RELATION ALGEBRAS OF SUGIHARA, BELNAP, MEYER, AND CHURCH

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Abstract. Algebras introduced by, or attributed to, Sugihara, Belnap, Meyer, and Church are representable as algebras of binary relations with set-theoretically defined operations. They are definitional reducts or subreducts of proper relation algebras. The representability of Sugihara matrices yields sound and complete set-theoretical semantics for R-mingle.

1. Introduction

Sugihara's matrix, described by A. Anderson and N. Belnap [4, pp. 335–6], was introduced by T. Sugihara in 1955 [69]. A smaller one, obtained by using only one element per integer instead of two, is taken by Anderson and Belnap as "the Sugihara matrix". R. K. Meyer introduced finite Sugihara matrices for his proof that they are complete for the Dunn-McCall logic R-mingle, or RM [4, SS29.3.2]. Various algebras, including all Sugihara matrices and perhaps others, are representable as algebras of binary relations. Their operations are defined set-theoretically, and need not be specified by tables. Since their operations are definable in the similarity type of relation algebras, they are definitional reducts or subreducts of proper relation algebras. This was proved already in [55] for finite Sugihara matrices of even cardinality. In this paper we extend this result to all finite Sugihara matrices plus Sugihara's original infinite matrix and two others described by Anderson and Belnap. We also show it for Belnap's M_0 and for matrices of Meyer and Church. These algebras may be represented by a list of relations on a set, together with some operations on relations selected from Table 1. We start with Belnap.

2. Belnap

Belnap's M_0 was first introduced in 1960 [11] by matrices for binary operations $\vee$, $\wedge$, $\to$, $\sim$, and unary operations N and M, on an eight-element set. From the matrices for $\wedge$ and $\vee$ it is apparent that the eight values appearing in them, namely $-3, -2, -1, -0, +0, +1, +2$, and $+3$ (the last four are the designated values), form a lattice isomorphic to the lattice of subsets of the 3-element set $\{-1, +0, -2\}$, with $+3$ at the top and -3 at the bottom, where $\wedge$ and $\vee$ are interpreted as intersection and union. This observation does not occur in [11], but in subsequent literature M_0 is usually portrayed this way, by a Hasse diagram along with tables for $\to$ and

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- identity relation on U, $\mathsf{Id} = \{\langle x,x\rangle : x \in U\}$,
- diversity relation on U, $\mathsf{Di} = \{\langle x,y\rangle : x,y \in U,\ x \neq y\}$,
- universal relation on U, $U^2 = \{\langle x,y\rangle : x,y \in U\}$,
- union, $A \cup B = \{\langle x,y\rangle : \langle x,y\rangle \in A$ or $\langle x,y\rangle \in B\}$,
- intersection, $A \cap B = \{\langle x,y\rangle : \langle x,y\rangle \in A$ and $\langle x,y\rangle \in B\}$,
- converse, $A^{-1} = \{\langle x,y\rangle : \langle y,x\rangle \in A\}$,
- complement, $\overline{A} = \{\langle x,y\rangle : x,y \in U,\ \langle x,y\rangle \notin A\}$,
- converse-complement, ${\sim}A = \{\langle x,y\rangle : x,y \in U,\ \langle y,x\rangle \notin A\}$,
- relative product,
  $A|B = \{\langle x,y\rangle :$ for some $z \in U$, $\langle x,z\rangle \in A$ and $\langle z,y\rangle \in B\}$,
- residual,
  $A \to B = \{\langle x,y\rangle :$ for all $z \in U$, if $\langle z,x\rangle \in A$ then $\langle z,y\rangle \in B\}$,
- relativized converse-complement,
  ${\sim}'A = \{\langle x,y\rangle : \langle y,x\rangle \in \mathsf{Di}$ and $\langle y,x\rangle \notin A\}$,
- relativized relative product, $A|'B = \{\langle x,y\rangle : \langle x,y\rangle \in \mathsf{Di}$, and for some $z \in U$, $\langle x,z\rangle \in A \cap \mathsf{Di}$ and $\langle z,y\rangle \in B \cap \mathsf{Di}\}$,
- relativized residual, $A \to' B = \{\langle x,y\rangle : \big(\langle x,y\rangle \in \mathsf{Di}$, and for all $z \in U$, if $\langle z,x\rangle \in A \cap \mathsf{Di}$ and $\langle z,y\rangle \in \mathsf{Di}$ then $\langle z,y\rangle \in B\big)\}$.

TABLE 1. Some relations on a set U and some operations on relations.

$\sim$. See, for example, [4, SS18.4, SS22.1.3], [5, SSSS34.1–2], [67, p. 178], or [70, pp. 101–2]. It is described in [14, p. 117], [54], and [55, Theorem 4.1] as an algebra

(1) $\mathsf{M}_0 = \langle M_0, \cup, \cap, \to, \sim\rangle, \qquad M_0 = \{\emptyset, <, >, =, \neq, \leq, \geq, \mathbb{Q}^2\},$

whose universe M_0 consists of eight binary relations on the rational numbers $\mathbb{Q}$: the empty relation $\emptyset$, the less-than relation $<$, the greater-than relation $>$, the identity relation $=$, the diversity relation $\neq$, less-than-or-equal $\leq$, greater-than-or-equal $\geq$, and the universal relation $\mathbb{Q}^2$. These eight relations are the unions of subsets of $\{<, >, =\}$. The Hasse diagram for M_0 is shown in Figure 1. The operations of M_0 are union $\cup$, intersection $\cap$, residuation $\to$, and converse-complementation $\sim$, defined in Table 1 with $U = \mathbb{Q}$. The logic called BM [70, p. 128] is defined by an explicit finite axiomatization. By [70, Theorem 9.8.6] and its corollary, M_0 is characteristic for the logic BM. Because it has a single finite characteristic structure, BM is a complete decidable logic. The universe M_0 of M_0 is also closed under complementation $^-$, conversion $^{-1}$, and relative multiplication $|$, and contains the empty relation $\emptyset$, universal relation $\mathbb{Q}^2$, and identity relation Id on $\mathbb{Q}$. Therefore M_0 is the universe of an algebra

$$\mathfrak{M}_0 = \langle M_0, \cup, \cap, {}^-, \emptyset, \mathbb{Q}^2, |, {}^{-1}, \mathsf{Id}\rangle.$$

We refer to $\mathfrak{M}_0$ as **Belnap's relation algebra**. It is a proper relation algebra on the set of rational numbers. Proper relation algebras were first defined in [46], [48, Definition 4.23], and [50, SS2].

Definition 1. *For any equivalence relation E, let*

$$\mathfrak{Sb}(E) = \langle \wp(E), \cup, \cap, {}^-, \emptyset, E, |, {}^{-1}, \mathsf{Id}\rangle,$$

where $\wp(E)$ is the set of all subsets of E, $\cup$ is union, $\cap$ is intersection, $^-$ is complementation with respect to E, $|$ is relative multiplication, $^{-1}$ is conversion, $\emptyset$ is the

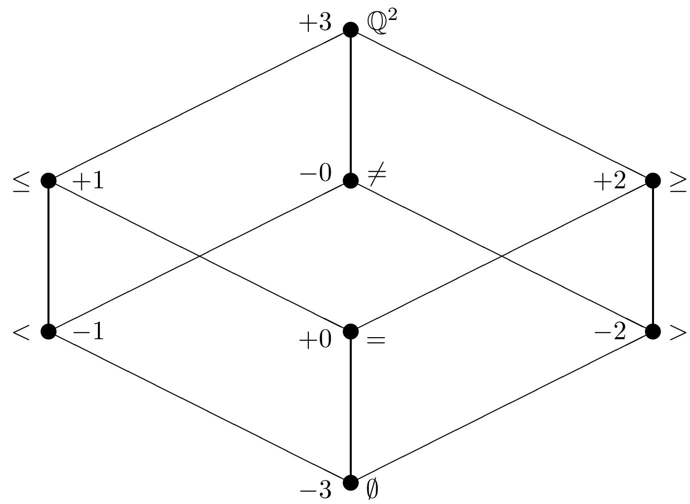

FIGURE 1. Lattice of relations in M_0.

empty relation, and $\mathsf{Id} = \{\langle u, u\rangle : \langle u, u\rangle \in E\}$ *is the identity relation on the field of* E. $\mathfrak{Sb}(E)$ *is the* ***algebra of subrelations of*** E. *A* ***proper relation algebra*** *is any subalgebra of the algebra of subrelations of an equivalence relation. An algebra is* ***representable*** *if it is isomorphic to a proper relation algebra. For any set* U, *let*

$$\mathfrak{Re}(U) = \mathfrak{Sb}(U^2).$$

$\mathfrak{Re}(U)$ *is the* ***algebra of relations on*** U. *A* ***proper relation algebra on*** U *is any subalgebra of* $\mathfrak{Re}(U)$.

$\mathfrak{Re}(U)$ is the prototypical example of a relation algebra. Tarski's original axioms [72] were chosen because they are true in $\mathfrak{Re}(U)$. Tarski's axiom XII implies simplicity, that is, any algebra satisfying XII has no non-trivial homomorphic images. This axiom was later dropped in [24, 46] so that all proper relation algebras would satisfy the axioms for relation algebras. It was noticed very early that Belnap's M_0 is a lattice with additional operations. What required nearly half a century after its introduction in 1960 was the realization, first mentioned in 2007 [54], that M_0 is a definitional reduct of $\mathfrak{M}_0$ (see Definition 5). The proper relation algebra $\mathfrak{M}_0$ was already known to Lyndon in 1956 [51]. In footnote 13, p. 307, Lyndon says,

> "Every relation algebra without zero divisors that is of order not exceeding 8 (there are 13 such) is commutative and isomorphic to a complex algebra of either the additive rationals or a cyclic group of order not exceeding 13."

In the numbering system of [53], the 13 relation algebras without zero divisors of order 8 or less are algebra 1_1 of order 2, algebras 1_2 and 2_2 of order 4, and the ten algebras 1_3, 2_3, 3_3, 1_7, 2_7, 3_7, 4_7, 5_7, 6_7, and 7_7 of order 8. Algebras 1_1, 1_2, 2_3, 1_7, and 5_7 are isomorphic to proper relation algebras on sets of size

1, 2, 3, 4, and 5, respectively, but on no larger or smaller sets. Algebras 2_2, 2_7, 4_7, 6_7, and 7_7 are isomorphic to proper relation algebras on sets of size at least 3, 6, 9, 8, and 9, respectively, and also on sets of all larger sizes. Algebra 4_7 shows up in SS8, where it is called $\mathfrak{Ch}$. It has Church's diamond as a definitional reduct. Algebra 3_7 is isomorphic to proper relation algebras on all sets of even cardinality 6 or larger. Algebra 3_3 also shows up in SS8, where it is called $\mathfrak{Rm}$. It has Meyer's algebra RM84 as a definitional reduct. Algebra 3_3 is isomorphic to proper relation algebras on sets of cardinality 7, and 9 or more, but not on sets of size 8 [6, Theorem 4.2]. The representation on 7 elements appears in SS8. Finally, algebra 1_3 is isomorphic to Belnap's relation algebra $\mathfrak{M}_0$. It is isomorphic to proper relation algebras on sets of every infinite cardinality. All the representations of these algebras on the smallest possible sets are unique [6]. The representation on the rationals $\mathbb{Q}$ mentioned by Lyndon is unique because of Cantor's theorem on the categoricity of dense linear orderings without endpoints on countable sets. It is the one relation algebra mentioned by Lyndon that requires the "additive rationals".

Starting with [1, 2, 3], an extensive literature developed in the 1980s in which $\mathfrak{M}_0$ is known as the **Point Algebra**, because among its eight relations are the three ways that two points on the rational number line can be related to each other: either they are equal (=), or the first point is to the left of the second point (<), or to the right (>). The Point Algebra and similar algebras based on the relationships that hold between various combinations of points and regions are widely used in computer science for spatial and temporal reasoning, and for constraint satisfaction problems. Consult [49], where references to some of the early work can be found. More recent papers that explicitly mention the Point Algebra include [8, 9, 15, 16, 17, 19, 20, 25, 34, 35, 36, 40, 44, 45, 63].

3. Sugihara

A **lattice** is an algebra $\langle S, \vee, \wedge\rangle$ with binary operations $\vee$ and $\wedge$ that are associative, commutative, and idempotent, such that the absorption laws $A\wedge(A\vee B) = A = A\vee(A\wedge B)$ hold. A lattice is a **chain** if $A\wedge B$ is always either A or B, *i.e.*, the ordering $\leq$ is linear, where $A \leq B$ iff $A\wedge B = A$.

Definition 2. $\mathbf{S} = \langle S, \vee, \wedge, \to, \sim\rangle$ *is a* ***Sugihara chain*** *if* $\langle S, \vee, \wedge\rangle$ *is a chain,* $\sim$ *is an involution that reverses the ordering, i.e.,*

$$\sim\sim A = A, \qquad A \leq B \;\textit{ iff }\; \sim B \leq \sim A,$$

and $A \to B = \sim A \vee B$ *if* $A \leq B$*, otherwise* $A \to B = \sim A \wedge B$*. An element* $A \in S$ *is said to be* ***designated*** *if* $\sim A \leq A$*.*

Ten examples of Sugihara chains residing in Belnap's M_0 are shown in Table 2, specified by their relations and operations. The chains $\{<, \leq\}$ and $\{>, \geq\}$ appear in Belnap's original proof [11, p. 145] of the variable-sharing property for the logic E of Anderson-Belnap [4, SS21.1], which says that if $A \to B$ is a theorem of E then A and B share at least one propositional variable. The same proof applies to the logic R. Axioms (R1)–(R13) for R are shown in Table 6; see [4, SS27.1.1] or [5, pp. xxiii–xxvi]. The axioms of R are valid in M_0, and the rules of deduction preserve validity, so $A \to B$ is not a theorem of R whenever A and B share no variable.

For every finite cardinality there is exactly one Sugihara chain having that cardinality. Because $\sim$ is order-reversing, finite Sugihara chains of odd cardinality must have an element that is a fixed point for $\sim$, the one in the middle. Such an

Universe	Operations
$\{<, \leq\}$, $\{>, \geq\}$, $\{\emptyset, \mathbb{Q}^2\}$,	$\cup$, $\cap$, $\rightarrow$, $\sim$,
$\{\emptyset, <, \leq, \mathbb{Q}^2\}$, $\{\emptyset, >, \geq, \mathbb{Q}^2\}$,	$\cup$, $\cap$, $\rightarrow$, $\sim$,
$\{<\}$, $\{>\}$, $\{\emptyset, \neq\}$,	$\cup$, $\cap$, $\rightarrow'$, $\sim'$,
$\{\emptyset, <, \neq\}$, $\{\emptyset, >, \neq\}$,	$\cup$, $\cap$, $\rightarrow'$, $\sim'$.

TABLE 2. Ten examples of Sugihara chains in M_0

element would be assigned as a truth value to a formula that is equivalent to its own negation. Sugihara chains without fixed points under negation are called "normal" by Meyer [4, p. 400], so odd Sugihara chains are not normal. Sugihara chains with even cardinality have no elements fixed by $\sim$, which interchanges the top and bottom halves while reversing their order. Sugihara chains with even cardinality were used by Meyer [4, p. 413, Corollary 3.1] to prove that the theorems of RM are exactly those formulas that are valid in all finite Sugihara chains. His result was used to prove [55, Theorem 6.2(iii)], which says that a formula is a theorem of RM if and only if it is valid in every finite algebra in K_{RM} (see Definition 9).

Infinite Sugihara chains are not determined by cardinality alone; see [7]. In what Anderson and Belnap "have accordingly come to think of ... as *the* Sugihara matrix" [4, p. 337], the universe is the set $\mathbb{Z}^* = \{n \colon 0 \neq n \in \mathbb{Z}\}$ of non-zero integers and $\sim(i) = -i$ for every non-zero $i \in \mathbb{Z}^*$. This Sugihara chain was named $\mathbf{S}_{\mathbb{Z}^*}$ by Meyer [4, p. 414], who proved that the theorems of RM are exactly those formulas valid in $\mathbf{S}_{\mathbb{Z}^*}$ [4, p. 414, Corollary 3.5]. Having described $\mathbf{S}_{\mathbb{Z}^*}$, Anderson and Belnap suggest, "Or one might insert 0 between -1 and $+1$, counting it designated" [4, p. 337]. The resulting chain was called $\mathbf{S}_{\mathbb{Z}}$ by Meyer [4, p. 414]. It has a fixed point for $\sim$, namely $0 = \sim 0$. No such fixed point occurs in the original chain of Sugihara [69]. In this chain, the ordering is isomorphic to two copies of the integers, one after the other, so we call it $\mathbf{S}_{\mathbb{Z}+\mathbb{Z}}$ (with + denoting ordinal addition). In more detail, the elements are s_i and t_j for integers $i, j \in \mathbb{Z}$, and the ordering is defined by $s_i < s_j$ and $t_i < t_j$ whenever $i < j$, and $s_i < t_j$ for any i and j. The Sugihara chains $\mathbf{S}_{\mathbb{Z}^*}$, $\mathbf{S}_{\mathbb{Z}}$, and $\mathbf{S}_{\mathbb{Z}+\mathbb{Z}}$ are countably infinite but not isomorphic. We turn now to the construction of proper relation algebras that have these and all finite Sugihara matrices as definitional reducts.

4. Definition of $\mathcal{S}_I$

For an arbitrary index set $I \subseteq \mathbb{Z}$ of integers, $\mathcal{S}_I$ is a set of relations on ${}^{\mathbb{Z}}\mathbb{Q}$, where ${}^{\mathbb{Z}}\mathbb{Q}$ be the set of functions $q : \mathbb{Z} \to \mathbb{Q}$ that map the integers $\mathbb{Z}$ to the rationals $\mathbb{Q}$. By Theorem 1 in the next section, $\mathcal{S}_I$ is the universe of a proper relation algebra called $\mathfrak{S}_I$. When $I = \{0\}$, Belnap's M_0 is a definitional reduct of $\mathfrak{S}_{\{0\}}$ (which is isomorphic to Belnap's relation algebra), and, when $I = \mathbb{Z}$, Sugihara's original matrix $\mathbf{S}_{\mathbb{Z}+\mathbb{Z}}$ is a definitional subreduct (but not a definitional reduct) of $\mathfrak{S}_{\mathbb{Z}}$.

Definition 3. *Let $I \subseteq \mathbb{Z}$.*

(i) *If $q \in {}^{\mathbb{Z}}\mathbb{Q}$, we say that q is* ***eventually zero*** *if there exists some integer $n \in \mathbb{Z}$ such that $q_i = 0$ for every integer $i > n$.*

(ii) *Let U_I be the set of functions, called* ***sequences****, that map $\mathbb{Z}$ to $\mathbb{Q}$, are eventually zero, and are non-zero only on I:*

$$U_I = \{q\colon q \in {}^{\mathbb{Z}}\mathbb{Q},\ (\exists_{n\in\mathbb{Z}})(\forall_{i>n})(q_i = 0),\ (\forall_{i\in\mathbb{Z}})(i \notin I \Rightarrow q_i = 0)\}.$$

(iii) *Define the identity and diversity relations*

$$\mathsf{Id}_I = \{\langle q, q\rangle\colon q \in U_I\}, \qquad \mathsf{Di}_I = \{\langle q, r\rangle\colon q, r \in U_I,\ q \neq r\},$$

and, for every $n \in \mathbb{Z}$,

$$L_n = \{\langle q, r\rangle\colon q, r \in U_I,\ q_n < r_n,\ \text{and } q_i = r_i \text{ whenever } n < i\},$$
$$R_n = \{\langle q, r\rangle\colon q, r \in U_I,\ q_n > r_n,\ \text{and } q_i = r_i \text{ whenever } n < i\}.$$

(iv) *Define a set of relations and its set of unions*

$$\mathcal{A}t_I = \{\mathsf{Id}_I\} \cup \bigcup_{n\in I}\{L_n,\ R_n\}, \qquad \mathcal{S}_I = \left\{\bigcup \mathcal{X}\colon \mathcal{X} \subseteq \mathcal{A}t_I\right\}.$$

By Theorem 1 below, $\mathcal{A}t_I$ is the set of atoms of a complete atomic proper relation algebra called $\mathfrak{S}_I$. Let $I = \emptyset$. Then $U_\emptyset$ is the set consisting of just the one sequence $q = \langle \cdots, 0, 0, 0, \cdots\rangle$ that is always zero. Notice that for every $n \in \mathbb{Z}$, $L_n = \emptyset$ iff $R_n = \emptyset$ iff $n \notin I$. Hence $\mathcal{A}t_\emptyset = \{\mathsf{Id}_\emptyset\} = \{\langle q, q\rangle\}$ and $\mathcal{S}_\emptyset = \{\emptyset, \{q\}\}$, so $\mathfrak{S}_\emptyset$ is isomorphic to the proper relation algebra $\mathfrak{Re}(\{q\})$.

Suppose $I = \{0\}$. In this case $U_{\{0\}}$ is the set of $\mathbb{Z}$-indexed sequences of rational numbers having 0 everywhere except possibly at index 0. There is a bijection between $U_{\{0\}}$ and $\mathbb{Q}$ that maps $q \in U_{\{0\}}$ to $q_0 \in \mathbb{Q}$. Setting $I = \{0\}$ in Definition 3 gives

$$\mathcal{A}t_{\{0\}} = \{\mathsf{Id}_{\{0\}}, L_0,\ R_0\},$$
$$\mathcal{S}_{\{0\}} = \left\{\emptyset,\ \mathsf{Id}_{\{0\}},\ L_0,\ R_0,\ \mathsf{Id}_{\{0\}} \cup L_0,\ \mathsf{Id}_{\{0\}} \cup R_0,\ L_0 \cup R_0,\ (U_{\{0\}})^2\right\}.$$

For every relation $X \subseteq (U_{\{0\}})^2$, let $f(X) = \{\langle q_0, r_0\rangle\colon \langle q, r\rangle \in X\}$. Applying f to the relations in $\mathcal{A}t_{\{0\}}$ produces the relations in M_0:

$$\begin{aligned}
f(\emptyset) &= \emptyset,\\
f(\mathsf{Id}_{\{0\}}) &= \{\langle x, y\rangle\colon x, y \in \mathbb{Q},\ x = y\},\\
f(L_0) &= \{\langle x, y\rangle\colon x, y \in \mathbb{Q},\ x < y\},\\
f(R_0) &= \{\langle x, y\rangle\colon x, y \in \mathbb{Q},\ x > y\},\\
f(\mathsf{Id}_{\{0\}} \cup L_0) &= \{\langle x, y\rangle\colon x, y \in \mathbb{Q},\ x \leq y\},\\
f(\mathsf{Id}_{\{0\}} \cup R_0) &= \{\langle x, y\rangle\colon x, y \in \mathbb{Q},\ x \geq y\},\\
f(L_0 \cup R_0) &= \{\langle x, y\rangle\colon x, y \in \mathbb{Q},\ x \neq y\},\\
f(U_{\{0\}}) &= \mathbb{Q}^2.
\end{aligned}$$

In fact, f is an isomorphism from $\mathfrak{S}_{\{0\}}$ to Belnap's relation algebra, so $\mathfrak{S}_{\{0\}}$ is also called "Belnap's relation algebra". It contains copies of the ten Sugihara chains in Table 2.

When $I = \mathbb{Z}$ the set $\mathcal{S}_\mathbb{Z}$ contains far more than is needed for the original Sugihara chain. $\mathcal{A}t_\mathbb{Z}$ has countably many relations, so the cardinality of $\mathcal{S}_\mathbb{Z}$ is same as that of the real numbers. By Theorem 1 in the next section, $\mathcal{S}_\mathbb{Z}$ is the universe of a relation algebra $\mathfrak{S}_\mathbb{Z}$, called **Sugihara's relation algebra**. By Theorem 2 in the section after that, $\mathfrak{S}_\mathbb{Z}$ contains countable chains isomorphic to $\mathbf{S}_{\mathbb{Z}+\mathbb{Z}}$.

5. Structure of $\mathfrak{S}_I$

In this section we show $\mathcal{S}_I$ is the universe of the complete atomic proper relation algebra $\mathfrak{S}_I$. First we review Definition 3 for easy reference. For every set of integers $I \subseteq \mathbb{Z}$, U_I is the set of functions from $\mathbb{Z}$ to $\mathbb{Q}$ that are eventually zero and non-zero only on I, $\mathcal{A}t_I = \{\mathsf{Id}_I\} \cup \bigcup_{n\in I}\{L_n,\, R_n\}$, and $\mathcal{S}_I = \{\bigcup \mathcal{X} \colon \mathcal{X} \subseteq \mathcal{A}t_I\}$, where $\mathsf{Id}_I = \{\langle q, q\rangle \colon q \in U_I\}$, and for every $n \in I$,

$$L_n = \{\langle q, r\rangle \colon q, r \in U_I,\ q_n < r_n, \text{ and } q_i = r_i \text{ whenever } n < i\},$$
$$R_n = \{\langle q, r\rangle \colon q, r \in U_I,\ q_n > r_n, \text{ and } q_i = r_i \text{ whenever } n < i\}.$$

Theorem 1. *$\mathcal{A}t_I$ is a partition of $(U_I)^2$. $\mathcal{A}t_I$ is the set of atoms of the complete atomic proper relation algebra*

$$\mathfrak{S}_I = \langle \mathcal{S}_I, \cup, \cap, ^-, \emptyset, (U_I)^2,\ |, ^{-1}, \mathsf{Id}_I\rangle.$$

Proof. To see that the relations in $\mathcal{A}t_I$ are pairwise disjoint and their union is $(U_I)^2$, note that any two sequences $q, r \in U_I$ are either equal everywhere (are in the identity relation Id_I), or differ somewhere, in which case there is a *largest* integer n where they differ, since they are both eventually zero. The pair $\langle q, r\rangle$ cannot be in L_m or R_m if $n < m$ since q and r agree at every such m by the choice of n, and $\langle q, r\rangle$ cannot be in L_m or R_m if $n > m$ since q and r differ somewhere larger than m, namely at n. Since q and r differ at n, one of them is not zero at n, so $n \in I$. Since the ordering of the rationals is linear, either $q_n < r_n$ or $q_n > r_n$ but not both, so the pair $\langle q, r\rangle$ must be in the relation L_n or R_n but not both. Therefore we have a disjoint union:

$$(U_I)^2 = \bigcup_{X\in\mathcal{A}t_I} X.$$

Since the relations in $\mathcal{A}t_I$ form a partition of $(U_I)^2$, the unions of arbitrary subsets of $\mathcal{A}t_I$ form a complete atomic Boolean algebra whose set of atoms is $\mathcal{A}t_I$. Thus $\mathcal{S}_I$ is closed under union, intersection, complementation, and contains $\emptyset$, $(U_I)^2$, and Id_I. It remains to be verified that $\mathcal{S}_I$ is closed under conversion and relative multiplication. From their definitions it follows that R_n and L_n are converses of each other. The converse of the identity relation is itself. Conversion distributes over arbitrary unions of relations. We therefore have the following rules. For all $n \in I$ and $\mathcal{X} \subseteq \mathcal{A}t_I$,

$$(2) \qquad (L_n)^{-1} = R_n, \qquad \mathsf{Id}_I^{-1} = \mathsf{Id}_I, \qquad \left(\bigcup \mathcal{X}\right)^{-1} = \bigcup_{X\in\mathcal{X}} X^{-1}.$$

From (2) it follows that $\{\bigcup \mathcal{X} \colon \mathcal{X} \subseteq \mathcal{A}t_I\}$ is closed under conversion. For closure under relative multiplication, we reason as follows. The relative product of two unions of sets of atoms is, by distributivity, the union of the relative products of the atoms in the two sets. More exactly, if $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{A}t_I$ then

$$\bigcup \mathcal{X} | \bigcup \mathcal{Y} = \bigcup \{X|Y \colon X \in \mathcal{X},\, Y \in \mathcal{Y}\}.$$

The relative product is again a union of atoms if the relative product of any two atoms is a union of atoms. As we will see, the relative product of any two atoms is an atom in every case except the relative product of a diversity atom and its converse, in which case the relative product is the union of the identity relation and all the diversity atoms with smaller index; see (10).

Assume q, r, and s are distinct sequences in U_I. Each sequence must differ from the other two, so the cardinality of $\{q_n, r_n, s_n\}$ cannot be 1 for every $n \in \mathbb{Z}$. On the other hand, since q, r, and s are all eventually zero, the number of elements of $\{q_n, r_n, s_n\}$ will eventually be constantly 1, since $\{q_n, r_n, s_n\} = \{0\}$ whenever n is large enough. Hence there is an integer n at which $\{q_n, r_n, s_n\}$ contains either exactly three or exactly two elements (hence $n \in I$ because they can't all be zero) and $|\{q_i, r_i, s_i\}| = 1$ for all $i > n$ (hence q, r, and s all agree beyond n). Although any two of q, r, and s are equal beyond n, any pair of them could also agree beyond an integer strictly smaller than n. In the first case, when $\{q_n, r_n, s_n\}$ has exactly three elements, those elements must form a chain under the dense linear ordering $<$ on the rationals, and, since q, r, and s all agree beyond n, we may choose x, y, z so that $\{x, y, z\} = \{q, r, s\}$ and $x\ L_n\ y\ L_n\ z$ (and $x\ L_n\ z$). This is listed as case (3) below. If there are exactly two elements in $\{q_n, r_n, s_n\}$, then one of them differs from the other two, and the other two coincide. Therefore, for some x, y, z such that $\{x, y, z\} = \{q, r, s\}$, we have $x_n \neq y_n = z_n$ and $1 = |\{x_i, y_i, z_i\}|$ for every $i > n$. If $x_n < y_n = z_n$ then $x\ L_n\ y$ and $x\ L_n\ z$, while if $x_n > y_n = z_n$ then $x\ R_n\ y$ and $x\ R_n\ z$. Now y and z are distinct, but they agree beyond n and also agree at n. Hence they disagree at some $j < n$, and agree beyond j, in which case $y\ L_j\ z$ or $y\ R_j\ z$. We may assume x, y, z were chosen so that $y\ L_j\ z$. This yields the remaining two cases (4) and (5). Thus, given any three distinct $q, r, s \in U_I$, there are $x, y, z \in U_I$ and $n \in I$ such that $\{x, y, z\} = \{q, r, s\}$, $|\{q_n, r_n, s_n\}| > 1$, $|\{q_i, r_i, s_i\}| = 1$ for all $i > n$, and one of these three cases holds:

(3) $$x\ L_n\ y\ L_n\ z \text{ and } x\ L_n\ z,$$

(4) $$x\ L_n\ y\ L_j\ z \text{ and } x\ L_n\ z \text{ for all } j < n,$$

(5) $$x\ R_n\ y\ L_j\ z \text{ and } x\ R_n\ z \text{ for all } j < n.$$

From the fact that these are the only possible cases, we will be able to deduce the rules for computing relative products of pairs of relations in $\mathcal{A}t_I$. First we consider the relative products with the identity relation.

(6) $$\mathsf{Id}_I|\mathsf{Id}_I = \mathsf{Id}_I, \qquad L_n|\mathsf{Id}_I = \mathsf{Id}_I|L_n = L_n, \qquad R_n|\mathsf{Id}_I = \mathsf{Id}_I|R_n = R_n.$$

We will only prove $\mathsf{Id}_I|L_n = L_n$. The other equations have similar proofs. Assume $\langle q, r\rangle \in \mathsf{Id}_I|L_n$. Then there is some s such that $\langle q, s\rangle \in \mathsf{Id}_I$ and $\langle s, r\rangle \in L_n$. The latter two statements tell us that $q = s$ and $s\ L_n\ r$, from which we conclude $q\ L_n\ r$ by the fact that equal objects have the same properties, hence $\langle q, r\rangle \in L_n$, showing that $\mathsf{Id}_I|L_n \subseteq L_n$. For the opposite inclusion, we assume $\langle q, r\rangle \in L_n$ and note that by choosing $s = q$ we get $\langle q, s\rangle \in \mathsf{Id}_I$ and $\langle s, r\rangle \in L_n$, hence $\langle q, r\rangle \in \mathsf{Id}_I|L_n$. Thus $\mathsf{Id}_I \subseteq \mathsf{Id}_I|L_n$. Combining this with $\mathsf{Id}_I|L_n \subseteq \mathsf{Id}_I$, we obtain the desired equality.

Next we introduce notation for special relations in $\mathcal{S}_I$ that arise from relative products of diversity atoms. For any $n, m \in I$ let

(7) $$L_{[n,m]} = \bigcup\{L_k \colon n \leq k \leq m,\ k \in I\}, \qquad L_{(-\infty,n]} = \bigcup\{L_k \colon n \geq k \in I\},$$

(8) $$L_{[n,\infty)} = \bigcup\{L_k \colon n \leq k \in I\}, \qquad L_{(-\infty,\infty)} = \bigcup\{L_k \colon k \in I\}.$$

Note that $L_{[n,m]} = \emptyset$ if $n > m$, and $L_{[n,n]} = L_n$. The same notation is used with converses (change L to R in the equations above). The rules (2) imply

$$\big(L_{[n,m]}\big)^{-1} = R_{[n,m]}, \qquad \big(L_{(\infty,m]}\big)^{-1} = R_{(\infty,m]},$$
$$\big(L_{[n,\infty)}\big)^{-1} = R_{[n,\infty)}, \qquad \big(L_{(-\infty,\infty)}\big)^{-1} = R_{(-\infty,\infty)}.$$

The relative product of diversity atoms $L_m, R_n \in \mathcal{A}t_I$ can be computed according to four basic rules. Rule (9) says that the relative product of a diversity atom with itself is itself. Rule (10) says that the relative product of a diversity atom with its converse is the union of the identity relation and all the diversity atoms having equal or smaller index. Rules (11) and (12) say that the relative product of two diversity atoms with distinct indices n and m is the one with the larger index.

$$L_n|L_n = L_n, \qquad R_n|R_n = R_n, \tag{9}$$

$$R_n|L_n = L_n|R_n = \mathsf{Id}_I \cup L_{(-\infty,n]} \cup R_{(-\infty,n]}, \tag{10}$$

$$L_m|L_n = L_n|L_m = R_m|L_n = L_n|R_m = L_n \quad \text{if } m < n, \tag{11}$$

$$R_m|R_n = R_n|R_m = R_n|L_m = L_m|R_n = R_n \quad \text{if } m < n. \tag{12}$$

To prove (9), assume $\langle q, r\rangle \in L_n|L_n$, so there is some $s \in U_I$ such that $\langle q, s\rangle \in L_n$ and $\langle s, r\rangle \in L_n$. It follows that $q_n < s_n$, $s_n < r_n$, and q, r, and s all agree beyond n. We also have $q_n < r_n$ by the transitivity of the ordering $<$ on $\mathbb{Q}$, so $\langle q, r\rangle \in L_n$. This shows $L_n|L_n \subseteq L_n$. For the opposite inclusion, assume $\langle q, r\rangle \in L_n$. Then $q_n < r_n$, and q and r agree beyond n. By the density of $<$, we may choose $s \in U_I$ so that s agrees with q and r beyond n and has some value s_n between q_n and r_n (such as the average of q_n and r_n) so that $q_n < s_n < r_n$. The values of s on arguments in I and smaller than n are arbitrary. This completes the proof of the first equation in (9). The second equation has a similar proof.

To prove (10), assume $\langle q, r\rangle \in R_n|L_n$. If $q = r$ then $\langle q, r\rangle$ is in Id_I, one of the relations in the union on the right side of (10), as desired, so assume $q \neq r$. By the definition of $|$ there is some $s \in U_I$ such that $\langle q, s\rangle \in R_n$, $\langle s, r\rangle \in L_n$, and q, r, and s agree beyond n. From $\langle s, q\rangle \in L_n$, $\langle s, r\rangle \in L_n$, and $q \neq r$ we conclude that we are in case (3) or (4) with $s = x$ and $\{q, r\} = \{y, z\}$. Since $q \neq r$ and $\{q, r\} = \{y, z\}$, $\langle q, r\rangle$ is in some diversity atom whose index must be either n, as in case (3), or some smaller integer $j < n$, which occurs in case (4). In either case, depending on how q and r match up with y and z, we have $\langle q, r\rangle \in L_{(-\infty,n]} \cup R_{(-\infty,n]}$. The pair $\langle q, r\rangle$ thus belongs to one of the relations on the right, proving

$$R_n|L_n \subseteq \mathsf{Id}_I \cup L_{(-\infty,n]} \cup R_{(-\infty,n]}.$$

For the converse, assume $\langle q, r\rangle \in \mathsf{Id}_I \cup L_m \cup R_m$ and $m \leq n$. We will find $s \in U_I$ such that $\langle q, s\rangle \in R_n$ and $\langle s, r\rangle \in L_n$. Since q and r agree beyond m, they agree beyond n as well. Choose values for $s \in U_I$ so that s agrees with q and r beyond n. The values of s at arguments that are in I and smaller than n may be anything. At n, choose a rational s_n that is strictly smaller than both q_n and r_n, such as $s_n = \min(q_n, r_n) - 1$. Here we are using the fact that the ordering of the rationals does not have any endpoints. From $s_n < q_n$, $s_n < r_n$, and the agreement of q, r, and s beyond n we get $\langle s, q\rangle \in L_n$ and $\langle s, r\rangle \in L_n$, hence $\langle q, s\rangle \in R_n$, so $\langle q, r\rangle \in R_n|L_n$. This shows $R_n|L_n \supseteq \mathsf{Id}_I \cup L_{(-\infty,n]} \cup R_{(-\infty,n]}$, completing the proof of one of the two equations in (10). The other equation may be proved similarly.

The proofs for (11) and (12) are somewhat simpler. We first show that if $m < n$ then $(L_m \cup R_m)|L_n \subseteq L_n$. Assume $\langle q, r\rangle \in (L_m \cup R_m)|L_n$. Then there must exist some $s \in U_I$ such that $\langle q, s\rangle \in L_m \cup R_m$ and $\langle s, r\rangle \in L_n$. It follows from $\langle q, s\rangle \in L_m \cup R_m$ that $q_m \neq s_m$ and q and s agree beyond m. Since $m < n$, this tells us that $q_n = s_n$ and q and s agree beyond n. From $\langle s, r\rangle \in L_n$ we know $s_n < r_n$ and s and r agree beyond n. We conclude that $q_n = s_n < r_n$ and q, r, and s agree beyond n, hence $\langle q, r\rangle \in L_n$. Assume $\langle q, r\rangle \in L_n$ and $m < n$. Then $q_n < r_n$

and q and r agree beyond n. Let $s \in U_I$ have completely arbitrary entries up to s_m and agree with q beyond m. Since $m < n$, any such s agrees with r beyond n and $s_n = q_n < r_n$, so $\langle s, r\rangle \in L_n$. Since q and s agree beyond m, their relationship depends on the relation between q_m and s_m. If $q_m > s_m$ then $\langle q, s\rangle \in R_m$, and if $q_m < s_m$ then $\langle q, s\rangle \in L_m$. Both kinds of s exist, so $L_n \subseteq R_m|L_n \cap L_m|L_n$. From the two inclusions we have proved, it follows that $L_n = R_m|L_n = L_m|L_n$. The other equations in (11) and (12) can be proved similarly.

Rules (2), (6), (9), (10), (11), and (12) show that relative products of atoms are unions of atoms, hence the set of unions of sets of atoms is closed under relative multiplication, completing the proof of Theorem 1. □

Lemma 1. *Relative multiplication is commutative in* $\mathfrak{S}_I$.

Proof. Relative multiplication distributes over arbitrary unions, so if $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{A}t_I$ then

$$\begin{aligned}\bigcup\mathcal{X}|\bigcup\mathcal{Y} &= \bigcup\{X|Y \colon X \in \mathcal{X},\, Y \in \mathcal{Y}\}\\ &= \bigcup\{Y|X \colon X \in \mathcal{X},\, Y \in \mathcal{Y}\} && (6),\ (10)\text{–}(12)\\ &= \bigcup\mathcal{Y}|\bigcup\mathcal{X}.\end{aligned}$$

□

Chains constructed in the next section will be shown in Theorem 2 to be Sugihara chains by means of the following computational rules.

Lemma 2. *Let* $I \subseteq \mathbb{Z}$. *For all* $n, m \in I$,

$$L_{(-\infty,n]}|L_{(-\infty,m]} = L_{(-\infty,n]} \cup L_{(-\infty,m]}, \tag{13}$$

$$R_{(-\infty,n]}|R_{(-\infty,m]} = R_{(-\infty,n]} \cup R_{(-\infty,m]}, \tag{14}$$

$$L_{[n,\infty)}|L_{[m,\infty)} = L_{[n,\infty)} \cap L_{[m,\infty)}, \tag{15}$$

$$R_{[n,\infty)}|R_{[m,\infty)} = R_{[n,\infty)} \cap R_{[m,\infty)}, \tag{16}$$

$$R_{(-\infty,\infty)}|R_{[m,\infty)} = R_{[m,\infty)}, \tag{17}$$

$$R_{(-\infty,\infty)}|L_{(-\infty,m]} = R_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup L_{(-\infty,m]}, \tag{18}$$

$$\textit{if } n < m \textit{ then} \quad L_{(-\infty,n]}|R_{[m,\infty)} = R_{[m,\infty)}, \tag{19}$$

$$\textit{if } n \geq m \textit{ then} \quad L_{(-\infty,n]}|R_m = R_{(-\infty,m]} \cup \mathsf{Id}_I \cup L_{(-\infty,n]}, \tag{20}$$

$$L_{(-\infty,n]}|R_{[m,\infty)} = \begin{cases} R_{[m,\infty)} & \textit{if } n < m,\\ R_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup L_{(-\infty,n]} & \textit{if } n \geq m.\end{cases} \tag{21}$$

Proof. In the computations proving (13)–(21) we use (2), (6)–(12), and the fact that relative multiplication distributes over arbitrary unions of relations. (13) holds because

$$\begin{aligned}L_{(-\infty,n]}|L_{(-\infty,m]} &= \bigcup\{L_k|L_\ell \colon n \geq k \in I,\, m \geq \ell \in I\}\\ &= \bigcup\{L_{\max(k,\ell)} \colon n \geq k \in I,\, m \geq \ell \in I\} && (11)\\ &= L_{(-\infty,\max(n,m)]}\\ &= L_{(-\infty,n]} \cup L_{(-\infty,m]}.\end{aligned}$$

Taking converses of both sides in (13) gives (14). For (15),

$$\begin{aligned} L_{[n,\infty)}|L_{[m,\infty)} &= \bigcup\{L_k|L_\ell \colon n \le k \in I,\, m \le \ell \in I\} \\ &= \bigcup\{L_{\max(k,\ell)} \colon n \le k \in I,\, m \le \ell \in I\} && (11) \\ &= L_{[\max(n,m),\infty)} \\ &= L_{[n,\infty)} \cap L_{[m,\infty)}. \end{aligned}$$

Applying conversion to (15) gives (16). For (17),

$$\begin{aligned} R_{(-\infty,\infty)}|R_{[m,\infty)} &= \bigcup\{R_k|R_\ell \colon k \in I,\, m \le \ell \in I\} \\ &= \bigcup\{R_{\max(k,\ell)} \colon k \in I,\, m \le \ell \in I\} && (12) \\ &= R_{[m,\infty)}. \end{aligned}$$

For (18), by (7), (8), and distributivity we have

$$R_{(-\infty,\infty)}|L_{(-\infty,m]} = \bigcup\{R_k|L_\ell \colon k \in I,\, m \ge \ell \in I\}. \tag{22}$$

Assume $k \in I$ and $m \ge \ell \in I$. If $k \ne \ell$ then

$$\begin{aligned} R_k|L_\ell &\subseteq R_k \cup L_\ell && (11), (12) \\ &\subseteq R_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup L_{(-\infty,m]}, \end{aligned}$$

while if $k = \ell \le m$ then

$$\begin{aligned} R_k|L_\ell &= R_k|L_k \\ &= R_{(-\infty,k]} \cup \mathsf{Id}_I \cup L_{(-\infty,k]} && (10) \\ &\subseteq R_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup L_{(-\infty,m]}. \end{aligned}$$

Along with (22), this shows

$$R_{(-\infty,\infty)}|L_{(-\infty,m]} \subseteq R_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup L_{(-\infty,m]}. \tag{23}$$

For the other direction, note that $R_m \subseteq R_{(-\infty,\infty)}$ and $L_m \subseteq L_{(-\infty,m]}$ since $m \in I$, hence, by (10),

$$R_{(-\infty,\infty)}|L_{(-\infty,m]} \supseteq R_m|L_m = R_{(-\infty,m]} \cup \mathsf{Id}_I \cup L_{(-\infty,m]}.$$

What remains is to show $R_{(-\infty,\infty)}|L_{(-\infty,m]} \supseteq R_k$ whenever $k \in I$ and $k > m$. From $k, m \in I$ we get $R_k \subseteq R_{(-\infty,\infty)}$ and $L_m \subseteq L_{(-\infty,m]}$, so

$$\begin{aligned} R_k &= R_k|L_m && k > m \\ &\subseteq R_{(-\infty,\infty)}|L_{(-\infty,m]}, \end{aligned}$$

completing the proof of (18). For (19), if $n < m$ then

$$\begin{aligned} L_{(-\infty,n]}|R_{[m,\infty)} &= \bigcup\{L_k|R_\ell \colon k \in I,\, k \le n < m \le \ell \in I\} \\ &= \bigcup\{R_\ell \colon k \in I,\, k \le n < m \le \ell \in I\} && (12) \\ &= R_{[m,\infty)}. \end{aligned}$$

For (20), if $n \ge m$ then, by (10)–(12),

$$L_{(-\infty,n]}|R_m = \bigcup_{n \ge k \in I} L_k|R_m$$

$$
\begin{aligned}
&= \Big(\bigcup_{n\ge k\in I,\,k<m} L_k|R_m\Big) \cup \big(L_m|R_m\big) \cup \Big(\bigcup_{n\ge k\in I,\,k>m} L_k|R_m\Big)\\
&= \Big(\bigcup_{n\ge k\in I,\,k<m} R_m\Big) \cup \big(R_{(-\infty,m]} \cup \mathsf{Id}_I \cup L_{(-\infty,m]}\big) \cup \Big(\bigcup_{n\ge k\in I,\,k>m} L_k\Big)\\
&= R_{(-\infty,m]} \cup \mathsf{Id}_I \cup L_{(-\infty,n]}.
\end{aligned}
$$

Finally we prove (21). The first case, in which $n<m$, holds by (19). If $n\ge m$ then

$$
\begin{aligned}
L_{(-\infty,n]}|R_{[m,\infty)} &= L_{(-\infty,n]}|R_{[m,n]} \cup L_{(-\infty,n]}|R_{[n+1,\infty)}\\
&= L_{(-\infty,n]}|R_{[m,n]} \cup R_{[n+1,\infty)} && (19)\\
&= \Big(\bigcup_{m\le \ell\le n,\,\ell\in I} L_{(-\infty,n]}|R_\ell\Big) \cup R_{[n+1,\infty)}\\
&= \bigcup_{m\le \ell\le n,\,\ell\in I} \big(\mathsf{Id}_I \cup L_{(-\infty,n]} \cup R_{(-\infty,\ell]}\big) \cup R_{[n+1,\infty)} && (20)\\
&= L_{(-\infty,n]} \cup \mathsf{Id}_I \cup R_{(-\infty,n]} \cup R_{[n+1,\infty)}\\
&= R_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup L_{(-\infty,n]}.
\end{aligned}
$$

□

6. Sugihara chains

Definition 4. *For every $I\subseteq\mathbb{Z}$, let*

$$
\begin{aligned}
\mathcal{C}_I &= \{S^I_n\colon -n\in I\} \cup \{T^I_n\colon n\in I\},\\
\mathcal{C}'_I &= \{S^I_n\colon -n\in I\} \cup \{\hat{T}^I_n\colon n\in I\},
\end{aligned}
$$

where, for every $n\in\mathbb{Z}$,

$$
S^I_n = R_{[-n,\infty)}, \qquad T^I_n = R_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup L_{(-\infty,n-1]},
$$
$$
\hat{T}^I_n = R_{(-\infty,\infty)} \cup L_{(-\infty,n-1]}.
$$

It follows from (7) and (8) that the relations in $\mathcal{C}_I$ and $\mathcal{C}'_I$ form chains under inclusion. They are shown in Theorem 2 to be the universes of Sugihara chains. When $I=\mathbb{Z}$, the order types of $\mathcal{C}_\mathbb{Z}$ and $\mathcal{C}'_\mathbb{Z}$ are the same as Sugihara's original $\mathbf{S}_{\mathbb{Z}+\mathbb{Z}}$, and the resulting Sugihara chains are both isomorphic to $\mathbf{S}_{\mathbb{Z}+\mathbb{Z}}$:

($\mathcal{C}_\mathbb{Z}$) $\cdots\subseteq S^I_{-2}\subseteq S^I_{-1}\subseteq S^I_0\subseteq S^I_1\subseteq\cdots \quad \cdots\subseteq T^I_{-1}\subseteq T^I_0\subseteq T^I_1\subseteq T^I_2\subseteq\cdots$,

($\mathcal{C}'_\mathbb{Z}$) $\cdots\subseteq S^I_{-2}\subseteq S^I_{-1}\subseteq S^I_0\subseteq S^I_1\subseteq\cdots \quad \cdots\subseteq \hat{T}^I_{-1}\subseteq \hat{T}^I_0\subseteq \hat{T}^I_1\subseteq \hat{T}^I_2\subseteq\cdots$.

In $\mathbf{S}_{\mathbb{Z}+\mathbb{Z}}$, the designated elements are the ones in the second, larger copy of $\mathbb{Z}$. In the Sugihara chain with universe $\mathcal{C}_\mathbb{Z}$, the designated relations are the ones that contain the identity relation on $U_\mathbb{Z}$, but all the relations in $\mathcal{C}'_\mathbb{Z}$ are disjoint from the identity relation.

If $I=\emptyset$ then $U_\emptyset$ is a singleton containing just the function that is constantly zero, and $\mathcal{C}_\emptyset=\mathcal{C}'_\emptyset=\emptyset$. If $I=\{0\}$ then $\mathfrak{S}_{\{0\}}$ is isomorphic to Belnap's relation algebra and

$$
\mathcal{C}_{\{0\}} = \{S^{\{0\}}_0, T^{\{0\}}_0\}, \qquad \mathcal{C}'_{\{0\}} = \{S^{\{0\}}_0\},
$$
$$
S^{\{0\}}_0 = R_0 = \hat{T}^{\{0\}}_0, \qquad T^{\{0\}}_0 = R_0 \cup \mathsf{Id}_{\{0\}}.
$$

We thus obtain the two Sugihara chains $\mathcal{C}_{\{0\}} = \{R_0, R_0 \cup \mathsf{Id}_{\{0\}}\}$ and $\mathcal{C}'_{\{0\}} = \{R_0\}$, which match up with the Sugihara chains $\{<, \leq\}$ and $\{<\}$ in Table 2, under the isomorphism f defined after Definition 3. For a final example, if $I = \{0, 1\}$, then

$$\begin{aligned} &\mathcal{C}_{\{0,1\}} = \{S^{\{0,1\}}_{-1}, S^{\{0,1\}}_{0}, T^{\{0,1\}}_{0}, T^{\{0,1\}}_{1}\}, && \mathcal{C}'_{\{0,1\}} = \{S^{\{0,1\}}_{-1}, \hat{T}^{\{0,1\}}_{0}, \hat{T}^{\{0,1\}}_{1}\}, \\ &S^{\{0,1\}}_{-1} = R_1, && S^{\{0,1\}}_{0} = \hat{T}^{\{0,1\}}_{0} = R_0 \cup R_1, \\ &T^{\{0,1\}}_{0} = R_0 \cup R_1 \cup \mathsf{Id}_{\{0,1\}}, && T^{\{0,1\}}_{1} = L_0 \cup R_0 \cup R_1 \cup \mathsf{Id}_{\{0,1\}}, \\ & && \hat{T}^{\{0,1\}}_{1} = L_0 \cup R_0 \cup R_1. \end{aligned}$$

Note that $\mathcal{C}_{\{0,1\}}$ and $\mathcal{C}'_{\{0,1\}}$ can be extended by adding the empty relation at one end, and the universal relation to $\mathcal{C}_{\{0,1\}}$, or the diversity relation to $\mathcal{C}'_{\{0,1\}}$, at the other end (or both), thus creating Sugihara chains of sizes 5 and 6. There are four relations in $\mathfrak{S}_{\{0,1\}}$ that are fixed by $\sim'$, namely $L_0 \cup L_1$, $R_0 \cup L_1$, $L_0 \cup R_1$, and $R_0 \cup R_1$. Two of these relations appear in the middle of two Sugihara chains of length 5. The union of these two chains forms a definitional reduct of $\mathfrak{S}_{\{0,1\}}$ that is isomorphic to the crystal lattice in SS8.

Theorem 2. *For every $I \subseteq \mathbb{Z}$, $\langle \mathcal{C}_I, \cup, \cap, \to, \sim\rangle$ and $\langle \mathcal{C}'_I, \cup, \cap, \to', \sim'\rangle$ are Sugihara chains. In particular, $\langle \mathcal{C}_{\mathbb{Z}}, \cup, \cap, \to, \sim\rangle$ and $\langle \mathcal{C}'_{\mathbb{Z}}, \cup, \cap, \to', \sim'\rangle$ are isomorphic to the original $\mathbf{S}_{\mathbb{Z}+\mathbb{Z}}$.*

Proof. By (2) and (8),

$$\sim S^I_n = \overline{(S^I_n)^{-1}} = \overline{\left(R_{[-n,\infty)}\right)^{-1}} = \overline{L_{[-n,\infty)}} = R_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup L_{(-\infty,-n-1]} = T^I_{-n},$$

so

$$\sim' S^I_n = \sim S^I_n \cap \mathsf{Di}_I = T^I_{-n} \cap \mathsf{Di}_I = \hat{T}^I_{-n}. \tag{24}$$

It is straightforward to verify that $\sim$ is an order-reversing involution on all relations, and that $\sim'$ is an order-reversing involution on relations included in Di_I. Therefore we have $\sim T^I_{-n} = S^I_n$, and $\sim'(\hat{T}^I_{-n}) = S^I_n$ since $S^I_n \cup \hat{T}^I_{-n} \subseteq \mathsf{Di}_I$. It follows that $\mathcal{C}_I$ and $\mathcal{C}'_I$ are closed under converse-complementation $\sim$ and relativized converse-complementation $\sim'$, respectively. Since $\sim\emptyset = (U_I)^2$ and $\sim'\emptyset = \mathsf{Di}_I$, converse-complementation and relativized converse-complementation are also order-reversing involutions on $\mathcal{C}_I \cup \{\emptyset, (U_I)^2\}$ and $\mathcal{C}'_I \cup \{\emptyset, \mathsf{Di}_I\}$, respectively. Turning to relative products, we show for all $n, m \in I$,

$$S^I_n | S^I_m = S^I_n \cap S^I_m, \tag{25}$$

$$S^I_n | T^I_m = T^I_m | S^I_n = \begin{cases} S^I_n & \text{if } n \leq -m, \\ T^I_m & \text{if } n > -m, \end{cases} \tag{26}$$

$$T^I_m | T^I_n = T^I_m \cup T^I_n, \tag{27}$$

$$S^I_n | \hat{T}^I_m = \hat{T}^I_m | S^I_n = \begin{cases} S^I_n & \text{if } n \leq -m, \\ \hat{T}^I_m \cup \mathsf{Id}_I & \text{if } n > -m, \end{cases} \tag{28}$$

$$\hat{T}^I_m | \hat{T}^I_n = \hat{T}^I_m \cup \hat{T}^I_n \cup \mathsf{Id}_I. \tag{29}$$

For (25) and (26) we have

$$\begin{aligned} S^I_n | S^I_m &= R_{[-n,\infty)} | R_{[-m,\infty)} \\ &= R_{[-n,\infty)} \cap R_{[-m,\infty)} \qquad (16) \end{aligned}$$

$$
\begin{aligned}
&= S^I_n \cap S^I_m, \\
T^I_m|S^I_n &= \left(R_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup L_{(-\infty,m-1]}\right)|R_{[-n,\infty)} \\
&= R_{(-\infty,\infty)}|R_{[-n,\infty)} \cup \mathsf{Id}_I|R_{[-n,\infty)} \cup L_{(-\infty,m-1]}|R_{[-n,\infty)} \\
&= R_{[-n,\infty)} \cup R_{[-n,\infty)} \cup L_{(-\infty,m-1]}|R_{[-n,\infty)} && (17),\ (6) \\
&= \begin{cases} R_{[-n,\infty)} & \text{if } m-1 < -n \\ R_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup L_{(-\infty,m-1]} & \text{if } m-1 \geq -n \end{cases} && (21) \\
&= \begin{cases} S^I_n & \text{if } n \leq -m, \\ T^I_m & \text{if } n > -m. \end{cases}
\end{aligned}
$$

For (27) we start with the observation that

$$T^I_m|T^I_n = \left(R_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup L_{(-\infty,m-1]}\right) | \left(R_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup L_{(-\infty,n-1]}\right).$$

Multiplying this out yields these nine relative products.

$$
\begin{aligned}
R_{(-\infty,\infty)}|R_{(-\infty,\infty)} &= R_{(-\infty,\infty)}, && (9) \\
R_{(-\infty,\infty)}|\mathsf{Id}_I &= R_{(-\infty,\infty)}, && (6) \\
R_{(-\infty,\infty)}|L_{(-\infty,n-1]}, &= R_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup L_{(-\infty,n-1]}, && (18) \\
\mathsf{Id}_I|R_{(-\infty,\infty)} &= R_{(-\infty,\infty)}, && (6) \\
\mathsf{Id}_I|\mathsf{Id}_I &= \mathsf{Id}_I, && (6) \\
\mathsf{Id}_I|L_{(-\infty,n-1]} &= L_{(-\infty,n-1]}, && (6) \\
L_{(-\infty,m-1]}|R_{(-\infty,\infty)} &= R_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup L_{(-\infty,m-1]}, && (18),\ \text{Lemma 1} \\
L_{(-\infty,m-1]}|\mathsf{Id}_I &= L_{(-\infty,m-1]}, && (6) \\
L_{(-\infty,m-1]}|L_{(-\infty,n-1]} &= L_{(-\infty,m-1]} \cup L_{(-\infty,n-1]}. && (13)
\end{aligned}
$$

Taking the union of the relations on the right gives us

$$T^I_m|T^I_n = R_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup L_{(-\infty,m-1]} \cup L_{(-\infty,n-1]} = T^I_m \cup T^I_n,$$

so (27) holds. The proofs of (28) and (29) are somewhat simpler. They can be obtained from the computations just given by deleting references to Id_I on the left sides of the equations, and expressing the relations on the right sides in terms of Id_I and the relations in $\mathcal{C}'_I$. Recall from Definition 2 that in a Sugihara chain, $\to$ is defined by

$$A \to B = \begin{cases} {\sim}A \vee B & \text{if } A \leq B, \\ {\sim}A \wedge B & \text{if } A > B. \end{cases}$$

Substitute ${\sim}B$ for B and apply $\sim$ to both sides. The double negation and De Morgan laws for $\wedge$, $\vee$, and $\sim$ hold in every Sugihara chain, so

$${\sim}(A \to {\sim}B) = \begin{cases} A \wedge B & \text{if } A \leq {\sim}B, \\ A \vee B & \text{if } A > {\sim}B. \end{cases}$$

To show residuation and relativized residuation act like Sugihara's $\to$, we will use the latter equation. By some elementary calculations starting from the definitions in Table 1 of relative multiplication, converse-complementation, residuation and their relativized counterparts, we get

(30) $\qquad {\sim}(A \to {\sim}B) = B|A, \qquad {\sim}'(A \to' {\sim}'B) = B|'A,$

hence all we need to show is that for any $A, B \in \mathcal{C}_I$,

$$B|A = \begin{cases} A \cap B & \text{if } A \subseteq {\sim}B, \\ A \cup B & \text{if } A \supset {\sim}B, \end{cases} \tag{31}$$

and for any $A, B \in \mathcal{C}'_I$,

$$B|'A = \begin{cases} A \cap B & \text{if } A \subseteq {\sim'}B, \\ A \cup B & \text{if } A \supset {\sim'}B. \end{cases} \tag{32}$$

Proof of (31). Because of commutativity (Lemma 1), there are just three cases that arise by substituting into (31) when $n, m \in I$, $A \in \{S^I_n, T^I_n\}$, and $B \in \{S^I_m, T^I_m\}$.

Case 1. $A = S^I_n$, $B = S^I_m$, ${\sim}B = T^I_{-m}$. The first case in (31) applies because $S^I_n \subseteq T^I_{-m}$. By (25), $B|A = S^I_m|S^I_n = S^I_n \cap S^I_m = A \cap B$. This agrees with (31), and shows that (31) holds.

Case 2. $A = T^I_n$, $B = T^I_m$, ${\sim}B = S^I_{-m}$. The second case in (31) applies since $T^I_n \supset S^I_{-m}$. By (27) we have $B|A = T^I_m|T^I_n = T^I_n \cup T^I_m = A \cup B$, as required for (31) to hold.

Case 3. $A = S^I_n$, $B = T^I_m$, ${\sim}B = S^I_{-m}$. Since $A \subset B$ in this case, (31) simplifies into the form proved below.

$$\begin{aligned} B|A = T^I_m|S^I_n &= \begin{cases} S^I_n & \text{if } n \leq -m \\ T^I_m & \text{if } n > -m \end{cases} && (26) \\ &= \begin{cases} S^I_n & \text{if } S^I_n \subseteq S^I_{-m} \\ T^I_m & \text{if } S^I_n \supset S^I_{-m} \end{cases} \\ &= \begin{cases} A & \text{if } A \subseteq {\sim}B, \\ B & \text{if } A \supset {\sim}B. \end{cases} \end{aligned}$$

Proof of (32). Again there are three cases.

Case 1. $A = S^I_n$, $B = S^I_m$, ${\sim'}B = \hat{T}^I_{-m}$. The first case in (32) applies because $S^I_n \subseteq \hat{T}^I_{-m}$. By (25) and $S^I_n \cup S^I_m \subseteq \mathsf{Di}_I$, $B|'A = S^I_m|S^I_n \cap \mathsf{Di}_I = S^I_n \cap S^I_m = A \cap B$, which agrees with (32).

Case 2. $A = \hat{T}^I_n$, $B = \hat{T}^I_m$, ${\sim'}B = S^I_{-m}$. By $\hat{T}^I_n \supset S^I_{-m}$, the second case in (32) applies. By (27) and $\hat{T}^I_n \cup \hat{T}^I_m \subseteq \mathsf{Di}_I$, $B|'A = \hat{T}^I_m|\hat{T}^I_n \cap \mathsf{Di}_I = \hat{T}^I_n \cup \hat{T}^I_m = A \cup B$, so (32) holds.

Case 3. $A = S^I_n$, $B = \hat{T}^I_m$, ${\sim'}B = S^I_{-m}$. In this case $A \subset B$, so for (32) we need only show

$$\begin{aligned} B|'A = \hat{T}^I_m|S^I_n \cap \mathsf{Di}_I &= \begin{cases} S^I_n & \text{if } n \leq -m \\ \hat{T}^I_m & \text{if } n > -m \end{cases} && (28),\ \hat{T}^I_m \cup S^I_n \subseteq \mathsf{Di}_I \\ &= \begin{cases} A & \text{if } A = S^I_n \subseteq S^I_{-m} = {\sim'}B, \\ B & \text{if } A = S^I_n \supset S^I_{-m} = {\sim'}B. \end{cases} \end{aligned}$$

This completes the proof that $\langle \mathcal{C}_I, \cup, \cap, \to, {\sim} \rangle$ and $\langle \mathcal{C}'_I, \cup, \cap, \to', {\sim'} \rangle$ are Sugihara chains. Because of the match between the order types when $I = \mathbb{Z}$, illustrated in the remarks preceding Theorem 2, we also conclude that

$$\mathbf{S}_{\mathbb{Z}+\mathbb{Z}} \cong \langle \mathcal{C}_\mathbb{Z}, \cup, \cap, \to, {\sim} \rangle \cong \langle \mathcal{C}'_\mathbb{Z}, \cup, \cap, \to', {\sim'} \rangle,$$

which completes the proof of Theorem 2. □

The set of converses of a Sugihara chain is another Sugihara chain. Applying this observation to $\mathcal{C}_I$, we let

$$\begin{aligned}\breve{S}^I_n &= L_{[-n,\infty)},\\ \breve{T}^I_n &= L_{(-\infty,\infty)} \cup \mathsf{Id}_I \cup R_{(-\infty,n-1]},\\ \breve{\mathcal{C}}_I &= \{\breve{S}^I_n\colon -n \in I\} \cup \{\breve{T}^I_n\colon n \in I\}.\end{aligned}$$

Then $\breve{\mathcal{C}}_I$ is the other copy of the Sugihara chain $\mathcal{C}_I$ in $\mathfrak{S}_I$. Observe that

$$\mathcal{X}_I = \mathcal{C}_I \cup \breve{\mathcal{C}}_I \cup \{\mathsf{Id}_I, \mathsf{Di}_I, \emptyset, U^2\}$$

is closed under union, intersection, converse-complementation, and residuation. In addition, $\mathcal{X}_I$ is closed and commutative under relative multiplication. All the relations in $\mathcal{X}_I$ are dense. The only non-transitive relation in $\mathcal{X}_I$ is Di_I. When $I = \{0\}$, $\mathcal{X}_{\{0\}}$ coincides with the entire universe of $\mathfrak{S}_{\{0\}}$, reflecting the fact that the diversity relation $\neq$ is the only non-transitive relation in M_0.

7. Reducts, relation algebras, and atom structures

Definition 5. *A **definitional reduct** of an algebra $\mathfrak{A}$ is obtained by omitting some of the fundamental operations of $\mathfrak{A}$ and adding some operations that are term-definable in $\mathfrak{A}$. A **definitional subreduct** is a subalgebra of a definitional reduct.*

For example, when defined as in (1), Belnap's M_0 is a definitional reduct of Belnap's relation algebra $\mathfrak{M}_0$, but not conversely. They both have the same universe, but $\mathfrak{M}_0$ has operations not definable from the operations of M_0. We use two methods to obtain definitional subreducts, called **direct** and **relativized**. They apply to all relation algebras, although we will be primarily interested in applying them to proper relation algebras (Definition 1), so we review basic definitions and facts about relation algebras. Good resources for relation algebras are [37, 38, 43, 53], especially the first two.

Definition 6. *A **relation algebra** is an algebra $\mathfrak{A} = \langle A, +, \cdot, ^-, 0, 1, ;, ^\smile, 1'\rangle$, consisting of a set A, binary operations $+$ and $;$ on A, unary operations $^-$ and $^\smile$ on A, and a distinguished element $1' \in A$, called the **identity element** of $\mathfrak{A}$, such that $\langle A, +, \cdot, ^-, 0, 1\rangle$ is a Boolean algebra and $\mathfrak{A}$ satisfies the axioms*

$$(x;y);z = x;(y;z), \tag{r1}$$

$$(x+y);z = (x;z) + (x;z), \tag{r2}$$

$$x = x;1', \tag{r3}$$

$$\breve{\breve{x}} = x, \tag{r4}$$

$$(x+y)^\smile = \breve{x} + \breve{y}, \tag{r5}$$

$$(x;y)^\smile = \breve{y};\breve{x}, \tag{r6}$$

$$\breve{x};\overline{x;y} \leq \overline{y}, \tag{r7}$$

*where $x \leq y$ iff $x + y = y$. Define the **diversity element** by $0' = \overline{1'}$. An element $a \in A$ is an **atom** if $a \neq 0$ and for all $x \in A$, if $x \leq a$ then $x = 0$ or $x = a$. $\mathfrak{A}$ is **atomic** if for every non-zero element $x \in A$ there is an atom $a \in A$ such that $a \leq x$. An element $x \in A$ is **symmetric** if $\breve{x} = x$, **dense** if $x \leq x;x$, and **transitive** if*

*$x;x \leq x$. The algebra $\mathfrak{A}$ is **symmetric** if all its elements are symmetric, **dense** if its elements are dense, **commutative** if it satisfies $x;y = y;x$, and **Boolean** if $1' = 1$.*

Proper relation algebras are, indeed, relation algebras. Boolean relation algebras are symmetric and also satisfy $x;y = x \cdot y$. Their name is based on the observation, made after [47, Theorem 4.35], that if $\langle A, +, \cdot, ^-, 0, 1\rangle$ is a Boolean algebra and $\breve{x}$ is defined to be x, then $\langle A, +, \cdot, ^-, 0, 1, \cdot, ^\smile, 1\rangle$ is a relation algebra. Each of the identities $1' = 1$ and $x;y = x \cdot y$ characterizes Boolean relation algebras [38, Lemma 3.1]. Boolean relation algebras are representable [37, Theorem 17.5]. A relation algebra $\mathfrak{A}$ is **simple** (has no non-trivial homomorphic images) if and only if $1;x;1 = 1$ whenever $0 \neq x \in A$ [47, Theorem 4.10]. A relation algebra is **integral** (has no zero divisors) if and only if 1' is an atom [47, Theorem 4.17]. In every relation algebra, the **converse** $\breve{a}$ of an atom a is an atom [47, Theorem 4.3(xii)], which allows the following definition of atom structure. The definition of complex algebra is the special case of [47, Definition 3.8] that applies to relation algebras.

Definition 7. [52, Definitions 2.1, 3.2]

(i) *If $\mathfrak{U} = \langle U, R, f, I\rangle$ is a structure where $R \subseteq U^3$, $f\colon U \to U$, and $I \subseteq U$, then the **complex algebra** of $\mathfrak{U}$ is*

$$\mathfrak{Cm}(\mathfrak{U}) = \langle \wp(U), \cup, \cap, ^-, \emptyset, U, ;, ^\smile, I\rangle,$$

where $\wp(U)$ is the powerset of U, $\langle \wp(U), \cup, \cap, ^-, \emptyset, U\rangle$ is the Boolean algebra of all subsets of U, and for all $X, Y \subseteq U$, $\breve{X} = \{\breve{x} : x \in X\}$ and $X;Y = \{z : x;y \geq z \in At$ for some $x \in X$ and $y \in Y\}$.

(ii) *The **atom structure** of an atomic relation algebra $\mathfrak{A}$ is $\langle At, R, ^\smile, I\rangle$, where At is the set of atoms of $\mathfrak{A}$, $R = \{\langle x, y, z\rangle : x, y, z \in At,\ x;y \geq z\}$, and $I = \{u : 1' \geq u \in At\}$.*

The next theorem is the relation algebraic case of [47, Theorem 3.9]. The specific conditions were first stated earlier in [50, SS4] in a slightly different but equivalent form.

Theorem 3. [52, Theorem 2.2] *The complex algebra $\mathfrak{Cm}(\mathfrak{U})$ of a structure $\mathfrak{U} = \langle U, R, f, I\rangle$ is a complete and atomic Boolean algebra with operators, and $\mathfrak{Cm}(\mathfrak{U})$ is a relation algebra if and only if, for all $x, y, z \in U$,*

(i) *if $\langle x, y, z\rangle \in R$ then $\langle fx, z, y\rangle \in R$,*
(ii) *if $\langle x, y, z\rangle \in R$ then $\langle z, fy, x\rangle \in R$,*
(iii) *$x = y$ iff there is some $u \in I$ such that $\langle x, u, y\rangle \in R$,*
(iv) *if $\langle v, w, x\rangle, \langle x, y, z\rangle \in R$ then for some $u \in U$, $\langle v, u, z\rangle, \langle w, y, u\rangle \in R$.*

It follows from just (i), (ii), and (iii) that f is an **involution** on U ($ffx = x$ for all $x \in U$), $fx = x$ for all $x \in I$, R is the union of **cycles** [50, (1), p. 710], which are sets of the form

$$(33) \qquad [x, y, z] = \{\langle x, y, z\rangle, \langle z, fy, x\rangle, \langle fz, x, fy\rangle, \langle fy, fx, fz\rangle, \langle y, fz, fx\rangle, \langle fx, z, y)\rangle\},$$

and finally, the complex algebra satisfies axioms (r2)–(r8). The associative law (r1) is the only axiom that may fail, and (r1) holds if and only if (iv) holds. Every relation algebra has a complete and atomic extension, called its **perfect extension**, **canonical extension**, or **canonical embedding algebra** [48, Theorem 4.21], from which we get the following special case of [47, Theorem 3.10 (Representation Theorem)] that includes the appropriate conditions for relation algebras.

Theorem 4. [52, Theorems 3.13, 4.3] *A relation algebra is complete and atomic if and only if it is isomorphic to the complex algebra of its atom structure. An algebra* $\mathfrak{A} = \langle A, +, \cdot, ^-, 0, 1, ;, ^\smile, 1' \rangle$ *is a relation algebra if and only if it is isomorphic to a subalgebra of the complex algebra of a structure satisfying conditions* (i)*,* (ii)*,* (iii)*, and* (iv) *in Theorem* 3*.*

For an arbitrary relation algebra $\mathfrak{A}$, this structure may be constructed directly from $\mathfrak{A}$ as follows [56, Theorem 2.11]. An **ultrafilter** is a maximal proper subset $X \subseteq A$ such that $x \cdot y \in X$ whenever $x, y \in X$ and $x + y \in X$ whenever $x \in X$ and $y \in A$. Let U be the set of ultrafilters of $\mathfrak{A}$. Let R be the set of triples $\langle X, Y, Z \rangle$ of ultrafilters such that $X ; Y \subseteq Z$, let $f \colon U \to U$ be defined by $fX = \{\breve{x} : x \in X\}$, and let I be the set of ultrafilters that contain 1'. Then the desired **canonical atom structure** is $\langle U, R, f, I \rangle$. When $\mathfrak{A}$ is complete and atomic, the canonical atom structure is isomorphic to the atom structure of $\mathfrak{A}$.

Definition 8. *Let* $\mathfrak{A} = \langle A, +, \cdot, ^-, 0, 1, ;, ^\smile, 1' \rangle$ *be a relation algebra. Then* $\mathfrak{A}_r = \langle A, +, \cdot, \to, \sim \rangle$ *is the algebra obtained from* $\mathfrak{A}$ *by deleting* $^-$*,* 0*,* 1*,* ;*,* $^\smile$*, and* 1'*, retaining* $+$ *and* $\cdot$*, and adding operations* $\to$ *and* $\sim$*, defined by*

$$x \to y = \overline{\breve{x} ; \overline{y}}, \qquad \sim x = \overline{\breve{x}},$$

and $\mathfrak{A}'_r = \langle A, +, \cdot, \to', \sim' \rangle$ *is the algebra obtained from* $\mathfrak{A}$ *by deleting* $^-$*,* 0*,* 1*,* ;*,* $^\smile$*, and* 1'*, retaining* $+$ *and* $\cdot$*, and adding operations* $\to'$ *and* $\sim'$*, defined by*

$$x \to' y = \overline{(x \cdot 0')^\smile ; \overline{y \cdot 0'}} \cdot 0', \qquad \sim' x = \overline{(x \cdot 0')^\smile} \cdot 0'.$$

$\mathfrak{A}_r$ *is the* ***direct reduct*** *of* $\mathfrak{A}$*, and* $\mathfrak{A}'_r$ *is the* ***relativized reduct*** *of* $\mathfrak{A}$*. An algebra is a* ***direct subreduct*** *of* $\mathfrak{A}$ *if it is a subalgebra of the direct reduct of* $\mathfrak{A}$*, and a* ***relativized subreduct*** *of* $\mathfrak{A}$ *if it is a subalgebra of the relativized reduct of* $\mathfrak{A}$*.*

When applied to proper relation algebras, the operations $\to$, $\sim$, $\to'$, and $\sim'$ are the ones (with the same names) defined in Table 1. For example, when defined as in (1), Belnap's M_0 is the direct reduct of the proper relation algebra $\mathfrak{M}_0$. The next theorem is the main result of this paper. The part asserting that every finite Sugihara chain of even cardinality is isomorphic to a direct subreduct of a proper relation algebra was already proved in [55, Theorem 6.2]. The two innovations that allow us to extend this result to infinite Sugihara chains and to finite Sugihara chains of odd cardinality are sequences that are eventually zero and relativized subreducts.

Theorem 5. *For every* $I \subseteq \mathbb{Z}$*,* $\mathfrak{S}_I$ *is a proper relation algebra such that*

(i) *the Sugihara chain* $\langle \mathcal{C}_I, \cup, \cap, \to, \sim \rangle$ *is a direct subreduct of* $\mathfrak{S}_I$*, and*
(ii) *the Sugihara chain* $\langle \mathcal{C}'_I, \cup, \cap, \to', \sim' \rangle$ *is a relativized subreduct of* $\mathfrak{S}_I$*.*
(iii) $\mathbf{S}_{\mathbb{Z}^*}$*, Sugihara's original* $\mathbf{S}_{\mathbb{Z}+\mathbb{Z}}$*, and all finite Sugihara chains of even cardinality are isomorphic to both direct subreducts and relativized subreducts of the proper relation algebra* $\mathfrak{S}_{\mathbb{Z}}$*.*
(iv) $\mathbf{S}_{\mathbb{Z}}$ *and all Sugihara chains of odd cardinality are isomorphic to relativized subreducts of the proper relation algebra* $\mathfrak{S}_{\mathbb{Z}^+}$*, where* $\mathbb{Z}^+ = \{n : 0 < n \in \mathbb{Z}\}$*.*

Proof. Recall that $\mathcal{S}_I$ is the universe of the Sugihara relation algebra $\mathfrak{S}_I$. By Definition 3, (7), (8), and Definition 4, we have $\mathcal{C}_I \subseteq \mathcal{S}_I$ and $\mathcal{C}'_I \subseteq \mathcal{S}_I$. Furthermore, $\mathcal{C}_I$ is closed under $\cup$, $\cap$, $\rightarrow$, and $\sim$, and $\mathcal{C}'_I$ is closed under $\cup$, $\cap$, $\rightarrow'$, and $\sim'$ since $\langle \mathcal{C}_I, \cup, \cap, \rightarrow, \sim\rangle$ and $\langle \mathcal{C}'_I, \cup, \cap, \rightarrow', \sim'\rangle$ are Sugihara chains by Theorem 2. It follows by Definition 8 that $\langle \mathcal{C}_I, \cup, \cap, \rightarrow, \sim\rangle$ is a direct subreduct of $\mathfrak{S}_I$ and $\langle \mathcal{C}'_I, \cup, \cap, \rightarrow', \sim'\rangle$ is a relativized subreduct of $\mathfrak{S}_I$. Thus (i) and (ii) hold. By Theorem 2,

$$\langle \mathcal{C}_{\mathbb{Z}}, \cup, \cap, \rightarrow, \sim\rangle \cong \langle \mathcal{C}'_{\mathbb{Z}}, \cup, \cap, \rightarrow', \sim'\rangle \cong \mathbf{S}_{\mathbb{Z}+\mathbb{Z}},$$

so $\mathbf{S}_{\mathbb{Z}+\mathbb{Z}}$ is isomorphic to both a direct and a relativized subreduct of $\mathfrak{S}_{\mathbb{Z}}$. All of the subalgebras of $\mathbf{S}_{\mathbb{Z}+\mathbb{Z}}$ are therefore also isomorphic to direct and relativized subreducts of $\mathfrak{S}_{\mathbb{Z}}$. This includes $\mathbf{S}_{\mathbb{Z}^*}$ and all finite Sugihara chains of even cardinality, thus proving (iii).

For the Sugihara chains of odd cardinality we proceed differently. Suppose I has a minimum element $m \in I \subseteq \mathbb{Z}$. This means that $\{n : m > n \in I\} = \emptyset$. Therefore $L_{(-\infty,m-1]} = \bigcup \emptyset = \emptyset$ by (7) and $R_{[m,\infty)} = R_{(-\infty,\infty)}$ by (8), so by (24),

$$S^I_{-m} = R_{[m,\infty)} = R_{(-\infty,\infty)} \cup L_{(-\infty,m-1]} = \hat{T}^I_m = {\sim'}(S^I_{-m}).$$

Thus the relation S^I_{-m} is fixed by $\sim'$. If I is also infinite, then $\langle \mathcal{C}'_I, \cup, \cap, \rightarrow', \sim'\rangle$ is isomorphic to $\mathbf{S}_{\mathbb{Z}}$, by an isomorphism that sends the fixed point $S^I_{-m} = \hat{T}^I_m$ to 0, which is the fixed point of negation in $\mathbf{S}_{\mathbb{Z}}$. Taking $I = \mathbb{Z}^+$, we see that $\mathbf{S}_{\mathbb{Z}}$ is isomorphic to a relativized subreduct of $\mathfrak{S}_{\mathbb{Z}^+}$. Every subalgebra of $\mathbf{S}_{\mathbb{Z}}$ is also isomorphic to a relativized subreduct of $\mathfrak{S}_{\mathbb{Z}^+}$. This includes all finite Sugihara chains of odd cardinality, so (iv) holds. □

Theorem 5 shows that every finite Sugihara chain $\mathbf{S}$ is isomorphic to a chain of binary relations closed under the relevant operations. If the identity relation is included in the relations occurring in the top half of this chain (this is the direct method), then there cannot be a relation fixed by $\sim$ and $\mathbf{S}$ has even cardinality. If $\mathbf{S}$ is odd it can be represented by the relativized method purely with diversity relations. The element in the middle of $\mathbf{S}$ is mapped to a relation that is its own relativized converse-complement. For example, Belnap's relation algebra $\mathfrak{S}_{\{0\}}$ has two Sugihara chains of length 3, namely $\{\emptyset, <, \neq\}$ and $\{\emptyset, >, \neq\}$ (see Figure 1 and Table 2). Note how the relations in the middle, namely $<$ and $>$, are fixed by $\sim'$. The Sugihara chains of length 3 are isomorphic to RM3, described on [4, p. 470] and [67, p. 92]. This is yet another algebra of relevance logic that can be represented as an algebra of binary relations. Sugihara chains of even cardinality are subalgebras of those with odd cardinality. Thus the normal Sugihara chains are isomorphic to definitional subreducts of proper relation algebras by both the direct and relativized methods, while non-normal ones need the relativized method.

From Theorem 5 we know that $\mathbf{S}_{\mathbb{Z}^*}$ ("the Sugihara matrix" of Anderson and Belnap) is isomorphic to a direct subreduct of $\mathfrak{S}_{\mathbb{Z}^+}$. In [14] there is a computation intended to show that this is not possible. On [14, p. 123],

> "Figure 5 shows some components of the canonical embedding algebra of the Sugihara matrix $\mathbf{S}_{\mathbb{Z}^*}$ ... Unfortunately, this Boolean algebra is not a relation algebra, let alone a transitive or a representable one. To show that (r6) is not true, we give a concrete counterexample."

The ensuing computation at the bottom of [14, p. 122] ends with $\{[i) : i \leq -2\}$, but should end with $\{[i) : i \geq -2\}$. When corrected, it confirms an instance of axiom

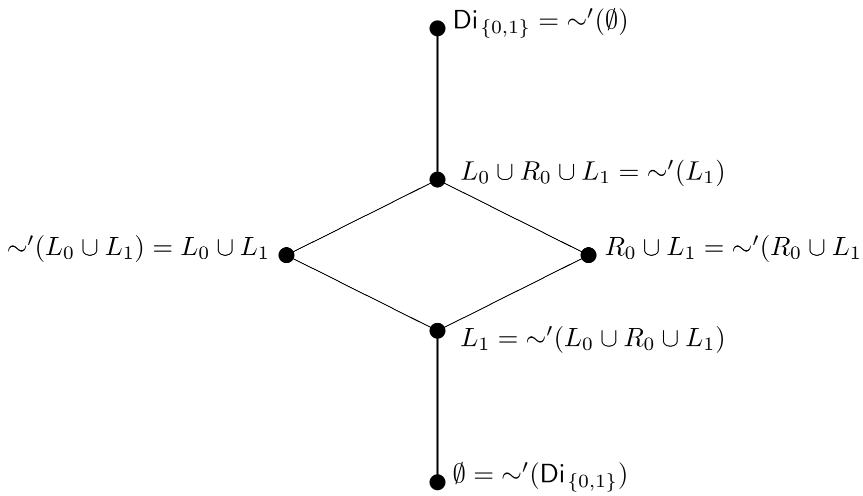

FIGURE 2. The crystal lattice

(r6) in Definition 6. It was reasonable to suspect this equation may not hold, because it corresponds to a property of atom structures of relation algebras not shared by the model structures of R. That property, identified and called "tagging" by Dunn [30], says that if $\langle x, y, z\rangle \in R$ then $\langle fy, fx, fz\rangle \in R$. In Theorem 3, either one of (i) and (ii) can be replaced by tagging. None of these three conditions necessarily holds in a model structure for R. Such structures do satisfy the condition that if $\langle x, y, z\rangle \in R$ then $\langle fz, x, fy\rangle \in R$. This condition, together with any one of of the three conditions (i), (ii), and tagging, can be used in Theorem 3 instead of (i) and (ii), because any of these combinations are enough to prove that R is a union of cycles (33). To obtain relevant model structures for logics like R, one must add the conditions expressing density, that $\langle x, x, x\rangle \in R$ for all x, and commutativity, that if $\langle x, y, z\rangle \in R$ then $\langle y, x, z\rangle \in R$.

By Theorem 5, the canonical embedding algebra of the Sugihara matrix $\mathbf{S}_{\mathbb{Z}^*}$ is, in fact, isomorphic to the complete atomic proper relation algebra $\mathfrak{S}_{\mathbb{Z}^+}$, hence also isomorphic to the complex algebra of the canonical atom structure of $\mathfrak{S}_{\mathbb{Z}^+}$. Although $\mathfrak{S}_{\mathbb{Z}^+}$ is commutative and dense, not all of its elements are transitive. For example, the diversity relation $\mathsf{Di}_{\mathbb{Z}^+}$ is not transitive. However, $\mathfrak{S}_{\mathbb{Z}^+}$ does have subsets that contain only transitive (and dense) relations and are closed under the relevant operations. As we have seen, a copy of the Sugihara matrix $\mathbf{S}_{\mathbb{Z}^*}$ is among them.

8. THE CRYSTAL LATTICE, CHURCH'S DIAMOND, AND MEYER'S RM84

The crystal lattice. The crystal lattice first appears in Routley [65], where it is attributed to R. K. Meyer; see [18, pp. 65–6], [67, p. 250], and [70, pp. 95–7]. By [70, Theorems 9.8.1, 9.8.3], the crystal lattice $\mathbf{Cr}$ is characteristic for the finitely axiomatized logic CL [70, p. 114]. We can obtain the crystal lattice from $\mathfrak{S}_{\{0,1\}}$, which is isomorphic to the relation algebra 2_{83}, the second of 83 relation algebras listed in [53, Ch. 6, SSSS62–3].

Theorem 6. *The crystal lattice is isomorphic to a relativized subreduct of* $\mathfrak{S}_{\{0,1\}}$.

$X \to' Y$	$\mathsf{Di}_{\{0,1\}}$	$L_0 \cup R_0 \cup L_1$	$L_0 \cup L_1$	$R_0 \cup L_1$	L_1	$\emptyset$
$\mathsf{Di}_{\{0,1\}}$	$\mathsf{Di}_{\{0,1\}}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$L_0 \cup R_0 \cup L_1$	$\mathsf{Di}_{\{0,1\}}$	L_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$L_0 \cup L_1$	$\mathsf{Di}_{\{0,1\}}$	$L_0 \cup L_1$	$L_0 \cup L_1$	$\emptyset$	$\emptyset$	$\emptyset$
$R_0 \cup L_1$	$\mathsf{Di}_{\{0,1\}}$	$R_0 \cup L_1$	$\emptyset$	$R_0 \cup L_1$	$\emptyset$	$\emptyset$
L_1	$\mathsf{Di}_{\{0,1\}}$	$L_0 \cup R_0 \cup L_1$	$L_0 \cup L_1$	$R_0 \cup L_1$	L_1	$\emptyset$
$\emptyset$	$\mathsf{Di}_{\{0,1\}}$	$\mathsf{Di}_{\{0,1\}}$	$\mathsf{Di}_{\{0,1\}}$	$\mathsf{Di}_{\{0,1\}}$	$\mathsf{Di}_{\{0,1\}}$	$\mathsf{Di}_{\{0,1\}}$

$X \vert' Y$	L_0	R_0	L_1	R_1
L_0	L_0	$L_0 \cup R_0$	L_1	R_1
R_0	$L_0 \cup R_0$	R_0	L_1	R_1
L_1	L_1	L_1	L_1	$L_0 \cup R_0 \cup L_1 \cup R_1$
R_1	R_1	R_1	$L_0 \cup R_0 \cup L_1 \cup R_1$	R_1

TABLE 3. Tables for the crystal lattice and $\mathfrak{S}_{\{0,1\}}$

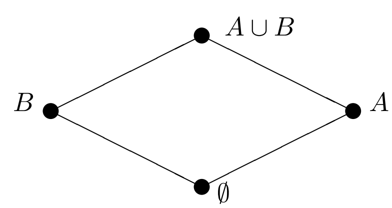

FIGURE 3. The Church diamond

Proof. For a copy of the crystal lattice in $\mathfrak{S}_{\{0,1\}}$, let

$$Cr = \{\emptyset,\ L_1,\ L_0 \cup L_1,\ R_0 \cup L_1,\ L_0 \cup R_0 \cup L_1,\ \mathsf{Di}_{\{0,1\}}\},$$
$$\mathbf{Cr} = \langle Cr, \cup, \cap, \to', \sim' \rangle.$$

Inspection shows Cr is closed under union, intersection, relativized residuation, and relativized converse-complementation. Comparison with [67, p. 250] or [70, pp. 95–7] shows $\mathbf{Cr}$ is the crystal lattice. The Hasse diagram and the action of $\sim'$ are shown in Figure 2, while $\to'$ is given in Table 3. Cr is the union of two Sugihara chains of length 5 that intersect in all but one relation. To get these two chains, delete either $L_0 \cup L_1$ or $R_0 \cup L_1$ from Cr. Cr is also a set of generators for $\mathfrak{S}_{\{0,1\}}$ (since conversion and complementation are allowed). Table 3 shows the relativized relative products of the diversity atoms of $\mathfrak{S}_{\{0,1\}}$. □

$\mathbf{Cr}$ is used in [67, Theorem 3.22] for a proof of the variable-sharing property that is simpler because it uses a smaller algebra, with only six elements instead of eight, and the 2-element chains $\{<, \leq\}$ and $\{>, \geq\}$ in Belnap's proof are replaced by singletons $\{L_0 \cup L_1\}$ and $\{R_0 \cup L_1\}$.

The Church lattice. The Church lattice [67, p. 379] is also called Church's diamond [68, p. 277].

Theorem 7. *The Church lattice is the relativized reduct of a proper relation algebra on any set with 9 or more elements.*

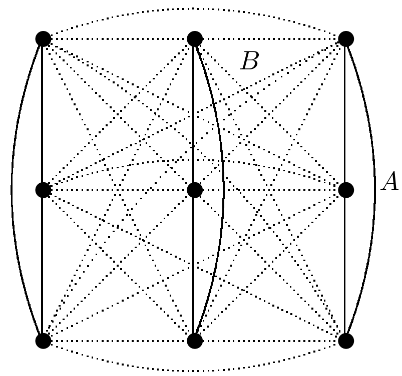

FIGURE 4. Atoms of Church's relation algebra: A = pairs connected by solid lines, B = pairs connected by dotted lines

$X \to' Y$	$A \cup B$	A	B	$\emptyset$	$\sim'$
$A \cup B$	$A \cup B$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
A	$A \cup B$	A	B	$\emptyset$	B
B	$A \cup B$	$\emptyset$	A	$\emptyset$	A
$\emptyset$	$A \cup B$	$A \cup B$	$A \cup B$	$A \cup B$	$A \cup B$

$X\|Y$	Id	A	B
Id	Id	A	B
A	A	$\mathsf{Id} \cup A$	B
B	B	B	$\mathsf{Id} \cup A \cup B$

TABLE 4. Tables for Church's relation algebra

Proof. On any set U with at least 9 elements, let V_1, V_2, and V_3 be a partition of U into pairwise disjoint sets, each containing at least 3 elements. In the 9-element case, V_1, V_2, and V_3 are arranged in three columns as in Figure 4. Let

$$U = V_1 \cup V_2 \cup V_3, \quad \mathsf{Id} = \{\langle u, u\rangle : u \in U\}, \quad \mathsf{Di} = \{\langle u, v\rangle : u, v \in U,\, u \neq v\},$$

$$A = \mathsf{Di} \cap \big((V_1)^2 \cup (V_2)^2 \cup (V_3)^2\big), \quad B = \bigcup\{V_i \times V_j : 1 \leq i, j \leq 3,\, i \neq j\}.$$

Then $\{\mathsf{Id}, A, B\}$ is a partition of U^2 into relations that are symmetric. The eight unions of subsets of $\{\mathsf{Id}, A, B\}$ form a proper relation algebra $\mathfrak{Ch}$ with $\{\mathsf{Id}, A, B\}$ as its set of atoms. As noted in SS2, $\mathfrak{Ch}$ is isomorphic to relation algebra 4_7 [53, Ch. 6, SS56.13]. The relative products of atoms are shown in Table 4. Unions of symmetric relations are symmetric, so $\mathfrak{Ch}$ is a symmetric proper relation algebra, called **Church's relation algebra**. Symmetric relation algebras are commutative because, by axiom (r6) and the symmetry of both the factors and the relative product, $x;y = (x;y)^{\smile} = \breve{y};\breve{x} = y;x$. Not all symmetric relation algebras are dense, but $\mathfrak{Ch}$ is dense. The four diversity relations form Church's diamond,

$$Ch = \{A \cup B, A, B, \emptyset\},$$

$$\mathbf{Ch} = \langle Ch, \cup, \cap, \to', \sim' \rangle,$$

with a Hasse diagram in Figure 3. Tables for $\to'$ and $\sim'$ are in Table 4. □

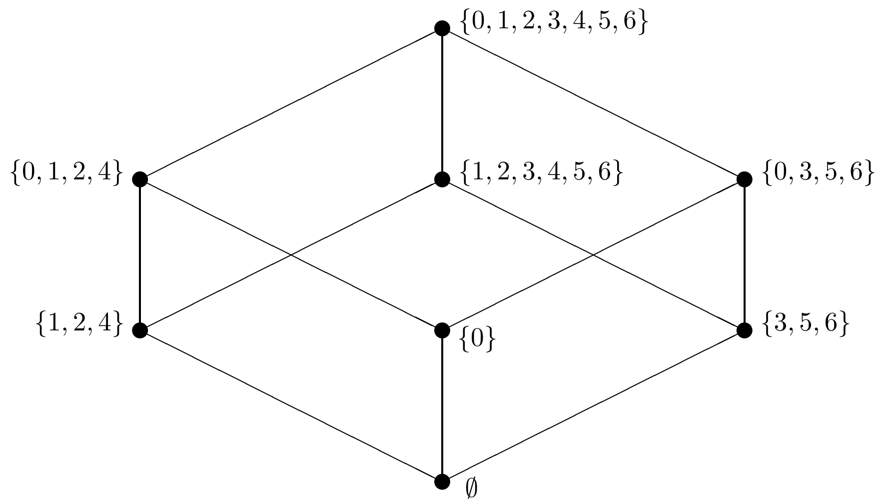

Figure 5. Hasse diagram for RM84

The Church lattice **Ch** validates the logic KR, which is axiomatized by axioms (R1)–(R13) in Table 6 along with $(X \wedge {\sim}X) \to Y$. The Lindenbaum algebra of KR is a relation algebra [14, Lemma 6.7]. (The method for creating what is here called a "Lindenbaum" algebra is due to Tarski [71], [73, Ch. XII]; see [39], [58, p. 122, footnote 7], [42, p. 85, footnote 4], [41, p. 169, footnote 2].) **Ch** shows that KR is "crypto-relevant" [67, p. 379], which means that the variable sharing property holds for a formula $X \to Y$ if the only connective appearing is $\to$. To show this, assign the variables in X to $A \cup B$ and the variables in Y to A. Then X and Y are mapped to $A \cup B$ and A since these are fixed by the operation $\to'$, but $(A \cup B) \to' A = \emptyset$ and the designated elements are $A \cup B$ and A, so $X \to Y$ is not valid in **Ch**.

Meyer's RM84. Anderson and Belnap [4, p. 334] present Meyer's lattice, but they do not give it a name. Instead, "RM84" is their name for Meyer's theorem [4, p. 417], which says that if $X \to Y$ is a theorem of RM then either X and Y share a variable or both ${\sim}X$ and Y are theorems of RM. When Routley, Plumwood, Meyer, and Brady [67, p. 253] present Meyer's lattice, they call it "RM84", as is done here. In [67, Theorem 3.26] they show RM84 verifies all theorems of R, but fails to satisfy any of eight particular formulas that happen to be theorems of RM. The proper relation algebra $\mathfrak{Rm}$, described here by subsets of the cyclic group of order 7 and called **Meyer's relation algebra**, is isomorphic to relation algebra 3_3 [53, Ch. 6, SS58.8].

Theorem 8. [55, Theorem 4.2] RM84 *is the relativized reduct the proper relation algebra* $\mathfrak{Rm}$*.*

Proof. Let $U = \{0,1,2,3,4,5,6\}$, $D = \{1,2,3,4,5,6\}$, and

$$\mathcal{R} = \{U, D, \{0,1,2,4\}, \{0,3,5,6\}, \{1,2,4\}, \{3,5,6\}, \{0\}, \emptyset\}.$$

We use $\mathcal{R}$ as an index set for eight binary relations on U. For $x, y \in U$, let $x \equiv_7 y$ mean $y - x$ is divisible by 7, and for every $X \subseteq U$, define a relation on U by

$$\rho(X) = \{\langle y, z\rangle \colon y, z \in U,\ z + x \equiv_7 y \text{ for some } x \in X\}.$$

X	$\sim X$
$\emptyset$	U
$\{3,5,6\}$	$\{0,3,5,6\}$
$\{1,2,4\}$	$\{0,1,2,4\}$
$D = \{1,2,3,4,5,6\}$	$\{0\}$
$\{0\}$	D
$\{0,1,2,4\}$	$\{1,2,4\}$
$\{0,3,5,6\}$	$\{3,5,6\}$
$U = \{0,1,2,3,4,5,6\}$	$\emptyset$

$X\|Y$	$\{0\}$	$\{1,2,4\}$	$\{3,5,6\}$
$\{0\}$	$\{0\}$	$\{1,2,4\}$	$\{3,5,6\}$
$\{1,2,4\}$	$\{1,2,4\}$	D	U
$\{3,5,6\}$	$\{3,5,6\}$	U	D

$X \to Y$	$\emptyset$	$\{3,5,6\}$	$\{1,2,4\}$	D	$\{0\}$	$\{0,1,2,4\}$	$\{0,3,5,6\}$	U
$\emptyset$	U	U	U	U	U	U	U	U
$\{3,5,6\}$	$\emptyset$	$\{0\}$	$\emptyset$	$\{0,3,5,6\}$	$\emptyset$	$\emptyset$	$\{0\}$	U
$\{1,2,4\}$	$\emptyset$	$\emptyset$	$\{0\}$	$\{0,1,2,4\}$	$\emptyset$	$\{0\}$	$\emptyset$	U
D	$\emptyset$	$\emptyset$	$\emptyset$	$\{0\}$	$\emptyset$	$\emptyset$	$\emptyset$	U
$\{0\}$	$\emptyset$	$\{3,5,6\}$	$\{1,2,4\}$	D	$\{0\}$	$\{0,1,2,4\}$	$\{0,3,5,6\}$	U
$\{0,1,2,4\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{1,2,4\}$	$\emptyset$	$\{0\}$	$\emptyset$	U
$\{0,3,5,6\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{3,5,6\}$	$\emptyset$	$\emptyset$	$\{0\}$	U
U	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	U

TABLE 5. Tables for RM84

Then $\{\rho(X) \colon X \in \mathcal{R}\}$ is the universe of the proper relation algebra $\mathfrak{Rm}$, which is an 8-relation subalgebra of $\mathfrak{Re}(U)$. Figure 5 shows the Hasse diagram for sets in $\mathcal{R}$ and their images under ρ. The images of $\{0\}$, $\{1,2,4\}$, and $\{3,5,6\}$ are atoms of $\mathfrak{Rm}$. The converse-complements and relative products in Table 5 are stated in terms of sets in $\mathcal{R}$. The entry for $X, Y \in \mathcal{R}$ is the set $Z \in \mathcal{R}$ such that $\rho(Z) = \rho(X)|\rho(Y)$. Converse-complements and relative products can also be computed directly by the rules $\sim X = \{0 -_7 x : x \notin X, x \in U\}$ and $X|Y = \{x +_7 y \colon x \in X, y \in Y\}$, where $-_7$ and $+_7$ are subtraction and addition *modulo* 7. As described here, RM84 is the direct reduct of $\mathfrak{Rm}$. □

9. A relational completeness theorem for R-mingle

The logic R-mingle, or RM, was created by Dunn and McCall from Anderson and Belnap's relevance logic R by adding the mingle axiom $A \to (A \to A)$; see [4, SS8.15, SS27.1.1]. The rules of deduction for both R and RM are *Adjunction* (infer $A \wedge B$ from A and B) and *modus ponens* (infer B from $A \to B$ and A). An axiom set for RM is shown in Table 6; see [4, p. 341] or [5, pp. xxiii–xxvi].

If **S** is a Sugihara chain and the connectives of RM are interpreted as the corresponding operations (with the same names) in **S**, then any function from the propositional variables to elements of **S** extends uniquely to a homomorphism from the algebra of formulas to elements of **S**. A formula is **valid in S** if it is sent to a designated element by every such homomorphism. Meyer [4, pp. 413–4, Corollaries

(R1)	$A \to A$
(R2)	$(A \to B) \to ((B \to C) \to (A \to C))$
(R3)	$A \to ((A \to B) \to B)$
(R4)	$(A \to (A \to B)) \to (A \to B)$
(R5)	$(A \wedge B) \to A$
(R6)	$(A \wedge B) \to B$
(R7)	$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
(R8)	$A \to (A \vee B)$
(R9)	$B \to (A \vee B)$
(R10)	$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
(R11)	$(A \wedge (B \vee C)) \to ((A \wedge B) \vee C)$
(R12)	$(A \to {\sim}B) \to (B \to {\sim}A)$
(R13)	${\sim}{\sim}A \to A$
(R14)	$A \to (A \to A)$

TABLE 6. Axioms of RM

3.1, 3.5] proved that the theorems of RM are the formulas valid in all finite Sugihara chains, and that the theorems of RM are the formulas valid in $\mathbf{S}_{\mathbb{Z}^*}$. These results, together with Theorem 2, imply that RM is complete with respect to the following class of algebras.

Definition 9. *Let* $\mathfrak{K} = \langle K, \cup, \cap, \to, \sim \rangle$*, where*

(k1) *K is a non-empty set of binary relations on a set U, called the* ***base*** *of $\mathfrak{K}$,*
(k2) *K is closed under the operations $\cup$, $\cap$, $\to$, and $\sim$, defined in Table 1 using the base U.*

A formula A is ***valid in*** *the algebra $\mathfrak{K}$ if, for every homomorphism h from the algebra of formulas to $\mathfrak{K}$, $h(A)$ contains the identity relation on the base of $\mathfrak{K}$. Let K_{RM} be the class of algebras $\mathfrak{K} = \langle K, \cup, \cap, \to, \sim \rangle$ such that* (k1)*,* (k2)*, and*

(k3) *$A|B = B|A$ for all $A, B \in K$,*
(k4) *$A \subseteq A|A$ for all $A \in K$,*
(k5) *$A|A \subseteq A$ for all $A \in K$.*

A formula is ***valid in*** *K_{RM} if it is valid in every algebra in K_{RM}.*

Since K is not empty by (k1), the algebra $\mathfrak{K}$ determines the base according to the formula $U = \{a : \langle a, a \rangle \in A \cup {\sim}A,\ A \in K\}$. Condition (k2) implies that $\mathfrak{K}$ is also closed under $|$, since $A|B = {\sim}(B \to {\sim}A)$ by (30). From Theorem 2 and Meyer's results we get the following completeness theorem.

Theorem 9 ([55, Theorem 6.2(iii)])**.** *The theorems of* RM *are the formulas valid in K_{RM}.*

Proof. The axioms of RM are valid in K_{RM} by Theorem 10 below. Validity is preserved by Adjunction, for if $\mathsf{Id} \subseteq A$ and $\mathsf{Id} \subseteq B$ then $\mathsf{Id} \subseteq A \cap B$, and validity is preserved by *modus ponens*, for if $\mathsf{Id} \subseteq A \to B$ and $\mathsf{Id} \subseteq A$, then $\mathsf{Id} = \mathsf{Id}|\mathsf{Id} \subseteq$

$A|(A \to B) \subseteq B$ by Lemma 4(ii) below. Therefore all theorems of RM are valid in K_{RM}. For the converse, suppose X is not a theorem of RM. By Meyer's theorems, X fails in $\mathbf{S}_{\mathbb{Z}^*}$, $\mathbf{S}_{\mathbb{Z}+\mathbb{Z}}$, and in every sufficiently large finite normal Sugihara chain. By Theorem 2, these algebras are isomorphic to algebras in K_{RM}. Therefore there are algebras in K_{RM} in which X fails to be valid. $\square$

We assume for the rest of this section that U is a set and $\mathfrak{K} = \langle K, \cup, \cap, \to, \sim \rangle$ is an algebra satisfying conditions (k1) and (k2) of Definition 9. In each formula, we interpret the connectives $\vee$, $\wedge$, $\to$, and $\sim$ as the operations $\cup$, $\cap$, $\to$, and $\sim$, respectively. Thus every formula denotes a relation that depends on the interpretation of its variables. A formula is valid in $\mathfrak{K}$ if it denotes a relation that contains Id, the identity relation on U, no matter how its variables are interpreted. Implications are analyzed as inclusions because of the following lemma.

Lemma 3 ([55, Theorem 5.1(17)]). *For all $A, B \subseteq U^2$, $\mathsf{Id} \subseteq A \to B$ iff $A \subseteq B$.*

According to Lemma 3, the validity of each axiom of RM can be equivalently expressed as an inclusion between binary relations. For example, (R1) is valid in $\mathfrak{K}$ just because the inclusion $A \subseteq A$ always holds. Evidently (R1), (R5), (R6), (R7), (R8), (R9), (R10), (R11), and (R13) are true under the set-theoretical meanings assigned to the connectives, by [55, Theorem 5.1(32), (33), (34), (35), (36), (37), (38), (39), (40)], respectively. To analyze the remaining axioms of RM we recall some other results from [55].

Lemma 4. [55, Theorem 5.1(18)(19)(21)(22)] *For all $A, B, C \subseteq U^2$,*

(i) $A \to (B \to C) = (B|A) \to C$,
(ii) $A|(A \to B) \subseteq B$,
(iii) *if* $A \subseteq B$ *then* $B \to C \subseteq A \to C$ *and* $C \to A \subseteq C \to B$.

Axioms (R1), (R5)–(R11), and (R13) are valid for all binary relations. The remaining five axioms do not hold for all relations, but will hold under conditions on the relations that occur in them, and in some cases are equivalent to those conditions. We now analyze (R2), (R3), (R4), (R12), and (R14). By [55, Theorem 5.1(55)], (R2) holds whenever $B \to C$ and $A \to B$ commute, but (R2) also holds under the weaker hypothesis of Lemma 5 below, because inclusion in only one direction is needed. (R2) holds if $\mathfrak{K}$ is commutative under relative multiplication, but fails in some non-commutative examples that have 16 relations. On the other hand, (R2) is valid when recast as a rule of inference, for if $A \to B$ contains the identity relation then so does $(B \to C) \to (A \to C)$ [55, Theorem 5.1(29)]. This also follows immediately from Lemma 3 and Lemma 4(iii).

Lemma 5. *For all $A, B, C \subseteq U^2$, if $(B \to C)|(A \to B) \subseteq (A \to B)|(B \to C)$ then $A \to B \subseteq (B \to C) \to (A \to C)$ and* (R2) *is valid, but the converse may fail.*

Proof. First we prove the assumption implies the validity of (R2).

$(A\|(B \to C))\|(A \to B)$	
$= A\|((B \to C)\|(A \to B))$	$\|$ is associative
$\subseteq A\|((A \to B)\|(B \to C))$	assumption, $\|$ is monotonic
$= (A\|(A \to B))\|(B \to C)$	$\|$ is associative
$\subseteq B\|(B \to C)$	Lemma 4(ii), $\|$ is monotonic

$$\subseteq C, \qquad \text{Lemma 4(ii)}$$

hence

$$\begin{aligned} C \to C &\subseteq ((A|(B \to C))|(A \to B)) \to C && \text{Lemma 4(iii)} \\ &= (A \to B) \to ((A|(B \to C)) \to C) && \text{Lemma 4(i)} \\ &= (A \to B) \to ((B \to C) \to (A \to C)). && \text{Lemma 4(i)} \end{aligned}$$

Since $\mathsf{Id} \subseteq C \to C$, it follows that (R2) is valid, *i.e.*,

$$\mathsf{Id} \subseteq (A \to B) \to ((B \to C) \to (A \to C)).$$

By Lemma 3, $A \to B \subseteq (B \to C) \to (A \to C)$. However, the assumption and this conclusion are not equivalent. To see this, let $C = U^2$. Then $A \to C = U^2$, hence

$$(B \to C) \to (A \to C) = (B \to C) \to U^2 = U^2,$$

so the conclusion holds for all A and B. Since $A \to C = U^2$, the assumption becomes $U^2|\,(A \to B) \subseteq (A \to B)\,|U^2$. But this inclusion will fail whenever $A \to B$ is not empty and has a domain that is not all of U. □

By [55, Theorem 5.1(54)], (R3) holds whenever A and $A \to B$ commute. In fact, it holds under a weaker hypothesis to which it is not equivalent.

Lemma 6. *If* $(A \to B)|A \subseteq A|(A \to B)$ *then* (R3) *is valid. The converse may fail.*

Proof. We have $(A \to B)|A \subseteq B$ by the hypothesis and Lemma 4(ii). Let $C = A \to B$, so that $C|A \subseteq B$. This formula can be rewritten as $\overline{B} \cap (C|A) = \emptyset$. This is equivalent to $(C^{-1}|\overline{B}) \cap A = \emptyset$, which is in turn equivalent to $A \subseteq \overline{C^{-1}|\overline{B}}$, but $\overline{C^{-1}|\overline{B}} = C \to B$, so $A \subseteq C \to B$. Hence $A \subseteq (A \to B) \to B$ and (R3) is valid by Lemma 3. This conclusion does not imply the hypothesis, for if $B = U^2$, then $A \to B = \overline{A^{-1}|\overline{B}} = \overline{A^{-1}|\overline{U^2}} = U^2$, so the hypothesis is equivalent to $U^2|A \subseteq A|U^2$, which fails if A is a relation on U whose domain is not all of U. On the other hand, the conclusion of Lemma 6 holds since $(A \to B) \to B = \overline{(U^2)^{-1}|\overline{U^2}} = U^2$. □

Lemma 7. (R4) *is valid if and only if* A *is dense.*

Proof. By [55, Theorem 5.1(56)], (R4) is valid whenever A is a dense relation, for if $A \subseteq A|A$ then $A \to (A \to B) \subseteq A \to B$. Suppose (R4) is valid when $B = {\sim}\mathsf{Id}$. Since $A \to {\sim}\mathsf{Id} = \overline{A^{-1}|\overline{{\sim}\mathsf{Id}}} = \overline{A^{-1}|\mathsf{Id}^{-1}} = {\sim}A$, (R4) is equivalent to $(A \to {\sim}A) \to {\sim}A$, which is valid if and only if $A \to {\sim}A \subseteq {\sim}A$, by Lemma 3. This last inclusion can be equivalently transformed by the definitions of $\sim$ and $\to$ first into $\overline{A^{-1}|\overline{{\sim}A}} \subseteq \overline{A^{-1}}$, then $A^{-1} \subseteq A^{-1}|A^{-1}$, and finally $A \subseteq A|A$, which asserts that A is dense. □

The contraposition axiom (R12) is valid whenever $A|B = B|A$ by [55, Theorem 5.1(53)], but it is actually equivalent to $A|B \subseteq B|A$.

Lemma 8. (R12) *is valid if and only if* $A|B \subseteq B|A$.

Proof. By Lemma 3, (R12) is valid if and only if $A \to {\sim}B \subseteq B \to {\sim}A$ for all $A, B \subseteq U^2$. Since $A \to {\sim}B = \overline{A^{-1}|\overline{{\sim}B}} = \overline{A^{-1}|B^{-1}}$ and $B \to {\sim}A = \overline{B^{-1}|A^{-1}}$, this inclusion is equivalent to $B^{-1}|A^{-1} \subseteq A^{-1}|B^{-1}$. Taking converses of both sides, we get the equivalent inclusion $A|B \subseteq B|A$. □

By [55, Theorem 5.1(63)], (R14) holds if A is a transitive relation, but (R14) is actually equivalent to the transitivity of A.

Lemma 9. (R14) *is valid if and only if A is transitive.*

Proof. By Lemma 3, (R14) is valid if and only if $A \subseteq A \to A$. This inclusion can be equivalently restated first as $A \subseteq \overline{A^{-1}|\overline{A}}$, then $A \cap (A^{-1}|\overline{A}) = \emptyset$, then $A|A \cap \overline{A} = \emptyset$, and finally $A|A \subseteq A$, which asserts that A is transitive. □

The following theorem gathers together the observations above and confirms that the axioms of RM are valid in K_{RM}, completing the proof of Theorem 9.

Theorem 10. *Let $\mathfrak{K} = \langle K, \cup, \cap, \to, \sim \rangle$ be an algebra satisfying conditions* (k1) *and* (k2) *in Definition 9. Then*

(i) (R1), (R5), (R6), (R7), (R8), (R9), (R10), (R11), (R13) *are valid in $\mathfrak{K}$,*
(ii) (R2) *and* (R3) *are valid in $\mathfrak{K}$ if* (k3)*, but neither is equivalent to* (k3)*,*
(iii) (R4) *is valid in $\mathfrak{K}$ if and only if* (k4)*,*
(iv) (R12) *is valid in $\mathfrak{K}$ if and only if* (k3)*,*
(v) (R14) *is valid in $\mathfrak{K}$ if and only if* (k5)*.*

10. Interpreting formulas as relations

Theorems 9 and 10 suggest an alternative approach to RM. Instead of adopting 14 axioms and two rules, simply define RM as the set of formulas valid in K_{RM}. It is then a theorem that RM can be axiomatized by (R1)–(R14) and the rules of Adjunction and *modus ponens*.

To explore the theorems and rules of RM, assume that $\mathfrak{K} = \langle K, \cup, \cap, \to, \sim \rangle$ is an algebra satisfying (k1) and (k2) in Definition 9. Even if $\mathfrak{K}$ does not satisfy (k3), (k4), or (k5), nine of the axioms of RM are valid in $\mathfrak{K}$ by Theorem 10(i). Since $A \cup {\sim}A$ always contains the identity relation on U, $A \vee {\sim}A$ is also valid in $\mathfrak{K}$. Thus $A \vee {\sim}A$ is a theorem of RM.

Simple counterexamples show that $B \to (A \vee {\sim}A)$ and $(A \wedge {\sim}A) \to C$ need not be valid in $\mathfrak{K}$ and are not theorems of RM. Counterexamples for relations in general can be found on a 2-element set, but for RM it is more appropriate to use the Sugihara chain $\{\emptyset, <, \leq, \mathbb{Q}^2\}$ from Table 2, which is an algebra in K_{RM}. Just let A, B, and C be $<$, $\mathbb{Q}^2$, and $\emptyset$, respectively. The formula $((A \vee C) \wedge {\sim}A) \to C$ expressing Extensional Disjunctive Syllogism also fails under the same assignment, so it is also not a theorem of RM.

The proof of Theorem 9 shows the rules of Adjunction and *modus ponens* preserve validity. Unlike the corresponding axiom, Extensional Disjunctive Syllogism (to infer B from $A \vee B$ and ${\sim}A$) is admissible. To show this, assume $\mathsf{Id} \subseteq A \cup B$ and $\mathsf{Id} \subseteq {\sim}A$. The second hypothesis is equivalent to $\mathsf{Id} \subseteq \overline{A^{-1}}$. Taking the converse of both sides, we get $\mathsf{Id} \subseteq \overline{A}$. By the first hypothesis, $\mathsf{Id} \subseteq \overline{A} \cap (A \cup B) = \overline{A} \cap B \subseteq B$. Intensional Disjunctive Syllogism is to infer B from $A + B$ and ${\sim}A$, where $A + B$ is intensional disjunction. Since $A + B$ is defined as ${\sim}A \to B$ [4, SS27.1.4], this rule is an instance of *modus ponens*. The E-rule [70, p. 8], also called BR1 [67, p. 289] and R5 [70, p. 193], is to infer $(A \to B) \to B$ from A. Assume $\mathsf{Id} \subseteq A$. Then $\overline{B} = \mathsf{Id}^{-1}|\overline{B} \subseteq A^{-1}|\overline{B} = \overline{A \to B}$, hence $A \to B \subseteq B$, so $\mathsf{Id} \subseteq (A \to B) \to B$ by Lemma 3. The admissibility of Suffixing, Contraposition, and several other rules can be proved similarly at this point, without any appeal to commutativity, density, or transitivity.

To achieve RM, assume that $\mathfrak{K}$ also satisfies (k3), (k4), and (k5), so that $\mathfrak{K} \in K_{\mathsf{RM}}$. Then axioms (R2), (R3), (R4), (R12), and (R14) are also valid in $\mathfrak{K}$ by Theorem 10(ii)(iii)(iv)(v). Alternate proofs of the admissibility of various rules, such as the E-rule, Contraposition, and Suffixing, are possible using commutativity. A significant example of a theorem of RM that requires all three hypotheses of commutativity, density, and transitivity is $(A \to B) \vee (B \to A)$. Meyer called this formula RM64, "Simple order". Discussing its significance, he wrote [4, pp. 397–8],

> "RM63 and RM64, in fact, decide that RM represents a much longer step in the direction of classical logic (and, for that matter, in the direction of an extensional approach to sentential logic) than one would have thought from the heuristic considerations by which we motivated its axioms and rules. ...
>
> "RM64 leaves shattered in the dust much of the motivation to which previous opponents of the paradoxes have appealed. But this just goes to show that one can have many reasons for disliking the paradoxes; one very plausible ground for disliking them is that they turn every minor inconsistency into a catastrophe. From this charge, RM is yet free. If in other respects it moves in the direction of classical logic, there is as yet no reason to rue that fact."

The following lemma gives a relational proof that $(A \to B) \vee (B \to A)$ is a theorem of RM.

Lemma 10. *If* $\langle K, \cup, \cap, \to, \sim\rangle \in K_{\mathsf{RM}}$ *then* $\mathsf{Id} \subseteq (A \to B) \cup (B \to A)$ *for all* $A, B \in K$.

Proof. Note that K is closed under $|$ by (30). Assume $A, B \in K$, and let $C = (A \to A) \cap (B \to B)$. Apply $\sim$ to both sides and use (30) to get

$$\sim C = \sim(A \to A) \cup \sim(B \to B) = (\sim A|A) \cup (\sim B|B).$$

It follows that $\sim A|A \subseteq \sim C$ and $\sim B|B \subseteq \sim C$, so $(\sim A|A)|(\sim B|B) \subseteq \sim C|\sim C$ by the monotonicity of $|$. By the associativity of $|$ and our assumption that $|$ is commutative on relations in K, $(\sim A|B)|(\sim B|A) \subseteq \sim C|\sim C$. Let $D = (\sim A|B) \cap (\sim B|A)$. From $D, \sim C \in K$ it follows that D is dense and $\sim C$ is transitive, so by the monotonicity of $|$,

$$D \subseteq D|D \subseteq (\sim A|B)|(\sim B|A) \subseteq \sim C|\sim C \subseteq \sim C.$$

By applying $\sim$ to both sides, (30), and Lemma 3, we conclude that

$$\mathsf{Id} \subseteq C \subseteq \sim D = (B \to A) \cup (A \to B).$$

□

RM64 is one reason given by Anderson and Belnap for the title of [4, SS29.5], "Why we don't like mingle." They describe how to prove RM64, using axioms and rules, from the "unhappy theorem" $A \to (\sim A \to A)$. Their suggestions for proving the latter formula include the mingle axiom, contraposition, and permutation, thereby invoking both transitivity and commutativity, but commutativity is not needed. By Lemma 3, $A \to (\sim A \to A)$ is valid if $A \subseteq \sim A \to A$ for every relation $A \in K$. Since $\sim A \to A = \sim(\sim A|\sim A)$, this inclusion is equivalent to $\sim A|\sim A \subseteq \sim A$, which asserts that $\sim A$ is transitive, as is indeed the case for every $A \in K$.

The Routley-Meyer semantics are called relational because every relevant model structure contains a ternary relation. Instead of a ternary relation, the RM model structures of Dunn [29] have a binary accessibility relation, which corresponds to the inclusion relation in the Sugihara chain $\mathcal{C}_I$ [29, SS7]. In both cases, the elements of the model structures are objects with no further structure. Formulas are interpreted as sets of unstructured objects. This feature is advantageous because it provides more general interpretations than semantics with special objects. For example, Dunn [31] asks,

> "What could be more natural than to interpret *Rabc* as that in the context of the information a, the information b is relevant to the information c?"

Three more examples (with variations) are presented by the eleven authors of [10], based on three ways of grouping the arguments of R, called Modal (Absence-of-Counterexample) Conditionals: $Rx\langle yz\rangle$, Conditionals as Operators: $R\langle xy\rangle z$, and Conditional Logics: $Rx\rangle y\langle z$. These interpretations address some issues, explained on [10, p. 599].

> "The story goes like this: whereas the binary relation invoked by Kripke in the semantics of modal logics has several philosophically interesting and revealing interpretations (as relative possibility, or as a temporal ordering, or as the relation of being-morally-ideal-from-the-point-of-view-of, or ...), the ternary relation invoked by Routley and Meyer has no such standardly accepted interpretations/applications. 'Sure,' the objector says (it helps here to imagine the hint of a sneer), 'there are mathematical structures of the sort described by Routley and Meyer, and those structures bear important and interesting relations to the logics described by Anderson and Belnap, but these logics were supposed to tell us something interesting about *conditionality*, or at least some important kind of conditionality, and it would take more than just abstract mathematical structures to tell us *that.* I want to know what it is that *instantiates* these structures that has anything to do with conditionals.' "

Although "we say nothing about negation" and "this paper isn't about negation" [10, footnote 4], elsewhere Dunn [31] observed,

> "The 'Routley-Star' has come under a lot of criticism both from those within and outside of the relevance logic community, and was more of a focus of Copeland's [26] critical review than the ternary accessibility relation."

This is reflected by van Benthem [74], in his review of Copeland [26].

> "Relevance logic is a subject whose motivation has turned out to be surprisingly difficult to capture in an enlightening and convincing semantics. ...the only general approach to date is a rather abstract possible-worlds framework, proposed by R. and V. Routley. Here, relevant implication is explicated through some ternary 'perspective' relation among worlds, while the account of negation employs an additional 'reversal' operation upon worlds. It fails to satisfy the 'requirements which distinguish an illuminating and philosophically significant semantics from a merely formal model theory.' "

Much later in the review, van Benthem observes,

> "Postulated operations in model structures may be viewed as theoretical terms, truth-definitions rather as some kind of correspondence principles. Any demand for 'realism' ought to take these different roles into account.
>
> Even in this more charitable perspective, the Routley semantics still has to prove its mettle. On the realistic side, its model structures ought to admit of, if not a natural linguistic anchoring, then at least one mathematical 'standard example', providing some food for independent reflection."

In our analysis, formulas are interpreted as sets of objects that *do* have structure. Each object is a binary relation. Theorem 9 says that a formula is a theorem of RM if and only if it contains the identity relation when regarded as a relation belonging to a set of transitive dense binary relations that commute under relative multiplication. We call this the formulas as relations approach.

Suppose $\mathfrak{A}$ is an atomic proper relation algebra on a set U. Let $\mathfrak{U} = \langle At, R, {}^{-1}, I\rangle$ be the atom structure of $\mathfrak{A}$. If $\mathfrak{A}$ happens to be commutative and dense, then $\mathfrak{U}$ is a relevant model structure. The commutative dense atomic proper relation algebras discussed in this paper are Belnap's relation algebra $\mathfrak{M}_0$ in SS2, Sugihara's relation algebra $\mathfrak{S}_{\mathbb{Z}}$ in SS4 and, more generally $\mathfrak{S}_I$ for any $I \subseteq \mathbb{Z}$ in Theorem 1, Church's relation algebra $\mathfrak{Ch}$ in Theorem 7, and Meyer's relation algebra $\mathfrak{Rm}$ in Theorem 8. In any case, the atoms in At are relations on U, the unary operation ${}^{-1}$ of $\mathfrak{U}$ produces the converse of each atom, the distinguished element I of $\mathfrak{U}$ is the set of atoms contained in the identity relation of U, and the ternary relation R of $\mathfrak{U}$ is the set of triples $\langle a, b, c\rangle$ of relations that are atoms of $\mathfrak{A}$ and satisfy $a|b \supseteq c$. In brief, the ternary relation is set-theoretically defined as $a|b \supseteq c$ and the Routley star * is conversion ${}^{-1}$. These interpretations of R and * are complete for RM according to Theorem 9. They may satisfy the objector described in [10], who wants "to know what it is that *instantiates* these structures". For RM at least, we could echo Dunn and ask, what could be more natural than to interpret $Rabc$ as $a|b \supseteq c$? For van Benthem and Copeland we suggest interpreting the "additional 'reversal' operation" as conversion. In the light of Theorem 9, RM could serve as a mathematical "standard example" sought by van Benthem.

11. Summary and problems

Table 7 summarizes our results that some finite lattices with operators and all subalgebras of three countably infinite Sugihara chains are isomorphic to definitional subreducts of proper relation algebras. The widespread occurrence of representability where it was not previously suspected could lead to further thoughts of a fundamental, perhaps philosophical nature. This idea is elaborated in Problem 4, and there is further mathematical work proposed in Problems 1, 2, and 3.

Problem 1. *What other algebras in the relevance logic literature are isomorphic to definitional subreducts of proper relation algebras? For example, is every uncountable Sugihara chain isomorphic to a definitional subreduct of a proper relation algebra?*

Remarks on Problem 1. Each of the logics CL, BM, and RM is characterized by a single lattice.

Name	Method	Type	PRA	Full PRA
Belnap M_0	Direct	reduct	$\mathfrak{S}_{\{0\}}$	$\mathfrak{Re}(U_{\{0\}})$
Sugihara $\mathbf{S}_{\mathbb{Z}+\mathbb{Z}}$, $\mathbf{S}_{\mathbb{Z}^*}$, finite even	Direct	subreduct	$\mathfrak{S}_{\mathbb{Z}}$	$\mathfrak{Re}(U_{\mathbb{Z}})$
Sugihara $\mathbf{S}_{\mathbb{Z}+\mathbb{Z}}$, $\mathbf{S}_{\mathbb{Z}}$, $\mathbf{S}_{\mathbb{Z}^*}$, finite	Relativized	subreduct	$\mathfrak{S}_{\mathbb{Z}}$	$\mathfrak{Re}(U_{\mathbb{Z}})$
crystal **Cr**	Relativized	subreduct	$\mathfrak{S}_{\{0,1\}}$	$\mathfrak{Re}(U_{\{0,1\}})$
Church **Ch**	Relativized	reduct	$\mathfrak{Ch}$	$\mathfrak{Re}(9)$
Meyer RM84	Direct	reduct	$\mathfrak{Rm}$	$\mathfrak{Re}(7)$

TABLE 7. Definitional subreducts of proper relation algebras (PRAs)

- The crystal lattice **Cr** is characteristic for CL,
- Belnap's M_0 is characteristic for BM, and
- each of the Sugihara chains $\mathbf{S}_{\mathbb{Z}^*}$, $\mathbf{S}_{\mathbb{Z}}$, and $\mathbf{S}_{\mathbb{Z}+\mathbb{Z}}$ is characteristic for RM.

These lattices can be represented as algebras of subsets of relevant model structures. This was done for CL by two relevant model structures, one with 45 triples of elements of $\{\mathsf{T}, \mathsf{T}^*, \mathsf{a}, \mathsf{a}^*\}$, and the other with 49, the largest possible number of triples that can be used for this purpose [70, pp. 95–100]. Both structures produce the table on [70, p. 97]. The relevant model structure in Table 3 is

$$\langle\{L_0, R_0, L_1, R_1\}, C, {}^{-1}, \emptyset\rangle,$$
$$C = \{\langle a, b, c\rangle\colon a, b, c \in \{L_0, R_0, L_1, R_1\},\ a|'b \supseteq c\}.$$

It has only 24 triples, and is isomorphic to the restriction of the canonical atom structure of $\mathfrak{S}_{\{0,1\}}$ to the diversity atoms. The table for $\to'$ coincides with the table on [70, p. 97] when $\mathsf{T} = L_1$, $\mathsf{T}^* = R_1$, $\mathsf{a} = L_0$, and $\mathsf{a}^* = R_0$. Other numbers of triples besides 24, 45, and 49 work, but the choice made here has the feature that the ternary relation holds among binary relations, instead of unstructured objects. The ternary relation is set-theoretically defined as "the relativized relative product of the first two contains the third", and the Routley star * is conversion $^{-1}$. Similarly, BM was characterized in [70, pp. 100–104] by a single finite relevant model structure with 13 triples of elements of $\{\mathsf{T}, \mathsf{a}, \mathsf{a}^*\}$, which is isomorphic to the canonical atom structure of $\mathfrak{S}_{\{0\}}$. In the notation of SSSS2–3, that structure is

$$\langle\{<, >, =\}, C, {}^{-1}, \{=\}\rangle$$
$$C = \{\langle a, b, c\rangle\colon a, b, c \in \{<, >, =\},\ a|b \supseteq c\}.$$

The points T, a, and a^* match up with the binary relations $=$, $<$, and $>$ on the rationals. The Routley-Meyer ternary relation in this case is "the relative product of the first two contains the third", and the Routley star is conversion. The atom structure of the complete atomic proper relation algebra $\mathfrak{S}_{\mathbb{Z}}$ is

$$\langle\mathcal{A}t_{\mathbb{Z}}, C, {}^{-1}, \{\mathsf{Id}_{\mathbb{Z}}\}\rangle,$$
$$C = \{\langle a, b, c\rangle\colon a, b, c \in \mathcal{A}t_{\mathbb{Z}},\ a|b \supseteq c\}.$$

This relevant model structure is characteristic for RM, and its ternary relation is the product-inclusion relation.

In the previous examples, the ternary relation of the Routley-Meyer semantics is the product-inclusion relation, possibly relativized. The logic KR is different. The atom structure of the canonical extension of the free symmetric dense relation algebra on countably many generators is a relevant model structure characteristic for KR. The same structure can be constructed by letting the atoms be maximal

KR-theories. In both cases the atoms are not binary relations, nor can they be represented as relations, because there are symmetric dense relation algebras that are not representable. For example, there are three non-representable symmetric dense relation algebras with four atoms, but none smaller. The 65 symmetric relation algebras with four atoms are numbered 1_{65}–65_{65} in [53]. The three that are non-representable and dense are 36_{65}, 42_{65}, and 50_{65}.

On the other hand, many symmetric dense relation algebras *are* representable, such as Church's proper 3-atom relation algebra $\mathfrak{Ch}$. The atom structure of $\mathfrak{Ch}$ is a relevant model structure verifying KR that is characteristic for a complete decidable extension of KR. Once again, its ternary relation is the relativized product-inclusion relation and its Routley star is conversion. Problem 1 asks how far this kind of analysis can be extended. What other algebras can be represented with binary relations?

Problem 2. *If a relation algebra is finite, integral, possibly commutative, and every one of its diversity atoms is dense, transitive, and distinct from its converse, must that algebra be representable?*

Remarks on Problem 2. There are three relation algebras with five atoms that contain the crystal lattice. In the numbering system of [53], they are 2_{83}, 29_{83}, and 43_{83} (the second, twenty-ninth, and forty-third algebras in a list of 83 algebras in [53, Ch. 6, SSSS62-3] whose atoms are the identity element 1', plus two diversity atoms r, s, and their converses $\breve{r}$, $\breve{s}$). Algebras 2_{83} and 43_{83} are commutative, but 29_{83} is not commutative. The two commutative algebras are representable. In fact, 2_{83} is isomorphic to $\mathfrak{S}_{\{0,1\}}$. What about the non-commutative algebra 29_{83}? Is it representable? This is the smallest particular instance of Problem 2.

Problem 3. *Explore the structure of algebras in K_{RM}. Does the traditional axiomatic approach to* RM *yield a finite equational axiomatization for the variety generated by K_{RM}?*

Remarks on Problem 3. $\mathfrak{S}_I \in K_{\mathsf{RM}}$ for every $I \subseteq \mathbb{Z}$. Preliminary investigation shows K_{RM} has many algebras that are not linearly ordered. What else is in K_{RM}?

Problem 4. *Do the product-inclusion relations $A|B \supseteq C$ and $A|'B \supseteq C$ on binary relations have any bearing on the concepts of relevance and conditionality? Do the residuations $A \to B$ and $A \to' B$ have any bearing on entailment? How do the De Morgan negations $\sim A$ and $\sim' A$ compare and contrast with the Boolean negation $\overline{A}$? How are relative multiplication and conversion related to fusion and the Routley star? Is K_{RM} a mathematical standard example of Routley-Meyer semantics?*

Remarks on Problem 4. We will comment on each of the questions in Problem 4 in the order they occur. We start with a ternary relation on ordered pairs. On [10, p. 599], the authors say,

> "[I]n the semantics given by Routley and Meyer, the crucial ternary relation R is involved in the semantics as follows: for any sentences A and B at any point x in any model M:

(R) $x \models_M A \to B$ iff for all y, z such that $Rxyz$, if $y \models_M A$ then $z \models_M B$.

> . . . In order to provide a philosophically illuminating semantics of the relevant conditional, we need to say more about what these models are: what the points are, what the ternary relation R

> is, and why compound sentences—in particular conditionals—are evaluated in the way that they are. What's more, this explication had better make it clear how these models relate to conditionality; otherwise the semantics can be fairly accused of arbitrariness, or of ad hocness, or of simply copying the phenomenon to be explained. In short, the semantics are 'merely formal' and philosophically unilluminating—at least if we want to understand the *meaning* of a conditional. So more is required."

Later, on [10, p. 601], they suggest,

> "What's going on (or what may be seen as such) is that our conditional calls for a broader perspective on our universe of candidate counterexamples; it calls us to recognize 'pair points' in addition to our 'old' points."

We follow their lead and use ordered pairs as pair points. Let R be the set of triples of the form $\langle\langle a,b\rangle,\langle c,a\rangle,\langle c,b\rangle\rangle$, and let $x=\langle a,b\rangle$, $y=\langle c,a\rangle$, and $z=\langle c,b\rangle$. Then $Rxyz$ holds, so the phrase "such that $Rxyz$" may be deleted from (R), and "for all y,z" can be replaced by "for all c". We also replace x,y,z with the pairs $\langle a,b\rangle$, $\langle c,a\rangle$, and $\langle c,b\rangle$, respectively, and, to reduce the clutter, we drop the angle brackets, the commas, and the subscript M. The result is

(→) $ab\models A\to B$ iff for all c, if $ca\models A$ then $cb\models B$.

Clauses for the other connectives are

(∨) $ab\models A\vee B$ iff $ab\models A$ or $ab\models B$,

(∧) $ab\models A\wedge B$ iff $ab\models A$ and $ab\models B$,

(∼) $ab\models{\sim}A$ iff $ba\not\models A$.

Commutativity, density, and transitivity, which are required for RM, can be added in three more clauses,

(comm) if $ab\models A$ and $bc\models B$ then, for some d, $ad\models B$ and $dc\models A$,

(dense) if $ab\models A$ then, for some c, $ac\models A$ and $cb\models A$,

(trans) if $ab\models A$ and $bc\models A$ then $ac\models A$.

If we wish to have connectives $\neg$ for Boolean negation and * for Routley star, we add two more clauses.

(¬) $ab\models\neg A$ iff $ab\not\models A$,

(*) $ab\models A^*$ iff $ba\models A$.

Finally, add a clause defining validity in M (reinstated as a subscript),

(valid) $\models_M A$ iff for all a, $aa\models_M A$.

By Theorem 9, the clauses (→), (∨), (∧), (∼), (comm), (dense), and (trans) yield sound and complete semantics for RM. To get sound and complete semantics for "classical" RM in the sense of [60, 61], add connectives $\neg$ and * to the language, and add clauses (¬) and (*). The question is, what do these semantics say about conditionality, or entailment, or Boolean and De Morgan negation? Do they say anything about relevance?

To illustrate the connection between fusion and relative multiplication, recall that $\circ$ is defined by $A \circ B = {\sim}(A \to {\sim}B)$ [4, p. 269], [5, p. xxiii]. When formulas are regarded as relations, $A \circ B = B|A$ by (30). Consider the triple $\langle x, y, z\rangle = \langle ab, ca, cb\rangle \in R$, and note that $\{y\}|\{x\} = \{ca\}|\{ab\} = \{cb\} = \{z\}$, hence $\{x\}\circ\{y\} = \{z\}$, in conformity with the usual connection between fusion and the Routley-Meyer ternary relation [59, (v) p. 414]. Of course, the distinction between fusion $\circ$ and relative multiplication $|$ disappears under the assumption of commutativity.

Fusion appears as an associative and sometimes commutative operation in various algebras arising from relevance logics. Algebraization is mathematically illuminating, but it is open to the charge that "... algebraic characterizations ... are merely formal, exhibiting no connection with the intended meanings of the logical constants," [26, p. 405]. Algebras of subsets of relevant model structures do interpret $\vee$ and $\wedge$ as union and intersection, but the other connectives arise abstractly from the ternary relation R and the unary operation * according to (R) and $x \models {\sim}A$ iff $x^* \not\models A$. "If the only constraint on * is that the resulting theory should validate the right set of sentences, then we are indeed in the presence of merely formal model theory," [26, p. 410], and "... it is completely obscure what meaning is given to negation in the Routley-Meyer theory ...," [26, p. 408]. For RM, according to the interpretation of formulas as relations, the Routley star is conversion and the meaning of negation is ${\sim}A = \overline{A^{-1}}$. Anderson and Belnap ask [4, p. 345], "How then to interpret $\circ$? We confess puzzlement." For RM, the answer is $A \circ B = B|A$. Are these answers "merely formal"? Is the meaning of negation "completely obscure"? Do these answers help us understand fusion and star?

Perhaps the semantics of RM provided by K_{RM} is a mathematical "standard example". Maybe the semantics of BM and CL provided by $\mathfrak{S}_{\{0\}}$ and $\mathfrak{S}_{\{0,1\}}$ are also mathematical examples. The historical difference is that the logics BM and CL were built around M_0 and **Cr**, unlike RM, which arose entirely through choices of axioms based on purely logical considerations. Nevertheless, these choices led to K_{RM}, giving interpretations for fusion and Routley star drawn from nineteenth century algebraic logic, rather than simply constrained so that "the resulting theory should validate the right set of sentences." Unlike Belnap's M_0, the Point Algebra did not arise from relevance logic, and would have been intensely studied even if relevance logic never existed. Similarly, the definition of K_{RM} is independent of relevance logic, in spite of having been discovered by a careful analysis of RM.

Works that may be useful for Problem 4 include [4, 5, 10, 12, 13, 21, 22, 23, 26, 27, 28, 31, 32, 33, 60, 61, 64, 66, 75, 76, 77].

12. Concluding remarks

In the introduction [14, SS1], the concept of dynamic semantics is described.

> "Collections of *binary relations* can be viewed as a sort of dynamic interpretation for a logic, that is thought to describe the impact a sentence has on a situation via specifying a set of possible resulting situations. Special types of *dynamic semantics* are those in which the binary relations constitute a relation algebra or a relevant family of operations."

Comparison of definitions shows that a relevant family of relations ("operations" was a misprint) is a direct reduct of a proper relation algebra on a set. The concluding remarks begin [14, SS7],

> "As the reader has surely realized by now, constructing a dynamic semantics with certain closure properties is not a trivial enterprise because of the nonrepresentability result of Lyndon (1950)."

Of course, for non-representable relation algebras the construction of dynamic semantics for their logics is not possible. The first part of this sentence was an understatement for such cases. However, no non-representable relation algebra appears in [14], certainly not Lyndon's [50], nor is there any formula valid in proper relation algebras that is not also a theorem of R. For such formulas, consult [55, 62]. In other cases, such as the logic BM, dynamic semantics are possible because M_0 is the direct reduct of the Point Algebra, which was noted in [14]. As we have shown, there are dynamic semantics for the logic CL and the logic of Meyer's RM84 (which has not been axiomatized, so far as we know).

For any particular relevant model structure, it may not be readily apparent whether it is representable as a definitional reduct of a proper relation algebra. When it is, its logic has a dynamic semantics. Unless non-representability has been proved, it is not safe to assume that dynamic semantics cannot be found; see the remarks following Theorem 5. Similarly, any particular formula has a meaning if regarded as a statement about binary relations. In the theory of relation algebras, this meaning matters. It was the target of the axiomatizations by McKinsey in 1940 [57] and Tarski in 1941 [72], directed as they were at the Peirce-Schröder calculus of binary relations.

The concept of representability was present in the theory of relation algebras from its inception, as the title of the 1948 abstract [46] makes clear, but was absent from relevance logic until 2007. The representability of finite Sugihara matrices and the resulting dynamic semantics for RM are only a decade old. Even today it is possible to construct new and unsuspected dynamic semantics for rather old algebras and logics, as has been done here. The 1952 Jónsson-Tarski Representation Theorem [47, 3.10], stated here as Theorem 4, could be regarded as a successful application of the Routley-Meyer ternary relation and the Routley star, one that provides Routley-Meyer semantics for all relation algebras. But the Jónsson-Tarski Representation Theorem preceded the introduction of Routley-Meyer semantics by two decades. Shortly before his death, Meyer was informed of the relational completeness theorem for RM [55, Theorem 6.2]. His response was an email message that ended with "KEEP'EM COMING". In this paper we have tried to do so.

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References

[1] James F. Allen. An interval-based representation of temporal knowledge. In *Proceedings of the Seventh International Joint Conference on Artificial Intelligence, (IJCAI)*, pages 221–226, 1981.

[2] James F. Allen. Towards a general theory of action and time. *Artificial Intelligence*, 23(2):123–154, July 1984.

[3] James F. Allen. Maintaining knowledge about temporal intervals. *Communications of the Association for Computing Machinery*, 26(11):832–842, November 1983.

[4] Alan Ross Anderson and Nuel D. Belnap, Jr. *Entailment. The logic of relevance and necessity. Vol. I.* Princeton University Press, Princeton, N. J.-London, 1975.

[5] Alan Ross Anderson, Nuel D. Belnap, Jr., and J. Michael Dunn. *Entailment. The logic of relevance and necessity. Vol. II.* Princeton University Press, Princeton, NJ, 1992.
[6] Hajnal Andréka and Roger D. Maddux. Representations for small relation algebras. *Notre Dame J. Formal Logic*, 35(4):550–562, 1994.
[7] Arnon Avron. RM and its nice properties. In *J. Michael Dunn on information based logics*, volume 8 of *Outst. Contrib. Log.*, pages 15–43. Springer, [Cham], 2016.
[8] Silvana Badaloni and Massimiliano Giacomin. The algebra $\mathrm{IA}^{\mathrm{fuz}}$: a framework for qualitative fuzzy temporal reasoning. *Artificial Intelligence*, 170(10):872–908, 2006.
[9] Philippe Balbiani, Jean-Franccois Condotta, and Gérard Ligozat. On the consistency problem for the INDU calculus. *J. Appl. Log.*, 4(2):119–140, 2006.
[10] Jc Beall, Ross Brady, J. Michael Dunn, A. P. Hazen, Edwin Mares, Robert K. Meyer, Graham Priest, Greg Restall, David Ripley, John Slaney, and Richard Sylvan. On the ternary relation and conditionality. *J. Philos. Logic*, 41(3):595–612, 2012.
[11] Nuel D. Belnap, Jr. Entailment and relevance. *J. Symbolic Logic*, 25:144–146, 1960.
[12] K. Bimbo and J. M. Dunn. The emergence of set-theoretical semantics for relevance logics around 1970. *Proceedings of the Third Workshop, May 16–17, 2016, Edmonton, Canada, (IFCoLog Journal of Logics and Their Application)*, 4(3):557–589, 2017.
[13] Katalin Bimbó and J. Michael Dunn. *Generalized Galois logics*, volume 188 of *CSLI Lecture Notes*. CSLI Publications, Stanford, CA, 2008.
[14] Katalin Bimbó, J. Michael Dunn, and Roger D. Maddux. Relevance logics and relation algebras. *Rev. Symb. Log.*, 2(1):102–131, 2009.
[15] Manuel Bodirsky. Finite relation algebras with normal representations. In *Relational and algebraic methods in computer science. 17th international conference, RAMiCS 2018, Groningen, The Netherlands, October 29 – November 1, 2018. Proceedings*.
[16] Manuel Bodirsky and Hubie Chen. Qualitative temporal and spatial reasoning revisited. *J. Logic Comput.*, 19(6):1359–1383, 2009.
[17] Manuel Bodirsky and Jan Kára. A fast algorithm and Datalog inexpressibility for temporal reasoning. *ACM Trans. Comput. Log.*, 11(3):Art. 15, 21, 2010.
[18] Ross T. Brady. Depth relevance of some paraconsistent logics. *Stud. Log.*, 43:63–73, 1984.
[19] Mathias Broxvall. The point algebra for branching time revisited. In *KI 2001: Advances in artificial intelligence (Vienna)*, volume 2174 of *Lecture Notes in Comput. Sci.*, pages 106–121. Springer, Berlin, 2001.
[20] Mathias Broxvall and Peter Jonsson. Point algebras for temporal reasoning: algorithms and complexity. *Artificial Intelligence*, 149(2):179–220, 2003.
[21] John P. Burgess. Relevance: a fallacy? *Notre Dame J. Formal Logic*, 22(2):97–104, 1981.
[22] John P. Burgess. Common sense and "relevance". *Notre Dame J. Formal Logic*, 24(1):41–53, 1983.
[23] John P. Burgess. No requirement of relevance. In Stewart Shapiro, editor, *Oxford Handbook of Philosophy of Mathematics and Logic*, pages 727–750. Oxford University Press, 2005.
[24] Louise H. Chin and Alfred Tarski. Distributive and modular laws in the arithmetic of relation algebras. *Univ. California Publ. Math. (N.S.)*, 1:341–384, 1951.
[25] Jean-Franccois Condotta, Souhila Kaci, and Yakoub Salhi. Optimization in temporal qualitative constraint networks. *Acta Inform.*, 53(2):149–170, 2016.
[26] B. J. Copeland. On when a semantics is not a semantics: some reasons for disliking the Routley-Meyer semantics for relevance logic. *J. Philos. Logic*, 8(4):399–413, 1979.
[27] B. J. Copeland. The trouble Anderson and Belnap have with relevance. *Philos. Stud.*, 37(4):325–334, 1980.
[28] B. J. Copeland. What is a semantics for classical negation? *Mind*, 95(380):478–490, 1986.
[29] J. Michael Dunn. A Kripke-style semantics for R-mingle using a binary accessibility relation. *Stud. Log.*, 35:163–172, 1976.
[30] J. Michael Dunn. A representation of relation algebras using Routley-Meyer frames. In *Logic, Meaning and Computation*, volume 305 of *Synthese Lib.*, pages 77–108. Kluwer Acad. Publ., Dordrecht, 2001.
[31] J. Michael Dunn. The relevance of relevance to relevance logic. In *Logic and its applications*, volume 8923 of *Lecture Notes in Comput. Sci.*, pages 11–29. Springer, Heidelberg, 2015.
[32] Kit Fine. Models for entailment. *J. Philos. Logic*, 3:347–372, 1974.
[33] Dov M. Gabbay and Heinrich Wansing, editors. *What is negation?*, volume 13 of *Applied Logic Series*. Kluwer Academic Publishers, Dordrecht, 1999.

[34] Rosella Gennari. Temporal reasoning and constraint programming: a survey. *CWI Quarterly*, 11(2-3):163–214, 1998.
[35] Alfonso Gerevini. Incremental qualitative temporal reasoning: algorithms for the point algebra and the ORD-Horn class. *Artificial Intelligence*, 166(1-2):37–80, 2005.
[36] Alfonso E. Gerevini and Alessandro Saetti. Computing the minimal relations in point-based qualitative temporal reasoning through metagraph closure. *Artificial Intelligence*, 175(2):556–585, 2011.
[37] Steven Givant. *Advanced topics in relation algebras—Relation Algebras. Vol. 2*. Springer, Cham, 2017.
[38] Steven Givant. *Introduction to relation algebras—Relation Algebras. Vol. 1*. Springer, Cham, 2017.
[39] Steven R. Givant. A portrait of Alfred Tarski. *Math. Intelligencer*, 13(3):16–32, 1991.
[40] Gabrielle Assunta Grün. An efficient algorithm for the uniform maximum distance problem on a chain. *Discrete Math. Theor. Comput. Sci.*, 4(2):323–350, 2001.
[41] Leon Henkin, J. Donald Monk, and Alfred Tarski. *Cylindric algebras. Part I*, volume 64 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1985. With an introductory chapter: General theory of algebras, Reprint of the 1971 original.
[42] Leon Henkin and Alfred Tarski. Cylindric algebras. In *Proc. Sympos. Pure Math., Vol. II*, pages 83–113. American Mathematical Society, Providence, R.I., 1961.
[43] Robin Hirsch and Ian Hodkinson. *Relation algebras by games*, volume 147 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 2002. With a foreword by Wilfrid Hodges.
[44] Robin Hirsch, Marcel Jackson, and Tomasz Kowalski. Algebraic foundations for qualitative calculi and networks. *Theoret. Comput. Sci.*, 768:99–116, 2019.
[45] Yoichi Iwata and Yuichi Yoshida. Exact and approximation algorithms for the maximum constraint satisfaction problem over the point algebra. In *30th International Symposium on Theoretical Aspects of Computer Science*, volume 20 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages 127–138. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2013.
[46] Bjarni Jónsson and Alfred Tarski. Representation problems for relation algebras. *Bulletin of the American Mathematical Society*, 54:80 and 1192, 1948. Abstract 89.
[47] Bjarni Jónsson and Alfred Tarski. Boolean algebras with operators. I. *Amer. J. Math.*, 73:891–939, 1951.
[48] Bjarni Jónsson and Alfred Tarski. Boolean algebras with operators. II. *Amer. J. Math.*, 74:127–162, 1952.
[49] Peter B. Ladkin and Roger D. Maddux. On binary constraint problems. *J. Assoc. Comput. Mach.*, 41(3):435–469, 1994.
[50] Roger C. Lyndon. The representation of relational algebras. *Ann. of Math. (2)*, 51:707–729, 1950.
[51] Roger C. Lyndon. The representation of relation algebras. II. *Ann. of Math. (2)*, 63:294–307, 1956.
[52] Roger D. Maddux. Some varieties containing relation algebras. *Trans. Amer. Math. Soc.*, 272(2):501–526, 1982.
[53] Roger D. Maddux. *Relation Algebras*, volume 150 of *Studies in Logic and the Foundations of Mathematics*. Elsevier B. V., Amsterdam, 2006.
[54] Roger D. Maddux. Relevance logic and the calculus of relations (abstract). In *International Conference on Order, Algebra, and Logics, Vanderbilt University, June 13, 2007*, pages 1–3, 2007.
[55] Roger D. Maddux. Relevance logic and the calculus of relations. *Rev. Symb. Log.*, 3(1):41–70, 2010.
[56] Ralph Nelson Whitfield McKenzie. *THE REPRESENTATION OF RELATION ALGEBRAS*. ProQuest LLC, Ann Arbor, MI, 1966. Thesis (Ph.D.)–University of Colorado at Boulder.
[57] J. C. C. McKinsey. Postulates for the calculus of binary relations. *J. Symbolic Logic*, 5:85–97, 1940.
[58] J. C. C. McKinsey. A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology. *J. Symbolic Logic*, 6:117–134, 1941.

[59] Robert K. Meyer and Richard Routley. Algebraic analysis of entailment. I. *Logique et Analyse (N.S.)*, 15:407–428, 1972.
[60] Robert K. Meyer and Richard Routley. Classical relevant logics. I. *Studia Logica*, 32:51–68, 1973.
[61] Robert K. Meyer and Richard Routley. Classical relevant logics. II. *Studia Logica*, 33:183–194, 1974.
[62] Szabolcs Mikulás. Algebras of relations and relevance logic. *J. Logic Comput.*, 19(2):305–321, 2009.
[63] I. Navarrete, A. Sattar, R. Wetprasit, and R. Marin. On point-duration networks for temporal reasoning. *Artificial Intelligence*, 140(1-2):39–70, 2002.
[64] R. Routley, V. Routley, R. K. Meyer, and E. P. Martin. On the philosophical bases of relevant logic semantics. *J. Non-Classical Logic*, 1(1):71–105, 1982.
[65] Richard Routley. Alternative semantics for quantified first degree relevant logic. *Studia Logica*, 38(2):211–231, 1979.
[66] Richard Routley and Robert K. Meyer. The semantics of entailment. I. In *Truth, syntax and modality (Proc. Conf. Alternative Semantics, Temple Univ., Philadelphia, Pa., 1970)*, volume 68 of *Studies in Logic and the Foundations of Mathematics*, pages 199–243. North-Holland, Amsterdam, 1973.
[67] Richard Routley, Val Plumwood, Robert K. Meyer, and Ross T. Brady. *Relevant logics and their rivals. Part I*. Ridgeview Publishing Co., Atascadero, CA, 1982. The basic philosophical and semantical theory.
[68] Eric Schechter. *Classical and nonclassical logics*. Princeton University Press, Princeton, NJ, 2005.
[69] Takeo Sugihara. Strict implication free from implicational paradoxes. *Memoirs of the Faculty of Liberal Arts, Fukei University*, Series I:55–59, 1955.
[70] Richard Sylvan, Robert Meyer, Val Plumwood, and Ross Brady. *Relevant logics and their rivals. Vol. II*, volume 59 of *Western Philosophy Series*. Ashgate Publishing Limited, Aldershot, 2003. A continuation of the work of Richard Sylvan, Robert Meyer, Val Plumwood and Ross Brady, Edited by Brady.
[71] Alfred Tarski. Grundzüge des Systemenkalküls. I. *Fundam. Math.*, 25:503–526, 1935.
[72] Alfred Tarski. On the calculus of relations. *J. Symbolic Logic*, 6:73–89, 1941.
[73] Alfred Tarski. *Logic, semantics, metamathematics*. Hackett Publishing Co., Indianapolis, IN, second edition, 1983. Papers from 1923 to 1938, Translated by J. H. Woodger, Edited and with an introduction by John Corcoran.
[74] Johan van Benthem. Review: On when a semantics is not a semantics: Some reasons for disliking the Routley-Meyer semantics for relevance logic, by B. J. Copeland. *The Journal of Symbolic Logic*, 49(3 Sep., 1984):994–995.
[75] Johan van Benthem. *Language in action*. MIT Press, Cambridge, MA; copublished with North-Holland, Amsterdam, 1995. Categories, lambdas and dynamic logic, Revised reprint of the 1991 original.
[76] Johan van Benthem. *Exploring logical dynamics*. Studies in Logic, Language and Information. CSLI Publications, Stanford, CA; FoLLI: European Association for Logic, Language and Information, Amsterdam, 1996.
[77] Heinrich Wansing, editor. *Negation. A notion in focus. Proceedings volume of an interdisciplinary workshop held during the conference Analyomen 2 of the Gesellschaft für Analytische Philosophie (GAP), Leipzig, Germany, September 7–10, 1994*. Berlin: De Gruyter, 1996.

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