CONTROLLABILITY NEAR A HOMOCLINIC BIFURCATION

FRITZ COLONIUS
INSTITUTE OF MATHEMATICS, UNIVERSITY OF AUGSBURG, AUGSBURG, GERMANY , AMANI HASAN
SCHOOL OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, COLLEGE OF SCIENCE, UNIVERSITY OF TEHRAN, IRAN & FACULTY OF MECHANICAL AND ELECTRICAL ENGINEERING, DAMASCUS UNIVERSITY, SYRIA , AND GHOLAM REZA ROKNI LAMOUKI
SCHOOL OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, COLLEGE OF SCIENCE, UNIVERSITY OF TEHRAN, IRAN

Abstract: Controllability properties are studied for control-affine systems depending on a parameter α and with constrained control values. The uncontrolled systems in dimension two and three are subject to a homoclinic bifurcation. This generates two families of control sets depending on a parameter in the involved vector fields and the size of the control range. A new parameter β given by a split function for the homoclinic bifurcation determines the behavior of these control sets. It is also shown that there are parameter regions where the uncontrolled equation has no periodic orbits, while the controlled systems have periodic solutions arbitrarily close to the homoclinic orbit.

Key words: controllability, control set, homoclinic bifurcation
MSC 2020: 93C15, 37G15, 93B05, 34C37

1. Introduction. Complete controllability is a rare occurrence for nonlinear systems with restricted control range. Hence the (maximal) regions in the state space where complete controllability holds, i.e., control sets, are of interest, cf. Definition 3.1. A basic reference is Colonius and Kliemann [2]. The present paper studies control sets for parameter dependent systems in dimension two and three near a homoclinic bifurcation.

Several results for control sets near local bifurcations are available. For transcritical and pitchfork bifurcations in the one-dimensional case and for Hopf bifurcations, cf. [2, Section 8.3 and Section 9.3], also for applications to physically relevant systems and further references. Lamb, Rasmussen, and Rodrigues [14] develop a topological bifurcation theory for minimal invariant sets (which coincide with invariant control sets) of set-valued dynamical systems. The only contribution for control sets near a homoclinic bifurcation is due to Häckl and Schneider [10] who study systems when the uncontrolled two-dimensional system is obtained by the universal unfolding of a Takens-Bogdanov singularity. The relation of our results to [10] is discussed in more detail in Remark 4.4 and Remark 4.10. Control sets near homoclinic and heteroclinic orbits are also of relevance in the study of models for ship roll motion, cf. Gayer [6, 7] and Colonius, Kreuzer, Marquardt, and Sichermann [3]. While in the latter references the uncontrolled and unperturbed system is Hamiltonian, the present paper considers non-Hamiltonian cases. We use the monograph Kuznetsov [13] as a basic reference for homoclinic bifurcations, cp. also Guckenheimer and Holmes [8] and Wiggins [19].

We consider control-affine systems in $\mathbb{R}^d$ of the form

$$\dot{x}(t) = f_0(\alpha, x(t)) + \sum_{i=1}^{m} u_i(t) f_i(\alpha, x(t)), u(t) \in U \text{ with } 0 \in U \subset \mathbb{R}^m, \tag{1.1}$$

with parameter $\alpha \in \mathbb{R}$. The term $u(\cdot)$ can be interpreted as a control or as a time-dependent deterministic perturbation. The invariant control sets are also of relevance for the analysis of associated degenerate Markov diffusion processes, where u is replaced by a random disturbance, cf. Kliemann [12]. Furthermore, control sets are of interest in connection with minimal data rates for control systems, since their invariance entropy can be computed, cf. Kawan and Da Silva [11].

For the uncontrolled system $\dot{x} = f_0(\alpha_0, x)$ and dimension $d = 2$, a classical theorem due to Andronov and Leontovich completely describes the bifurcation of an orbit homoclinic to an hyperbolic equilibrium x_0. The saddle quantity σ_0 (the sum of the eigenvalues of $\frac{\partial f_0}{\partial x}(\alpha_0, x_0)$) and the sign of a parameter β given by a split function determine the direction of the bifurcation and the stability properties of the periodic orbits. It turns out that also the properties of control sets for (1.1) are determined by this new parameter β (instead of α). Furthermore, a main result of this paper shows that the qualitative behavior of the control system can be different from the behavior of the uncontrolled system: There are parameter regions where there is no homoclinic orbit and no limit cycle for the uncontrolled systems while there exist periodic orbits of the control system arbitrarily close to the homoclinic orbit. The analysis of homoclinic bifurcations of systems in $\mathbb{R}^3$ goes back to the work by L.P. Shil'nikov. We will only consider the cases, where unique periodic orbits bifurcate, the much more complicated case where, in particular, countably many periodic orbits occur is left for future work.

The contents of this paper are as follows. In Section 2, we introduce notation used for homoclinic bifurcations and cite relevant results in dimension two and three. Section 3 recalls properties of control sets and their parameter dependence, when the control range is perturbed or an external parameter occurs in the vector fields. Section 4 starts with a discussion of the control sets near an orbit homoclinic to a hyperbolic equilibrium in dimension $d = 2$. In dimension $d = 3$, we analyze the cases where the equilibrium is a saddle, and a saddle-focus with saddle quantity $\sigma_0 < 0$. Section 5 presents an example including numerical results which are based on Häckl's algorithm (Häckl [9]). We remark that an alternative for computing control sets are set oriented methods, cf. Szolnoki [17]. Finally, Section 6 draws some conclusions.

Notation. The Hausdorff distance between two compact subsets $A, B \subset \mathbb{R}^d$ is $d_H(A, B) = \max(\max_{a \in A} \min\{\|a - b\| \,|b \in B\}, \max_{b \in B} \min\{\|a - b\| \,|a \in A\})$. The ball of radius $\delta > 0$ around $x \in \mathbb{R}^d$ is $\mathbf{B}(x, \delta) = \{y \in \mathbb{R}^d \,|\|x - y\| < \delta\}$. It is convenient to write (as Kuznetsov [13]) $\Gamma_0 \cup x_0$ for the union of $\{x_0\}$ with an orbit Γ_0 homoclinic to x_0.

2. Bifurcation of orbits homoclinic to hyperbolic equilibria. This section introduces some notation and cites results on the bifurcation of orbits which are homoclinic to hyperbolic equilibria. This is done for planar systems in the first subsection and for three-dimensional systems in the second subsection We rely on the presentation in Kuznetsov [13, Chapter 6].

Consider a parameter dependent family of ordinary differential equations in $\mathbb{R}^d$ of the form

$$\dot{x}(t) = f(\alpha, x(t)), \tag{2.1}$$

where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a smooth (C^∞-)function. We assume that for every $\alpha \in \mathbb{R}$ and every initial value $x \in \mathbb{R}^d$ there exists a unique solution $\psi^\alpha(t, x), t \in \mathbb{R}$, and that all maps $\psi^\alpha(t, \cdot) : \mathbb{R}^d \to \mathbb{R}^d, t \in \mathbb{R}$, are continuous. An orbit $\Gamma^\alpha_{x_0} := \{\psi^\alpha(t, x) \,|t \in \mathbb{R}\}$ is called homoclinic to an equilibrium point x_0 (i.e., $f(\alpha, x_0) = 0$) if $\psi^\alpha(t, x) \to x_0$ as

$t \to \pm\infty$. Let

$$W^{\alpha,s}(x_0) = \{y \in \mathbb{R}^d \,|\, \psi^\alpha(t,y) \to x_0 \text{ for } t \to \infty\},$$
$$W^{\alpha,u}(x_0) = \{y \in \mathbb{R}^d \,|\, \psi^\alpha(t,y) \to x_0 \text{ for } t \to -\infty\},$$

denote the stable and the unstable manifold, resp., of x_0.

2.1. The planar case. In this subsection we cite a classical theorem due to Andronov and Leontovich on the bifurcation in the plane of orbits which are homoclinic to hyperbolic equilibria.

Suppose that system (2.1) is planar ($d = 2$) having for $\alpha_0 = 0$ a saddle equilibrium $x_0 = 0$, i.e., $f_x(0,0) = \frac{\partial}{\partial x} f(0,0)$ has a positive and a negative eigenvalue, $\lambda_1(0) < 0 < \lambda_2(0)$. Assume that Γ_0 is an orbit which is homoclinic to x_0. For α sufficiently close to $\alpha_0 = 0$, the implicit function theorem implies that there are saddle equilibria x_α with eigenvalues $\lambda_1(x_\alpha) < 0 < \lambda_2(x_\alpha)$ depending continuously on α.

Let Σ be a (one-dimensional) local cross-section to the stable manifold near the saddle. Select a coordinate $\xi \in \mathbb{R}$ along Σ such that the point of its intersection with the stable manifold $W^{\alpha_0,s}(x_0)$ corresponds to $\xi = 0$. This coincides with the point of intersection with the unstable manifold $W^{\alpha_0,u}(x_0) = W^{\alpha_0,s}(x_0) = \Gamma_0$. For all α sufficiently close to $\alpha_0 = 0$, Σ is also a local transversal section to the unstable manifolds $W^{\alpha,u}(x_\alpha)$. Denote by $\xi^u(\alpha)$ the ξ-value of the intersection of $W^{\alpha,u}(x_\alpha)$ with Σ. The scalar function $\alpha \mapsto \beta(\alpha) := \xi^u(\alpha)$ which is defined on a neighborhood of $\alpha_0 = 0$ is called a *split function*. The function $\beta(\cdot)$ is smooth and it is injective if $\beta'(0) \neq 0$.

In the planar case considered here, the homoclinic bifurcation is characterized by the following theorem due to Andronov and Leontovich, cf. Kuznetsov [13, Theorem 6.1]. Other references include Guckenheimer and Holmes [8, Theorem 6.1.1], Wiggins [19, Theorem 3.2.11].

THEOREM 2.1. *Consider a parameter dependent two-dimensional system of the form (2.1) having at $\alpha_0 = 0$ an orbit Γ_0 which is homoclinic to a saddle $x_0 = 0$ with eigenvalues $\lambda_1(0) < 0 < \lambda_2(0)$. Assume that the following conditions hold:*

(H1) $\sigma_0 = \lambda_1(0) + \lambda_2(0) \neq 0$;

(H2) $\beta^{'}(0) \neq 0$, where $\beta(\alpha)$ is a split function.

Then, there exist $\overline{\alpha} > 0$ and a neighborhood U_0 of $\Gamma_0 \cup x_0$ in which for all $|\alpha| < \overline{\alpha}$ a unique limit cycle $L_{\beta(\alpha)}$ bifurcates from Γ_0. The limit cycle exists and is asymptotically stable for $\beta > 0$ if $\sigma_0 < 0$, and exists and is unstable for $\beta < 0$ if $\sigma_0 > 0$.

(H1) is a nondegeneracy condition. Due to (H2) the split function $\alpha \mapsto \beta(\alpha)$ is injective for $|\alpha|$ small enough, hence the inverse $\alpha(\beta)$ exists for β in a neighborhood of 0 and β can be considered as a new parameter. Thus the unique limit cycle L_β exists for sufficiently small $|\beta|$. The homoclinic orbit is called "splitting down" if $\beta < 0$ and "splitting up" if $\beta > 0$. We remark (cf. Kuznetsov [13, formula (6.25) on p. 232]) that $\beta^{'}(0) \neq 0$ is equivalent to the Melnikov condition

$$M_{\alpha_0}(0) = \int_{-\infty}^{+\infty} \exp\left[-\int_0^t \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) d\tau\right] \left(f_1 \frac{\partial f_2}{\partial \alpha} - f_2 \frac{\partial f_1}{\partial \alpha}\right) dt \neq 0, \qquad (2.2)$$

where all expressions involving $f(0,x_1,x_2) = (f_1(0,x_1,x_2), f_2(0,x_1,x_2))^\top$ are evaluated along the homoclinic solution of (2.1) at $\alpha_0 = 0$. Hence (H2) is a transversality condition for the intersection of the stable and unstable manifolds.

2.2. The three-dimensional case. As exposed in Kuznetsov [13, Section 6.3] a three-dimensional state space gives rise to a wider variety of homoclinic bifurcations. We will discuss results for hyperbolic equilibria which are saddles and saddle-foci. Taking into account also the sign of σ_0 there are four main cases, cf. [13, p. 214]. We will only treat the three simpler cases.

Consider an equation in $\mathbb{R}^3$ of the form (2.1) having at $\alpha_0 = 0$ an orbit Γ_0 homoclinic to a hyperbolic equilibrium point $x_0 = 0$. It is also possible to define a split function in this case, cf. [13, p. 199]. Suppose that the unstable manifold W^u of x_0 is one-dimensional, introduce a two-dimensional cross-section Σ and let the point ξ^u correspond to the intersection of W^u with Σ. Then a split function $\beta = \xi^u$ can be defined as before in the planar case. Its zero $\beta = 0$ gives a condition for a homoclinic bifurcation in $\mathbb{R}^3$.

The case of a saddle is described in [13, Theorem 6.3 and Theorem 6.5] as follows.

THEOREM 2.2. *Consider system (2.1) in $\mathbb{R}^3$ having at $\alpha_0 = 0$ an orbit Γ_0 homoclinic to a saddle $x_0 = 0$ with real eigenvalues $\lambda_1(0) > 0 > \lambda_2(0) > \lambda_3(0)$. Assume that the following conditions hold:*

(H1) Γ_0 returns to x_0 along the eigenspace for $\lambda_2(0)$;

(H2) $\beta'(0) \neq 0$, where $\beta(\alpha)$ is a split function.

(i) Suppose that $\sigma_0 = \lambda_1(0) + \lambda_2(0) < 0$. Then, there exists a neighborhood U_0 of $\Gamma_0 \cup x_0$ in which the system has a unique and asymptotically stable limit cycle L_β for all sufficiently small $\beta > 0$. There are no periodic orbits if $\beta \leq 0$.

(ii) Suppose that $\sigma_0 = \lambda_1(0)+\lambda_2(0) > 0$ and, additionally, Γ_0 is simple or twisted. Then there exists a neighborhood U_0 of $\Gamma_0 \cup x_0$ in which for all sufficiently small $|\beta|$ a unique saddle limit cycle L_β bifurcates from Γ_0. The cycle exists for $\beta < 0$ if Γ_0 is simple, and for $\beta > 0$ if Γ_0 is twisted. In the first case there are no periodic orbits if $\beta \geq 0$ and in the second case there are no periodic orbits if $\beta \leq 0$.

The assumption in (ii) needs some explanation. Here we suppose that the two-dimensional stable manifold $W^{\alpha_0,s}(x_0)$ intersects itself near the saddle along the two exceptional orbits on $W^{\alpha_0,s}(x_0)$ that approach the saddle along the eigenspace for $\lambda_3(0)$ (this is called the *strong inclination property*). This yields a two-dimensional nonsmooth submanifold which is topologically equivalent to either a simple band or a twisted band called a Möbius band (cf. also Wiggins [18, Section 4.8A]). In the first case, Γ_0 is called simple, in the second case twisted.

The case of a saddle-focus with $\sigma_0 < 0$ is described in [13, Theorem 6.4] as follows.

THEOREM 2.3. *Consider system (2.1) in $\mathbb{R}^3$ having at $\alpha_0 = 0$ an orbit Γ_0 homoclinic to a saddle-focus $x_0 = 0$ with eigenvalues satisfying $\lambda_1(0) > 0 > \operatorname{Re}\lambda_{2,3}(0)$ and $\lambda_2(0) \neq \lambda_3(0)$. Assume that the following conditions hold:*

(H1) $\beta'(0) \neq 0$, where $\beta(\alpha)$ is a split function;

(H2) $\sigma_0 = \lambda_1(0) + \operatorname{Re}\lambda_{2,3}(0) < 0$.

Then exists a neighborhood U_0 of $\Gamma_0 \cup x_0$ in which the system has a unique and asymptotically stable limit cycle L_β for all sufficiently small $\beta > 0$. There are no periodic orbits if $\beta \leq 0$.

The remaining case of a saddle-focus with $\sigma_0 > 0$ is much more complicated and leads, among others, to infinitely many saddle limit cycles, cf. [13, Theorem 6.6].

3. Control sets and their parameter dependence. We consider control-affine systems in $\mathbb{R}^d$ of the form

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \tag{3.1}$$
$$u \in \mathcal{U} := \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m) \,|u(t) \in U \text{ for almost all } t \in \mathbb{R}\},$$

where $f_0, f_1, \ldots, f_m$ are smooth vector fields on $\mathbb{R}^d$ and the control range $U \subset \mathbb{R}^m$ is compact and convex with $0 \in \mathrm{int} U$. We assume that for every initial state $x \in \mathbb{R}^d$ and every control function $u \in \mathcal{U}$ there exists a unique solution $\varphi(t, x, u), t \in \mathbb{R}$, with $\varphi(0, x, u) = x$ of (3.1) depending continuously on x. The system with $u \equiv 0$ given by

$$\dot{x}(t) = f_0(x(t)) \tag{3.2}$$

is called the uncontrolled system. It generates a continuous flow $\psi(t, \cdot), t \in \mathbb{R}$, on $\mathbb{R}^d$.

The set of points reachable from $x \in \mathbb{R}^d$ and controllable to $x \in \mathbb{R}^d$ up to time $T > 0$ are defined by

$$\mathcal{O}^+_{\leq T}(x) := \{y \in \mathbb{R}^d \,|\ \text{there are } 0 \leq t \leq T \text{ and } u \in \mathcal{U} \text{ with } y = \varphi(t, x, u)\},$$
$$\mathcal{O}^-_{\leq T}(x) := \{y \in \mathbb{R}^d \,|\ \text{there are } 0 \leq t \leq T \text{ and } u \in \mathcal{U} \text{ with } x = \varphi(t, y, u)\},$$

resp. Furthermore, the reachable set (or positive orbit) from x and the set controllable to x (or negative orbit of x) are

$$\mathcal{O}^+(x) = \bigcup\nolimits_{T>0} O^+_{\leq T}(x), \quad \mathcal{O}^-(x) = \bigcup\nolimits_{T>0} O^-_{\leq T}(x),$$

resp. The system is called locally accessible in x, if $\mathcal{O}^+_{\leq T}(x)$ and $\mathcal{O}^-_{\leq T}(x)$ have nonvoid interior for all $T > 0$. This is guaranteed by the accessibility rank condition

$$\dim \mathcal{LA}\{f_0, f_1, \ldots, f_m\}(x) = d \text{ for all } x \in \mathbb{R}^d, \tag{3.3}$$

where the left hand side denotes the dimension of the subspace of $\mathbb{R}^d$ corresponding to the vector fields evaluated in x in the Lie algebra $\mathcal{LA}\{f_0, f_1, \ldots, f_m\}$ generated by the vector fields $f_0, f_1, \ldots, f_m$ (cf. Sontag [16, Theorem 9, p. 156]).

The following definition introduces subsets of complete approximate controllability which are of primary interest in the present paper.

Definition 3.1. *A set $D \subset \mathbb{R}^d$ is called a control set of system (3.1) if it has the following properties: (i) for all $x \in D$ there is a control function $u \in \mathcal{U}$ such that $\varphi(t, x, u) \in D$ for all $t \geq 0$, (ii) for all $x \in D$ one has $D \subset \mathrm{cl}\mathcal{O}^+(x)$, and (iii) D is maximal with these properties, that is, if $D' \supset D$ satisfies conditions (i) and (ii), then $D' = D$.*

A control set $D \subset \mathbb{R}^d$ is called an invariant control set if $\mathrm{cl}D = \mathrm{cl}\mathcal{O}^+(x)$ for all $x \in D$. All other control sets are called variant.

If the intersection of two control sets is nonvoid, the maximality property (iii) implies that they coincide. If the system is locally accessible in all $y \in \mathrm{int}D$, then $\mathrm{int}D \subset \mathcal{O}^+(x)$ for all $x \in D$ and $D = \mathrm{cl}\mathcal{O}^+(x) \cap \mathcal{O}^-(y)$ for all $x, y \in \mathrm{int}D$. For these properties and further discussion of control sets, we refer to Colonius and Kliemann [2, Chapters 3 and 4].

Next we will discuss the dependence of control sets on parameters. The parameters change the size of the control range or the involved vector fields. First we analyze families of control systems of the form

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \quad u(t) \in U^\rho := \rho U, \tag{3.4}$$

where $\rho > 0$ and $u \in \mathcal{U}^\rho := \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m) \,|\, u(t) \in U^\rho \text{ for almost all } t \in \mathbb{R}\}$. We suppose that the assumptions on (3.1) are satisfied. Obviously, the accessibility rank condition (3.3) is independent of $\rho > 0$.

A subset $K \subset \mathbb{R}^d$ is called invariant for the uncontrolled system (3.2) if $\psi(t,x) \in K$ for all $x \in K$ and $t \in \mathbb{R}$. An invariant set $K \subset \mathbb{R}^d$ is called chain transitive if for all $x, y \in K$ and every $\varepsilon > 0$ and $T > 0$, there exist $n \in \mathbb{N}$, points $x = x_0, x_1, \dots, x_n = y \in K$ and times $t_0, \dots, t_{n-1} > T$ such that $d(\psi(t_i, x_i), x_{i+1}) < \varepsilon$ for $i = 0, \dots, n-1$. It is easy to show that an equilibrium, a limit cycle, and an orbit homoclinic to an equilibrium x_0 together with x_0 are compact chain transitive sets, but they need not be maximal (with respect to inclusion).

The following result describes the behavior of control sets for small control ranges.

THEOREM 3.2. *Consider a family of control-affine systems of the form (3.4). Let $K \subset \mathbb{R}^d$ be a compact maximal chain transitive set for the flow of the uncontrolled system (3.2), assume that the accessibility rank condition (3.3) holds, and the following inner pair condition holds for all $(x, 0) \in K \times \mathcal{U}$: there is $T > 0$ with $\psi(T, x) = \varphi(T, x, 0) \in \mathrm{int}\mathcal{O}^+(x)$. Then there is an increasing family of control sets D^ρ of (3.4) with parameter $\rho > 0$ such that*

$$K \subset \mathrm{int} D^\rho \text{ and } K = \bigcap\nolimits_{\rho > 0} D^\rho.$$

If K is an asymptotically stable equilibrium or periodic orbit, then the control sets are invariant for $\rho > 0$, small enough.

Proof. The first assertion is proved in Colonius and Kliemann [2, Corollary 4.7.2]. The invariance of the control sets follows from [2, Corollary 4.1.13]. □

By [2, Proposition 4.5.19], the inner pair condition in $(x, 0)$ is satisfied, if for some $T > 0$ the following condition holds in $y = \varphi(T, x, 0)$:

$$\mathrm{span}\{f_0(y), \mathrm{ad}_{f_0}^k f_i(y) \,|\, i = 1, \dots, m, \ k = 0, 1, \dots\} = \mathbb{R}^d. \tag{3.5}$$

Here the ad-operator is given by iterated Lie brackets, $\mathrm{ad}_{f_0}^0 f_i = f_i$ and $\mathrm{ad}_{f_0}^{k+1} f_i = [f_0, \mathrm{ad}_{f_0}^k f_i]$ for $k \geq 0$.

Further results on the dependence of control sets, in particular, their boundaries, on the parameter ρ are given in Gayer [6].

Next we analyze the behavior of control sets under changes of an external parameter α. Consider the following family of control systems on $\mathbb{R}^d$ with $\alpha \in A \subset \mathbb{R}^k$,

$$\dot{x}(t) = f_0(\alpha, x(t)) + \sum_{i=1}^{m} u_i(t) f_i(\alpha, x(t)), \quad u \in \mathcal{U}, \tag{3.6}$$

with smooth maps $f_i : \mathbb{R}^k \times \mathbb{R}^d \to \mathbb{R}^d$, $i \in \{0, 1, \dots, m\}$. We assume that for every $\alpha \in A$ the system satisfies the assumptions on (3.1). For $x \in \mathbb{R}^d$ and $u \in \mathcal{U}$ the solutions are denoted by $\varphi^\alpha(t, x, u), t \in \mathbb{R}$.

The following theorem describes how the control sets change under parameter variation. Recall that a set-valued map $x \mapsto F(x)$ between metric spaces is lower semicontinuous at a point x_0 if for every open set O with $F(x_0) \cap O \neq \varnothing$ it follows that $F(x) \cap O \neq \varnothing$ for all x in a neighborhood of x_0; cf. Aubin and Frankowska [1, Definition 1.4.2].

Theorem 3.3. *For the family of systems (3.6) fix a parameter value $\alpha_0 \in \mathrm{int}A$. Assume that with α_0 the accessibility rank condition (3.3) is fulfilled and consider a control set D^{α_0} .*

(i) Let $K \subset \mathrm{int}D^{\alpha_0}$ be a compact set. Then there is $\delta_K > 0$ such that for all α with $\|\alpha - \alpha_0\| < \delta_K$ there is a unique control set D^α_K with $K \subset \mathrm{int}D^\alpha_K$ for system (3.6) with parameter value α.

(ii) There are $\delta_0 > 0$ and a unique family of control sets D^α for all α with $\|\alpha - \alpha_0\| < \delta_0$ with the following property: For every compact set $K \subset \mathrm{int}D^{\alpha_0}$ there is a $\delta_K \in (0, \delta_0)$ so that $K \subset \mathrm{int}D^\alpha$ for every α with $\|\alpha - \alpha_0\| < \delta_K$. The set-valued maps $\alpha \mapsto D^\alpha$ and $\alpha \mapsto \mathrm{cl}D^\alpha$ are lower semicontinuous at $\alpha = \alpha_0$.

This is a special case of Colonius and Lettau [4, Theorem 3.6] (in our case, the "worlds" $W^\alpha = \mathbb{R}^d$).

Remark 3.4. *The proof of Theorem 3.3(i) provides the following more precise information. Let $\varphi^{\alpha_0}(T, x, u) = y$ for $x, y \in K$. Then for every $\varepsilon > 0$ the trajectories of the system with parameter α satisfying $\varphi^\alpha(T^\alpha, x, u^\alpha) = y$ may be chosen with Hausdorff distance $d_H(\{\varphi^\alpha(t, x, u^\alpha) \,|t \in [0, T^\alpha]\}, \{\varphi^{\alpha_0}(t, x, u) \,|t \in [0, T]\}) < \varepsilon$ for $\|\alpha - \alpha_0\| < \delta$. Here δ may be chosen independently of $x, y \in K$.*

The following definition of local control sets replaces the global maximality property of control sets by a local property, cf. Colonius and Spadini [5, Definition 2.2] slightly generalized here.

Definition 3.5. *A bounded set $D_{loc} \subset \mathbb{R}^d$ is called a local control set of system (3.1) if there exists a neighborhood V of $\mathrm{cl}D_{loc}$ with the following properties: (i) for all $x \in D_{loc}$ there is a control $u \in \mathcal{U}$ such that $\varphi(t, x, u) \in D_{loc}$ for all $t \geq 0$, (ii) for all $x, y \in D_{loc}$ there exist $T > 0$ and a control u such that $\varphi(t, x, u) \in V$ for all $t \in [0, T]$ and $d(\varphi(T, x, u), y) < \varepsilon$, and (iii) D_{loc} is maximal with these properties.*

The results above for control sets remain valid for local control sets. In the proofs, one simply has to restrict the attention to the isolating neighborhood V of $\mathrm{cl}D_{loc}$.

The linearization of (3.4) in an equilibrium $(x_0, 0) \in \mathbb{R}^d \times \mathbb{R}^m$ with $0 = f_0(x_0)$ is the control system

$$\dot{y}(t) = Ay(t) + Bv(t) \text{ with } A := \frac{df_0(x_0)}{dx}, \quad B := [f_1(x_0), \ldots, f_m(x_0)]\,. \tag{3.7}$$

This system is controllable if and only if $\mathrm{rank}[B\ AB\ \ldots\ A^{d-1}B] = d$.

Local control sets with small control ranges satisfy the following uniqueness property, cf. [5, Theorem 5.1].

Theorem 3.6. *Consider for a family of control-affine systems of the form (3.4) a hyperbolic equilibrium x_0 of the uncontrolled system (3.2) and assume that the system linearized in $(x_0, 0) \in \mathbb{R}^d \times \mathbb{R}^m$ is controllable. Then there exist $\rho_0 > 0$ and $\delta_0 > 0$ such that for all $\rho \in (0, \rho_0)$ the ball $\mathbf{B}(x_0, \delta_0)$ contains exactly one local control set D^ρ_{loc} with nonvoid interior.*

4. Controllability near homoclinic bifurcations. In this section we analyze the control sets that occur near a homoclinic bifurcation of the uncontrolled system. We consider control-affine systems in $\mathbb{R}^2$ and $\mathbb{R}^3$ of the form

$$\dot{x}(t) = f_0(\alpha, x(t)) + \sum_{i=1}^m u_i(t) f_i(\alpha, x(t)), \quad u(t) \in U^\rho := \rho U, \tag{4.1}$$

where $\alpha \in A \subset \mathbb{R}$, $0 \in \mathrm{int}U$ with $U \subset \mathbb{R}^m$ compact and convex, and $\rho > 0$. We assume that for every $\alpha \in A$ the system satisfies the assumptions on (3.1).

These systems depend on the two parameters $(\alpha, \rho) \in A \times (0, \infty)$. The corresponding control sets will be denoted by $D^{\alpha,\rho}$ and an analogous notion is used for all other objects. The dependence of control sets on the parameter α is described in Theorem 3.3 and the dependence on ρ is described in Theorem 3.2. Throughout this section, the nominal parameter value α_0 will be taken as $\alpha_0 = 0$.

4.1. The planar case. The following theorem analyzes the control sets when the uncontrolled planar system undergoes a homoclinic bifurcation in the sense of Theorem 2.1, and hence for $\alpha_0 = 0$ it has an orbit Γ_0 homoclinic to a saddle equilibrium point $x_0 = 0$ with saddle quantity $\sigma_0 = \lambda_1(0) + \lambda_2(0) \neq 0$, a split function $\beta(\alpha)$ with $\beta'(0) \neq 0$, and bifurcating limit cycles L_β. Recall that we may write $\alpha = \alpha(\beta)$ for $|\beta|$ small enough and $\alpha(0) = 0$. We use the notation from Theorem 2.1 and, more explicitly, we assume that the limit cycle L_β exists for $0 < |\beta| < \overline{\beta}$ or, equivalently, for $0 < |\alpha| < \overline{\alpha} := \alpha(\overline{\beta})$. The following theorem shows that here two families of control sets are generated depending on the two parameters ρ and β.

THEOREM 4.1. *Consider a two-parameter family of control-affine systems in $\mathbb{R}^2$ of the form (4.1). Suppose that the uncontrolled and unperturbed system $\dot{x} = f_0(0, x)$ satisfies the assumptions of Theorem 2.1 and $\Gamma_0 \cup x_0$ is a maximal chain transitive set. Furthermore, assume that the accessibility rank condition (3.3) holds for $\alpha_0 = 0$ and the following inner pair condition holds for all β with $0 < |\beta| < \overline{\beta}$ and all $x \in \mathbb{R}^2$:*

$$\text{For all } \rho > 0 \text{ there is } T > 0 \text{ such that } \varphi^{\alpha(\beta)}(T, x, 0) \in \mathrm{int}\mathcal{O}^{\alpha(\beta),\rho,+}(x). \tag{4.2}$$

(i) Then there is a family of control sets $D_0^{\alpha(\beta),\rho}$, defined for $\rho > 0$ and $\beta \in (-\beta_0(\rho), \beta_0(\rho))$ with $\beta_0(\rho) \in (0, \overline{\beta})$, satisfying for all ρ and β

$$\Gamma_0 \cup x_0 \subset \mathrm{int}D_0^{\alpha(\beta),\rho} \text{ and } \Gamma_0 \cup x_0 = \bigcap\nolimits_{\rho>0} D_0^{0,\rho}. \tag{4.3}$$

(ii) If $\sigma_0 < 0$ there is a family of control sets $D_1^{\alpha(\beta),\rho}$, defined for $\rho > 0$ and $\beta \in (0, \overline{\beta})$, satisfying for all ρ and β

$$L_\beta \subset \mathrm{int}D_1^{\alpha(\beta),\rho} \text{ and } L_\beta = \bigcap\nolimits_{\rho>0} D_1^{\alpha(\beta),\rho}. \tag{4.4}$$

(iii) If $\sigma_0 > 0$ there is family of control sets $D_1^{\alpha(\beta),\rho}$ defined for $\rho > 0$ and $\beta \in (-\overline{\beta}, 0)$, such that (4.4) holds for all ρ and β.

Proof. (i): Since the set $\Gamma_0 \cup x_0$ is a maximal chain transitive set and the inner pair condition (4.2) holds, Theorem 3.2 shows that there is an increasing family of control sets $D_0^{0,\rho}, \rho > 0$, of (4.1) with $\alpha_0 = 0$ such that

$$\Gamma_0 \cup x_0 \subset \mathrm{int}D_0^{0,\rho} \text{ and } \Gamma_0 \cup x_0 = \bigcap\nolimits_{\rho>0} D_0^{0,\rho}.$$

Since the accessibility rank condition (3.3) holds for $\alpha_0 = 0$, Theorem 3.3 shows that for every $\rho > 0$ and some $\alpha_0(\rho) > 0$ there is a unique lower semicontinuous family of control sets $D_0^{\alpha,\rho}$ with parameters $|\alpha| < \alpha_0(\rho)$ containing $\Gamma_0 \cup x_0$ in the interior. With $\beta_0(\rho) = \beta(\alpha_0(\rho))$ assertion (i) follows.

(ii) and (iii)*:* By Theorem 2.1 there is a neighborhood U_0 of $\Gamma_0 \cup x_0$ in which a unique limit cycle $L_\beta, |\beta| \in (0, \overline{\beta})$, bifurcates from Γ_0. If $\sigma_0 < 0$ the limit cycle exists and is asymptotically stable for $\beta > 0$, and if $\sigma_0 > 0$ it exists and is unstable for $\beta < 0$. The limit cycle L_β is a maximal chain transitive set for the uncontrolled equation $\dot{x} = f_0(\alpha(\beta), x)$, hence Theorem 3.2 shows that for every limit cycle L_β there is an increasing family of control sets $D_1^{\alpha(\beta),\rho}, \rho > 0$, of (4.1) with (4.4). ∎

REMARK 4.2. *Theorem 2.1 does not yield any information about the behavior of the uncontrolled system outside of some neighborhood of the homoclinic orbit. Hence $\Gamma_0 \cup x_0$ may be a maximal chain transitive set only in an isolating neighborhood. In that case, the sets $D_0^{\alpha(\beta),\rho}$ will only be local control sets, cf. Definition 3.5.*

REMARK 4.3. *Theorem 4.1 shows that we may consider (β, ρ) as the parameters which determine the behavior of the control sets. In assertions (i)-(iii), for $i = 1, 2$ the maps $\rho \mapsto D_i^{\alpha(\beta),\rho}$ are increasing for every β and by Theorem 3.3 the maps $\beta \mapsto D_i^{\alpha(\beta),\rho}$ are lower semicontinuous for every ρ.*

REMARK 4.4. *Häckl and Schneider [10] consider control sets near a Takens-Bogdanov singularity, analytically and numerically, for*

$$\dot{x} = y, \ \dot{y} = \lambda_1 + \lambda_2 x + x^2 + xy + u(t), \quad u(t) \in [-\rho, \rho].$$

Here for all parameters $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ the bifurcation behavior of the uncontrolled equation is known. For parameters (λ_1, λ_2) in a subset $k_S \subset \mathbb{R}^2$ a homoclinic bifurcation occurs and one obtains a control set containing the homoclinic orbit, cf. [10, Figure 4] (and an invariant control set around the asymptotically stable focus surrounded by the homoclinic orbit). For (λ_1, λ_2) in $C \subset \mathbb{R}^2$ an unstable periodic orbit has bifurcated from the homoclinic orbit. It is contained in a variant control set, cf. [10, Figure 2].

REMARK 4.5. *While in Theorem 4.1 the homoclinic orbit vanishes for $\beta \neq 0$, the implicit function theorem implies that there are hyperbolic equilibria $x_{\alpha(\beta)}$ for the uncontrolled system which depend continuously on β. The local behavior near these equilibria will play a certain role in the proof of Theorem 4.9.*

REMARK 4.6. *The Index Theorem (see Wiggins [18, Corollary 6.0.2]) implies that inside any limit cycle L_β of the uncontrolled system there is at least one fixed point $x_2^{\alpha(\beta)}$. Under the inner pair condition, one finds by Theorem 3.2 control sets with $x_2^{\alpha(\beta)} \in \mathrm{int} D_2^{\alpha(\beta),\rho}$.*

Theorem 4.1 does not answer the question, when the control sets $D_0^{\alpha(\beta),\rho}$ and $D_1^{\alpha(\beta),\rho}$ coincide, in their common range of definition. The following corollary shows, in particular, how this equality depends on the relation between the parameters β and ρ. For simplicity we suppose that $\sigma_0 < 0$. Thus both control sets $D_0^{\alpha(\beta),\rho}$ and $D_1^{\alpha(\beta),\rho}$ exist for $\rho > 0$ and $\beta \in (0, \beta_0(\rho))$.

COROLLARY 4.7. *Let the assumptions of Theorem 4.1 be satisfied and assume that $\sigma_0 < 0$.*

(i) For every $\beta \in (0, \overline{\beta})$ there is $\rho_1(\beta) > 0$ such that for all $\rho \in (0, \rho_1(\beta)]$ the set $D_1^{\alpha(\beta),\rho}$ is an invariant control set.

(ii) For every $\rho > 0$ there is $\underline{\beta}(\rho) > 0$ such that for all $\beta \in (0, \underline{\beta}(\rho))$ the control sets coincide, $D_0^{\alpha(\beta),\rho} = D_1^{\alpha(\beta),\rho}$.

(iii) If $D_0^{\alpha(\beta),\rho}$ is a variant control set for some $\beta \in (0, \overline{\beta}), \rho \in (0, \rho_1(\beta)]$, then $D_0^{\alpha(\beta),\rho} \neq D_1^{\alpha(\beta),\rho}$.

Proof. (i) This follows from Theorem 3.2, since the periodic orbits L_β are asymptotically stable.

(ii) Fix $\rho > 0$. Since $\Gamma_0 \cup x_0 \subset \mathrm{int} D_0^{0,\rho}$ and the periodic orbits $L_\beta, \beta > 0$, bifurcate from this homoclinic orbit, it follows that there is $\beta'(\rho) > 0$ such that $L_\beta \subset \mathrm{int} D_0^{0,\rho}$

for all $\beta \in (0, \beta'(\rho)]$. Define a compact set $K \subset D_0^{0,\rho}$ by

$$K := (\Gamma_0 \cup x_0) \cup \bigcup_{\beta \in (0,\beta'(\rho)]} L_\beta.$$

By Theorem 3.3 it follows that there is $\beta''(\rho) \in (0, \beta_0(\rho))$ such that for all $\beta \in (0, \beta''(\rho))$ the inclusion $K \subset \mathrm{int} D_0^{\alpha(\beta),\rho}$ holds. Thus for $\underline{\beta}(\rho) := \min\{\beta'(\rho), \beta''(\rho)\}$ it follows that

$$L_\beta \subset \mathrm{int} D_0^{\alpha(\beta),\rho} \text{ and hence } D_0^{\alpha(\beta),\rho} = D_1^{\alpha(\beta),\rho} \text{ for all } \beta \in (0, \underline{\beta}(\rho)).$$

(iii) By assertion (i) $D_1^{\alpha(\beta),\rho}$ is an invariant control set, hence it cannot coincide with the variant control set $D_0^{\alpha(\beta),\rho}$. □

Remark 4.8. *Corollary 4.7 reveals the subtle relation between the size of the control range determined by ρ and the bifurcation parameter β. In assertion (i), the control set $D_1^{\alpha(\beta),\rho}$ around the asymptotically stable periodic orbit L_β is invariant for small $\rho > 0$; here one will expect $\rho_1(\beta) \to 0$ for $\beta \to 0$. In assertion (ii), $\rho > 0$ is fixed and the homoclinic orbit is contained in the interior of the control set $D_0^{0,\rho}$. Since $L_\beta \to \Gamma_0 \cup x_0$ for $\beta \to 0$, it follows that $\underline{L_\beta} \subset D_0^{0,\rho}$ for β small enough; here $\underline{\beta}(\rho) \to 0$ for $\rho \to 0$. In assertion (iii), $\beta \in (0, \overline{\beta}), \rho \in (0, \rho_1(\beta)]$ is small enough to guarantee by (i) that $D_1^{\alpha(\beta),\rho}$ is invariant. If $D_0^{\alpha(\beta),\rho}$ is variant, this implies that the control sets cannot coincide. In view of (ii), this can only happen if $\beta \geq \underline{\beta}(\rho)$, hence ρ must be small enough. The assumption that $D_0^{\alpha(\beta),\rho}$ is variant appears to be mild, since the hyperbolic equilibrium satisfies $x_0 \in \mathrm{int} D_0^{\alpha(\beta),\rho}$ for $\rho > 0$ and $\beta \in [0, \beta_0(\rho))$, and hence all points on the unstable manifold of x_0 can be reached from $D_0^{\alpha(\beta),\rho}$. See the example in Section 5 for an illustration.*

The next theorem shows that the qualitative behavior of the control system can be different from the behavior of the uncontrolled system. More precisely, we find parameter regions where there is no homoclinic orbit and no limit cycle for the uncontrolled system while there exist periodic orbits of the control system which are arbitrarily close to the homoclinic orbit.

Theorem 4.9. *Let the assumptions of Theorem 4.1 be satisfied and assume, additionally, that the system with $\alpha_0 = 0$ linearized in $(0,0) \in \mathbb{R}^d \times \mathbb{R}^m$ is controllable. Then there is a neighborhood U_1 of the homoclinic orbit $\Gamma_0 \cup x_0$ such that for every $\delta > 0$ there are a nonvoid parameter region $A \subset \mathbb{R}$ and $\rho_0 > 0$ such that*

(i) for $\alpha \in A$ and $\rho \in (0, \rho_0)$ there are periodic orbits $\varphi^\alpha(\cdot, y, u) \subset D_0^{\alpha,\rho}, u \in \mathcal{U}^\rho$, with Hausdorff distance $d_H(\varphi^\alpha(\cdot, y, u), \Gamma_0 \cup x_0) < \delta$;

(ii) for $\alpha \in A$ the uncontrolled system $\dot{x} = f_0(\alpha, x)$ has no homoclinic orbit or periodic solution in U_1 except for the hyperbolic equilibrium x_α.

Proof. Recall that the hyperbolic equilibrium x_0 yields for α near 0 hyperbolic equilibria x_α which depend continuously on α. Suppose first that $\sigma_0 < 0$. Theorem 2.1 shows that in a neighborhood U_0 of $\Gamma_0 \cup x_0$ for $|\alpha|$ small enough a unique limit cycle $L_{\beta(\alpha)}$ bifurcates from Γ_0. It exists if and only if $\beta(\alpha) > 0$. We will consider α with $\beta(\alpha) < 0$, hence the uncontrolled equation $\dot{x} = f_0(\alpha, x)$ has no periodic solution except for the equilibrium x_α and assertion (ii) holds. For the proof of (i) it is convenient to suppose that $\alpha(\beta) > 0$ for $\beta < 0$ (otherwise, we replace α by $-\alpha$).

By Theorem 4.1(i) with $\beta = 0$ there is for $\rho > 0$ a control set $D_0^{0,\rho}$ satisfying

$$\Gamma_0 \cup x_0 \subset \mathrm{int} D_0^{0,\rho} \text{ and } \Gamma_0 \cup x_0 = \bigcap_{\rho > 0} D_0^{0,\rho}.$$

By Theorem 3.6 there are unique local control sets $D_{loc}^{0,\rho}$ such that the equilibrium x_0 of the uncontrolled system satisfies

$$x_0 \in \mathrm{int} D_{loc}^{0,\rho} \text{ and } \{x_0\} = \bigcap\nolimits_{\rho>0} D_{loc}^{0,\rho}.$$

We can choose $\rho > 0$ so small that

$$\sup\left\{ \|z - x_0\| \,\middle|\, z \in D_{loc}^{0,\rho} \right\} < \delta. \tag{4.5}$$

Since $D_{loc}^{0,\rho} \cap D_0^{0,\rho} \neq \varnothing$, it follows that $D_{loc}^{0,\rho} \subset D_0^{0,\rho}$ and for $\delta > 0$, small enough, $D_{loc}^{0,\rho} \neq D_0^{0,\rho}$.

Let $y \in \Gamma_0 \cap \mathrm{int} D_{loc}^{0,\rho}$. Since $\varphi^0(t, y, 0) \to x_0 \in \mathrm{int} D_{loc}^{0,\rho}$ for $t \to \pm\infty$, there is $T > 0$ such that the homoclinic trajectory satisfies $\varphi^0(t, y, 0) \in \mathrm{int} D_{loc}^{0,\rho}$ for all $|t| \geq T$ and $\varphi^0(\tau, y, 0)$ is not in the isolating neighborhood of $D_{loc}^{0,\rho}$ for some $\tau \in (0, T)$. By continuous dependence of the solution on the parameter α, there is $\alpha_0(\rho) > 0$ such that $\varphi^\alpha(T, y, 0) \in \mathrm{int} D_{loc}^{0,\rho}$ for all $\alpha \in (0, \alpha_0(\rho)]$. Choose $\alpha_0(\rho)$ small enough such that the Hausdorff distance

$$d_H\left(\{\varphi^\alpha(t, y, 0) \,|\, t \in [0, T]\}, \Gamma_0 \cup x_0\right) < \delta \text{ for } \alpha \in (0, \alpha_0(\rho)], \tag{4.6}$$

where we use $\left\{\varphi^0(t, y, 0) \,|\, t \in \mathbb{R}\right\} = \Gamma_0$. The compact set

$$K := \{y\} \cup \{\varphi^\alpha(T, y, 0) \,|\, \alpha \in [0, \alpha_0(\rho)]\}$$

is contained in $\mathrm{int} D_{loc}^{0,\rho}$. Theorem 3.3 applied to local control sets implies that there is $\alpha_1(\rho) \in (0, \alpha_0(\rho)]$ such that for all $\alpha \in [0, \alpha_1(\rho)]$

$$\{y\} \cup \{\varphi^\alpha(T, y, 0) \,|\, \alpha \in [0, \alpha_1(\rho)]\} \subset K \subset \mathrm{int} D_{loc}^{\alpha,\rho}.$$

There are a control $u^0 \in \mathcal{U}^\rho$ and a time $T^0 > 0$ such that $\varphi^0(T^0, \varphi^\alpha(T, y, 0), u^0) = y$. Then Remark 3.4 implies that one may choose $\alpha_2(\rho) \in (0, \alpha_1(\rho)]$ such that for all $\alpha \in (0, \alpha_2(\rho))$ there are $u^\alpha \in \mathcal{U}^\rho$ and $T^\alpha > 0$ with $\varphi^\alpha(T^\alpha, \varphi^\alpha(T, y, 0), u^\alpha) = y$ and trajectories $\varphi^\alpha(t, \varphi^\alpha(T, y, 0), u^\alpha)$, $t \in [0, T^\alpha]$, arbitrarily close to $\varphi^0(t, \varphi^\alpha(T, y, 0), u^0), t \in [0, T^0]$, hence contained in $D_{loc}^{0,\rho}$. By (4.5) and (4.6) this implies that the Hausdorff distance of the resulting controlled periodic orbit to $\Gamma_0 \cup x_0$ is smaller than δ, hence assertion (i) holds in the case $\sigma_0 < 0$..

For $\sigma_0 > 0$ consider α with $\beta(\alpha) > 0$, where the uncontrolled equation $\dot{x} = f_0(\alpha, x)$ has no periodic solution except for the equilibrium x_α. Then the assertion is proved analogously. □

REMARK 4.10. *In their analysis of the Takens-Bogdanov equation, Häckl and Schneider [10, Theorem 4.7] prove that there exist parameter values and control ranges such that the control system has an at least doubly connected control set while for all constant controls only equilibrium points exist as limit sets. Here they use that the control directly affects the bifurcation parameter* λ_1.

4.2. The three-dimensional case. The following theorems analyze the control sets in $\mathbb{R}^3$ when the uncontrolled system undergoes a homoclinic bifurcation in the situation of Theorem 2.2 and Theorem 2.3.

If the uncontrolled system $\dot{x} = f_0(0, x)$ satisfies the hypotheses of Theorem 2.2(i), it has an orbit Γ_0 homoclinic to a saddle equilibrium point $x_0 = 0$, and $\dot{x} = f_0(\alpha, x)$ undergoes a homoclinic bifurcation with saddle quantity $\sigma_0 = \lambda_1(0) + \lambda_2(0) < 0$, a split function $\beta(\alpha)$ with $\beta'(0) \neq 0$, and bifurcating unique and asymptotically stable limit cycles L_β defined for $0 < |\beta| < \overline{\beta}$ and we may write $\alpha = \alpha(\beta)$. We use the notation from Theorem 2.2.

THEOREM 4.11. *Consider a family of control-affine systems in $\mathbb{R}^3$ of the form (4.1) and suppose that the accessibility rank condition (3.3) holds for $\alpha_0 = 0$ and that the control system satisfies the inner pair condition (4.2) for all $x \in \mathbb{R}^3$. Assume that the uncontrolled system $\dot{x} = f_0(0, x)$ has an orbit Γ_0 homoclinic to a saddle $x_0 = 0$ with real eigenvalues $\lambda_1(0) > 0 > \lambda_2(0) > \lambda_3(0)$, that $\Gamma_0 \cup x_0$ is a maximal chain transitive set and the assumptions of Theorem 2.2(i) are satisfied.*

(i) Then there is a family of control sets $D_0^{\alpha(\beta),\rho}$, defined for $\rho > 0$ and $\beta \in (-\beta_0(\rho), \beta_0(\rho))$ with $\beta_0(\rho) \in (0, \overline{\beta})$, satisfying for all ρ and β

$$\Gamma_0 \cup x_0 \subset \mathrm{int} D_0^{\alpha(\beta),\rho} \text{ and } \Gamma_0 \cup x_0 = \bigcap\nolimits_{\rho>0} D_0^{0,\rho}. \tag{4.7}$$

(ii) There is a family of control sets $D_1^{\alpha(\beta),\rho}$, defined for $\rho > 0$ and $\beta \in (0, \overline{\beta})$, satisfying for all ρ and β

$$L_\beta \subset \mathrm{int} D_1^{\alpha(\beta),\rho} \text{ and } L_\beta = \bigcap\nolimits_{\rho>0} D_1^{\alpha(\beta),\rho}. \tag{4.8}$$

Furthermore, for every $\beta \in (0, \overline{\beta})$ there is $\rho_1(\beta)$ such that for every $\rho \in (0, \rho_1(\beta)]$ the set $D_1^{\alpha(\beta),\rho}$ is an invariant control set.

Proof. (i) The set $\Gamma_0 \cup x_0$ is a maximal chain transitive set for the uncontrolled equation $\dot{x} = f_0(0, x)$. Theorem 3.2 shows that there is an increasing family of control sets $D_0^{0,\rho}$, $\rho > 0$, of (4.1) with $\alpha_0 = 0$ such that

$$\Gamma_0 \cup x_0 \subset \mathrm{int} D_0^{0,\rho} \text{ and } \Gamma_0 \cup x_0 = \bigcap\nolimits_{\rho>0} D_0^{0,\rho}.$$

Theorem 3.3 shows that for every $\rho > 0$ and some $\alpha_0(\rho) > 0$ there is a unique lower semicontinuous family of control sets $D_0^{\alpha,\rho}$ with parameters $|\alpha| < \alpha_0(\rho)$ containing $\Gamma_0 \cup x_0$ in the interior. With $\beta_0(\rho) = \beta(\alpha_0(\rho))$ assertion (i) follows.

(ii) By Theorem 2.2(i) there is a neighborhood U_0 of $\Gamma_0 \cup x_0$ in which a unique and asymptotically stable limit cycle $L_\beta, \beta \in (0, \overline{\beta})$, bifurcates from Γ_0. The limit cycle L_β is a maximal chain transitive set for the uncontrolled equation $\dot{x} = f_0(\alpha(\beta), x)$, hence Theorem 3.2 shows that for every limit cycle L_β there is an increasing family of control sets $D_1^{\alpha(\beta),\rho}$, $\rho > 0$, of (4.1) with

$$L_\beta \subset \mathrm{int} D_1^{\alpha(\beta),\rho} \text{ and } L_\beta = \bigcap\nolimits_{\rho>0} D_1^{\alpha(\beta),\rho}.$$

Theorem 3.2 shows that the control sets $D_1^{\alpha(\beta),\rho}$ containing the asymptotically stable limit cycle L_β are invariant for $\rho > 0$, small enough. □

Similarly one obtains the following result if the assumptions of Theorem 2.2(ii), in particular, $\sigma_0 > 0$, are satisfied, and hence a unique saddle limit cycle L_β bifurcates from the homoclinic orbit Γ_0 in a neighborhood U_0 of $\Gamma_0 \cup x_0$.

THEOREM 4.12. *In the situation of Theorem 4.11 suppose that the uncontrolled system $\dot{x} = f_0(0, x)$ satisfies the assumptions of Theorem 2.2(ii).*

(i) Then there is a family of control sets $D_0^{\alpha(\beta),\rho}$ defined for $\rho > 0$ and $\beta \in (-\beta_0(\rho), \beta_0(\rho))$ with $\beta_0(\rho) \in (0, \overline{\beta})$ such that (4.7) holds for all ρ and β.

(ii) If Γ_0 is simple, there is a family of control sets $D_1^{\alpha(\beta),\rho}$ defined for $\rho > 0$ and $\beta \in (-\overline{\beta}, 0)$ such that (4.8) holds for all ρ and β.

(iii) If Γ_0 is twisted, there is a family of control sets $D_1^{\alpha(\beta),\rho}$ defined for $\rho > 0$ and $\beta \in (0, \overline{\beta})$ such that (4.8) holds for all ρ and β.

Proof. The proof of this theorem follows the same steps as the proof of Theorem 4.11. One has to use that for simple Γ_0 the bifurcating limit cycles L_β exist for $\beta < 0$ and for twisted Γ_0 they exist for $\beta > 0$. ◻

The remarkable result here is that the direction of bifurcation for the control sets $D_1^{\alpha(\beta),\rho}$ depends on topological property if the stable manifold W^s of x_0 is simple or twisted.

Finally, we obtain the following result for a homoclinic bifurcation of a saddle-focus with $\sigma_0 < 0$ as described in Theorem 2.3.

THEOREM 4.13. *Consider a family of control-affine systems in $\mathbb{R}^3$ of the form (4.1) and suppose that the accessibility rank condition (3.3) holds for $\alpha_0 = 0$ and that the control system satisfies the inner pair condition (4.2) for all $x \in \mathbb{R}^3$. Assume that the uncontrolled system $\dot{x} = f_0(0, x)$ has an orbit Γ_0 homoclinic to a saddle-focus $x_0 = 0$ satisfying the assumptions of Theorem 2.3, and $\Gamma_0 \cup x_0$ is a maximal chain transitive set.*

(i) Then there is a family of control sets $D_0^{\alpha(\beta),\rho}$ defined for $\rho > 0$ and $\beta \in (-\beta_0(\rho), \beta_0(\rho))$ with $\beta_0(\rho) \in (0, \overline{\beta})$ such that (4.7) holds for all ρ and β.

(ii) There is a family of control sets $D_1^{\alpha(\beta),\rho}$ defined for $\rho > 0$ and $\beta \in (0, \overline{\beta})$ such that (4.8) holds for all ρ and β. Furthermore, for every $\beta \in (0, \overline{\beta})$ there is $\rho_1(\beta)$ such that for every $\rho \in (0, \rho_1(\beta)]$ the set $D_1^{\alpha(\beta),\rho}$ is an invariant control set.

Proof. The proof of this theorem follows the same steps as the proof of Theorem 4.11.One also has to use that the bifurcating limit cycles $L_\beta, \beta < 0$, are asymptotically stable, hence in assertion (ii) one obtains invariant control sets. ◻

REMARK 4.14. *In all situations analyzed in Theorems 4.11, Theorem 4.12, and Theorem 4.13, one can obtain results analogous to Corollary 4.7, and to Theorem 4.9 on the existence of controlled homoclinic orbits in parameter regions where no periodic solutions exist for the uncontrolled system. This holds, since the corresponding proofs do not use that the dimension of the state space is two.*

5. An example. Consider the following planar control system

$$\begin{aligned} \dot{x} &= -x + 2y + x^2 \\ \dot{y} &= (2 - \alpha)x - y - 3x^2 + \frac{3}{2}xy + u(t) \end{aligned} \tag{5.1}$$

with $u(t) \in U = [-\rho, \rho]$, $\rho > 0$. This is a special case of (4.1) with

$$f_0(\alpha, x, y) = \begin{bmatrix} -x + 2y + x^2 \\ (2 - \alpha)x - y - 3x^2 + \frac{3}{2}xy \end{bmatrix}, \quad f_1(x, y) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For $u = 0$ one obtains Sandstede's example of a homoclinic bifurcation, cf. Sandstede [15], Kuznetsov [13, Example 6.1]. For an application of Theorem 4.1, we first check that this uncontrolled system suffers a homoclinic bifurcation according to Theorem 2.1.

The origin $(x_0, y_0) = (0, 0)$ is an equilibrium for all α and it is a saddle for sufficiently small $|\alpha|$. For $\alpha_0 = 0$ one obtains $\sigma_0 = \lambda_1(0) + \lambda_2(0) = 1 - 3 = -2 < 0$. One can show that there is a homoclinic orbit contained in the set of all (x, y) with

$$x^2(1 - x) - y^2 = 0,$$

hence $y = \pm x\sqrt{1-x}$ for all points (x,y) on the homoclinic orbit. As noted above, the condition $\beta'(0) \neq 0$ is equivalent to the Melnikov condition (2.2), which here has the form

$$M_{\alpha_0}(0) = -\int_{-\infty}^{\infty} \exp\left[-\int_0^t \left(-2 + \frac{7}{2}x\right) d\tau\right] x\dot{x}dt.$$

Write the first component of the homoclinic trajectory for $y > 0$ as $x(t) = x^+(t)$ and for $y < 0$ as $x(t) = x^-(t)$. Thus the equation for $x(\cdot)$ can be written as

$$\dot{x}^+ = x(x - 1 + 2\sqrt{1-x}) > 0, \ \dot{x}^- = -x(x - 1 - 2\sqrt{1-x}) < 0.$$

Observe that $(x_1, y_1) = (1, 0)$ is on the homoclinic orbit and we may suppose that the homoclinic solution satisfies $x(0) = x_1 = 1$. Define $h(t) = \exp\left[-\int_0^t \left(-2 + \frac{7}{2}x\right) d\tau\right]$, $t \in \mathbb{R}$. Then the integral for $M_\alpha(0)$ can be written as

$$\int_{-\infty}^0 h(t)x\dot{x}dt + \int_0^\infty h(t)x\dot{x}dt = \int_0^1 h(t^+(x^+))x^+dx^+ + \int_0^1 h(t^-(x^-))x^-dx^- > 0,$$

showing that $M_\alpha(0) \neq 0$. Hence Theorem 2.1 implies that the uncontrolled equation has an asymptotically stable limit cycle for $\alpha > 0$.

Next we check the assumptions of Theorem 4.1. One computes for $\alpha_0 = 0$

$$ad_{f_0}f_1(x,y) = [f_0, f_1](x,y) = -\begin{bmatrix} \frac{\partial f_{01}}{\partial x} & \frac{\partial f_{01}}{\partial y} \\ \frac{\partial f_{02}}{\partial x} & \frac{\partial f_{02}}{\partial y} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 2 \\ -1 + \frac{3}{2}x \end{bmatrix},$$

where $f_0 = (f_{01}, f_{02})^\top$. One finds that $f_1(x,y) = (0,1)^\top$ and $ad_{f_0}f_1(x,y)$ are linearly independent for all α. Thus condition (3.5) holds implying the inner pair condition (4.2) and the accessibility rank condition (3.3) for all $(x,y) \in \mathbb{R}^2$. Furthermore, also the controllability condition in Theorem 4.9 holds, since the control system with $\alpha_0 = 0$ linearized in $(x,y) = (0,0), u = 0$ is controllable,

$$A := \left[\frac{\partial f_0(\alpha_0,0,0)}{\partial x}, \frac{\partial f_0(\alpha_0,0,0)}{\partial y}\right] = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}, \quad B := f_1(0,0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

hence $\operatorname{rank}[B, AB] = \operatorname{rank}\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} = 2.$

Theorem 4.1 implies that for $\rho > 0$ there is $\beta_0(\rho) > 0$ such that there is a family of control sets $D_0^{\alpha(\beta),\rho}$ satisfying for all $\rho > 0$ and $\beta \in (-\beta_0(\rho), \beta_0(\rho))$ assertion (4.3). Since $\sigma_0 < 0$ it also follows that there is a family of control sets $D_1^{\alpha(\beta),\rho}$ such that for all $\rho > 0$ and $\beta \in (0, \overline{\beta})$ assertion (4.4) holds. Corollary 4.7 shows that for every $\beta \in (0, \overline{\beta})$ there is $\rho_1(\beta)$ such that $D_1^{\alpha(\beta),\rho}$ is an invariant control set for $\rho \in (0, \rho_1(\beta)]$. For $\alpha_0 = 0, u = 0$ one finds the unique equilibrium $(\bar{x}, \bar{y}) = (\frac{2}{3}, \frac{1}{9})$ in the interior of the region bounded by the homoclinic orbit, cf. Remark 4.6. It is an unstable focus, which for $\rho > 0$ small enough is contained in the interior of an open control set, since one can check the inner pair condition (4.2).

Figures 1 – 5 indicate phase portraits of the uncontrolled systems and present numerical approximations of the control sets. These results are based on Häckl's algorithm for the computation of reachable and controllable sets, cf. Häckl [9], Colonius and Kliemann [2, Appendix C]. A reachable set is the union of all solutions from an initial state corresponding to admissible control functions. The solutions are approximated via discrete-time systems (obtained here by a Runge-Kutta method RK5(4)). A space discretization via a grid (here of size 150×150 cells) allows us to keep track of those cells that have already been reached by some computed solution. The controllable sets are obtained via time reversal. The implementation of Häckl's algorithm is based on MATLAB.

For the parameter values $\alpha_0 = 0$ and $\rho = 0.01$, Figure 1 shows approximations of the control sets $D_0^{\alpha_0,\rho}$ around the homoclinic orbit Γ_0 and $D_2^{\alpha_0,\rho}$ around the unstable focus. The control set $D_0^{\alpha_0,\rho}$ is obtained by the intersection of the reachable and controllable sets,

$$D_0^{\alpha_0,\rho} = \mathrm{cl}\mathcal{O}^+(x_1, y_1) \cap \mathcal{O}^-(x_2, y_2) \text{ for any } (x_1, y_1), (x_2, y_2) \in \Gamma_0 \subset \mathrm{int} D_1^{\alpha_0,\rho}.$$

Hence an approximation of $D_0^{\alpha,\rho}$ is obtained by the intersection of numerical approximations for the reachable and controllable sets. The control set $D_2^{\alpha_0,\rho}$ around the unstable focus $(\bar{x}, \bar{y})$ is computed as the controllable set $\mathcal{O}^-(\bar{x}, \bar{y})$.

For $\alpha = -0.017241$, $\rho = 0.01$, Figure 2 shows the control set $D_0^{\alpha,\rho}$ around Γ_0 and the control set $D_2^{\alpha,\rho}$ around the unstable focus. For this negative α-value, no periodic orbit has bifurcated from the homoclinic orbit of the uncontrolled equation. This illustrates Theorem 4.9. For $\alpha = 0.01$, $\rho = 0.01$, Figure 3 shows the control set $D_2^{\alpha,\rho}$ around the unstable focus, and the control set $D_0^{\alpha,\rho} = D_1^{\alpha,\rho}$ containing the homoclinic orbit Γ_0 and the periodic orbit $L_{\beta(\alpha)}$, cf. Corollary 4.7(ii). For $\alpha = 0.03$, $\rho = 0.01$ Figure 4 shows the control set $D_2^{\alpha,\rho}$ around the unstable focus, the control set $D_0^{\alpha,\rho}$ containing Γ_0, and the invariant control set $D_1^{\alpha,\rho}$ containing the stable periodic orbit $L_{\beta(\alpha)}$ computed as the reachable set. This illustrates Corollary 4.7(iii). Finally, for $\alpha = 0.07$, $\rho = 0.01$, Figure 5 shows the invariant control set $D_1^{\alpha,\rho}$ around the stable periodic orbit $L_{\beta(\alpha)}$, the control set $D_2^{\alpha,\rho}$ around the unstable focus, and the control set $D_0^{\alpha,\rho}$ which has collapsed to a control set around the saddle close to the local control sets $D_{loc}^{0,\rho}$ used in the proof of Theorem 4.9.

6. Conclusions and open problems. Our results show, in particular, that sometimes a homoclinic bifurcation may lead to invariant control sets (for $d = 2$ this is the case in Corollary 4.7 and for $d = 3$ in Theorem 4.11 and Theorem 4.13). Invariant control sets are also of interest beyond control and deterministic perturbations, since they are the supports of invariant densities for associated Markov diffusion processes (cf. Kliemann [12]). We did not include the case of a bifurcation in $\mathbb{R}^3$ for an orbit homoclinic to a saddle-focus with saddle quantity $\sigma_0 > 0$. This bifurcation results in an infinite number of saddle limit cycles, cf. Kuznetsov [13, Theorem 6.6]. It certainly would be of great interest to study the controllability properties in this situation and also for general system in $\mathbb{R}^d$.

REFERENCES

[1] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, 1990.
[2] F. Colonius and W. Kliemann, *The Dynamics of Control*, Birkhäuser 2000.
[3] F. Colonius, E. Kreuzer, A. Marquardt and W. Sichermann, *A numerical study of capsizing: comparing control set analysis and Melnikov's method,* International Journal of Bifurcation and Chaos, 18(5) (2008), pp. 1503–1514.
[4] F. Colonius and R. Lettau, *Relative controllability properties*, IMA Journal of Mathematical Control and Information, 33 (2016), pp. 701-722.
[5] F. Colonius and M. Spadini, *Uniqueness of local control sets,* J. Dynamical and Control Systems, 9(4) (2003), pp. 513-530.

[6] T. Gayer, *Control sets and their boundaries under parameter variation,* J. Differential Equations, 201 (2004), pp. 177-200.

[7] ———, *Controllability and invariance properties of time-periodic systems,* International Journal of Bifurcation and Chaos, 15(4) (2005), pp. 1361–1375.

[8] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer Science + Business Media, New York, 1983.

[9] G. Häckl, *Numerical approximation of reachable sets and control sets,* Random and Computational Dynamics, 1(4) (1992-93), pp. 371-394.

[10] G. Häckl and K.R. Schneider, *Controllability near Takens-Bogdanov points,* J. Dynamical and Control Systems, 2(4) (1996), pp. 583-598.

[11] C. Kawan and A. Da Silva, *Invariance entropy of hyperbolic control sets*, Discrete Contin. Dynam. Systems, 30(1) (2016), pp. 97-136.

[12] W. Kliemann, *Recurrence and invariant measures for degenerate diffusions*, Ann. Probab., 15 (1987), pp. 690–707.

[13] Y.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, 2nd ed., Springer-Verlag, New York, 1998.

[14] J.S.W. Lamb, M. Rasmussen, and C.S. Rodrigues, *Topological bifurcations of minimal invariant sets for set-valued dynamical systems,* Proc. Amer. Math. Soc., 143(9) (2015), pp. 3927-3937.

[15] B. Sandstede, *Constructing dynamical systems having homoclinic bifurcation points of codimension two*, J. Dynamics Differential Equations 9 (1997), pp. 269–288.

[16] E. Sontag, *Mathematical Control Theory*, 2nd ed., Springer-Verlag 1998.

[17] D. Szolnoki, *Set oriented methods for computing reachable sets and control sets,* Discrete Contin. Dynam. Systems–B, 3 (2003), pp. 361–382.

[18] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos,* 2nd ed., Springer-Verlag, 2003.

[19] ———, *Global Bifurcations and Chaos. Analytical Methods,* Springer-Verlag 1988.

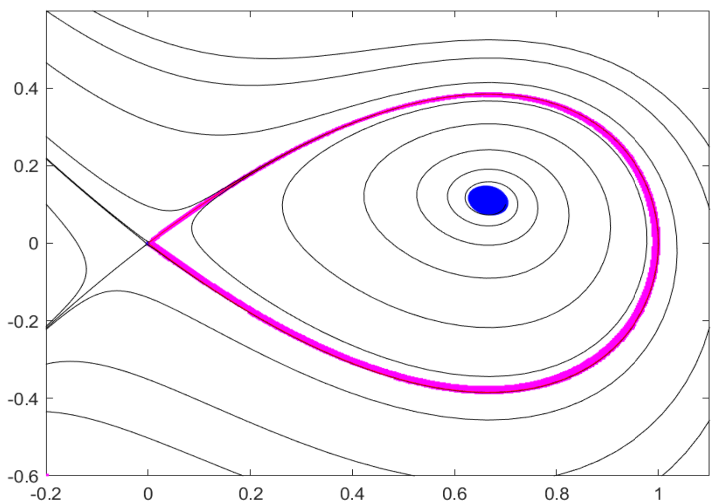

Fig. 1: Phase portrait and control sets D_0 around the homoclinic orbit Γ_0 and D_2 around the unstable focus for $\alpha_0 = 0.0$ and $\rho = 0.01$

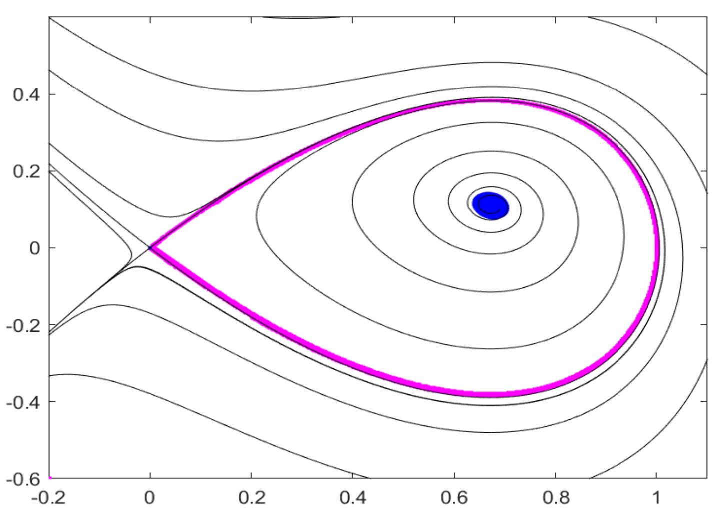

Fig. 2: Phase portrait and control sets D_0 around the homoclinic orbit Γ_0 and D_2 around the unstable focus for $\alpha = -0.017241,\ \rho = 0.01$

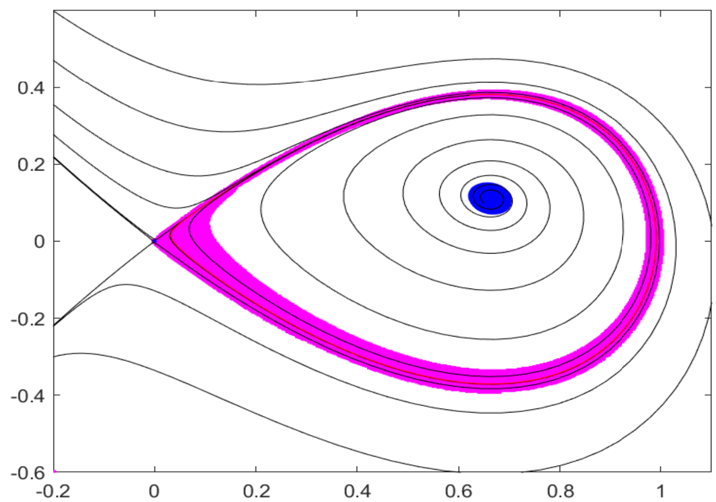

Fig. 3: Phase portrait and control sets $D_0 = D_1$ around Γ_0 and the periodic orbit, and D_2 around the unstable focus for $\alpha = 0.01,\ \rho = 0.01$

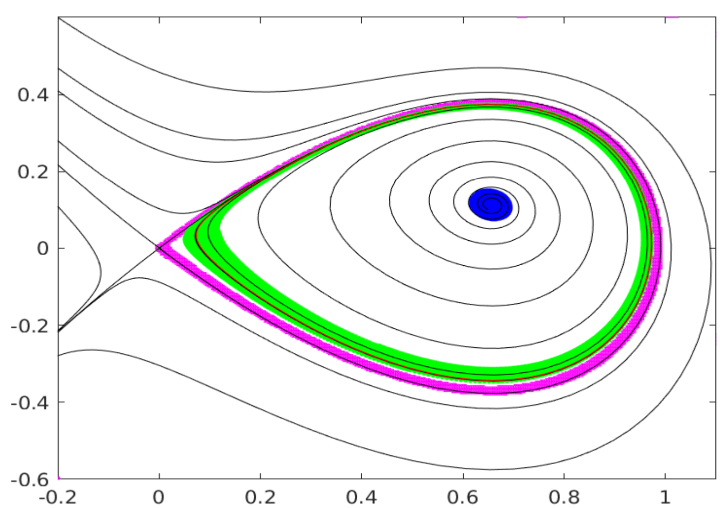

Fig. 4: Phase portrait and control sets D_0 around Γ_0, D_1 around the periodic orbit, and D_2 around the unstable focus for $\alpha = 0.03,\ \rho = 0.01$

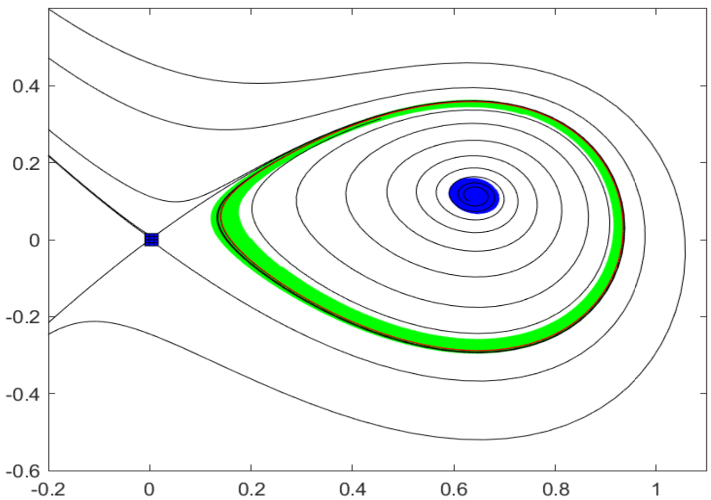

Fig. 5: Phase portrait and control sets D_0 around the saddle, D_1 around the periodic orbit, and D_2 around the unstable focus for $\alpha = 0.07,\ \rho = 0.01$