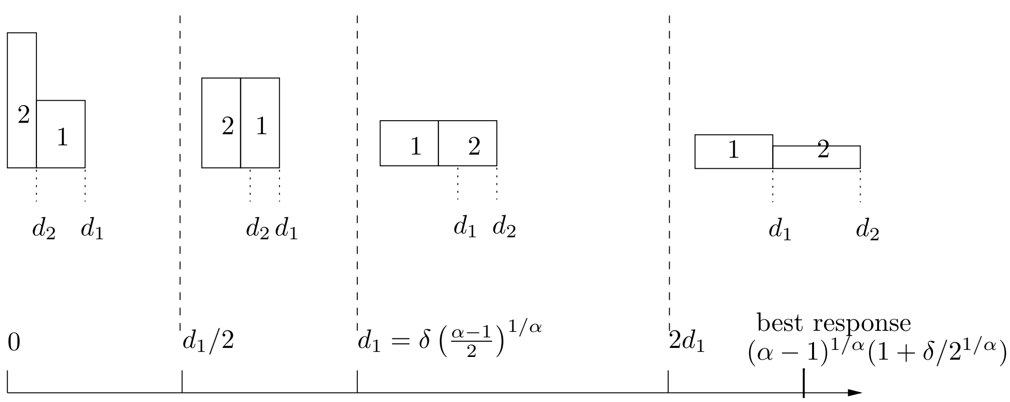

Figure 3: Best response for player 2.

In that case the penalty would be

$$d_1 + (\alpha - 1)^{\frac{1-\alpha}{\alpha}} + (\alpha - 1)^{1/\alpha} = (\alpha - 1)^{1/\alpha}(1 + \delta/2^{1/\alpha} + 1/(\alpha - 1)). \tag{6}$$

By comparing (6) with the remaining case, we show that it is indeed the best choice for player 2.

Range of h_3: The derivative of h_3 in d_2 is

$$1 - 2^\alpha(\alpha-1)d_2^{-\alpha},$$

which is zero for $d_2 = 2(\alpha-1)^{1/\alpha}$ and negative for $d_2 < 2(\alpha-1)^{1/\alpha}$. As

$$2(\alpha-1)^{1/\alpha} > 2\delta\left(\frac{\alpha-1}{2}\right)^{1/\alpha} = 2d_1,$$

the minimum penalty in this range is attained at the right boundary of the interval, i.e., for $d_2 = 2d_1 = 2\delta\left(\frac{\alpha-1}{2}\right)^{1/\alpha}$. But at this point the function h, which is continuous, coincides with h_4, which is decreasing in $\left(2d_1, d_1 + (\alpha-1)^{1/\alpha}\right)$. Hence $2d_1$ is not a local minimum of h.

Range of h_2: h_2 is an increasing function, and therefore it attains a minimum value only at the lower end of its range, which is $d_1/2$ in this case. Clearly, $d_1/2$ is a local minimum of h if and only if $h_1'(d_1/2) \leq 0$ and $h_2'(d_1/2) \geq 0$. However, we have $h_2'(d_1/2) = h_1'(d_1/2) = 1$, so $d_2 = d_1/2$ is not a local minimum of h either.

Range of h_1: We will show that h_1 is strictly larger than (6).

Since $\delta < 2^{1/\alpha}$, (6) is at most

$$(\alpha-1)^{1/\alpha}(2 + 1/(\alpha-1)) = (\alpha-1)^{1/\alpha-1}(2\alpha-1).$$

To lower bound h_1 we use the strict convexity of the function $x \mapsto x^{1-\alpha}$, which implies

$$2\left(\frac{d_2^{1-\alpha}}{2} + \frac{(d_1-d_2)^{1-\alpha}}{2}\right) > 2\left(\frac{d_1}{2} + \frac{d_1-d_2}{2}\right)^{1-\alpha} = 2^\alpha d_1^{1-\alpha}.$$

Note that $d_1 < (\alpha-1)^{1/\alpha}$ implies $d_1^{1-\alpha} > (\alpha-1)^{1/\alpha-1}$. Combining these, we can finally strictly lower bound the difference between h_1 and the value in (6) by

$$\begin{aligned}
& d_2 + (2^\alpha-1)d_1^{1-\alpha} - (\alpha-1)^{1/\alpha-1}(2\alpha-1) \\
> \;& d_2 + (2^\alpha-1)(\alpha-1)^{1/\alpha-1} - (\alpha-1)^{1/\alpha-1}(2\alpha-1) \\
= \;& d_2 + (\alpha-1)^{1/\alpha-1}(2^\alpha - 2\alpha),
\end{aligned}$$

which is non-negative since $2^\alpha \geq 2\alpha$ whenever $\alpha \geq 2$.

□

From now on we assume that player 2 chooses $d_2 = d_1 + (\alpha-1)^{1/\alpha} = (\alpha-1)^{1/\alpha}(1+2^{-1/\alpha}\delta)$. What is the best response for player 1?

Lemma 4. *Given the second player's choice d_2, the penalty of the first player as a function of his choice d_1 is given by $h(d_1, d_2)$ and the best response for player 1 is*

$$d_1 = \delta'\left(\frac{\alpha-1}{2}\right)^{1/\alpha},$$

for some $\delta' \in (1, \delta)$.

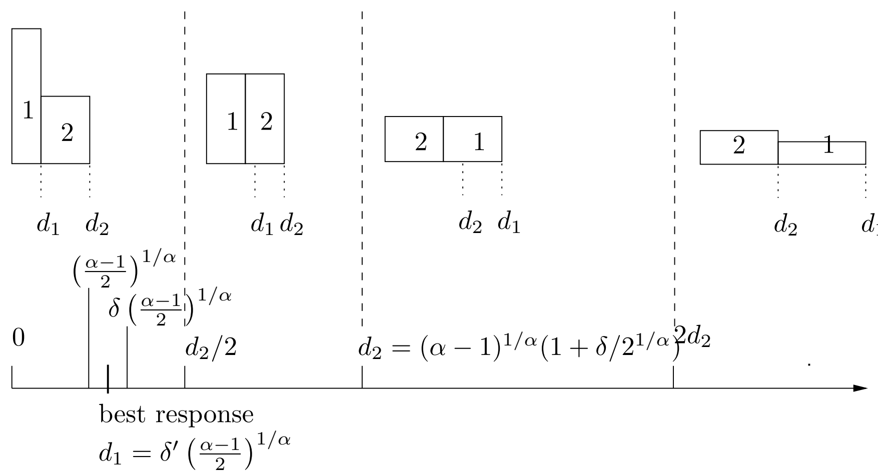

Figure 4: Best response for player 1.

Proof. Again player 1 best response is analyzed through a case analysis, similar to the previous one. Figure 4 illustrates the best response for player 1 when player 2 chose $d_2 = (\alpha-1)^{1/\alpha}(1+2^{-1/\alpha}\delta)$ for some $\delta > 1$. As in the previous proof, for convenience we omit parameter d_2 in each function h_i.

Range of h_1: The first derivative of h_1 in d_1 is

$$h_1'(d_1) = 1 + (\alpha-1)((d_2-d_1)^{-\alpha} - d_1^{-\alpha})$$

And the second derivative is

$$h_1''(d_1) = \alpha(\alpha-1)((d_2-d_1)^{-\alpha-1} + d_1^{-\alpha-1})$$

which is positive, implying that the penalty is convex in d_1.

Now, we show that we have a local minimum for some $1 < \delta' < \delta$ at some value

$$\delta'\left(\frac{\alpha-1}{2}\right)^{1/\alpha}.$$

For this purpose we analyze the interval

$$\left(\frac{\alpha-1}{2}\right)^{1/\alpha} \leq d_1 \leq \delta\left(\frac{\alpha-1}{2}\right)^{1/\alpha}.$$

First we evaluate h_1' at the lower end $d_1 = \left(\frac{\alpha-1}{2}\right)^{1/\alpha}$. In this case

$$d_2 - d_1 = \left(\frac{\alpha-1}{2}\right)^{1/\alpha}(2^{1/\alpha}+\delta-1).$$

This means that

$$\begin{aligned} h_1'(d_1) &= 1 + (\alpha-1)\frac{2}{\alpha-1}(2^{1/\alpha}+\delta-1)^{-\alpha} - (\alpha-1)\frac{2}{\alpha-1} \\ &= 2(2^{1/\alpha}+\delta-1)^{-\alpha} - 1 < 0\ , \end{aligned}$$

as $\delta > 1$ and $\alpha > 1$.

Secondly, we evaluate the h_1' at the upper end

$$d_1 = \delta \left(\frac{\alpha - 1}{2}\right)^{1/\alpha} , \tag{7}$$

then by $d_2 - d_1 = (\alpha - 1)^{1/\alpha}$ we obtain

$$1 + (\alpha - 1)\frac{1}{\alpha - 1} - (\alpha - 1)\frac{2}{\alpha - 1}\delta^{-\alpha} = 2 - 2/\delta^{\alpha} > 0 \ ,$$

as $\delta < 2^{1/\alpha}$ and $\alpha > 1$. Together with the continuity of the penalty function, it implies that there is a value $1 < \delta' < \delta$ such that

$$d_1 = \delta' \left(\frac{\alpha - 1}{2}\right)^{1/\alpha}$$

is a local minimum.

To conclude we compare this local minimum with the remaining cases, showing the dominance of this value.

Range of h_2: h_2 is an increasing function and therefore is minimum at $d_1 = d_2/2$. Again, $d_1/2$ is a local minimum of h if and only if $h_1'(d_2/2) \leq 0$ and $h_2'(d_2/2) \geq 0$. However, we have $h_2'(d_2/2) = h_1'(d_2/2) = 1$, so $d_1 = d_2/2$ is not a local minimum of h either.

Range of h_3: In this case, the derivative of the penalty function is

$$1 - (\alpha - 1)2^{\alpha} d_1^{-\alpha}$$

which is zero at

$$d_1 = 2(\alpha - 1)^{1/\alpha}. \tag{8}$$

Note that the second derivative

$$h_3''(d_1) = \alpha(\alpha - 1)2^{\alpha} d_1^{-\alpha-1} > 0$$

for $\alpha > 1$, so the penalty is convex, and (8) is a local minimum. It is greater than d_2 by $\delta < 2^{1/\alpha}$ and smaller than $2d_2$ by $\delta > 0$. Therefore the local minimum belongs to the range considered in this case.

The penalty at $d_1 = 2(\alpha - 1)^{1/\alpha}$ is

$$\begin{aligned} & 2(\alpha - 1)^{1/\alpha} + 2^{\alpha} \cdot 2^{1-\alpha}(\alpha - 1)^{1/\alpha - 1} - (\alpha - 1)^{1/\alpha - 1}(1 + \delta/2^{1/\alpha})^{1-\alpha} \\ = \ & (\alpha - 1)^{1/\alpha - 1}\left(2\alpha - (1 - \delta/2^{1/\alpha})^{1-\alpha}\right) . \end{aligned}$$

We claim that this penalty is larger than h_1 evaluated at (7), eliminating $2(\alpha - 1)^{1/\alpha}$ for a best response. For this purpose we need to show

$$\begin{aligned} h_1\left(\delta\left(\frac{\alpha - 1}{2}\right)^{1/\alpha}\right) &< h_3(2(\alpha - 1)^{1/\alpha}) \\ \Leftrightarrow \frac{\delta(\alpha - 1)}{2^{1/\alpha}} + \frac{2^{1-1/\alpha}}{\delta^{\alpha-1}} + 1 &< 2\alpha \\ \Leftrightarrow \delta^{\alpha}(\alpha - 1) + 2 &< (2\alpha - 1)2^{1/\alpha}\delta^{\alpha-1}. \end{aligned}$$

This holds since

$$\delta^{\alpha-1}(2\alpha-1)2^{1/\alpha} - \delta^{\alpha}(\alpha-1) = \delta^{\alpha-1}\left(2^{1/\alpha}(2\alpha-1) - \delta(\alpha-1)\right)$$
$$> 2^{1/\alpha}(2\alpha-1) - \delta(\alpha-1) > 2^{1/\alpha}\alpha > 2,$$

for any $\alpha > 1$ because $2^{1/\alpha}\alpha$ increases with $\alpha > \ln 2$ and equals 2 when $\alpha = 1$.

Range of h_4: The derivative of h_4 in d_2 is

$$1 - (\alpha-1)(d_1 - d_2)^{-\alpha},$$

which is zero exactly for the value

$$d_2 + (\alpha-1)^{1/\alpha}.$$

As the second derivative $h_1''(d_1) = \alpha(\alpha-1)(d_1 - d_2)^{-\alpha-1}$ is positive, the choice $d_1 = d_2 + (\alpha-1)^{1/\alpha}$ minimizes the penalty among $d_2 \geq 2d_1$. However, h_1 has a local minimum if

$$(\alpha-1)^{1/\alpha} \geq d_2 = (\alpha-1)^{1/\alpha}(1 + 2^{-1/\alpha}\delta),$$

which is a contradiction by $\delta 2^{-1/\alpha} > 0$

□

This concludes the proof of Theorem 2. □

4.3 Bounding total charge

In this section we bound the total cost share for the Marginal Cost Sharing Scheme, by showing that it is at least $E(\mathrm{OPT}(d))$ and at most α times this value. In fact we show a stronger claim for individual cost shares.

Theorem 3. *For every player i, its marginal costshare is at least its proportional costshare and at most α times the proportional costshare.*

Proof. Fix a player i, and denote by S_{-i} the schedule obtained from $\mathrm{OPT}(d)$ when all executions of i are replaced by idle times. Clearly we have the following inequalities.

$$E(\mathrm{OPT}(d_{-i})) \leq E(S_{-i}) \leq E(\mathrm{OPT}(d))$$

Then the marginal cost share of player i can be lower bounded by

$$E(\mathrm{OPT}(d)) - E(\mathrm{OPT}(d_{-i})) \geq E(\mathrm{OPT}(d)) - E(S_{-i}).$$

According to [16] the schedule OPT can be obtained by the following iterative procedure. Let P be the support of a partial schedule. For every interval $[t,t')$ we define its domain $I_{t,t'} := [t,t')\backslash P$, the set of included jobs $J_{t,t'} := \{j : [r_j, d_j) \subseteq [t,t')\}$, and the density $\sigma_{t,t'} := \sum_{j\in J_{t,t'}} w_j/|I_{t,t'}|$. The procedure starts with $P = \emptyset$, and while not all jobs are scheduled, selects an interval $[t,t')$ with maximal density, and schedules all jobs from $J_{t,t'}$ in earliest deadline order in $I_{t,t'}$ at speed $\sigma_{t,t'}$, then adds $I_{t,t'}$ to P.

For the upper bound, let $t_1 < t_2 < \ldots < t_\ell$ be the sequence of all release times and deadlines for some $\ell \leq 2n$. For convenience, we denote $S = \mathrm{OPT}(d)$ and $S' = \mathrm{OPT}(d_{-i})$. Clearly both schedules run at uniform speed in every interval $[t_{k-1}, t_k)$. For every $1 \leq k \leq n$ let s_k be the speed of S in $[t_{k-1}, t_k)$, and s'_k the speed of S' in the same interval.

From the iterative procedure described above it follows that every job is scheduled at constant speed. Let s_0 be the speed at which job i is scheduled in S. It also follows that if $s_k > s_0$, then $s'_k = s_k$, and if $s_k \leq s_0$, then $s'_k \leq s_k$.

We establish the following upper bound.

$$\begin{aligned}
E(\mathrm{OPT}(d)) - E(\mathrm{OPT}(d_{-i})) &= \sum_{k=1}^{\ell} s_k^\alpha (t_k - t_{k-1}) - s_k'^\alpha (t_k - t_{k-1}) \\
&= \sum_{k=1}^{\ell} (t_k - t_{k-1})(s_k^\alpha - (s_k - (s_k - s'_k))^\alpha) \\
&= \sum_{k=1}^{\ell} (t_k - t_{k-1}) s_k^\alpha \left(1 - \left(1 - \frac{s_k - s'_k}{s_k}\right)^\alpha\right) \\
&\leq \sum_{k=1}^{\ell} (t_k - t_{k-1}) s_k^\alpha \left(1 - \left(1 - \alpha \frac{s_k - s'_k}{s_k}\right)\right) \\
&= \sum_{k=1}^{\ell} (t_k - t_{k-1}) \alpha s_k^{\alpha-1} (s_k - s'_k) \\
&\leq \alpha s_a^{\alpha-1} \sum_{k=1}^{\ell} (t_k - t_{k-1})(s_k - s'_k) \\
&= \alpha s_a^{\alpha-1} w_i \\
&= \alpha (E(\mathrm{OPT}(d)) - E(S_{-i})).
\end{aligned}$$

The first inequality uses the generalized Bernoulli inequality, and the last one the fact that for all k with $s_k \neq s'_k$ we have $s_k \leq s_a$.

The theorem follows from the fact that $s_a^{\alpha-1} w_i$ is precisely the proportional cost share of job i in $\mathrm{OPT}(d)$. □

A tight example is given by n jobs, each with workload $1/n$, release time 0 and deadline 1. Clearly the optimal energy consumption is 1 for this instance. The marginal cost share for each player is $1 - (1 - 1/n)^\alpha$. Finally we observe that the total marginal cost share tends to α, i.e.

$$\lim_{n \to +\infty} n - n(1 - 1/n)^\alpha = \alpha.$$

5 Final remarks

5.1 Cross-monotonicity

The *cross-monotonicity* is a property of cost sharing games, stating that whenever new players enter the game, the cost share of any fixed player does not increase. This property is useful for stability in the game, and is the key to the Moulin carving algorithm [14], which selects a set of players to be served for specific games.

In the game that we consider, the minimum energy of an optimal schedule for a set S of jobs contrasts with many studied games, where serving more players becomes more cost effective, because the considered equipment is better used.

Consider a very simple example of two identical players, submitting their respective jobs with the same deadline 1. Suppose the workload of each job is w, then the minimum energy necessary to schedule one job is w^α, while the cost to serve both jobs is $(2w)^\alpha$, meaning that the cost share increase whenever a second player enters the game. Therefore the marginal cost sharing scheme is not cross-monotonic.

5.2 Uniqueness of Nash equilibria

In this paper we showed that the deadline game with the marginal cost sharing mechanism always admits a pure Nash equilibrium. However the Nash equilibrium may not be unique. Here, a simple example is an instance with n identical players where $n!$ Nash symmetric equilibria are admitted. For arbitrary instances, the uniqueness of Nash Equilibrium raises two questions. The first question concerns the comparison of different Nash equilibria in terms of social cost. If the divergence is significant, then it means that the outcome of the game can be arbitrary, and may indicate the need for another mechanism, which smooths the possible resulting Nash equilibria. The second question concerns the characterization of job sets which lead to a unique Nash equilibrium.

We are interested in this last question, already in the 2 player setting. Here we fixed normalized quantities $p_1 = 1, w_1 = 1$ for the first player, and leave the quantities of the second player variable. Which points (p_2, w_2) do admit a unique Nash equilibrium?

The 2 potential Nash equilibria are the following strategy profiles

S21 $d_2 = \ell_2^*, d_1 = \ell_2^* + \ell_1^*$

S12 $d_1 = \ell_1, d_2 = \ell_1 + \ell_2$,

where the lengths $\ell_1, \ell_2, \ell_1^*, \ell_2^*$ are defined as follows.

$$\ell_2^* = w_2 \frac{(\alpha-1)^{1/\alpha}}{(p_1+p_2)^{1/\alpha}} \qquad \ell_1^* = w_1 \frac{(\alpha-1)^{1/\alpha}}{(p_1)^{1/\alpha}} \tag{9}$$

$$\ell_1 = w_1 \frac{(\alpha-1)^{1/\alpha}}{(p_1+p_2)^{1/\alpha}} \qquad \ell_2 = w_2 \frac{(\alpha-1)^{1/\alpha}}{(p_2)^{1/\alpha}}. \tag{10}$$

To break the symmetry we consider only points where the social cost of S21 is minimal among the two profiles. In [10] we showed that these are precisely the points that satisfy

$$w_2 \le w_1 \frac{(p_1+p_2)^{(\alpha-1)/\alpha} - p_1^{(\alpha-1)/\alpha}}{(p_1+p_2)^{(\alpha-1)/\alpha} - p_2^{(\alpha-1)/\alpha}}.$$

For such points, we ask whether one of the players wants to deviate from S12, i.e., the other potential equilibrium, with larger social cost. Our experiments indicate that, for each player $j \in \{1, 2\}$, there is a threshold t_j, which depends on p_2, such that player j wants to deviate if and only if $w_2 \le t_j$. Figure 5 depicts the plots of these threshold functions, determined numerically. We were unable to rigorously prove their existence, obtain their closed forms, or relate them to functions studied in [5]. However by choosing $\alpha = 2$, $p_1 = w_1 = 1$ and (p_2, w_2) inside the shaded region of Figure 5, one can easily verify that S21 is a unique pure Nash equilibrium.

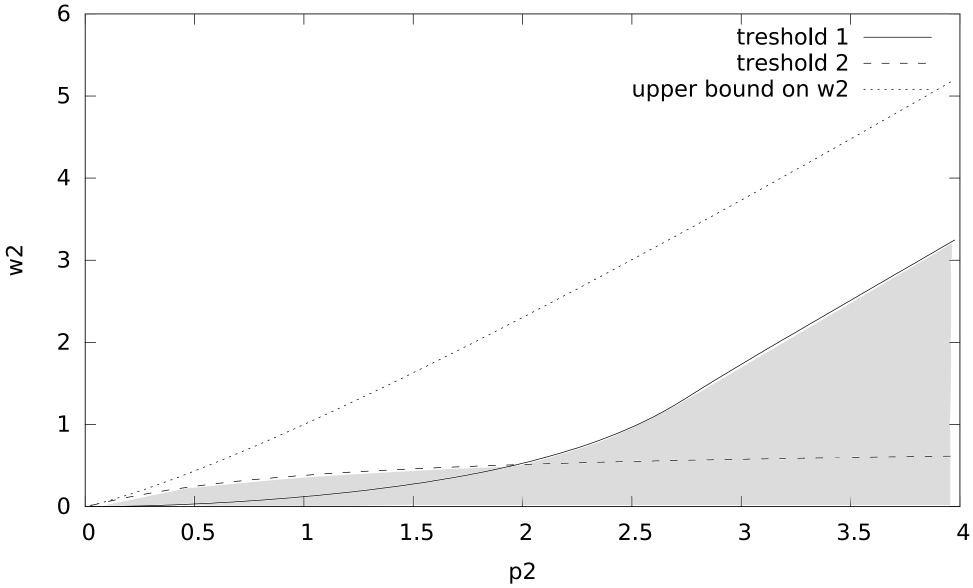

Figure 5: Experimental results for $\alpha = 2$. Horizontal axis is p_2, and vertical axis is w_2. Depicted are the upper bound for w_2 and the thresholds t_1, t_2.

Acknowledgements

We would like to thank anonymous referees for remarks and suggestions on an earlier version of this paper.

Christoph Dürr and Oscar C. Vásquez were partially supported by grant ANR-11-BS02-0015. Oscar C. Vásquez was partially supported by FONDECYT grant 11140566. Łukasz Jeż was partially supported ERC consolidator grant 617951, Israeli Centers of Research Excellence (I-CORE) program, Center No.4/11, NCN grant DEC-2013/09/B/ST6/01538, and FNP Start scholarship.

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